The AdS/CFT Correspondence

Ranveer Kumar Singh[a]

[a] *NHETC and Department of Physics and Astronomy, Rutgers University, 126 Frelinghuysen Rd., Piscataway NJ 08855, USA*

E-mail: ranveer.singh@rutgers.edu

Abstract: We review AdS/CFT correspondence and provide two examples. The first example is a scalar field coupled to graviton in AdS space and a conformal field theory on the boundary. The second example is the original example of the correspondence of Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory in 4 dimensions first put forward by Juan Maldacena.

Contents

1 Introduction

The Anti-DeSitter/Conformal field theory correspondence is a duality between a theory of quantum gravity in AdS space and a conformal field theory on the boundary of AdS space. This duality is an example of holography, that is equivalence of a theory in d dimensions with a theory in $d-1$ dimensions. The AdS/CFT correspondence originally developed by Maldacena in [7] posits the equivalence of Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills in 4 dimensions. This equivalence rests on the dual interpretation of certain dynamical objects in string theory called D-branes. This is an extremely powerful correspondence since as we will see, it related a strongly coupled theory ($\mathcal{N} = 4$ super Yang-Mills) to a weakly coupled theory (Type IIB string theory on $AdS_5 \times S^5$) and hence non perturbative calculations can be done using perturbative string theory. Although the correspondence has not been rigorously proven, substantial amount of evidence has been demonstrated in favour of the correspondence and hence the AdS/CFT correspndence remains the AdS/CFT conjecture.

In this short review, we discuss this correspndence. We start by discussing aspects of AdS spaces and conformal field theory. We provide a toy example of the correspondence using scalar fields coupled to gravity in AdS space and a conformal field theory in Minkowski space. Finally we discuss the original Maldacena's proposal of the correspondence.

2 The Anti-deSitter Space

We begin by defining the AdS space.

2.1 Definition of AdS_{d+1}

The easiest way to describe the AdS space is by embedding it in $d + 2$ dimensional Minkowski space. Let $\left(\mathbb{R}^{d,2}, \eta_{\mu\nu}\right)$ be the $d + 2$ dimensional Minkowski space with coordinates $(T_1, T_2, X_1, \ldots, X_d)$. The metric is

$$ds^2 = -dT_1^2 - dT_2^2 + \sum_{i=1}^{d} dX_i^2.$$

We now consider the following function $f \in C^\infty(\mathbb{R}^{d,2})$:

$$f\left(T_1, T_2, X_1, \ldots, X_n\right) = T_1^2 + T_2^2 - \sum_{i=1}^{d} X_i^2 - \ell^2.$$

where $\ell \neq 0$ is a constant real number. Consider the zero set of this function:

$$f^{-1}(\{0\}) := \{p \in \mathbb{R}^{d,2} \mid f(p) = 0\}.$$

The Jacobian of this function is

$$\left[\frac{\partial f}{\partial T_1}\ \frac{\partial f}{\partial T_2}\quad \frac{\partial f}{\partial X_1} \cdots \frac{\partial f}{\partial X_n}\right] \equiv \left[\,2T_1\ 2T_2\ -2x_1\ \cdots\ -2x_n\,\right]$$

which is $\vec{0}$ only at $(0, 0, \ldots 0) \notin f^{-1}(\{0\})$. Thus $f^{-1}(\{0\})$ is a regular level set and by *Regular level set theorem*, $f^{-1}(\{0\})$ is a regular submanifold of $\mathbb{R}^{d,2}$ and dimension $d+2-1 = d-1$. The coordinates on $f^{-1}(\{0\})$ is given by $t, r, \theta_1, \ldots, \theta_{d-1}$ with the transformations given by

$$\begin{aligned}
&T_1 = \sqrt{r^2+\ell^2}\cos t, \quad T_2 = \sqrt{r^2+\ell^2}\sin t \\
&X_1 = r\cos\theta_1, \quad X_2 = r\sin\theta_1\cos\theta_2, \quad X_3 = r\sin\theta_1\sin\theta_2\cos\theta_3, \ldots
\end{aligned}$$

Clearly

$$\vec{X}^2 = \sum_i X_1^2 = r^2 \text{ and } \quad T_1^2 + T_2^2 = r^2 + \ell^2.$$

So these coordinates automatically satisfy the constraint and we only have $d+1$ coordinates to describe $f^{-1}(\{0\})$ as is appropriate for a $d+1$ dimensional space. We now induce the metric from the ambient Minkowski space to the submanifold $f^{-1}(\{0\})$ by simply substituting the above transformations:

$$\begin{aligned}
dT_1 &= \frac{r}{\sqrt{r^2+\ell^2}} dr \cos t - \sqrt{r^2+\ell^2} \sin t dt \\
dT_2 &= \frac{r}{\sqrt{r^2+\ell^2}} dr \sin t + \sqrt{r^2+\ell^2} \cos t dt \\
d\vec{X}^2 &= dr^2 + r^2 d\Omega_{d-1}^2
\end{aligned}$$

where $d\Omega_{d-1}^2$ is the metric on S^{d-1}. This is obvious from the fact that the transformation $X_i = X_i\,(r, \theta_1, \theta_2, \ldots, \theta_n)$ is simply the generalised spherical polar coordinate transformation. For example

$$\begin{aligned}
d &= 2,\ d\Omega_1^2 = d\theta \\
d &= 3,\ d\Omega_2^2 = d\theta^2 + \sin\theta d\phi^2
\end{aligned}$$

and so on. Thus the metric on $f^{-1}(\{0\})$ is given by

$$\begin{aligned}
ds^2 =& -\left(\frac{r}{\sqrt{r^2+R^2}} dr \cos t - \sqrt{r^2+\ell^2} \sin t dt\right)^2 - \left(\frac{r}{\sqrt{r^2+\ell^2}} dr \sin t + \sqrt{r^2+\ell^2} \cos t dt\right)^2 \\
& \qquad + dr^2 + r^2 d\Omega_{d-1}^2 \\
=& -\frac{r^2}{r^2+\ell^2} dr^2 \cos^2 t - \left(r^2+\ell^2\right) \sin^2 t dt^2 + 2r dr \cos t \sin t dt \\
& - \frac{r^2}{r^2+\ell^2} dr^2 \sin^2 t - \left(r^2+\ell^2\right) \cos^2 t dt^2 - 2r dr \cos t \sin t dt + dr^2 + r^2 d\Omega_{d-1}^2 \\
=& -\left(r^2+\ell^2\right) dt^2 + dr^2 \left(1 - \frac{r^2}{r^2+R^2}\right) + r^2 d\Omega_{d-1}^2 \\
=& -\left(r^2+\ell^2\right) dt^2 + dr^2 \left(\frac{\ell^2}{r^2+\ell^2}\right) + r^2 d\Omega_{d-1}^2 \\
=& -\left(r^2+\ell^2\right) dt^2 + \frac{dr^2}{1+\frac{r^2}{\ell^2}} + r^2 d\Omega_{d-1}^2.
\end{aligned}$$

So the metric on $f^{-1}(\{0\})$ is given by

$$ds^2 = -\left(r^2+\ell^2\right) dt^2 + \frac{dr^2}{1+\frac{r^2}{\ell^2}} + r^2 d\Omega_{d-1}^2. \tag{2.1}$$

This is the *intrinsic metric* on $f^{-1}(\{0\})$ and the coordinates $(r, t, \theta, \ldots, \theta_{d-1})$ are called *global coordinates* of $f^{-1}(\{0\})$. Note that there is an identification of t and $t+2\pi$. So to avoid repitition, we must restrict t to $[0, 2\pi)$. Clearly this cannot represent physical time as this restriction means that we can reach the past by going to the future. Thus $f^{-1}(\{0\})$ naturally has *closed timelike curves.* This can be overcome by considering the universal cover of $f^{-1}(\{0\})$. The space parametrised by $(t, r, \theta_1, \ldots, \theta_{n-1})$ with $t \in \mathbb{R}$ with the metric in (2.1) is the universal cover of AdS_{d+1} and this space is called the $(d+1)$-dimensional anti-Desitter space AdS_{d+1}. We write it explicitly below:

The AdS_{d+1} space is defined by the metric

$$\begin{aligned} ds^2 &= -\left(r^2+\ell^2\right)dt^2 + \frac{dr^2}{1+\frac{r^2}{\ell^2}} + r^2 d\Omega_{d-1}^2 \\ r &\in [0,\infty), \quad t \in \mathbb{R}, \end{aligned} \tag{2.2}$$

where $d\Omega_{d-1}^2$ is the round metric on S^{d-1} and $\ell \neq 0$ is the AdS radius.

2.2 Symmetries

The Minkowski space $\mathbb{R}^{d,2}$ has the isometry group $\mathrm{SO}(d,2) \ltimes \mathbb{R}^{d,2}$. Infact it is one of the *maximally symmetric spaces.* The AdS_{d+1} is another maximally symmetric space with isometry group $\mathrm{SO}(d,2)$. If is clear that $f^{-1}(\{0\})$ also has isometry group $\mathrm{SO}(d,2)$ since f remains invariant under such a transformation. But this not clear from the metric on $f^{-1}(\{0\})$. On taking the universal cover, the isometry group also becomes the cover of $\mathrm{SO}(d,2)$. Also AdS_{d+1} is one of the three *homogenous and isotropic spaces.* By homogenous, we mean that the isometry group acts transitively on the space and by isotropic we mean that the isometry group acts on the tangent space transitively fixing the point of tangency.

2.3 Geometry of AdS_{d+1}

To understand the geometry of AdS_{d+1}, we make further change of coordinates. Introduce $r = R\tan\rho$ with $r \in [0,\infty)$ so that $\rho \in [0,\pi/2)$. Then the metric becomes

$$ds^2 = \ell^2 \sec^2\rho \left(-dt^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2\right)$$

- Near $\rho = 0$, $\sec\rho \approx 1, \sin^2\rho \approx \rho^2$ so

  $$ds^2 \simeq \ell^2(-dt^2 + \underbrace{d\rho^2 + \rho^2 d\Omega_{d-1}^2)}_{\text{flat metric}}$$

  So near $\rho = 0$ (origin) the AdS metric is flat. Also note that ρ has finite range.

- Near $\rho = \frac{\pi}{2}$

  $$\sin\rho \approx 1, \quad \sec\rho \approx \frac{1}{\frac{\pi}{2}-\rho}.$$

  The metric becomes

  $$ds^2 \simeq \frac{\ell^2}{\left(\frac{\pi}{2}-\rho\right)^2}\left(-dt^2 + d\rho^2 + d\Omega_{d-1}^2\right).$$

  Upto the divergent scaling factor, the metric in (t,ρ) is the $1+1$ Minkowski metric along with the $(d-1)$-dimensional round metric.

With these two information, we can represent AdS_{d-1} as a cylinder shown below.

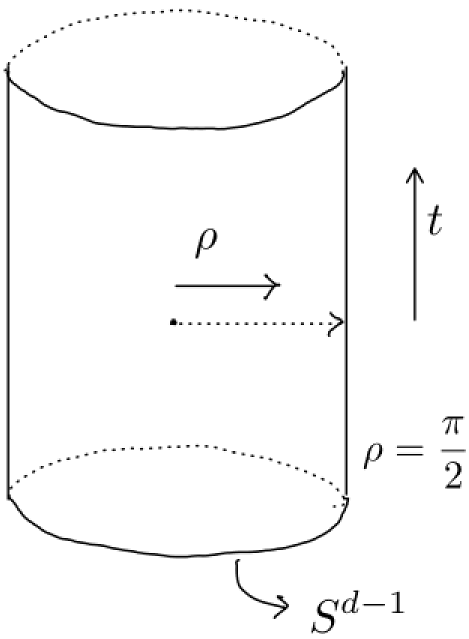

Figure 1. AdS_{d+1} as a cylinder

Note that although the range of ρ is finite, the radial distance from the origin to the boundary is

$$\ell \int_0^{\pi/2} \sec\rho \, d\rho = \infty.$$

But it turns out that although the boundary is infinitely far away from the origin, the AdS space behaves as a finite box. To see this, consider a radial null geodesic in AdS space. Null geodesics satisfy $ds^2 = 0$. Also since we are considering radial null geodesics $d\Omega^2_{d-1} = 0$. Thus $ds^2 = 0$ implies

$$dt = \pm d\rho.$$

Thus light ray travels at 45°. Now suppose we send a light ray from origin towards boundary and place a reflector at the boundary, so that the light ray bounces back and comes to origin again. From Fig. 2 below it is clear that, the light ray returns back after time π (dimensionless). Thus although the radial distance from origin to boundary is ∞, AdS behaves as a finite box.

