Donaldson's Invariants and Topological Twisting of $4d$ $\mathcal{N} = 2$ Super Yang-Mills Theory

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Abstract: The classification of four manifolds upto diffeomorphism is still a wide open problem. Significant progress was made in the 1980s by Simon Donaldson when he constructed new invariants of smooth structure on four manifold, called Donaldson invariants. Physics made contact with four manifolds classification program in 1988 when Edward Witten came up with a physical interpretation of Donaldson invariants. Witten introduced the idea of topological twisting of supersymmetric Yang-Mills theory and showed that certain correlators in the topologically twisted theory are precisely the Donaldson invariants. In this exposition, we discuss the topological twisting of the $4d$ $\mathcal{N} = 2$ pure supersymmetric Yang-Mills theory and sketch the relation of Donaldson invariants with the twisted theory.

Contents

1 Introduction

A topological n-manifold X is a topological space which locally looks like $\mathbb{R}^n$. More precisely, for every $x \in X$ there is an open set U such that $x \in U \subset X$ and a homeomorphism $\varphi : U \longrightarrow \varphi(U)$. We call (U, φ) a *chart*. The set of all charts (φ_i, U_i) such that $\cup_i U_i = X$ is called an *atlas* of X. A *smooth structure* on X is an atlas (φ_i, U_i) such that for overlapping charts $(U_i, \varphi_i), (U_j, \varphi_j)$ with $U_{ij} \equiv U_i \cap U_j \neq \emptyset$, the *transition functions* $\varphi_j \circ \varphi_i^{-1}, \varphi_i \circ \varphi_j^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is infinitely differentiable. A topological manifold with a smooth structure is called a *smooth manifold* or simply a manifold. One can define the notion of differentiability of maps between manifolds unambiguously using a smooth structure, that is smooth structure allows us to do calculus on a manifold. A diffeomorphism of manifolds is a bijective map which is differentiable along with its inverse.

Given a topological manifold, one can ask (1) whether it can be endowed with a smooth structure and if so, (2) how many non-diffeomorphic smooth structures can the manifold be endowed with. In dimensions other than four, any compact topological manifold has a finite number of non-diffeomorphic smooth structures [1]. In contrast, any compact 4-manifold can have countably infinite number of non-diffeomorphic smooth structures. A typical example of noncompact case is $\mathbb{R}^4$ itself which admits an uncountably many non-diffeomorphic smooth structure [1]. 4-manifolds are important for physics for various reasons, one obvious reason being general theory of relativity where we work with smooth 4 dimensional spacetime and we would like to know how many choices of smooth structures we have to work with.

In this exposition, we will be interested in the second question, since in physics we always work with smooth manifolds. Simon Donaldson's work gives some partial answers to the two questions [2–4]. In particular, Donaldson constructed new invariants of smooth structure on a 4-manifold [2, 5], called the Donaldson invariants, meaning that two diffeomorphic 4-manifolds must have the same Donaldson invariants. Equivalently, if two manifolds have different Donaldson invariants, they cannot be diffeomorphic. Thus Donaldson invariants can be used to distinguish between differentiable structures on a manifold. In the rest of the exposition, X will denote a compact smooth Riemannian 4-manifold without boundary. The construction of Donaldson invariants for X roughly goes as follows: we consider the space of all gauge fields A_μ which are 1-forms on X valued in the Lie algebra $\mathfrak{g}$ of the gauge group G (see Appendix C for precise definitions) such that the field strength $F_{\mu\nu}$ which is a 2-form is *anti-selfdual*, that is $F_{\mu\nu} + \widetilde{F}_{\mu\nu} = 0$ where $\widetilde{F}_{\mu\nu}$ is the Hodge dual of $F_{\mu\nu}$ (see (3.26)). These gauge fields are called *instantons*. We then restrict to gauge inequivalent instantons, this space is called the *moduli space of instantons* and is denoted by $\mathcal{M}_{\text{ASD}}$ (see Appendix D for more details). The next step is the construction of the *Donaldson map* $\mu_D : H_*(X) \longrightarrow H^*(\mathcal{M}_{\text{ASD}})$ from the homology[1] of X to the cohomology on $\mathcal{M}_{\text{ASD}}$ (see [7] for some details). For the gauge group SU(2), $\mu_D : H_j(X) \longrightarrow H^{4-j}(\mathcal{M}_{\text{ASD}})$. Donaldson invariants are then defined on $H_0(X) \oplus H_2(X)$: for $\ell, r \in \mathbb{N}$

$$(x, S) \longmapsto P_D(x^\ell S^r) \equiv \int_{\mathcal{M}_{\text{ASD}}} \mu_D(x)^\ell \mu_D(S)^r \tag{1.1}$$

where $\mu_D(x)^\ell \equiv \wedge^\ell \mu_D(x) \equiv \mu_D(x) \wedge \mu_D(x) \wedge \cdots \wedge \mu_D(x)$, ℓ number of times. Clearly the integral is nonvanishing only when $4\ell + 2r = \dim\mathcal{M}_{\text{ASD}}$. There is another technical point in the calculation of the invariants: we need to sum over *instanton numbers* of the instanton configurations and fix the *second Stiefel-Whitney class* of X while performing the integral. See Section 3.2 for more details. The stunning result of Donaldson says that $P_D(x^\ell S^r)$ are rational numbers and in most cases independent of the metric on X. So to distinguish between smooth structures on X, we need to calculate these invariants with the two smooth structures and if they are different, Donaldson's theorem implies that the two smooth structures are non-diffeomorphic. One cannot say anything if the invariants are the same, meaning that the Donaldson invariants are not *complete invariants* of the smooth structure on X.

[1]See [6, Chapter 6] for definition of homology and cohomology groups.

Up until now, our discussion was completely mathematical. Michael Atiyah (who was Simon Donaldson's advisor), realising the intimate role of Yang-Mills theory in the formulation of Donaldson invariants, posed a question to physicists: What is the physical interpretation of Donaldson invariants? Edward Witten rose up to the challenge and in 1988 came up with a physical formulation [8] of Donaldson invariants: these are simply the correlation functions of appropriate operators of a certain quantum field theory[2] (QFT), called the *Donaldson-Witten theory*, giving a path integral representation for the invariants. In this exposition, we will only describe the Donaldson-Witten theory but this is not the end of the story.

In his landmark paper [8], Witten introduced the idea of topological twisting to construct *topological quantum field theory* (TQFT). Roughly speaking, a TQFT is a QFT in which the partition function and some of the correlators do not depend on the metric. Topological twisting[3] when applied to certain 4d theories with $\mathcal{N} = 2$ supersymmetry give us TQFTs.

This exposition is organised as follows: In Section 2, we explain TQFTs and describe the topological twisting procedure. We then discuss the particular TQFT – Donaldson-Witten theory studied by Witten to relate to Donaldson invariants in Section 3. In the appendices, we describe some of the details of the construction, definitions and set up some notations.

2 Topological Quantum Field Theory

Let X be a compact, connected Riemannian 4-manifold with metric $g_{\mu\nu}$. Suppose we have a quantum field theory (QFT) with fields $\{\phi\}$ and action functional $S[\phi]$. The partition function of the theory is defined as

$$Z = \int [D\phi] e^{-\frac{1}{\hbar} S[\phi]} \tag{2.1}$$

The correlation function of operators $\mathcal{O}_1, \cdots, \mathcal{O}_n$ is

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \frac{1}{Z} \int [D\phi] \mathcal{O}_1(\phi) \cdots \mathcal{O}_n(\phi) e^{-\frac{1}{\hbar} S[\phi]} \tag{2.2}$$

In general, the action might depend on the metric $g_{\mu\nu}$. The quantity that measures this dependence is the energy momentum tensor[4]

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}. \tag{2.3}$$

[2] To be more precise topologically twisted $\mathcal{N} = 2$ pure SU(2) supersymmetric Yang-Mills theory. See Section 3.1 for more details.

[3] There are other ways of twisting in other dimensions and more number of supersymmetries but in this note we will restrict to twisting of $4d$ $\mathcal{N} = 2$ theories.

[4] In general relativity, the energy momentum tensor is defined with an extra factor of $4\pi/\sqrt{g}$ where $g = \det g_{\mu\nu}$. This will not be important for our discussion in this section.

In quantum theory the dependence of the action on the metric is measured by the expectation of the energy momentum tensor:

$$\langle T_{\mu\nu}\rangle = \frac{1}{Z}\int [D\phi]\frac{\delta S}{\delta g^{\mu\nu}}e^{-\frac{1}{\hbar}S[\phi]} = -\hbar\frac{\delta \log Z}{\delta g^{\mu\nu}}. \tag{2.4}$$

In general the expectation value $\langle T_{\mu\nu}\rangle \neq 0$ and hence the partition function also depends on the metric.