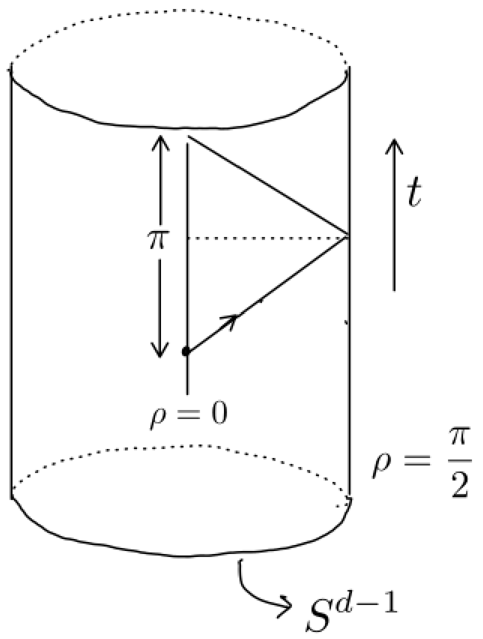

Figure 2. Null geodesics in AdS_{d+1}

The boundary of AdS_{d+1} is topologically $\mathbb{R} \times S^{d-1}$. We change coordinates so the boundary looks flat. This is done by introducing the *Poincaré coordinates*. Introduce the coordinates $y > 0$ and $(t, \vec{x}) \in \mathbb{R}^{d-1}$ via:

$$\begin{aligned} X_0 &= \frac{1}{2y}\left(1 + y^2\left(\ell^2 + \vec{x}^2 - t^2\right)\right) \\ X_d &= \ell y t \\ X_{d-1} &= \frac{1}{2y}\left(1 - y^2\left(\ell^2 - \vec{x}^2 + t^2\right)\right) \\ X_i &= \ell y x_i \end{aligned}$$

where $(i = 1 \ldots d-2)$ and $\vec{x}^2 = \sum_{i=1}^{d-2} x_i^2$. One then easily checks that in this coordinate, the AdS metric takes the form

$$ds^2 = \frac{\ell^2}{y^2} dy^2 + \frac{y^2}{\ell^2} \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.3}$$

where $x^\mu = (t, \vec{x})$. Substitution $u = \ell^2/y$ we have $dy^2 = \ell^4/u^4 du^2$ and $\ell^2/y^2 = u^2/\ell^2$ and we get

$$ds^2 = \frac{\ell^2}{u^2}\left(du^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right)$$

which is the metric in Poincaré coordinates. It is now clear that the AdS metric is conformally equivalent to the Minkowski metric.

2.4 Dynamincs

The AdS_{d+1} metric $g_{\mu\nu}$ satisfies Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.4}$$

where

$$T_{\mu\nu} = -\frac{d(d-1)}{16\pi G R^2} g_{\mu\nu}. \quad (d \geq 2).$$

The constant

$$\Lambda = -\frac{d(d-1)}{16\pi G \ell^2}$$

is called the *cosmological constant*. Indeed, the Einstein equation (2.4) can be obtained from the action

$$S_{AdS} = \frac{1}{16\pi G} \int d^d x \sqrt{-g}(R + \Lambda).$$

So AdS is a negative cosmological constant solution to Einstein's equation.

2.5 Quantisation of Scalar field on AdS space

Consider a scalar field $\phi : AdS_{d+1} \longrightarrow \mathbb{R}$. The corresponding action is

$$S = -\frac{1}{2} \int d^{d+1}x \ \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2.$$

The equation of motion for ϕ is

$$\Box \phi - m^2 \phi = 0$$

where

$$\Box \phi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right).$$

Using the AdS metric (2.1), we get

$$\Box \phi = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left[r^{d-1} \frac{\partial \phi}{\partial r} \left(1 + \frac{r^2}{\ell^2} \right) \right] + \frac{1}{r^2} \nabla^2_{S^{d-1}} \phi - \frac{1}{r^2 + \ell^2} \partial_t^2 \phi - m^2 \phi = 0,$$

where $\nabla^2_{S^{d-1}}$ is the Laplacian over S^{d-1}. For example, for $d = 3$

$$\nabla^2_{S^2} := \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

Inductively, one can define $\nabla^2_{S^{d-1}}$ as:

$$\nabla^2_{S^{d-1}} = \frac{1}{(\sin\theta_1)^{d-2}} \frac{\partial}{\partial\theta} \left((\sin\theta)^{d-2} \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \nabla^2_{S^{d-2}}$$

We now want to find the general solution to the equation of motion. To find the general solution, we use seperation of variables. To do this, we look at symmetries. Since the metric has manifest SO(d) symmetry, one factor of the solution is the solution to

$$\nabla^2_{S^{d-1}} Y_{l,\vec{m}}(\vec{\theta}) = -l(l + d - 2) Y_{l,\vec{m}}(\vec{\theta}).$$

where $Y_{l,\vec{m}}(\vec{\theta})$ are the spherical harmonics in d dimensions. The time translation symmetry asks for one factor of $e^{-i\omega t}$. We keep r-dependence explicit. So the trial solution is

$$\phi(r,t,\vec{\theta}) = e^{-i\omega t} Y_{l,\vec{m}}(\vec{\theta})\psi(r) \tag{2.5}$$

Plugging in this trial solution, we get

$$\left(1+\frac{r^2}{l^2}\right)\psi'' + \left[\frac{d-1}{r}\left(1+\frac{r^2}{l^2}\right)+\frac{2r}{l^2}\right]\psi' + \left[\frac{\omega^2}{r^2+l^2} - \frac{l(l+d-2)}{r^2} - m^2\right]\psi = 0.$$

Near $r=0$, the above equation reduces to

$$\psi'' + \frac{d-1}{r}\psi' - \frac{l(l+d-2)}{r^2}\psi = 0$$

Using the ansatz $\psi = cr^{\alpha}$, we get

$$\alpha(\alpha-1) + (d-1)\alpha - l(l+d-2) = 0$$

whose solutions are $\alpha = l, -l-d+2$. Note that $\alpha = -\ell - d + 2$ gives singular solution near $r=0$ and hence we discard it.

Near $r=\infty$, the above equation reduces to

$$\frac{r^2}{l^2}\psi'' + \frac{(d+1)}{l^2}r\psi' - m^2\psi = 0$$

Using the ansatz $\psi(r) = kr^{\beta}$ we get

$$\beta(\beta-1)\frac{1}{l^2} + \frac{(d+1)\beta}{l^2} - m^2 = 0$$

whase solutions are

$$\beta = -\frac{d}{2} \pm \frac{1}{2}\sqrt{d^2+4m^2l^2}.$$

Note that if we choose sign, $\beta > 0$ and hence the solution diverges. So the acceptable solution is

$$-\Delta \equiv \beta = -\frac{d}{2} - \frac{1}{2}\sqrt{d^2+4m^2l^2}.$$

Since the differential equation for ψ is a linear second order ordinary differential equation, it has two linearly independent solutions say $f_+$and f_-. Then ignoring boundary conditions

$$\psi(r) = Af_+(r) + Bf_-(r)$$

Boundary condition at $r=0$ fixes $\frac{A}{B}$ and boundary condition at $r=\infty$ also fixes $\frac{A}{B}$. We now need to make sure that these two conditions match. This is done using adjusting ω. The final result turns out to quantise to be a discrete quantity. To be precise ω is restricted to

$$\omega_{nl} = \Delta + l + 2n, \quad n \in \mathbb{N}\cup\{0\}.$$

The solution for ψ is then

$$\psi_{nl}(r) = C_{nl}(\sin\rho)^l(\cos\rho)^\Delta P_n^{\frac{d-1}{2},\Delta-\frac{d}{2}}(\cos 2\rho) \tag{2.6}$$

where as before $r = \tan\rho$, $C_{n,l}$ are constants and $P_n^{(\alpha,\beta)}(z)$ are the *Jacobi Polynomials* given by

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)\alpha\cdots(\alpha-n+2)}{n!}F_1\left(-n, 1+\alpha+\beta+n, \alpha+1, \frac{1}{2}(1-z)\right)$$

and ${}_2F_1$ are the Hypergeometric functions. The solution to the equation of motion is thus given by

$$f_{nl\vec{m}}(t,r,\vec{\theta}) = \psi_{nl}(r)Y_{l\vec{m}}(\vec{\theta})e^{-i\omega_{nl}t}$$

and the general solution is

$$\phi(t,r,\vec{\theta}) = \sum_{n,l,\vec{m}}\left[a_{nl\vec{m}}f_{nl\vec{m}}(t,r,\vec{\theta}) + a^\dagger_{nl\vec{m}}f^*_{nl\vec{m}}(t,r,\vec{\theta})\right].$$

We now define an inner product on the space of classical solutions. For any two solutions g, f to $\Box\phi - m^2\phi = 0$, define

$$\langle g, f\rangle = i\int_\Sigma d^dx\sqrt{\det\gamma}\; n^\mu\left(g^*\partial_\mu f - f\partial_\mu g^*\right)$$

Here Σ is a spacelike slice $t = t_0, \gamma^{\alpha\beta}$ is the metric on Σ, n^μ is the normal to Σ, that is $n^0 \neq 0$ and $n^i = 0$ which is also normalised such that that is

$$g_{\mu\nu}n^\mu n^\nu = -1 \implies g_{00}\left(n^0\right)^2 = -1$$

that is $n^0 = \sqrt{-g^{00}}$. The inner product then becomes

$$\langle g, f\rangle = i\int_\Sigma d^dx\sqrt{\det\gamma}\;\frac{1}{\sqrt{g_{00}}}\left(g^*\partial_t f - f\partial_t g^*\right). \tag{2.7}$$

One can show that $\langle g, f\rangle$ is independent of t_0 and Σ. With this inner product, the constants C_{nl} in (2.6) can be chosen so that [2]

$$\langle f_{nl\vec{m}}, f_{n'l'\vec{m}'}\rangle = \delta_{nn'}\delta_{ll'}\delta_{\vec{m}\vec{m}'}.$$

We now perform cannonical quantisation by imposing

$$[\phi(t,\vec{x}), \Pi(t,\vec{y})] = \delta^{(d)}(\vec{x}-\vec{y}),$$

where

$$\Pi(t,\vec{y}) = \frac{\partial\mathcal{L}}{\partial\left(\partial_t\phi(t,\vec{y})\right)}$$

This gives us the commutator of oscillators

$$\left[a_{nl\vec{m}}, a^\dagger_{n'l'\vec{m}'}\right] = \delta_{nn'}\delta_{ll'}\delta_{\vec{m}'\vec{m}'}$$

and other commutators are trivial. Now to define ground state, we need a time coordinate since the ground state is defined to be the lowest eigenvalue state of the time translation symmetry generator (the Hamiltonian). We already have a time coordinate and since $a_{nl\vec{m}}$ is the coefficient of $e^{i\omega_{nl}t}$, it annihilates the vacuum and $a^\dagger_{nl\vec{m}}$ creates states. Thus

$$a_{nl\vec{m}}|0\rangle = 0$$

and $a^\dagger_{nl\vec{m}}|0\rangle$ is a single particle state with energy ω_{nl}. The state $a^\dagger_{nl\vec{m}}a^\dagger_{n'l'm'}|0\rangle$ is a two particle state with energy $\omega_{nl} + \omega_{n'l'}$.

Remark 2.1. Although we are calling above states as one, two or multiparticle states but it is not so since for particle interpretation we need continuous momentum and energy which is not available here.

Other excited states can be written as

$$\prod_{i=1}^{k} a^\dagger_{n_i l_i \vec{m}_i}|0\rangle$$

with energy

$$\omega_k = \sum_{i=1}^{k} (\Delta + 2n_i + l_i)\,.$$

Note that the generators of manifest SO(d) symmetry and time translation symmetry are Noether charges of these symmetries which are quadratic in oscillators a and $a^\dagger$. After normal ordering, these generators annihilate the vacuum. The full SO($d,2$) invariance of the vacuum is more involved to prove.
Interactions can be treated perturbatively if the coupling constants are small. There are UV divergences but we will consider these theories as embedded in some string theory which is UV finite due to a cutoff.
Other fields like fermions, $A_\mu, g_{\mu\nu}$ can be treated similarly. In particular for A_μ and $g_{\mu\nu}$ one has to take care of gauge freedom and diffeomorphism invariance. For metric, we will usually take

$$g_{\mu\nu} \approx g^{\text{AdS}}_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ is a small perturbation around background metric $g^{\text{AdS}}_{\mu\nu}$. To do this we expand the Einstein-Hilbert action in $h_{\mu\nu}$ and treat the nonlinear terms as interactions. Later we will deal with black holes in AdS. We now list the observables in the quantum theory. First class of observables are the energies. Others are correlation functions.

$$\left\langle \prod_i \phi_i(x_i) \right\rangle \equiv \left\langle 0 \left| \prod_i \phi_i \left(t, r_i, \vec{\theta}_i\right) \right| 0 \right\rangle$$

These can be calculated in terms of $f^i_{nl\vec{m}}$ and f^{i*}_{nlm}. These are also restricted by SO($d,2$) invariance. We now face a problem. The correlation functions is a function of x_i. In absence of gravity, one can use distances and angles as arguments of functions which are invariant under general coordinate transformation. But in a quantum theory of gravity, the distances and angles do not have an invariant meaning. So in a quantum theory of gravity, correlation functions are not good observables. But we can construct observables using correlation functions. Quantisation does not allow metric to fluctuate near the boundary. The fluctuations $h_{\mu\nu} \to 0$ near boundary. So if we take $r \to \infty$ in the correlation functions, then we get exact observables. So the observables are

$$\lim_{r\to\infty} \prod_{i=1}^{k} r^{\Delta_i} \left\langle \prod_i \phi_i\left(t_i, r, \vec{\theta}_i\right) \right\rangle$$

where we attached r^{Δ_i} to get a finite number since

$$\phi(r, t, \vec{\theta}_i) \longrightarrow r^{-\Delta} f(t, \vec{\theta}) \quad \text{as} \quad r \to \infty.$$

Note that the observables above are functions of $t_i, \vec{\theta}_i$ where $\vec{\theta}_i$ are coordinates on S^{d-1} and $t_i \in \mathbb{R}$. Thus we have observables on the boundary $\mathbb{R} \times S^{d-1}$ of AdS_{d+1}. One can show that [2]

$$\lim_{r\to\infty} r^{2\Delta} \left\langle \phi(r, t, \vec{\theta}) \phi\left(r, t', \vec{\theta}'\right) \right\rangle \propto \left(\frac{1}{\cos\left(t - t'\right) - \cos\alpha} \right)^{\Delta} \tag{2.8}$$

where α is the geodesic distance between $\vec{\theta}$ and and $\vec{\theta}'$ on S^{d-1}. But there is another issue. Under a general SO$(d, 2)$ transformation

$$r \to F(t, \vec{\theta}) r$$

where $F(t, \vec{\theta})$ is some function. Thus the boundary correlation functions transform as

$$\lim_{r\to\infty} \left\langle \prod_i r^{\Delta_i} \phi_i\left(x_i\right) \right\rangle \xrightarrow{r\to\infty} \left(\prod_i F(t, \vec{\theta})^{\Delta_i} \right) \lim_{r\to\infty} \left\langle \prod_i r^{\Delta_i} \phi_i\left(x_i\right) \right\rangle.$$

We will discuss the implications of this later.