Definition 2.1. A topological quantum field theory (TQFT) is a special type of QFT in which the partition function and some of the correlators do not depend on the metric. In particular

$$\langle T_{\mu\nu}\rangle = 0. \tag{2.5}$$

There are two types of TQFTs:

1. *TQFT of Schwartz type*: the action does not depend on the metric (classically), that is, there is a classical symmetry associated to varying the metric. We construct these TQFTs in a way that this classical symmetry does not suffer anomaly meaning that the measure $[D\phi]$ remains invariant under a change in the metric.

2. *Cohomological TQFTs or TQFTs of Witten type*: the classical theory may have metric dependence but the partition function is metric independent. This is achieved by a highly nontrivial procedure introduced by Witten [8] called *topological twisting* and will be the main focus of this exposition.

2.1 Witten type TQFTs

We start with the following data for Witten type TQFTs:

Data:

1. A theory of fields $\{\phi\}$ which are a collection of Grassmann even and odd fields with action $S[\phi]$.

2. A scalar non-anomalous symmetry[5] δ of the action, that is $\delta S[\phi] = 0$.

Assumptions:

1. The symmetry δ is Grassmannian. That is it maps Grassmann even fields to Grassmann odd fields and vice versa. Moreover δ does not change the spin of the field. This QFT thus does not satisfy spin-statistics theorem since integral spin fields may be Grassmann odd.

2. $T_{\mu\nu} = \delta G_{\mu\nu}$ for some Grassmann odd field $G_{\mu\nu}$.

3. There is a set of operators $\{\mathcal{O}_i\}$ such that $\delta\mathcal{O}_i = 0$.

[5] By scalar symmetry we mean that the generator δ of the symmetry transforms as a scalar under the Lorentz group.

4. $\delta^2 = \mathcal{L}_V$ where $\mathcal{L}_V$ is some scalar symmetry of the theory: $\mathcal{L}_V S[\phi] = 0$. This notation just means that $\mathcal{L}_V$ is the Lie derivative along a vector field V.

Note that although we do not require the scalar symmetry $\delta^2 = 0$, in cohomology theory assumption 2 and 3 would simply be that $T_{\mu\nu}$ is δ exact and there are some δ-closed operators. Infact since $\delta^2 = \mathcal{L}_V$, we would be dealing with what is called *equivariant cohomology* but it is not the focus of this exposition. These assumptions have a remarkable consequence: The partition function Z and the correlators $\langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle$ of δ-closed operators is metric independent. Indeed

$$\begin{aligned}\frac{\delta Z}{\delta g^{\mu\nu}} &= \int [D\phi] e^{-\frac{1}{\hbar}S[\phi]} \left(-\frac{1}{\hbar} \frac{\delta S}{\delta g^{\mu\nu}} \right) = -\frac{1}{\hbar} \int [D\phi] e^{-\frac{1}{\hbar}S[\phi]} T_{\mu\nu} \\ &= -\frac{1}{\hbar} \int [D\phi] e^{-\frac{1}{\hbar}S[\phi]} \delta G_{\mu\nu} \\ &= -\frac{1}{\hbar} \int [D\phi] \delta \left(e^{-\frac{1}{\hbar}S[\phi]} G_{\mu\nu} \right) \\ &= 0,\end{aligned}$$

where we assumed that δ is non-anomalous and the fields die on the boundary. Similarly using the fact that $\delta \mathcal{O}_{i_k} = 0$ and Z is metric independent, we have

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0. \tag{2.6}$$

Suppose $\mathcal{O} = \delta\Sigma$ for some operator Σ, that is, $\mathcal{O}$ is δ-exact. Then it is easy to see that

$$\begin{aligned}\langle \mathcal{O} \rangle &= 0 \\ \langle \mathcal{O}\mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle &= 0\end{aligned} \tag{2.7}$$

for δ-closed operators $\mathcal{O}_{i_k}$. Thus any δ-exact operator is not relevant since all its correlators are zero. Thus the relevant operators are δ-closed operators modulo δ-exact i,e the cohomology class of δ:

$$\text{Topological operators} \equiv \frac{\text{Ker } \delta}{\text{Im } \delta}. \tag{2.8}$$

Note that this definition does not make sense unless $\text{Im}\delta \subseteq \text{Ker}\delta$. Since $\delta^2 = \mathcal{L}_V$, we only consider operators which are invariant under $\mathcal{L}_V$ to make sense of the definition above.

2.1.1 Descent procedure

Given $\mathcal{O}$ a topological operator and γ_n a representative from the n^{th} homology class of X, that is an n-dimensional submanifold of X. Then we can define a new topological operators $W^{\mathcal{O}}_{\gamma_n}$ using descent equation which we now describe. Recall that $T_{0\mu} = P_\mu$ is the momentum operator which generates translations on X. It is represented on fields by the partial derivative $P_\mu = -i\partial_\mu$. Put $G_\mu \equiv G_{0\mu}$ so that

$$P_\mu = \{\delta, G_\mu\}. \tag{2.9}$$

Define

$$\mathcal{O}^{(n)}_{\mu_1 \ldots \mu_n} = \left[G_{\mu_1}, \left[G_{\mu_2}, \cdots \left[G_{\mu_n}, \mathcal{O} \right]_\epsilon \right]_\epsilon \right]_\epsilon, \quad n = 1, \ldots, 4. \tag{2.10}$$

where $[.,.]_\epsilon$ is the graded commutator, that is, anticommutator if both the operators are fermionic ($\epsilon = -1$) and commutator otherwise ($\epsilon = 0$). Now since P_μ is Grassmann even, G_μ is Grassmann odd, which means $\mathcal{O}^{(n)}_{\mu_1\cdots\mu_n}$ is a completely antisymmetric tensor. Thus

$$\mathcal{O}^{(n)} \equiv \mathcal{O}^{(n)}_{\mu_1\cdots\mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{2.11}$$

is an n-form. Now

$$\begin{aligned}
d\mathcal{O}^{(n)} &= \partial_\mu \mathcal{O}^{(n)}_{\mu_1\cdots\mu_n} dx^\mu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} \\
&= i\left[P_\mu, \mathcal{O}^{(n)}_{\mu_1\cdots\mu_n}\right]_\epsilon dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\
&= i\left[\{\delta, G_\mu\}, \left[G_{\mu_1}, \left[G_{\mu_2}, \cdots \left[G_{\mu_n}, \mathcal{O}\right]_\epsilon\right]_\epsilon\right]_\epsilon\right]_\epsilon dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\
&= i\left[\delta, \left[G_\mu, \left[G_{\mu_1}, \cdots \left[G_{\mu_n}, \mathcal{O}\right]_\epsilon\right]_\epsilon\right]_\epsilon\right]_\epsilon dx^\mu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \\
&= i\delta\mathcal{O}^{(n+1)}
\end{aligned}$$

where we used the Jacobi identity multiple times. The equation

$$d\mathcal{O}^{(n)} = i\delta\mathcal{O}^{(n+1)} \tag{2.12}$$

is called the *descent equation*. Now define the operator

$$W^{\mathcal{O}}_{\gamma_n} \equiv \int_{\gamma_n} \mathcal{O}^{(n)}. \tag{2.13}$$

Using the descent equation, we have

$$\delta W^{\mathcal{O}}_{\gamma_n} = \int_{\gamma_n} \delta\mathcal{O}^{(n)} = -i\int_{\gamma_n} d\mathcal{O}^{(n-1)} = -i\int_{\partial\gamma_n} \mathcal{O}^{(n-2)} = 0.$$

since $\partial\gamma_n = \emptyset$ for the homology class. If $\widetilde{\gamma}_n = \gamma_n + \partial\gamma_{n+1}$ then by Stokes' theorem

$$\begin{aligned}
W^{\mathcal{O}}_{\widetilde{\gamma}_n} &= \int_{\gamma_n} \mathcal{O}^{(n)} + \int_{\partial\gamma_{n+1}} \mathcal{O}^{(n)} \\
&= W^{\mathcal{O}}_{\gamma_n} + \int_{\gamma_{n+1}} d\mathcal{O}^{(n)} \\
&= W^{\mathcal{O}}_{\gamma_n} + i\delta W^{\mathcal{O}}_{\gamma_{n+1}}.
\end{aligned} \tag{2.14}$$

Thus $W^{\mathcal{O}}_{\widetilde{\gamma}_n}$ is same as $W^{\mathcal{O}}_{\gamma_n}$ upto a δ-exact operator and hence they are in the same δ-cohomology class. Thus the operator $W^{\mathcal{O}}_{\gamma_n}$ is a topological operator. The descent equation thus gives us a one-to-one map from the homology classes on X to a family of topological operators of the TQFT on X. In the coming sections, we will see how the symmetry δ is intimately connected to supersymmetry via topological twisting procedure.