3 Conformal Field Theory

We will consider CFT in fixed background geometry without gravity. The background metric is not necessarily flat.

3.1 General Aspects of CFT

CFT is an ordinary QFT with the property that

$$T_{\mu\nu}(x) g^{\mu\nu}(x) = 0.$$

where $T_{\mu\nu}$ is the stress-energy tensor. That is the stress tensor is traceless. This is an operator statement which means this is required inside a correlation function. In a simplistic view of a CFT, suppose that the tracelessness of the stress tensor holds without the use of equation of motion. The stress tensor is given by

$$T_{\mu\nu} \propto \frac{\delta S}{\delta g^{\mu\nu}}$$

where S is the action of the theory. Suppose

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu} = e^{2\Omega(x)} g_{\mu\nu}$$

where $\Omega(x)$ is an infinitesimal function. Then it is easy to see that $g^{\mu\nu}T_{\mu\nu} = 0$ is equivalent to the action S being invariant under $g_{\mu\nu}(x) \longrightarrow e^{2\Omega(x)} g_{\mu\nu}$.
The correlation functions are invariant under conformal transformations

$$\left\langle \prod_i \mathcal{O}_i \left(x_i\right) \right\rangle_{g_{\mu\nu}} = \left\langle \prod_i \mathcal{O}_i \left(x_i\right) \right\rangle_{e^{2\Omega(x)} g_{\mu\nu}}$$

But in general, the stress tensor is not traceless at quantum level. Several subtle issues may arise. For one example, suppose $T_{\mu\nu}g^{\mu\nu} = 0$ after using equation of motion. Then the trace must be proportional to the equations of motion

$$T_{\mu\nu} g^{\mu\nu} = \sum_i C_i \left(\{\phi_i\}\right) \frac{\delta S}{\delta \phi_i}$$

where ϕ_i are the fields of the theory. Note that under

$$\phi_i \longrightarrow \phi_i + N C_i \left(\{\phi_i\}\right)$$

where N is some constant,

$$\begin{aligned} \delta S &= \sum_i \frac{\delta S}{\delta \phi_i} \delta \phi_i = N \sum_i c_i \left(\{\phi_i\}\right) \frac{\delta S}{\delta \phi_i} \\ &= N T_{\mu\nu} g^{\mu\nu}. \end{aligned}$$

Thus upon using equations of motion, S is invariant under

$$g_{\mu\nu} \to e^{2\Omega(x)} g_{\mu\nu} \text{ and } \phi_i \longrightarrow \phi_i + N C_i \left(\{\phi_i\}\right)$$

Thus in general

$$\left\langle \prod_i \mathcal{O}'_i \left(x_i\right) \right\rangle_{e^{2\Omega(x)} g_{\mu\nu}} = \left\langle \prod_i \mathcal{O}_i \left(x_i\right) \right\rangle_{g_{\mu\nu}}$$

where $\mathcal{O}'_i$ are transformed operator under $g_{\mu\nu} \to e^{2\Omega(x)} g_{\mu\nu}$. So the operators need to be transformed. Another issue is $T_{\mu\nu}g^{\mu\nu} = 0$ may hold only in flat background. This is called the *conformal anomaly.* In general

$$T_{\mu\nu} g^{\mu\nu} \propto \text{ terms involving Riemann tensor.}$$

For example in 2d, it is proportional to Ricci scalar. In general

$$\left\langle \prod_i \mathcal{O}'_i \left(x_i\right) \right\rangle_{e^{2\Omega(x)} g_{\mu\nu}} = \left\langle \prod_i \mathcal{O}_i \left(x_i\right) \right\rangle_{g_{\mu\nu}} \exp[A(g, \Omega)]$$

where the function A depends on terms appearing in $T_{\mu\nu}g^{\mu\nu}$. There are some special kind of operators called *conformal primaries*. Operators $\mathcal{O}_i$ are called conformal primary operators of conformal dimension Δ_i if

$$\left\langle \prod_i \mathcal{O}_i\left(x_i\right) \right\rangle_{e^{2\Omega(x)}g_{\mu\nu}} = \exp\left(-\sum_i \Delta_i \Omega(x_i)\right) \left\langle \prod_i \mathcal{O}_i\left(x_i\right) \right\rangle_{g_{\mu\nu}} . \tag{3.1}$$

Other operators which do not transform in this specific way are called *secondary operators*. Another important feature of a CFT is the *state-operator correspondence*. The state-operator correspondence asserts that there is a one-to-one correspondence between states of the theory and local operators. Thus we can use the operator algebra of the theory or the Hilbert space interchangeably. This will have implications in the AdS/CFT correspondence. In general, a classical field theory is conformally invariant if the action has no dimensionful couplings, since dimensionful couplings set a scale in the theory which breaks the scale invariance and hence breaking the conformal invariance. At the quantum level, things are more complicated. Due to loop corrections, the conformal invariance may be broken and the stress tensor may not remain traceless at the quantum level as discussed above. Indeed, for a theory to be conformally invariant, a necessary condition is the vanishing of the *renormalisation group beta functions* for every coupling g:

$$\beta_g(\mu_s) = \mu_s \frac{\partial g}{\partial \mu_s}.$$

3.2 CFT with flat Minkowski background

(3.1) is a relation between QFT in two different backgrounds. We want to see if we can get relations in same background. To proceed, we start with the flat metric $g_{\mu\nu} = \eta_{\mu\nu}$. Under a general coordinate transformation

$$\begin{aligned} x^\mu &\longrightarrow f^\mu\left(x'\right) \\ \eta_{\mu\nu} &\longrightarrow \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\eta_{\alpha\beta} =: g'_{\mu\nu} \end{aligned}$$

The correlators change as

$$\left\langle \prod_i \mathcal{O}_i\left(x_i\right) \right\rangle_{\eta_{\mu\nu}} = \left\langle \prod_i \mathcal{O}_i\left(x'_i\right) \right\rangle_{g'_{\mu\nu}} \tag{3.2}$$

where we are considering scalar operators $\mathcal{O}_i$ so that they do not change. Now under conformal transformations

$$g'_{\mu\nu} = e^{2\Omega(x)}\eta_{\mu\nu}$$

R.H.S of (3.2) becomes

$$\left\langle \prod_i O_i\left(x'_i\right) \right\rangle_{e^{2\Omega(x)}\eta_{\mu\nu}} .$$

If in addition, the theory is a CFT,

$$\left\langle \prod_i \mathcal{O}_i\left(x_i'\right)\right\rangle_{e^{2\Omega(x)}\eta_{\mu\nu}} = \exp\left(-\sum_i \Delta_i\Omega(x_i)\right)\left\langle \prod_i \mathcal{O}_i\left(x_i'\right)\right\rangle_{\eta_{\mu\nu}}.$$

For flat space, the independent conformal transformations are

1. Translations: $x^\mu \longrightarrow x^\mu + a^\mu$
2. Lorentz transformation: $x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$
3. Scaling: $x^\mu \longrightarrow \lambda x^\mu$
4. Special Conformal transforamtions:

$$x^\mu \longrightarrow \frac{x^\mu + a^\mu x^2}{1 + 2a\cdot x + a^2x^2} \tag{3.3}$$

Also the algebra is $\mathfrak{so}(d,2)$. Combining (3.2) and (3.1), we get

$$\left\langle \prod_i \mathcal{O}_i\left(x_i'\right)\right\rangle_{\eta_{\mu\nu}} = \exp\left(\sum_i \Delta_i\Omega\left(x_i'\right)\right)\left\langle \prod_i \mathcal{O}_i\left(x_i\right)\right\rangle_{\eta_{\mu v}} \tag{3.4}$$

For example, consider $x'^\mu = \lambda x^\mu$ for some constant λ. Then

$$g_{\mu\nu} = \lambda^{-2}\eta_{\mu\nu}$$

Thus (3.4) gives

$$\left\langle \prod_i \mathcal{O}_i\left(\lambda x_i\right)\right\rangle_{\eta_{\mu\nu}} = \lambda^{-\sum_i \Delta_i}\left\langle \prod_i \mathcal{O}_i\left(x_i\right)\right\rangle_{\eta_{\mu\nu}} \tag{3.5}$$

This fixes the two point function up to a constant. Poincaré invariance says that

$$\left\langle \mathcal{O}\left(x_1\right)\mathcal{O}\left(x_2\right)\right\rangle = f\left(\left(x_1 - x_2\right)^2\right) \tag{3.6}$$

where f is some function. Then (3.5) gives

$$f\left(\lambda^2\left(x_1 - x_2\right)^2\right) = \lambda^{-2\Delta} f\left(\left(x_1 - x_2\right)^2\right).$$

This implies

$$f\left(\left(x_1 - x_2\right)^2\right) = \frac{C}{\left(x_1 - x_2\right)^{2\Delta}}$$

where C is some constant.

Remark 3.1. Although we did not check the tranformation of correlation function under special conformal transformation, one can show that SCT constrains the conformal weights of the two operators in the 2-point function to be same.

4 The $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ Correspondence

Before we rigorously state the AdS/CFT conjecture, let us explore the examples we considered in Subsections 2.5 and 3.2, namely a scalar field with dynamical gravity in AdS_{d+1} space and a CFT with flat Minkowski space background.

4.1 Scalar Field in AdS_{d+1} space and CFT_d in flat Minkowski space

We start with the CFT correlator (3.6). One can make a Wick rotation $t_1 \to ix_1^0, t_2 \to ix_2^0$ to get

$$(x_1 - x_2)^2 = \left(x_1^0 - x_2^0\right)^2 + (\vec{x}_1 - \vec{x}_2)^2$$

since the Minkowski metric changes to Euclidean metric upon Wick rotation. The 2-point correlator has the form

$$\langle O(x_1)\, O(x_2)\rangle = \frac{c}{\left[\left(x_1^0 - x_2^0\right)^2 + (\vec{x}_1 - \vec{x}_2)^2\right]^{\Delta}}$$

We change to polar coordinates

$$\begin{aligned} x^0 &= \rho \cos\theta^1 \\ x^1 &= \rho \sin\theta^1 \cos\theta^2 \\ x^2 &= \rho \sin\theta^1 \sin\theta^2 \cos\theta^3 \end{aligned}$$

and so on. The metric is

$$ds^2 = d\rho^2 + \rho^2 d\Omega_{d-1}^2$$

We use the $SO(d)$ rotation symmetry of the metric to set

$$\begin{aligned} x_1 &= (\rho_1, 0, 0, \ldots 0) \\ x_2 &= \left(\rho_2, \theta_2^1, 0, \ldots 0\right) \end{aligned}$$

Then

$$\begin{aligned} (x_1 - x_2)^2 &= \left(\rho_1 - \rho_2 \cos\theta_2^1\right)^2 + \left(\rho_2 \sin\theta_2^1\right)^2 \\ &= \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\theta_2^1. \end{aligned}$$

Note that θ_2^1 can be interpreted as the geodesic seperation of x_1 and x_2 on S^{d-1} (see Figure 3 below). The 2-point correlator becomes

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\rangle = \frac{C}{\left(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\theta_2^1\right)^{\Delta}}$$

We now change coordinates to

$$\rho = e^{\tau}$$

keeping other variables same. Then

$$ds^2 = e^{2\tau} \underbrace{\left(d\tau^2 + d\Omega_{d-1}^2\right)}_{\text{metric on } \mathbb{R}\times S^{d-1}}$$

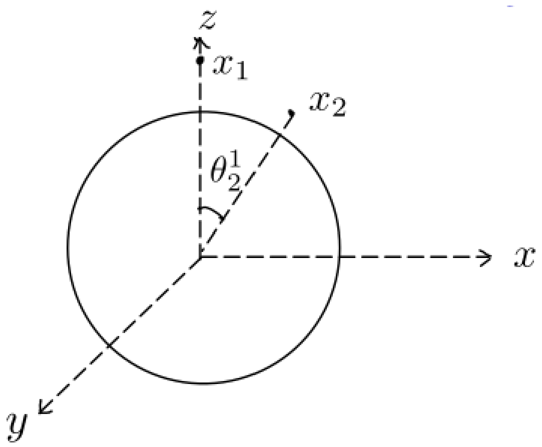

Figure 3. θ_2^1 as the geodesic distance between x^1 and x^2 on S^{d-1}

By (3.2) we have

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle_{(\mathbb{R}\times S^{d-1})e^{2\tau}} = \frac{C}{\left(e^{2\tau_1}+e^{2\tau_2}-2e^{\tau_1+\tau_2}\cos\theta_2^1\right)^{\Delta}}$$

Then using (3.4) we get

$$\begin{aligned}\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle_{\mathbb{R}\times S^{d-1}} &= \frac{Ce^{(\tau_1+\tau_2)\Delta}}{\left(e^{2\tau_1}+e^{2\tau_2}-2e^{\tau_1+\tau_2}\cos\theta_2^1\right)^{\Delta}}\\ &= \frac{C}{2^{\Delta}\left(\cosh(\tau_1-\tau_2)-\cos\theta_2^1\right)^{\Delta}}\end{aligned}$$

We now again analytically continue to complex τ and evaluate at $\tau = it$ to get to Minkowski metric. We get

$$\langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\rangle_{\mathbb{R}\times S^{d-1}} = \frac{C'}{\left(\cos(t_1-t_2)-\cos\theta_2^1\right)^{\Delta}}$$

This is exactly the same as the boundary two point correlator in (2.8) of a scalar field in AdS space upto a constant. We make a few remarks on this observation.

Remark 4.1. We make a few remarks about this example.