2.2 Topological Twisting of $\mathcal{N} = 2$ Theories

Suppose we want to put a QFT on a general curved Riemannian 4-manifold. The usual procedure to promote a theory from flat space to curved space is to define the action using the minimal coupling to gravity prescription in which partial derivative ∂_μ changes to covariant derivative with Levi-Civita connection or spin connection[6] depending on whether the field is a differential form or a spinor. But not every 4-manifold is spin (see Appendix C for definition of spin manifold) and hence any supersymmetric theory does not make sense on general Riemannian 4-manifold. Topological twisting is a procedure to construct a Witten type TQFT which can be defined on any general 4-manifold (with some constriants in some cases) from a supersymmetric theory (see Appendix A for details about supersymmetry) on $\mathbb{R}^4$. A supersymmetric QFT has a larger symmetry algebra, namely the super Poincaré algebra which is an extension of the Poincaré algebra by fermionic charges, called supercharges, satisfying the supersymmetry algebra. We are interested in $\mathcal{N}=2$ theories where we have a pair of supercharges $Q_{\alpha I},\ I=1,2$ and their conjugates $\overline{Q}_{\dot{\alpha} I},\ I=1,2$. We will be working with Euclidean[7] signature, so the Lorentz group is SO(4) and spinors are representations of Spin(4)$\cong$ SU$(2)_+\times$SU$(2)_-$. In particular α is an SU$(2)_-$ index and $\dot{\alpha}$ is an SU$(2)_+$ index in the fundamental representation of SU(2). The Euclidean supersymmetry[8] algebra is then

$$\begin{aligned}
&\{Q_{\alpha I},\overline{Q}_{\dot{\beta}J}\}=2i\epsilon_{IJ}\sigma^{\mu}_{\alpha\dot{\beta}}P_\mu, && \{Q_{\alpha I},Q_{\beta J}\}=0,\\
&[P_\mu,Q_{\alpha I}]=0, && [P_\mu,\overline{Q}_{\dot{\alpha}I}]=0,\\
&[M_{\mu\nu},Q_{\alpha J}]=-\left(\sigma_{\mu\nu}\right)_{\alpha}{}^{\beta}Q_{\beta J}, && [M_{\mu\nu},\overline{Q}^{\dot{\alpha}J}]=-\left(\overline{\sigma}_{\mu\nu}\right)^{\dot{\alpha}}{}_{\dot{\beta}}\overline{Q}^{\dot{\beta}J}
\end{aligned} \tag{2.15}$$

where $P_\mu, M_{\mu\nu}$ are the usual momentum and Lorentz generators, $\epsilon^{IJ}\left(\epsilon_{12}=1,\epsilon_{IJ}\epsilon^{JK}=\delta_I^K\right)$ is totally antisymmetric tensor and is used to lower and raise the capital indices I,J on the supercharges, $\sigma^{\mu}_{\alpha\dot{\beta}}=(\sigma^1,\sigma^2,\sigma^3,i\mathbb{1})_{\alpha\dot{\beta}}$ where

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \tag{2.16}$$

are the Pauli matrices. Finally

$$\left(\sigma_{\mu\nu}\right)_{\alpha}^{\ \beta}=\frac{1}{4}(\sigma_\mu\overline{\sigma}_\nu-\sigma_\nu\overline{\sigma}_\nu)_{\alpha}^{\ \beta},\quad \left(\overline{\sigma}_{\mu\nu}\right)^{\dot{\beta}}_{\ \dot{\alpha}}=\frac{1}{4}(\overline{\sigma}_\mu\sigma_\nu-\overline{\sigma}_\mu\sigma_\mu)^{\dot{\beta}}_{\ \dot{\alpha}} \tag{2.17}$$

where $\overline{\sigma}_\mu=(-\sigma^1,-\sigma^2,-\sigma^3,i\mathbb{1})$. The indices $\alpha,\dot{\alpha}$ are raised and lowered using $\epsilon_{\alpha\beta}=-\epsilon^{\alpha\beta}=\epsilon_{\dot{\alpha}\dot{\beta}}=-\epsilon^{\dot{\alpha}\dot{\beta}}=i(\sigma^2)_{\alpha\beta}$. There is another SU(2)$\times$U(1) symmetry where SU(2) acts on the indices I,J and U(1) acts by a global phase. This is called the R-symmetry. The total symmetry group of the theory is thus

$$\Gamma=\mathrm{SU}(2)_-\times\mathrm{SU}(2)_+\times\mathrm{SU}(2)_R\times\mathrm{U}(1)_R. \tag{2.18}$$