1. Here we see the first glimpse of AdS/CFT correspondence. Recall that the boundary of AdS_{d+1} has the topology of $\mathbb{R}\times S^{d-1}$ and hence we say that *quantum theory with gravity in the bulk of* AdS_{d+1} *is "equivalent" to a CFT without gravity on the boundary.* We will make this precise later.

2. Clearly, by equivalence we mean that all correlators in the two theories match. Usually one lists all boundary correlators of quantum gravity in AdS and conjectures that it constitutes a CFT on the boundary. Also note that the Δ and coefficients in correlator have to match in the two theories.

3. Roughly, we see that scalar primary operators $\mathcal{O}(t,\vec{x})$ is related to scalar field $\phi(t,r,\vec{\theta})$ as
$$\mathcal{O}(t,\vec{x}) = \lim_{r\to\infty} r^{\Delta}\phi(t,r,\vec{\theta})$$
Also since every CFT has a stress tensor $T_{\mu\nu}$ and every quantum gravity has a graviton $g_{\mu\nu}$, thus
$$T_{\mu\nu} \longleftrightarrow g_{\mu\nu}.$$

4. AdS/CFT correspondence also conjectures that the Hilbert space of quantum gravity + matter on AdS space is isomorphic to the Hilbert space of the CFT on the boundary. Thus in continuation to above point, by state-operator correspondence in CFT, all local operators in the CFT correspond to some state in the AdS theory. In the particular example of scalar field in AdS, the single particle states
$$a^{\dagger}_{nl\vec{m}}|0\rangle, \quad \text{with energy } \Delta + l + 2n$$
corresponds to the secondary operators
$$\lim_{r\to\infty} \partial_{\mu_1}\cdots\partial_{\mu_k} r^{\Delta}\phi(r,t,\vec{\theta}),$$
where the derivatives are along t and $\vec{\theta}$ and $k = l + 2n$. The permutations of these derivatives gives all single particle states in AdS with $l + 2n = k$.

5. The double particle states
$$a^{\dagger}_{nl\vec{m}}a^{\dagger}_{n'l'\vec{m}'}|0\rangle$$
for a free scalar field in AdS has energy $\Delta + l + 2n + \Delta + l' + 2n'$. But note that this relation is only true in free scalar field theory in AdS, once we add interactions, the this does not hold true. But physically, if we take large radius limit of AdS taking ℓ to be large, this relation becomes exact even in interacting theory. Thus we need to identify a parameter in the corresponding CFT which corresponds to this "large ℓ limit" of the AdS theory and then we can check if there are local operators in the CFT whose conformal dimension is $\Delta + l + 2n + \Delta + l' + 2n'$ in the relavant limit. One such example which we will explore in detail is the Type IIB string theory in $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills in 4d with $\mathrm{SU}(N)$ gauge group. In this correspondence, the large ℓ limit corresponds to the large N limit on the CFT side and there are indeed local operators whose conformal dimensions are the sum of conformal dimensions of other local operators.

6. Note that in the above example, we only need the boundary of the quantum gravity background space to have the geometry of $\mathbb{R} \times S^{d-1}$. But this can be achieved by allowing more general background spaces for the quantum gravity theory. Indeed, to get interesting examples of the correspondence, we allow for background metrics which approach the AdS metric near the boundary in a very specific sense [5]. Such spaces are called *asymptotically* AdS *spaces*. One can be more general by allowing the background to be asymptotically $\mathrm{AdS}_{d+1} \times M$ where M is some compact manifold

whose volume remains finite as the AdS radius $r \to \infty$. In this case, the correspondence works as follows: we start with scalar fields[1] $\phi : \mathrm{AdS}_{d+1} \times M \longrightarrow \mathbb{R}$. Let x, y be the coordinates on AdS_{d+1} and M respectively. Then we can expand the field in terms of some basis functions $\{f_\alpha(y)\}$:

$$\phi(x, y) = \sum_\alpha \phi_\alpha(x) f_\alpha(y). \tag{4.1}$$

Now ϕ_α are scalar field on AdS_{d+1} and we can again proceed as above.

4.2 Precise statement of the correspondence

We now state the AdS/CFT conjecture:

Conjecture 1. *Any conformal field theory on $\mathbb{R} \times S^{d-1}$ is equivalent to a theory of quantum gravity in asymptotically $\mathrm{AdS}_{d+1} \times M$ where M is some (possibly trivial) compact manifold.*

This correspondence raises immediate questions.

Question 4.1. What is the map between the observables on the two sides?

The answer to this question is called the *dictionary*. We still do not know the entire dictionary but many important entries of this dictionary have been worked out. Firstly, the dictionary means that we have a map between the Hilbert spaces of the quantum gravity theory and the CFT:

$$\Phi : \mathscr{H}_{\mathrm{AdS}} \longrightarrow \mathscr{H}_{CFT} \tag{4.2}$$

Next we must have that the unitary operators on the Hilbert space representing the symmetries of both the theories must commute with Φ:

$$\Phi \circ U_{\mathrm{AdS}} = U_{\mathrm{CFT}} \circ \Phi. \tag{4.3}$$

A solid evidence for why this may be true is the fact that the group of isometries of the AdS_{d+1} is $\mathrm{SO}(d, 2)$ which is also the conformal group in d dimensions as observed above.

5 The Correspondence of Type IIB String Theory on $\mathrm{AdS}_5 \times S^5$ with $\mathcal{N} = 4$ Super Yang-Mills in 4d

We will now present a concrete example of the AdS/CFT correspondence which appeared in the original paper of Maldacena in this subject [7]. We begin by reviewing the $\mathcal{N} = 4$ Super-Yang Mills (SYM) theory.

5.1 $\mathcal{N} = 4$ SYM Theory in 4d

We first quickly review the supersymmetry conventions and set up the stage to discuss the theory.

[1] one can also consider complex scalar fields.

5.1.1 Super-Poincaré algebra and BPS states

The Super-Poincaré algebra in 4d for $\mathcal{N}$ supercharges consists of the usual 10 generators $M^{\mu\nu}, P^{\mu}$ of the Poincaré algebra and a pair of $\mathcal{N}$ supercharges $Q^I_\alpha, \bar{Q}_{\dot\alpha I}$ with $I = 1, 2, \ldots, \mathcal{N}$ satisfying the Majorana condition

$$\bar{Q}_{\dot\alpha I} = (Q^I_\alpha)^\dagger. \tag{5.1}$$

The supercharges Q^I_α and $\bar{Q}_{\dot\alpha I}$ transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representation of $\mathrm{SL}(2, \mathbb{C})$. In addition to these spacetime symmetries, one can have an internal symmetry in the theory generated by the bosonic generator [2] $\{B_\ell\}$. The full algebra is then given by (see [6] for details):

$$\begin{aligned}
&\left\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\right\} = 2P_\mu \sigma^\mu_{\alpha\dot\beta}\delta^I_J \\
&\{Q^I_{\alpha,}, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \\
&\left\{\bar{Q}_{\dot\alpha I}, \bar{Q}_{\dot\beta J}\right\} = -\epsilon_{\dot\alpha\dot\beta}\bar{Z}_{IJ} \\
&\left[Q^I_\alpha, P^\mu\right] = [\bar{Q}^I_{\dot\beta}, P^\mu] = 0 \\
&\left[Q^I_\alpha, M_{\mu\nu}\right] = (\sigma_{\mu\nu})^{\beta}_{\alpha}\, Q^I_\beta \\
&\left[\bar{Q}^{\dot\alpha}_I, M_{\mu\nu}\right] = (\sigma_{\mu\nu})^{\beta}_{\dot\alpha}\, \bar{Q}^{\beta}_I \\
&[B_\ell, B_m] = f^{n}_{\ \ell m} B_n \\
&\left[Q^I_\alpha, B_\ell\right] = S^I_{\ell\ J} Q^J_\alpha \\
&[Q_{\dot\alpha I}, B_\ell] = (S^*)^J_{\ell I}\bar{Q}_{\dot\alpha J}.
\end{aligned} \tag{5.2}$$

along with the commutators of the Poincaré algebra

$$\begin{aligned}
&i\,[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\sigma\mu}M_{\rho\nu} + \eta_{\sigma\nu}M_{\rho\mu} \\
&i\,[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho \\
&i\,[P_\mu, P_\rho] = 0.
\end{aligned} \tag{5.3}$$

In this equation, σ^μ are the Pauli matrices with $\sigma^0 = \mathbf{1}$, $\epsilon = i\sigma^2$, Z^{IJ} is called the *central charges* since one can show that it commutes with all the generators of the algebra and $\bar{Z}_{IJ}$ are its complex conjugate, $\sigma^{\mu\nu}$ and $\bar\sigma^{\mu\nu}$ are the standard $\mathrm{SL}(2, \mathbb{C})$ generators, $f^{n}_{\ \ell m}$ are the structure constants of the internal symmetry algebra and $\{S_\ell\}$ are a representation of the internal symmetry algebra acting on the supercharges. An explicit automorphism of the algebra is the transformation of the supercharges by any unitary transformation U:

$$Q^I_\alpha \to U^I_J Q^J_\alpha, \quad \bar{Q}_{\dot\alpha I} \to (U^\dagger)^I_J \bar{Q}_{\dot\alpha J} \tag{5.4}$$

as long as the central charges transform accordingly. The group of transformations can be atmost $\mathrm{U}(\mathcal{N})$ and is called the R-symmetry group. The R-symmetry group for $\mathcal{N} = 4$ SYM turns out to be $\mathrm{SU}(4)$. This will be important later as this will correspond to the isometry group of S^5 since $\mathrm{SU}(4)$ is the double cover of $\mathrm{SO}(6)$.

[2] in the sense that they satisfy commutation relations rather that anticommutation relations.

To construct the irreducible representation of the Super-Poincaré algebra, called a supermultiplet, one first constructs the quadratic Casimirs of the algebra. It turns out that P^2 is still a Casimir but the Pauli-Lubanski pseudovector squared, which was a Casimir in the Poincaré algebra, no longer commutes with the supercharges and hence is not a Casimir. As a result the mass is constant in a supermultiplet but the spin can change resulting in a supermultiplet consisting of a bunch of particles of different spins. To construct these explicitly, we first define the *Clifford vacuum*. We now outline the steps of constructing the supermultiplet.

Vanishing central charge

For vanishing central charge, we define the annihilation and creation operators as

$$a^I_\alpha = \begin{cases} \frac{1}{\sqrt{2m}} Q^I_\alpha & P^2 = m^2 > 0 \\ \frac{1}{\sqrt{2}} Q^I_\alpha & P^2 = 0 \end{cases}, \quad a^\dagger_{\dot{\alpha} I} = \begin{cases} \frac{1}{\sqrt{2m}} \bar{Q}_{\dot{\alpha} I} & P^2 = m^2 > 0 \\ \frac{1}{\sqrt{2}} Q_{\dot{\alpha} I} & P^2 = 0. \end{cases} \tag{5.5}$$

One can show using the Super-Poincaré algebra that

$$\left\{ a^I_\alpha, a^\dagger_{\dot{\beta} J} \right\} = \delta^I_J \delta_{\alpha \dot{\beta}}. \tag{5.6}$$

The Clifford vacuum is defined to be annihilated by all annihilation operators $a_{\alpha I}$. Using the Casimirs of the algebra one shows that the Clifford vacuum for massive states is characterised by mass $m > 0$ and spin j with a total of $2j+1$ degrees of freedom and transforms in the usual spin-j representation of the Lorentz algebra and the Clifford vacuum of massless states is characterised by the helicity λ with two degrees of freedom. Then we construct the supermultiplet using the creation operators. For the massless supermultiplet, going to the rest frame one shows that $a^\dagger_{\dot{1} I} = 0$ and hence we only have $\mathcal{N}$ creation operators. Consequently a general massless state in the supermultiplet is obtained by applying $1 \leq k \leq \mathcal{N}$ number of creation operators and hence the total number of states is

$$\sum_{k=1}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}.$$

Next the algebra also shows that the creation operators raise the spin (helicity) by 1/2 and the annihilation operator decrease it by 1/2. Thus a supermultiplet constructed on a Clifford vacuum $|\lambda_0\rangle$ has the helicity content

$$|\lambda_0\rangle, \quad a^\dagger_{\dot{2} I}|\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I, \quad a^\dagger_{\dot{2} I} a^\dagger_{\dot{2} J}|\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ}, \quad \ldots, \quad a^\dagger_{\dot{2} 1} \ldots a^\dagger_{\dot{2} \mathcal{N}}|\lambda_0\rangle \equiv |\lambda_0 + \frac{\mathcal{N}}{2}\rangle \tag{5.7}$$

This is not CPT invariant unless

$$\lambda_0 = -\frac{\mathcal{N}}{4} \tag{5.8}$$

and hence in general we need to add the CPT conjugates. The massless supermultiplet for $\mathcal{N} = 4$ SYM is constructed on $\lambda_0 = -1$ and hence is CPT invariant. The supermultiplet has the content:

$$\left(-1, \mathbf{4} \times -\frac{1}{2}, \mathbf{6} \times 0, \mathbf{4} \times \frac{1}{2}, 1 \right). \tag{5.9}$$

The vector is a singlet under SU(4), the fermions transform in the fundamental representation and the six real scalars transform in the fundamental of SO(6). The massive supermultiplet is constructed similarly with the $2\mathcal{N}$ creation operators. The more interesting massive supermultiplet appears for non-vanishing central charge.