Under the symmetry group Γ, the supersymmetry generators $Q_{\alpha I}$ and $\overline{Q}_{\dot{\alpha}I}$ transform as $(\mathbf{2},\mathbf{1},\mathbf{2})^{-1}$ and $(\mathbf{1},\mathbf{2},\mathbf{2})^{1}$ respectively.[9] The superscript are the U$(1)_R$-charges and is also

[6] Spinors on a curved space couple to gravity via the spin connection. See [6, Chapter 7] for a more concrete discussion.

[7] Euclideanisation amounts to changing $v^0\to -iv^4$ of a 4-vector v^μ.

[8] We are considering supersymmetry with zero central charge, see Appendix A for details.

[9] The notation $(\mathbf{2},\mathbf{1},\mathbf{2})^{-1}$ means that $Q_{\alpha I}$ is scalar under SU$(2)_+$, vector under SU$(2)_-$, SU$(2)_R$ and has charge -1 under U$(1)_R$.

called the *ghost number*.

Suppose now we want to define the theory on a general smooth, compact, oriented Riemannian 4-manifold X. As we described above, it is not possible to do so if X is not spin. The idea of topological twisting is to change the coupling of various fields to gravity according to their R-symmetry transformation. That is the fields are coupled to the $\mathrm{SU}(2)_+$ spin connection according to their transformation under $\mathrm{SU}(2)_R$. This means that we identify the R-symmetry index I with the $\mathrm{SU}(2)_+$ index $\dot{\alpha}$ and change the covariant derivatives on the fields in the action accordingly. To be more precise, we identify a new rotation group $\Gamma' = \mathrm{SU}(2)_- \times \mathrm{SU}'(2)_+$ where $\mathrm{SU}'(2)_+ \cong \mathrm{Diag}(\mathrm{SU}(2)_+ \times \mathrm{SU}(2)_R)$. After the twisting the indices on the supercharges become $Q_{\alpha I} \to Q_{\alpha\dot{\beta}}$ and $\overline{Q}_{\dot{\alpha}I} \to \overline{Q}_{\dot{\alpha}\dot{\beta}}$. This means that the supercharges now transform as $(\mathbf{2},\mathbf{2})^{-1}$ and $(\mathbf{1},\mathbf{2}\otimes\mathbf{2})^{1}$. Thus $Q_{\alpha\dot{\beta}}$ transforms as a vector

$$Q_{\alpha\dot{\beta}} \to G'_\mu := (\overline{\sigma}_\mu)^{\dot{\beta}\alpha\dot{\beta}} Q_{\alpha\dot{\beta}} \tag{2.19}$$

and $Q_{\dot{\alpha}\dot{\beta}}$ can be decomposed into its symmetric and antisymmetric part. The scalar

$$\overline{\mathcal{Q}} := \epsilon^{\dot{\alpha}\dot{\beta}} \overline{Q}_{\dot{\alpha}\dot{\beta}} \tag{2.20}$$

is particularly important since it generates the scalar symmetry δ required in a TQFT of Witten type. If we define

$$G_\mu := -\frac{i}{4} G'_\mu \tag{2.21}$$

then using the supersymmetry algebra[10] we see that

$$\begin{aligned} \{\overline{\mathcal{Q}}, G_\mu\} &= -\frac{i}{4}(\overline{\sigma}_\mu)^{\dot{\gamma}\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\{\overline{Q}_{\dot{\alpha}\dot{\beta}}, Q_{\alpha\dot{\gamma}}\} = -\frac{1}{2}(\overline{\sigma}_\mu)^{\dot{\gamma}\alpha}\epsilon^{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\beta}\dot{\gamma}}(\sigma^\nu)_{\dot{\alpha}\alpha}P_\nu \\ &= \frac{1}{2}2\delta^\nu_\mu P_\nu = P_\mu, \end{aligned} \tag{2.22}$$

where we used $(\bar{\sigma}_\mu)^{\dot{\alpha}\beta}(\sigma_\nu)_{\beta\dot{\alpha}} = -2\delta_{\mu\nu}$. Moreover

$$\overline{\mathcal{Q}}^2 = 0. \tag{2.23}$$

Thus we have almost constructed a TQFT of Witten type. We will now implement this procedure to construct a specific TQFT called the *Donaldson-Witten theory* which gives a physical interpretation of the Donaldson invariants.