Non-Vanishing central charge

Using the algebra one can show that Z^{IJ} acts trivially on the massless supermultiplets and hence it remains the same even with non-vanishing central charge. For $m > 0$, the creation/annhiliation operators as defined in (5.5) do not satisfy the algebra in (5.6). To deal with this problem, we transform the supercharges by some U($\mathcal{N}$) transformation as in (5.4) such that the central charge transforms to

$$Z^{IJ} = U^I_K Z'^{KL} (U^T)^J_L \tag{5.10}$$

where Z'^{IJ} has the form

$$Z'^{IJ} = \begin{pmatrix} 0 & Z_1 & & & & & & \\ -Z_1 & 0 & & & & & & \\ & & 0 & Z_2 & & & & \\ & & -Z_2 & 0 & & & & \\ & & & & \cdots & \cdots & & \\ & & & & \cdots & \cdots & & \\ & & & & & & 0 & Z_{\mathcal{N}/2} \\ & & & & & & -Z_{\mathcal{N}/2} & 0 \end{pmatrix} \tag{5.11}$$

for even $\mathcal{N}$ and for odd $\mathcal{N}$, we add a zero row and column to Z'^{IJ} and treat the last supercharge as $\mathcal{N} = 1$ case. This is called the *Wess-Zumino* decomposition. In the above decomposition, Z_i are real and nonnegative. The R- symmetry index I can be now decomposed into a pair $I \equiv (A, a)$ where $1 \leq A \leq \frac{\mathcal{N}}{2}$ and $a = 1, 2$ where a is the matrix index of each block in the Wess-Zumino decomposition is $Z_A \epsilon^{ab}$. We then define creation annihilation operator as

$$\begin{aligned} a^A_\alpha &= \frac{1}{\sqrt{2}} \left[Q^{1A}_\alpha + \epsilon_{\alpha\beta} \bar{Q}_{\dot{\gamma}2A} (\bar{\sigma}^0)^{\dot{\gamma}\beta} \right], \\ \left(a^A_{\dot{\alpha}}\right)^\dagger &= \frac{1}{\sqrt{2}} \left[\bar{Q}_{\dot{\alpha}1A} + \epsilon_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}^0)^{\dot{\beta}\gamma} Q^{2A}_\gamma \right], \\ b^A_\alpha &= \frac{1}{\sqrt{2}} \left[Q^{1A}_\alpha - \epsilon_{\alpha\beta} \bar{Q}_{\dot{\gamma}2A} (\bar{\sigma}^0)^{\dot{\gamma}\beta} \right], \\ \left(b^A_{\dot{\alpha}}\right)^\dagger &= \frac{1}{\sqrt{2}} \left[\bar{Q}_{\dot{\alpha}1A} - \epsilon_{\dot{\alpha}\dot{\beta}} (\bar{\sigma}^0)^{\dot{\beta}\gamma} Q^{2A}_\gamma \right]. \end{aligned} \tag{5.12}$$

Then these operators satisfy the algebra

$$\begin{aligned} \left\{ a^A_\alpha, \left(a^B_{\dot{\beta}}\right)^\dagger \right\} &= (2m + Z_A)\, \sigma^0_{\alpha\dot{\beta}} \delta^B_A \\ \left\{ b^A_\alpha, \left(b^B_{\dot{\beta}}\right)^\dagger \right\} &= (2m - Z_A)\, \sigma^0_{\alpha\dot{\beta}} \delta^B_A. \end{aligned} \tag{5.13}$$

There are now $2\mathcal{N}$ creation operators. One can again show that $(a^A_{\dot 1})^\dagger, (b^A_{\dot 1})^\dagger$ lower m_s of the Clifford vacuum while $(a^A_{\dot 2})^\dagger, (b^A_{\dot 2})^\dagger$ raise it. Unitarity of the QFT demands

$$|Z_A| \leq 2m. \tag{5.14}$$

This is called the *BPS bound* and a state which saturates this bound is called a *BPS state*. BPS states are special because they are annihilated by some of the creation operators and hence have lesser degree of freedom than non-BPS states. Indeed if $Z_A = 2m$, then

$$\left\{b^A_\alpha, (b^B_{\dot\beta})^\dagger\right\} = 0 \tag{5.15}$$

and $(b_{\dot\alpha A})^\dagger|m, j\rangle$ is again vacuum since

$$b_{B\beta} b^\dagger_{A\alpha}|m, j\rangle = b^\dagger_{A\alpha} b_{B\beta}|m, j\rangle = 0 \tag{5.16}$$

for every B, β.

Let $0 \leq k \leq [\mathcal{N}/2]$ be the number of Z_A which saturate the BPS bound.

- $k = 0$, no states saturate BPS bound: The oscillators contribute to $2^{2\mathcal{N}}$ d.o.f of the spectrum and it is same as the vanishing central charge spectrum. This is called the *long multiplet*.

- $0 < k < [\mathcal{N}/2]$: k number of oscillators annihilate the multiplet, thus oscillators contribute to $2^{2(\mathcal{N}-k)}$ d.o.f of the spectrum. The spectrum is said to be $\frac{k}{\mathcal{N}}$-BPS multiplet. The multiplet we get is a *stort multiplet.*

- $k = [\mathcal{N}/2]$: We get an *ultrashort multiplet.* Here we have $2^{\mathcal{N}}$ d.o.f from the oscillators and is called the $\frac{1}{2}$-BPS multiplet.

5.1.2 The $\mathcal{N} = 4$ SYM Lagrangian and its symmetries

Now that we know that massless $\mathcal{N} = 4$ supermultiplet contains a vector boson, 4 Weyl fermions and 6 real scalars, we can try to construct an offshell field representing this supermultiplet. The superfield formalism is not very useful here. So we work in component formalism. The component fields are $A^\mu, \lambda^I_\alpha, \phi^I$ where I is the $\mathrm{SU}(4)_R$ index and α is the spinor index of the Weyl fermion. If we consider the $\mathcal{N} = 4$ SYM with gauge group $\mathrm{SU}(N)$ then each of the component fields have a gauge index a and each of the fields are Lie algebra valued as in the usual gauge theory. The Lagrangian of the theory can be obtained in atleast two different ways:

1. From $\mathcal{N} = 1$ Superfields: this requires three chiral superfields and a vector supefield.

2. Dimensional reduction: one can dimensionally reduce the $d = 10$ $\mathcal{N} = 1$ SYM to obtain the $4d, \mathcal{N} = 4$ SYM Lagrangian.

We refer to [1] for details. Using either of the two methods, the $4d, \mathcal{N}=4$ Lagrangian can be written as [3]

$$\begin{aligned}\mathcal{L} = \mathrm{Tr}\Bigg(& -\frac{1}{2g_{YM}^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{8\pi^2} F_{\mu\nu}(*F)^{\mu\nu} - i\sum_I \bar{\lambda}^I \bar{\sigma}^\mu D_\mu \lambda_I - \sum_I D_\mu \phi^I D^\mu \phi^I \\ & + g_{YM} \sum_{I,J,K} C_K^{IJ} \lambda_I \left[\phi^K, \lambda_J\right] + g_{YM} \sum_{I,J,K} \bar{C}_{IJK} \bar{\lambda}^I \left[\phi^J, \bar{\lambda}^K\right] + \frac{g_{YM}^2}{2} \sum_{I,J} \left[\phi^I, \phi^J\right]^2 \Bigg)\end{aligned} \tag{5.17}$$

Here g_{YM} is the coupling constant, θ is called the *instanton angle*, C^{IJK} is the structure constant of $\mathrm{SU}(4)_R$, D_μ is the gauge covariant derivative defined as

$$D_\mu = \partial_\mu - i g_{YM} A_\mu = \partial_\mu - i g_{YM} T^a A_\mu^a, \tag{5.18}$$

where T^a are the generators of $\mathrm{SU}(N)$ satisfying

$$[T^a T^b] = i f_c^{ab} T^c, \quad \mathrm{Tr}(T^a T^b) = \delta^{ab}. \tag{5.19}$$

$F^{\mu\nu}$ is the usual field strength defined by the commutator of D_μ:

$$F^{\mu\nu} = [D_\mu, D_\nu] = F_a^{\mu\nu} T^a \tag{5.20}$$

where

$$F^{a\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\nu^a + g_{YM} f^{abc} A_{b\mu} A_{c\nu}. \tag{5.21}$$

$(*F)^{\mu\nu}$ is the Hodge dual of $F^{\mu\nu}$. The trace in the Lagrangian is over the gauge index. We now list the symmetries of the lagrangian.

1. Poincaré invariance: the Lagrangian is easily seen to be invariant under Poincaré group. It is generated by $P^\mu, M^{\mu\nu}$ satisfying the Poincaré algebra (5.3).

2. Gauge invariance: the Lagrangian is gauge invariant since the gauge index is traced over.

3. Supersymmetry: under supersymmetry the component fields transform as [1]

$$\begin{aligned}&(\delta\phi^I)_\alpha^J \equiv [Q_\alpha^J, \phi^I] = C_K^{IJ} \lambda_\alpha^K \\ &(\delta\lambda_{\beta I})_\alpha^J = \left\{Q_\alpha^J, \lambda_{\beta I}\right\} = F_{\mu\nu}^+ (\sigma^{\mu\nu})_{\alpha\beta} \delta_I^J + \left[\phi^K, \phi^L\right] \epsilon_{\alpha\beta} (C_{KL})_I^J \\ &\left(\delta\bar{\lambda}_{\dot{\beta}}^I\right)_\alpha^J = \left\{Q_\alpha^J, \bar{\lambda}_{\dot{\beta}}^I\right\} = C_K^{JI} \sigma_{\alpha\dot{\beta}}^\mu D_\mu \phi^K \\ &(\delta A^\mu)_\alpha^I = \left[Q_\alpha^I, A^\mu\right] = \sigma_{\alpha\dot{\beta}}^\mu \bar{\lambda}^{\dot{\beta} I}\end{aligned} \tag{5.22}$$

   where $F_{\mu\nu}^+$ is the self-dual part of $F_{\mu\nu}$ given by

$$F_{\mu\nu}^+ = \frac{1}{2}(F_{\mu\nu} + (*F)_{\mu\nu}) \tag{5.23}$$

   and the constants $(C_{KL})_I^J$ is related to the bilinears in Clifford Dirac matrices of $\mathrm{SO}(6)_R$. This is generated by the 4 supercharges Q_α^I and its conjugate $\bar{Q}_{\dot{\beta} I}$.

4. Conformal invariance: using the mass dimensions of the fields $[A_\mu] = [\phi] = 1, [\lambda_\alpha] = 3/2$ and the mass dimension of the Lagrangian[3] $[\mathcal{L}] = 4$, it is easy to see that $[g_{YM}] = 0 = [\theta]$. Now since all the fields are massless, the action is classically scale invariant. Infact the Poincaré invariance combines to enhance the symmetry to full conformal invariance. Infact the theory is conformally invariant at quantum level as well since the renormalisation beta functions of the coupling constants vanish upto all orders in perturbation theory and hence quantum corrections do not introduce a mass scale in the quantum theory. This is generated by the the Poincaré algebra generators along with the genrator of scaling denoted by D and the special conformal transformation generator K_μ which acts on space as in (3.3). The full algebra is (5.3) along with

$$\begin{aligned} [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu} D - M_{\mu\nu}\right) \\ [K_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu\right) \end{aligned} \tag{5.24}$$

5. Superconformal invariance: the fact that the theory is invariant under conformal as well as supersymmetry transformation enlarges the symmetry of the theory to superconformal symmetry. For the algebra to close, we need to include another generator $S^I_\alpha, \bar{S}_{\dot\beta I}$ called the *conformal supersymmetry generator*. The algebra is

$$\begin{aligned} &\left\{S^I_\alpha, S^J_\beta\right\} = \left\{Q^I_\alpha, \bar{S}_{\dot\beta J}\right\} = 0 \\ &\left\{S^I_\alpha, \bar{S}_{\dot\beta J}\right\} = 2\sigma^\mu_{\alpha\dot\beta} K_\mu \delta^I_J \\ &\left\{Q^I_\alpha, S_{\beta J}\right\} = \epsilon_{\alpha\beta}\left(\delta^I_J D + T^I_J\right) + \frac{1}{2}\delta^I_J M_{\mu\nu}\left(\sigma^{\mu\nu}\right)_{\alpha\beta} \end{aligned} \tag{5.25}$$

where $\{T^A\}_{A=1}^{15}$ are the generators of the $\mathrm{SU}(4)_R$ symmetry. See [1, Appendix B.3.2] for the full algebra.

6. S-duality: $\mathcal{N} = 4$ SYM is invariant under the S-duality group $\mathrm{SL}(2, \mathbb{Z})$. To describe this duality, put

$$\tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2_{YM}}. \tag{5.26}$$

Then the theory is invariant under

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{5.27}$$

This duality is enormously important because it maps a weakly coupled theory to a strongly coupled theory, this is called the *strong-weak coupling duality*. Indeed under the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we see that

$$\tau \to -\frac{1}{\tau}. \tag{5.28}$$

[3]required for the action to be dimensionless.

When the instanton angle $\theta = 0$ then under this transformation

$$g_{YM} \to \frac{4\pi}{g_{YM}}. \tag{5.29}$$

This means that a strongly coupled theory gets mapped to a weakly coupled theory.

The above discussion reflects the fact that the full symmetry group of $\mathcal{N} = 4$ SYM theory is the supergroup $\mathrm{SU}(2,2|4)$.

5.1.3 Representations of the superconformal algebra

We want to construct local, gauge invariant operators in the theory. We restrict to operators which are polynomials in the local fields $A_\mu, \lambda_\alpha, \phi$ so that the scaling dimensions of the operators is well defined. Now a given local operator $\mathcal{O}$ is characterised by the scaling dimension Δ and the spin:

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0), \quad [M_{\mu\nu}, \mathcal{O}(0)] = -\mathcal{M}_{\mu\nu}\mathcal{O}(0) \tag{5.30}$$

where $\mathcal{M}_{\mu\nu}$ is the representation of the Lorentz generators $M_{\mu\nu}$ on fields. Other generators of the superconformal algebra act by raising or lowering the scaling dimension. For example Q^I_α raises the conformal dimension by 1/2 and S^I_α lowers the scaling dimension by 1/2 and as a result of the superconformal algebra, P^μ and K_μ raise and lower the scaling dimension by 1 respectively. Now since the conformal supersymmetry generator S^I_α lowers the scaling dimension, there exists operators which are annihilated by S^I_α, otherwise we would produce arbitrarily negative scaling dimension operators which break the unitarity of the theory. These operators are called the *superconformal primary operators.* Thus superconformal primary operators have to satisfy

$$[S^I_\alpha, \mathcal{O}\} = 0, \quad [\bar{S}_{\dot{\alpha}I}, \mathcal{O}\} = 0 \tag{5.31}$$

where the commutator or anticommutator depends on the operator $\mathcal{O}$ being bosonic or fermionic.

Remark 5.1. Recall that in conformal field theory, primary operators are those which are annihilated by K_μ. Because of the superconformal algebra, a superconformal primary operator is a conformal primary operator but the converse is not true.

Now the descendents can be constructed by applying any other generator of the superconformal algebra on the superconformal primaries. For example applying $[P^\mu, \mathcal{O}] = -\partial_\mu\mathcal{O}$ has scaling dimension $\Delta + 1$. In particular, the *superdescendents* are those obtained by applying the supercharge:

$$\mathcal{O}' = [Q^I_\alpha, \mathcal{O}\}. \tag{5.32}$$

Clearly

$$\Delta_{\mathcal{O}'} = \Delta_{\mathcal{O}} + \frac{1}{2}. \tag{5.33}$$

Superdescendents are conformal primaries as can be easily seen from the superconformal algebra. In particular note that each superdescendent gives rise to a Verma module and each such Verma module is related by supersymmetry transformation. On each such superconformal primary, we construct a tower of descendents and this gives an irreducible representation of the superconformal algebra. The full spectrum is the direct sum of all these *highest weight representations.*

Let us now apply this to $\mathcal{N} = 4$ SYM. We need to identify the superconformal primaries. Note that the Q-commutator of any field cannot be a superconformal primary since applying S lowers the scaling dimension by 1/2 and hence the Q-commutator is not annihilated by S. We now list all the Q-commutators:

$$\begin{aligned} \{Q, \lambda\} &= F^{+} + [\phi, \phi]; \qquad [Q, \phi] = \lambda \\ \{Q, \bar{\lambda}\} &= D\phi; \qquad [Q, F] = D\lambda \end{aligned} \tag{5.34}$$

Next the superconformal operators cannot be constructed out of only the gauge field as it is not gauge invariant. Thus the only left option is the scalar fields. Thus all superconformal primaries are polynomials of the scalar fields. To construct gauge invariant operators, we need to take trace over the gauge indices which symmetrises the R-symmetry indices. The simplest operator are the single trace operators of the form

$$\mathcal{O}_n = \mathrm{str}[\phi^{I_1} \dots \phi^{I_n}] \tag{5.35}$$

where "str" stands for symmetrised trace defined on the generators T^a as

$$\mathrm{str}[T^{a_1} \dots T^{a_n}] = \sum_{\sigma \in S_n} [T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}]. \tag{5.36}$$

One then defines the symmetrised trace of ϕ^I by expanding $\phi^I = \phi^I_a T^a$. Since the trace of generators of SU(N) is zero, the simplest single trace operator is the *Konishi multiplet*

$$\mathrm{str}[\phi^I \phi^I] = \mathrm{Tr}[\phi^I \phi^I] \qquad (\text{sum over } I) \tag{5.37}$$

and the *supergravity multiplet*

$$\mathrm{str}[\phi^I \phi^J] = \mathrm{Tr}[\phi^I \phi^J] - \frac{1}{6}\delta^{IJ}\mathrm{Tr}[\phi^K \phi^K] \qquad (\text{sum over } K). \tag{5.38}$$

BPS states

The states in the theory are representations of the SU(2,2|4) symmetry group. Thus the states are labelled by the quantum numbers corresponding to the Poincaré[4] group $\mathrm{SO}(1,3) \ltimes \mathbb{R}^{1,3}$, the $\mathrm{SU}(4)_R$ R-symmetry group and the dilatations $\mathrm{SO}(1,1)$. The massless representations of the Poincaré group are labelled by helicity s_+, s_-. The quantum number for dilatations are labeled $\Delta \geq 0$. The representations of SU(4) are labelled by the three *Dynkin labels* $[r_1, r_2, r_3]$. The dimension of the representation is given by

$$\dim[r_1, r_2, r_3] = \prod_{1 \leq i \leq j \leq 3} \frac{r_i - r_j + j - i}{j - i}. \tag{5.39}$$

[4]remember that we are looking at massless representations, which means that translations do not contribute to the labels.

The conjugate representation of $[r_1, r_2, r_3]$ is denoted by $[r_1, r_2, r_3]^*$ and given by

$$[r_1, r_2, r_3]^* = [r_3, r_2, r_1]. \tag{5.40}$$

Unitarity demands that the conformal dimension be bounded below by the spin quantum number and also the R-symmetry labels. A careful analysis of the representation labels based on the $\mathfrak{su}(2, 2|4)$ algebra gives [4]

$$\begin{aligned}
&(1) \quad \Delta = r_1 + r_2 + r_3 \\
&(2) \quad \Delta = \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3 \geq 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3; \quad r_1 \geq r_3 + 2 \\
&(3) \quad \Delta = \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \geq 2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3; \quad r_3 \geq r_1 + 2 \\
&(4) \quad \Delta \geq \mathrm{Max}\left[2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3; 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3\right]
\end{aligned} \tag{5.41}$$

Note that (2) and (3) are conjugates of each other. (4) is a kind of *BPS bound* which when saturated results in atleast one of the supercharges to commute with the corresponding primary operator and hence these primary operators do not create new states by acting on the vacuum in the theory and the multiplet is shortened. The multiplets we get when the BPS bound is saturated is called the *BPS-multiplet* in analogy to the supersymmetric BPS-multiplits discussed above. The first three cases are *discrete series of representations* and are clearly the BPS-multiplets. The importance of BPS-muliplets lies in the fact that the conformal dimension is given exactly in terms of SU(4) labels and hence group theoretic reasoning demands that the conformal dimension are *protected* again quantum corrections, that is the conformal dimensions of the BPS primary operators are not renormalised. This is important because we will see later that the AdS/CFT correspondence related the weak coupling limit of Type IIB string theory (which is a supergravity theory and calulations are easier to perform here) to the strong coupling limit of $\mathcal{N} = 4$ SYM (and hence it is hard to do perturbative calculations here). Since the conformal dimensions of BPS primary operators are non perturbative objects, they have an exact observable in the string theory.

The operators corresponding to (4) are non-BPS primary operators and the conformal dimensions of non-BPS operators are *unprotected*. We now list the quantum numbers of the BPS operators in a table. $\#Q$ represents the number of supercharges that commute with the operator.

Operator type	$\#Q$	spin range	$\mathrm{SU}(4)_R$ primary	dimension Δ
identity	16	0	$[0, 0, 0]$	0
1/2BPS	8	2	$[0, k, 0], k \geq 2$	k
1/4BPS	4	3	$[\ell, k, \ell], \ell \geq 1$	$k + 2\ell$
1/8BPS	2	7/2	$[\ell, k, \ell + 2m]$	$k + 2\ell + 3m, m \geq 1$
non-BPS	0	4	any	unprotected

Table 1. BPS and non-BPS primary operators

5.2 Type IIB String Theory and Supergravity

We now discuss the other side of the AdS/CFT corresponding, namely the Type IIB string theory on $\mathrm{AdS}_5 \times S^5$. We begin by a general discussion of string theories and superstring theories.

5.2.1 Bosonic String Theory

String theory is a theory of strings – a one dimensional object. As usual we start with an action, which in analogy with the action for a point particle, is the area of the surface called the *worldsheet* traced by the string. To write an expression for the action, we need to embed the string in a D-dimensional manifold $\mathcal{M}$ called the *target space* via an embedding $X : \mathbb{R} \times [0, \ell] \longrightarrow \mathcal{M}$ where L is the length of the string. Parameterising the worldsheet by $\sigma^\alpha \equiv (\tau, \sigma) \equiv \boldsymbol{\sigma}$ the embedding in the target space is given by coordinates $X^\mu(\tau, \sigma)$. The string action, called the *Nambu-Goto action* is given by

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\sigma d\tau \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)}, \tag{5.42}$$

where M is the surface traced by the string, α' is called the *Regge slope*. If we denote the intrinsic metric on the worldsheet by

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu. \tag{5.43}$$

The action can then be written as

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\sigma d\tau \mathcal{L}_{NG}, \quad \mathcal{L}_{NG} = \left[-\det(h_{\alpha\beta})\right]^{\frac{1}{2}}. \tag{5.44}$$

This action is not very easy to work with. We work with the *Polyakov action* which can be shown to be equivalent to the Nambu-Goto action. The Polyakov action is given by

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d\sigma d\tau \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu, \tag{5.45}$$

where $g = \det(g^{\alpha\beta})$ is a general worldsheet metric. Now the metric on the worldsheet is dynamical, so the Polyakov action can be considered as a bunch of scalars coupled to 2d gravity. This action possesses three different symmetries:

1. Reparametrization invariance: the parameters transform as $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha = \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$. The scalar fields X^μ transform as

   $$X^\mu(\sigma, \tau) \longrightarrow \widetilde{X}^\mu\left(\widetilde{\sigma}^\alpha\right) = X^\mu\left(\sigma^\alpha\right)$$

   and the worldsheet metric $g_{\alpha\beta}$ transforms in the usual way

   $$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta}\left(\widetilde{\sigma}^\alpha\right) = \frac{\partial\sigma^\gamma}{\partial\widetilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\widetilde{\sigma}^\beta} g_{\gamma\delta}(\sigma).$$

2. Poincaré Invariance: this is a global symmetry of the action.

   $$X^\mu \longrightarrow \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + \xi^\mu$$

   for some constant $\xi^\mu \in \mathbb{R}^{D-1,1}$ and $\Lambda^\mu{}_\nu \in \mathrm{SO}(D-1,1)$.

3. Weyl Invariance: there is another gauge invariance called Weyl symmetry. Under this $X^\mu \longrightarrow X^\mu$ and the metric transforms as

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \Omega^2(\boldsymbol{\sigma}) g_{\alpha\beta}$$

or infinitesimally if $\Omega^2(\boldsymbol{\sigma}) = e^{2\phi(\boldsymbol{\sigma})}$ then

$$\delta g_{\alpha\beta} = 2\phi(\boldsymbol{\sigma}) g_{\alpha\beta}.$$

This only works when the worldsheet is 2d.

Let us vary the action with respect to X^μ with $\delta X^\mu(\sigma, \tau_0) = \delta X^\mu(\sigma, \tau_1) = 0$ for some initial and final value τ_0, τ_1 respectively of the parameter τ. We get

$$\delta S_P = \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu + \underbrace{\frac{1}{2\pi\alpha'} \int_0^\ell d\sigma \left(\partial^\tau X^\mu\right) \delta X_\mu \Big|_{\tau_0}^{\tau_1}}_{= 0 \text{ as } \delta X^\mu(\sigma,\tau_0) = \delta X^\mu(\sigma,\tau_1) = 0} + \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \underbrace{\left(\partial^\sigma X^\mu \delta X_\mu\right) \Big|_0^\ell}_{\text{surface term}}. \tag{5.46}$$

To get the equations of motion, we need the surface term to go to zero. Physically we distinguish between two cases - the closed string and the open string. For the closed string $X^\mu(\sigma+\ell,\tau) = X^\mu(\sigma,\tau)$ and the surface term vanishes. For the open string, we can impose two different boundary conditions:

1. Dirichlet boundary condition: $\delta X_\mu = 0 \quad$ at $\sigma = 0, \ell$.
2. Neumann boundary condition: $\partial_\sigma X_\mu = 0$ at $\sigma = 0, \ell$.

The equations of motion are

$$\partial_\alpha \partial^\alpha X^\mu = 0 \tag{5.47}$$

which is the wave equation. We will briefly recall the quantisation of closed an open strings and discuss the derivation of the critical dimension.