3 Donaldson-Witten Theory

The topological twisting procedure described above applied to the $\mathcal{N} = 2$ pure supersymmetric Yang-Mills theory is called the *Donaldson-Witten twist*. As we will see, after twisting every field changes to bosonic field and hence makes sense on any general 4-manifold.

[10] Note that under a symmetry transformation whose classical generator is G_a and the conserved charge (that is the quantum generator) is Q_a, the transformation of fields is related by $\delta_\omega\Phi = -i\omega_a G_a\Phi$, $[Q_a, \Phi] = -iG_a\Phi$ where ω_a is the parameter of the transformation. See [9, Chapter 2] for more details.

3.1 Donaldson-Witten twist

The field content[11] of $\mathcal{N}=2$ pure supersymmetric Yang-Mills theory is a gauge field A_μ, two spinors $\lambda_{\alpha I},\ I=1,2$ and a complex scalar ϕ. There are three auxilary fields packed into the symmetric matrix D^{IJ}. Under twisting the fields change as[12]

$$\begin{aligned}
&A_\mu(\mathbf{2},\mathbf{2},\mathbf{1})^0 \to A_\mu(\mathbf{2},\mathbf{2})^0,\\
&\lambda_{\alpha I}(\mathbf{2},\mathbf{1},\mathbf{2})^1 \to \psi_{\alpha\dot\beta}(\mathbf{2},\mathbf{2})^1,\\
&\bar\lambda_{\dot\alpha I}(\mathbf{1},\mathbf{2},\mathbf{2})^{-1} \to \eta(\mathbf{1},\mathbf{1})^{-1},\quad \chi_{\dot\alpha\dot\beta}(\mathbf{1},\mathbf{3})^{-1},\\
&\phi(\mathbf{1},\mathbf{1},\mathbf{1})^{-2} \to \phi(\mathbf{1},\mathbf{1})^{-2},\\
&\phi^\dagger(\mathbf{1},\mathbf{1},\mathbf{1})^{2} \to \phi^\dagger(\mathbf{1},\mathbf{1})^{2},\\
&D_{IJ}(\mathbf{1},\mathbf{1},\mathbf{3})^0 \to D_{\dot\alpha\dot\beta}(\mathbf{1},\mathbf{3})^0,
\end{aligned} \tag{3.1}$$

On the right hand side the representation is with respect to $\Gamma'=\mathrm{SU}(2)_-\times\mathrm{SU}'(2)\times\mathrm{U}(1)_R$. Note that $\lambda_{\alpha I}$ which was a spinor in the untwisted theory is a vector $\psi_{\alpha\dot\beta}$ (written as a bispinor, that is $\psi_\mu\equiv(\bar\sigma_\mu)^{\alpha\dot\beta}\psi_{\alpha\dot\beta}$ is a vector, see [10, Appendix A] for details), the right handed spinor $\bar\lambda_{\dot\alpha I}$ turns into $\bar\lambda_{\dot\alpha\dot\beta}$ which decomposes as the antisymmetric piece η and the symmetric piece $\chi_{\dot\alpha\dot\beta}$:

$$\eta\equiv\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\alpha\dot\beta}=\bar\lambda_{[\dot1\dot2]}=-\bar\lambda_{[\dot2\dot1]},\quad \chi_{\dot\alpha\dot\beta}=\bar\lambda_{(\dot\alpha\dot\beta)}\equiv\frac{1}{2}\left(\bar\lambda_{\dot\alpha\dot\beta}+\bar\lambda_{\dot\beta\dot\alpha}\right) \tag{3.2}$$

$\chi_{\dot\alpha\dot\beta}$ gives rise to a 2-form[13] $\chi_{\mu\nu}\equiv(\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\chi_{\dot\alpha\dot\beta}$. Note that all fields on the right hand side are bosonic. Let us now write down the action for the twisted theory. The action is the same as (A.28) with some crucial changes:

1. Since we are on a general Riemannian manifold with metric $g_{\mu\nu}$, the integral measure changes
$$d^4x\longrightarrow d^4x\sqrt{g};\quad \sqrt{g}=\det\left(g_{\mu\nu}\right)^{\frac{1}{2}} \tag{3.3}$$
2. We follow minimal coupling to gravity and now since all fields are differential forms, the partial derivatives change to covariant derivative with Levi-Civita connection.