Closed String

Before we solve the equation of motion, we need to impose the constraint obtained from the equation of motion of $g_{\alpha\beta}$ since it was an auxiliary field. To simplify the constaint, we can use the reparametrization invariance and Weyl symmetry of the action to set $g_{\alpha\beta} = \eta_{\alpha\beta}$– this is called the *conformal gauge*. The equation of motion for $g_{\alpha\beta}$ is

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta} \partial_c X^\mu \partial^c X_\mu.$$

Also the energy momentum tensor for the Polyakov action is given by

$$T_{\alpha\beta} = -4\pi\alpha' \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} \partial_c X^\mu \partial^c X_\mu.$$

So the constraint is

$$T_{\alpha\beta}\big|_{g_{\alpha\beta}=\eta_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu = 0. \tag{5.48}$$

So we have to impose two constraints

$$\dot{X}^\mu X'_\mu = 0, \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0, \tag{5.49}$$

where the dot is derivative with respect to τ and prime is derivative with respect to σ. So the equation of motion is a wave equation along with the two constraints. The solution to the equations of motion is given by

$$X^\mu(\sigma,\tau) = X^\mu_L + X^\mu_R \tag{5.50}$$

where[5]

$$\begin{aligned} X^\mu_L\left(\sigma^+\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \widetilde{\alpha}^\mu_n e^{-in\sigma^+} \\ X^\mu_R\left(\sigma^-\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\sigma^-}, \end{aligned} \tag{5.51}$$

where $\sigma^\pm = \tau \pm \sigma$. The functions X^μ_L are called *left movers* and X^μ_R are called *right movers*. Reality of X^μ implies that

$$(\alpha^\mu_n)^\star = \alpha^\mu_{-n} \quad \text{and} \quad (\widetilde{\alpha}^\mu_n)^\star = \alpha^\mu_{-n} \quad \forall n \in \mathbb{Z}\backslash\{0\}.$$

Imposing the constraints gives

$$L_n := \frac{1}{2}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} = 0. \tag{5.52}$$

and $\widetilde{L}_n = 0$ for all $n \in \mathbb{Z}$ where $\widetilde{L}_n$ is defined with $\widetilde{\alpha}$ analogous to L_n. Here $\alpha^\mu_0 = \sqrt{\alpha'/2}p^\mu$. To quantise the closed string, we promote the modes α^μ_n and $\widetilde{\alpha}^\mu_n$ to operators and impose

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha^\mu_n, \alpha^\nu_m] &= m\eta^{\mu\nu}\delta_{m+n,0} = [\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n] \end{aligned} \tag{5.53}$$

The L_n's of (5.52) are now operators but to define L_0, we need to impose normal ordering on the modes. We put $\alpha^\mu_n, \quad n > 0$ to the right of $\alpha^\mu_n, \quad n < 0$. The algebra satisfied by L_n is then the *Virasoro algebra* given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0} \tag{5.54}$$

[5] the length of the string has been normalised to 2π.

and similarly for $\widetilde{L}_n$. Here c is called the *central charge* and is related to breaking of Weyl symmetry in the quantum theory. For D scalar fields $c = \eta^\mu_\mu = D$. Since this is a gauge symmetry of the theory, we want this *anomaly* to cancel in the quantum theory. Indeed in superstring theory which we will discuss soon, addition of worldsheet fermions contributes to central charge and the anomaly cancels when $D = 10$ giving the critical dimension of the superstring. The ground state of the theory is now defined by

$$\alpha^\mu_n|0;p^\mu\rangle = 0 = \widetilde{\alpha}^\mu_n|0;p^\mu\rangle \quad \text{for } \mu = 0, 1, \ldots, D-1; \; n > 0 \tag{5.55}$$

where p^μ is the momentum of string ground state. A general excitation of the string is

$$\left(\alpha^{\mu_1}_{-1}\right)^{n_{\mu_1}} \left(\alpha^{\mu_2}_{-2}\right)^{n_{\mu_2}} \cdots \left(\widetilde{\alpha}^{\nu_1}_{-1}\right)^{n_{\nu_1}} \left(\widetilde{\alpha}^{\nu_2}_{-2}\right)^{n_{\nu_2}} \cdots |0;p^\mu\rangle.$$

We now have negative-norm states in the theory. Indeed since $\eta^{00} = -1$, it can easily be checked that for $|\psi\rangle = \alpha^0_{-m}|0;p^\mu\rangle$, $\langle\psi|\psi\rangle = -m < 0$. It turns out that these states decouple from the theory if we choose $D = 26$ – the critical dimension of bosonic string (see [8] for proof). This is the *covariant quantisation* since it is manifestly Lorentz invariant. There is another quantisation scheme called the *lightcone quantisation*, which breaks Lorentz invariance and avoids negative norm states. But at the end when we enforce Lorentz invariance, we need to fix $D = 26$. For more details see [8]. It turns out that the string ground state is tachyonic – a red signal. The first excited state is massless and the particle content is

1. $g_{\mu\nu}(X)$: the traceless symmetric tensor field which we will identify with *graviton*.
2. $B_{\mu\nu}(X)$: the antisymmetric tensor field. This is sometimes called the *Kalb-Ramond field*.
3. $\Phi(X)$: the trace part of the tensor representations. This scalar field is called the *dilaton*.

All higher excited states are massive.

Open Strings and D-branes

The general solution to the wave equation (5.47) remains the same, we need to impose the boundary conditions to determine the specific form of the solution. In general one can apply different boundary conditions at the two ends of the open string. For example, one can have Neumann boundary condition at both ends (NN). Analogously other boundary conditions can be (DD), (ND) and (DN) where D stands for Dirichlet boundary condition. Also note that we can differentiate between the boundary conditions in different directions. For example we can apply NN for $0 \leq \mu \leq p$ and DD for $p+1 \leq \mu \leq D-1$. This means that

$$\partial_\sigma X^a = 0 \text{ for } a = 0, \cdots, p \text{ at } \sigma = 0, \pi$$
$$X^I(0,\tau) = c^I, \quad X^I(\pi,\tau) = d^I \quad \text{for } I = p+1, \cdots, D-1,$$

where we have normlaised the string length to π. This fixes the endpoints of the string in the $D-p-1$ directions and hence is constrained to move in the $(p+1)$-dimensional hypersurface. This hypersurface is usually called a *Dp-brane*. So a $D0$-brane is a particle, a $D1$-brane is itself a string, a $D2$-brane is a membrane and so on. In particular if $p = D-1$ then we get to NN case which means all space is a D-brane, that is we get space filling D-brane. Now the quantisation. In covariant quantisation, again a similar analysis as in closed string quantisation results in $D = 26$ independent of the boundary condition. The spectrum depends on the choice of boundary condition. The ground state is tachyonic, first excited state is massless and all higher excited state is massive. One can show that the maximum spin at level[6] n is n. Thus the first excited states are spin 1 particles which we can identify with photons.

	Open String	Closed String
Ground state	Tachyonic	Tachyonic
First excited state	A_μ	$g_{\mu\nu}, B_{\mu\nu}, \Phi$

It is worthwhile to note that the mass of the string excitations are inversely proportional to α' and hence in the low energy limit $\alpha' \to 0$, the mass of higher exited states go to infinity.

We now quickly review how gauge theories arise on the worldvolume of D-branes.

One Dp **Brane**: the boundary condition is

$$X^\mu(0,\tau) = c^I = X^\mu(\pi,\tau) \quad \mu = p+1, \dots, D-1.$$

Thus the ends of the string are constrained to lie on one Dp brane. The ground state is now defined by

$$\alpha_n^i \left|0; p^\mu\right\rangle = 0, \quad n > 0, \quad i = 1, 2, \cdots p-1, p+1, \cdots, D-1.$$

Note that the string momentum p^μ is actually only in $p+1$ directions. The $\mathrm{SO}(1, D-1)$ Lorentz group is broken into $\mathrm{SO}(1,p) \times \mathrm{SO}(D-P-1)$. As the first excited state has maximum spin 1, these states represent gauge fields. We introduce a gauge field A_i, $i = 0, \dots, p$ and its quanta represents spin 1 photons. The other oscillators are

$$\alpha_{-1}^I \left|0; p^\mu\right\rangle, \quad I = p+1, \dots, D-1.$$

These transform as scalar representations of $\mathrm{SO}(1,p)$ and hence we introduce $D-p-1$ scalar fields ϕ^I. Although ϕ^I transform as scalars under the $\mathrm{SO}(1,p)$ Lorentz group of the Dp-brane they transform as vectors as representations of the $\mathrm{SO}(D-p-1)$ rotation group. This appears as a global symmetry of the brane world volume. One can also consider ϕ^I as the Goldstone Bosons associated to the spontaneously broken translational symmetryor as *fluctuations of the D-brane itself*. this indicates that the D-branes itself are dynamical objects in string theory.

[6] excited states with n oscillators acting on the ground state

Two Dp-branes: This means that the string is stretched between two branes. The boundary condition is $X^\mu(0,\sigma) = X^\mu(\pi,\sigma), \quad \mu = p+1, \cdots, D-1$. One can show that the first excited states $\alpha^i_{-1}|0,p^i\rangle$ in this case are no longer massless.

In general we can stack N such Dp-branes on top of each other and denote the massless vector excitation as

$$\alpha^i_{-1}|k,\ell,p^i\rangle$$

where k,ℓ are labels which encode the Dp-branes on which the endpoints of the string end. These are called *Chan-Paton labels*. The resulting N^2 states can be embedded in an $N\times N$ matrix and expanded in a complete set of $N\times N$ matrices

$$\left|k,\ell;p^i\right\rangle = \lambda^a_{k\ell}|a;p^i\rangle, \quad a\in\left\{1,\cdots,N^2\right\},$$

where $\lambda^a_{k\ell}$ are called *Chan-Paton factors*. The resulting fields T^k_ℓ, $\left(\phi^I\right)^k_\ell$ and $(A^a)^k_\ell$ can be fit into Hermitian matrices. The diagonal fields arise from strings ending on same brane. In this way, $(A^a)^k_\ell$ can be identified with $\mathrm{U}(N)$ Yang-Mills gauge bosons and $\left(\phi^I\right)^k_\ell$ transform in the adjoint representation of $\mathrm{U}(N)$.

Low energy effective spacetime action

One can consider the string propagating in the background of its own massless fields. For the closed string, this is achieved by coupling the string with the graviton $g_{\mu\nu}(X)$, the Kalb-Ramond field $B_{\mu\nu}$ and the dilaton. The action can then be written as

$$\begin{aligned} S = &-\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu g_{\mu\nu}(X) \\ &\qquad -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X)-\sqrt{-h}\alpha' R^{(2)}\Phi(X)\right) \end{aligned} \tag{5.56}$$

where $R(2)$ is the Ricci scalar of the graviton $g_{\mu\nu}$ and $\epsilon^{\alpha\beta}$ is the Levi-Civita symbol. This is known as the *nonlinear σ-model*. We want to retain the reparametrization and Weyl symmetry of the Polyakov action. This can be made sure by looking at the renormalisation group beta functions for the couplings $g_{\mu\nu}, B_{\mu\nu}$ and Φ. To first order in α' we get

$$\begin{aligned} \beta^{(g)}_{\mu\nu} &= \alpha'\left(R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-\frac{1}{4}H_{\mu\lambda\rho}H_\nu^{\lambda\rho}\right) \\ \beta^{(B)}_{\mu\nu} &= \alpha'\left(-\frac{1}{2}\nabla^\lambda H_{\lambda\mu\nu}+\nabla^\lambda\Phi H_{\lambda\mu\nu}\right) \\ \beta^{(\Phi)} &= \alpha'\left(-\frac{1}{2}\nabla^2\Phi+\nabla_\mu\Phi\nabla^\mu\Phi-\frac{1}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right) \end{aligned} \tag{5.57}$$

where $H = dB$ is a 3-form field strength and $R_{\mu\nu}$ is the Ricci tensor for $g_{\mu\nu}$. Indeed in the low energy limit, only first order contributions are relevant. We see that the vanishing of the β-functions can be encoded in the equations of motion of a spacetime action, called the *low energy spacetime action of the closed string*, and given by

$$S_{\text{eff.}} = \int d^{26}x\sqrt{-\det g}e^{-2\Phi}\left(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\nabla_\mu\Phi\nabla^\mu\Phi\right) \tag{5.58}$$

In superstring theory, this method of deriving the spacetime action gives rise to supergravity theory as the low energy approximation of string theory.

5.2.2 Superstring theory and Supergravity

We now briefly discuss superstring theory. The bosonic string theory described in previous section is inherently incomplete because it does not contain fermions. Moreover, it is fraught with problems like the tachyonic ground state and the critical dimension being 26 which is far from 4 which we live in. So we introduce fermions in the theory. There are two equivalent formalisms of introducing fermions: the *Green-Schwartz formalism* (GS) where we introduce spacetime fermions in the target space and the *Ramond-Neveu-Schwartz formalism* (RNS) where we introduce worldsheet fermions on the 2d worldsheet. We follow the RNS formalism. The action is modified to

$$S = -\frac{1}{4\pi\alpha'}\int_M d\sigma d\tau\sqrt{-g}\partial_\alpha X^\mu\partial^\alpha X_\mu + i\bar{\Psi}^\mu\Gamma^\alpha\partial_\alpha\Psi_\mu \tag{5.59}$$

where Γ^α are the 2d gamma matrices satisfying

$$\{\Gamma^\alpha, \Gamma^\beta\} = 2\eta^{\alpha\beta} \tag{5.60}$$

where $\bar{\Psi}^\mu = i\Psi^\dagger\Gamma^0$. One can choose a simple representation of the gamma matrices such that when the fermion is decomposed into Weyl spinors $\Psi^\mu \equiv (\psi^\mu_+, \psi^\mu_-)$, the fermionic action takes the form

$$S_\Psi = -\frac{1}{2\pi\alpha'}\int d^2\sigma\left(\psi^\mu_-\partial_+\psi_{\mu-} + \psi^\mu_+\partial_-\psi_{\mu+}\right). \tag{5.61}$$

The equations of motion are

$$\partial_-\psi^\mu_+ = \partial_+\psi^\mu_- = 0. \tag{5.62}$$

But one needs to make sure that the boundary term vanishes while deriving this equation of motion. Indeed the boundary term has the form:

$$\delta S_\Psi|_{\text{boundary}} \propto \int d\tau\left(\psi^\mu_-\delta\psi_{-\mu} - \psi^\mu_+\delta\psi_{+\mu}\right)\Big|_{\sigma=0}^{\sigma=\pi} \tag{5.63}$$

This leads us to the following boundary conditions for open superstrings:

1. Ramond Sector (R): $\psi^\mu_+(\tau,\pi) = +\psi^\mu_-(\tau,\pi)$.
2. Neveu-Schwarz Sector (NS): $\psi^\mu_+(\tau,\pi) = -\psi^\mu_-(\tau,\pi)$.

We could have chosen $\psi^\mu_+(\tau,0) = \pm\psi^\mu_-(\tau,0)$, but these are redundant in the sense that they are not physically different, and so one imposes $\psi^\mu_+(\tau,0) = +\psi^\mu_-(\tau,0)$ by convention.

For the closed superstring, the boundary term vanishes because from the contributions of both $\psi^\mu_\pm \delta\psi_{\pm\mu}\big|_{\sigma=\pi}$ and $\psi^\mu_\pm \delta\psi_{\pm\mu}\big|_{\sigma=0}$. There are 4 different ways of doing this:

$$\psi^\mu_+(\tau,\sigma) = \pm\psi^\mu_+(\tau,\sigma+\pi)$$
$$\psi^\mu_-(\tau,\sigma) = \pm\psi^\mu_-(\tau,\sigma+\pi).$$

We thus have 4 sectors of the closed superstring theory: R-R, R-NS, NS-R, and NS-NS, where R refers to periodic and NS to anti-periodic boundary conditions.

The theory is quantised as usual by promoting the modes of the solutions of equations of motion to operators and imposing (anti)commutation relations. The Virasoro algebra is now extended to the *superconformal algebra* and the negative norm states are removed from the theory using this extended algebra and unitarity. The theory now has different vacuum for NS and R sectors, the vacuum of NS sector is a tachyon while the vacuum of the R sector is a *spacetime spinor*. Thus all spinors in the target space arise from the worldsheet spinor.

Since a fermion contributes to central charge $\frac{1}{2}$, the total centralc ahrge with D bosons X^μ and D fermions Ψ^μ is $D(1+\frac{1}{2})$. Next there are two different types of ghosts in the theory, the bc ghost coming from the gauge fixing of reparametrization invariance and the $\beta\gamma$ system coming from the definition of the BRST current. Both the ghosts contribute 11 to the central charge. Moreover the consistency of the theory requires the central charge to be $c = 26 - 11 = 15$. Thus we must have

$$D(1+\frac{1}{2}) = 15 \implies D = 10. \tag{5.64}$$

Thus the critical dimension of the superstring theories is 10.

Now in this *superconformal field theory*, the OPE of NS sector with the R sector has square root singularity and hence are nonlocal. Thus to make sense of the theory, we need to *project* to a subset of operators in the theory which are pairwise local. This process of projection is called the GSO projection. GSO projection projects out the NS vacuum and also one of the chiralities of the R vacuum. For the closed string, The NS-NS and R-R sectors have integral spin while the NS-R and R-NS sectors have half-integral spin. Also for the closed string, there are two inequivalent choices of the R vacuum giving rise to two string theories, the Type IIA and Type IIB. We will focus on the Type IIB. The massless spectrum of Type IIB is shown in table below.

RR	A_0, A_2, A_4^+
R-NS	Ψ^1_+, χ^1_-
NS-R	Ψ^2_+, χ^2_-
NS-NS	$\Phi, B_2, g_{\mu\nu}$

Table 2. Massless spectrum of Type IIB string theory

where A_n is an n-form (and $A_4^+$is self-dual), $\Psi_+^I(I=1,2)$ are right-handed dilatini, χ_-^I $(I=1,2)$ are left-handed gravitini, and the NS-NS sector is just the massless sector of the closed bosonic string spectrum. We see that the theory is chiral since the 2 dilatini and the 2 gravitini have the same chirality.

Type IIB superstring theory also has a low energy effective spacetime action and remarkably it turns out to be a theory of supergravity in $D=10$ with $\mathcal{N}=2$ supersymmetry. Moreover this the maximal supersymmetry with 32 supercharges one can have in $D=10$ supersymmetric theories containing gravitons. The full action cannot be written because of the self-dual form A_4^+ but once can write an action with $A_5=dA_4$ and other fields and then impose $*F_5=F_5$. The bosonic part of the action is

$$\begin{aligned} S_{\text{Type IIB}} = \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-\det g}e^{-2\Phi}\left(2R+8\nabla_\mu\Phi\nabla^\mu\Phi-|H_3|^2\right) \\ -\frac{1}{4\kappa^2}\int d^{10}x\left[\sqrt{-\det g}\left(|F_1|^2+|\tilde{F}_3|^2+\frac{1}{2}|\tilde{F}_5|^2\right)+A_4^+\wedge H_3\wedge F_3\right]+S_{\text{fermions}} \end{aligned} \tag{5.65}$$

where κ is the coupling constant, $F_n\equiv dA_{n-1}$ is the n-form field-strength, H_3 is the 3-form $H_3\equiv dB_2, \tilde{F}_3\equiv F_3-A_0H_3$ and $\tilde{F}_5\equiv F_5-\frac{1}{2}A_2\wedge H_3+\frac{1}{2}B_2\wedge F_3$ and the modulus squared is defined as

$$|F_n|^2=g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n}F_{\mu_1\dots\mu_n}F_{\nu_1\dots\nu_n}. \tag{5.66}$$

The Type IIB action exhibits a noncompact $\mathrm{SL}(2,\mathbb{R})$ symmetry. To describe it, transform the metric to the *Einstein frame* as follows:

$$g_{\mu\nu}\to e^{-\Phi/2}g_{\mu\nu}. \tag{5.67}$$

Combining the *axion* A_0 and the dilaton into a complex scalar $\tau=A_0+ie^{-\Phi}$, the symmetry transformation is given by:

$$\tau\to\frac{a\tau+b}{c\tau+d},\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{R}). \tag{5.68}$$

In the quantum theory there is a quantization condition $\tau\sim\tau+1$, and thus the symmetry group reduces to the subgroup $\mathrm{SL}(2,\mathbb{Z})$.

Branes in supergravity and superstring theory

We now introduce D-branes in supergravity. If we have a $p+1$ form A_{p+1}, then we can natually couple it to a $(p+1)$-dimensional hypersurface Σ_{p+1} of the spacetime via some coupling T_{p+1}. Indeed we can define the action of the coupling as

$$S_{p+1}=T_{p+1}\int_{\Sigma_{p+1}}A_{p+1}. \tag{5.69}$$

The action is clearly diffeomorphism invariant. Moreover this action is invariant under the gauge transformation

$$A_{p+1}\longrightarrow A_{p+1}+d\rho_p \tag{5.70}$$

where ρ_p is a p-form. The field strength of the $(p+1)$-form is the $(p+2)$-form $F_{p+2} = dA_{p+1}$.

Solutions to supergravity with nontrivial A_{p+1} charge are referred to as p-branes, after the space dimension of their geometry.

Each A_{p+1} gauge field has a *magnetic dual* A^{magn}_{D-3-p} which is a differential form field of rank $D-3-p$, whose field strength is related to that of A_{p+1} by Poincaré duality:

$$dA^{\text{magn}}_{D-3-p} \equiv *dA_{p+1} \tag{5.71}$$

Thus we see that each p-brane also has a magnetic dual, which is a $(D-4-p)$-brane and which now couples to the field A^{magn}_{D-3-p}. It is now clear that the p-branes in a supergravity theory depends on the form fields in the theory. For example, the low energy limit of Type IIB superstring theory, which is a supergravity theory in 10 spacetime dimensions, has the Kalb-Ramond field and hence it will have branes. The list of all branes in Type IIB superstring theory is listed in table below.

Brane	p-form	Magnetic Dual
D(−1)	$\tau = A_0 + ie^{-\Phi}$	D7
F1	B_2	NS5
D1	A_2	D5
D3	A_4^+	D3

Table 3. Branes in Type IIB superstring theory

In the above table, the terminology used is the following:

1. The brane corresponding to A_{p+1}-form is called a Dp-brane in analogy to the open string theory. It turns out that these Dp-branes are intimately related to the open string theory Dp-branes in the weak string coupling limit and that is why we use the same terminology for both of them. We refer the reader to [3] for details.

2. $D(-1)$ branes are called *instantons* since they are localised in space as well as time.

3. the 1-brane which corresponds to the 2-form NS-NS form field $B_{\mu\nu}$ is called the *fundamental string* F1. The NS in NS5 simply means that $B_{\mu\nu}$ is an NS-NS field.

We will focus on D3 branes since these are self-dual and important in AdS/CFT correspondence. Finally note that once we identify the D-branes in supergravity with the D-branes in open string theory, SU(N) gauge theories can be understood as the spectrum of stacks of N D-branes as explained at the end of Subsection 5.2.1. Moreover, it turns out that each brane solution to supergravity breaks half of the supersymmetry and hence one can recover $\mathcal{N} = 4$ super Yang-Mills with gauge group SU(N) from the spectrum of a stack of N D-branes in Type IIB superstring theory.

5.3 The Decoupling Argument and the Statement of the Correspondence

The decoupling argument due to Maldacena [7] is based on the two ways of interpretating D-branes in Type IIB superstring theory. The first interpretation is the one where D-branes are described as hypersurfaces in spacetime on which the ends of an open string are constrained by boundary conditions. The second interpretation is as solutions to the supergravity field equations in the low energy limits of the string theory. We begin by discussion these interpretations separately.

5.3.1 D-Branes as dynamical walls with open string excitations

As was indicated in Subsection 5.2.1, D-branes are dynamical objects and hence are described by an action S_{brane}. Now if we consider N D3 branes stacked over each other, then since D3 branes are 4 dimensional, as discussed before, we get 4d $\mathcal{N}=4$ super Yang-Mills theory with gauge group SU(N) from S_{brane}. The other dynamical aspects of the theory are the actions S_{bulk} which is the 10d supergravity theory with massive modes as well and the interaction between branes and supergravity S_{brane}. In the low energy limit $\alpha' \to 0$, the interactions vanish since the interaction term couples as Newton's contant $\sqrt{G_N} \sim g_s \alpha'^2$ where g_s is the string coupling constant. Thus in the low energy limit the theory decouples to two theories

$$(\mathcal{N}=4 \text{ SYM in 4d}) \oplus (\text{Type IIB Supergravity in 10d}). \tag{5.72}$$

5.3.2 D-Branes as Solutions in Supergravity

We now consider D-branes as solution to the supergravity field equations. In particular, we consider N D3 branes in supergravity. The gravitational part of the solution is given by [3]

$$ds^2 = \frac{1}{\sqrt{1+\frac{L^4}{y^4}}}\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{1+\frac{L^4}{y^4}}d\vec{y}^2 \tag{5.73}$$

where $L^4 = 4\pi g_s N \alpha'^2$, x^μ is coordinate on 4d D3-brane worldvolume and $\vec{y}$ covers Euclidean 6d space perpendicular to the brane. We need to take $L \sim \sqrt{\alpha'}(g_s N)^{\frac{1}{4}} \gg l_s \sim \sqrt{\alpha'}$ where l_s is the string length. This is because this guarantees that the curvature is large compared to the string length and hence supergravity description of the string theory is applicable and useful. But now note that this requires

$$\lambda := g_s N \gg 1. \tag{5.74}$$

Thus this description is in the opposite regime compared to the previous description in terms of gauge theory. We now consider the *Maldacena limit*. Writing

$$d\vec{y}^2 = dy^2 + y^2 d\Omega_5^2 \tag{5.75}$$

where $d\Omega_5^2$ is the round metric on S^5, the *near horizon limit* of the D3-brane solution (5.73) is

$$ds^2_{y\to 0} = \left(\frac{y^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{y^2}dy^2\right) + L^2 d\Omega_5^2 \tag{5.76}$$

This is called the Maldacena limit and was first discussed in [7]. Note that this metric is exactly the metric on $\mathrm{AdS}_5 \times S^5$ in Poincaré coordinates on the AdS space with radius L for both AdS and S^5. Moreover in the $y \to \infty$ limit the metric (5.73) becomes flat. Thus the energies at any point and at infinity are related as

$$E = \frac{1}{\sqrt{-g_{tt}}} E_\infty. \tag{5.77}$$

In particular, near the brane, we have

$$E_\infty = \frac{y}{L} E. \tag{5.78}$$

Thus for an observer at infinity, there are two decoupled theories:

1. At infintity, the theory is effectively a 10d supergravity since gravity becomes free at low energies/large distance.
2. Near the brane, the geometry of the background is $\mathrm{AdS}_5 \times S^5$ and the theory is the full Type IIB string theory.

Thus the theory decouples to

$$(\text{Type IIB String Theory on } \mathrm{AdS}_5 \times S^5) \oplus (\text{Type IIB Supergravity in 10d}). \tag{5.79}$$

Thus from (5.72), we can identify

$$\text{Type IIB String Theory on } \mathrm{AdS}_5 \times S^5 \cong \mathcal{N} = 4 \text{ SYM in 4d}. \tag{5.80}$$

The Precise Statement of the Correspondence is thus:

Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ *(both with radius* L*) with 5-form flux* N *and string coupling* g_s *is equivalent/dual to 4-dimensional* $\mathcal{N} = 4$ *SYM with gauge group* $\mathrm{SU}(N)$ *and coupling constant* g_{YM}*, where the couplings are identified as*

$$g_s = g_{YM}^2; \quad L^4 = 4\pi g_s N \alpha'^2. \tag{5.81}$$

Several checks have been performed on this correspondence. We refer the reader to [3] and references therein for the details of the checks.

References

[1] Martin Ammon, Johanna Erdmenger, *Gauge/Gravity Duality: Foundations and Applications*, Cambridge University Press, (2015).

[2] C.P. Burgess, C.A. Lutken, *Propagators and Effective Potentials in Anti-de Sitter Space*, Physics Letters B, **153**, 137–141 (1985).

[3] Eric D'Hoker and Daniel Z. Freedman, *Supersymmetric Gauge Theories and the AdS/CFT Correspondence*, ArXiv: 0201253 (2002).

[4] V. K. Dobrev and V. B. Petkova, *On The Group Theoretical Approach To Extended Conformal Supersymmetry: Classification Of Multiplets*, Lett. Math. Phys. **9**, 287 (1985).

[5] M. Henneaux and C. Teitelboim, *Asymptotically anti-De Sitter Spaces*, Commun. Math. Phys. **98** (1985) 391—424.

[6] Joseph D. Lykken, *Introduction to Supersymmetry*, ArXiv: 9612114 (1996).

[7] Juan Maldacena, *The Large N Limit of Superconformal field theories and supergravity*, ArXiv: 9711200.

[8] Ranveer K. Singh, *Introduction to String Theory*, Work in Progress, Link: https://ranveer14.github.io/String_Theory_notes.pdf.