The action takes the form

$$\begin{aligned}
S=\frac{1}{e^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\Big(&\nabla_\mu\phi\nabla^\mu\phi^\dagger+i\chi_{\dot\alpha\dot\beta}\bar\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\psi_\alpha^{\dot\beta}+i\eta\nabla^{\dot\alpha\alpha}\psi_{\alpha\dot\alpha}-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{4}D_{\dot\alpha\dot\beta}D^{\dot\alpha\dot\beta}\\
&-\frac{1}{2}\left[\phi,\phi^\dagger\right]^2-\frac{i}{\sqrt2}\chi^{\dot\alpha\dot\beta}\left[\phi^\dagger,\chi_{\dot\alpha\dot\beta}\right]+i\sqrt2\eta[\phi^\dagger,\eta]-\frac{i}{\sqrt2}\psi_{\alpha\dot\alpha}\left[\psi^{\alpha\dot\alpha},\phi^\dagger\right]\Big).
\end{aligned} \tag{3.4}$$

Let us see how we get this action from (A.28). The first term remains unchanged. The second term becomes

$$-i\lambda_\alpha^I\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\bar\lambda_{\dot\alpha I}\longrightarrow -i\psi_\alpha^{\dot\beta}\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\bar\lambda_{\dot\alpha\dot\beta} \tag{3.5}$$

[11] See Appendix A for the detailed construction.

[12] We call ϕ as a complex scalar just because $\phi,\phi^\dagger$ are two real degrees of freedom in Euclidean signature. They are *not* related by complex conjugation.

[13] To be more precise, a selfdual 2-form, see [10, Appendix A] for details

Now we have

$$\begin{aligned}
-i\psi^{\dot\beta}_{\alpha}\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\bar\lambda_{\dot\alpha\dot\beta} &= -i\psi^{\dot\beta}_{\alpha}\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\left(\bar\lambda_{(\dot\alpha\dot\beta)}+\bar\lambda_{[\dot\alpha\dot\beta]}\right)\\
&= -i\psi^{\dot 2}_{\alpha}\sigma^{\mu\dot 1\alpha}\nabla_\mu\bar\lambda_{[\dot 1\dot 2]} - i\psi^{\dot 1}_{\alpha}\sigma^{\mu\dot 2\alpha}\nabla_\mu\bar\lambda_{[\dot 2\dot 1]} - i\psi^{\dot\beta}_{\alpha}\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\chi_{\dot\alpha\dot\beta}\\
&= -i\epsilon^{\dot\alpha\dot 2}\psi_{\alpha\dot\alpha}\sigma^{\mu\dot 1\alpha}\nabla_\mu\eta + i\epsilon^{\dot\alpha\dot 1}\psi_{\alpha\dot\alpha}\sigma^{\mu\dot 2\alpha}\nabla_\mu\eta - i\psi^{\dot\beta}_{\alpha}\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\chi_{\dot\alpha\dot\beta}\\
&= -i\psi_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha}\eta - i\psi^{\dot\beta}_{\alpha}\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\chi_{\dot\alpha\dot\beta}.
\end{aligned} \tag{3.6}$$

Similarly we can check all other terms. Another thing to note is that $\nabla_\mu\chi_{\dot\alpha\dot\beta}$ contains the Levi-Civita connection for a 2-form and $\nabla^{\alpha\dot\alpha}\eta$ only has the gauge connection since $\chi_{\dot\alpha\dot\beta}$ is a 2-form and η is a scalar. We now list the action of the topological charge $\overline{\mathcal{Q}}$ on the fields. This can be obtained using the supersymmetry transformations of the fields given in (A.27)

$$\begin{aligned}
&[\overline{\mathcal{Q}},\phi]=0, \quad [\overline{\mathcal{Q}},A_\mu]=\psi_\mu,\\
&[\overline{\mathcal{Q}},\phi^\dagger]=2\sqrt{2}i\eta, \quad \{\overline{\mathcal{Q}},\eta\}=[\phi,\phi^\dagger],\\
&\{\overline{\mathcal{Q}},\chi_{\dot\alpha\dot\beta}\}=i(F^+_{\dot\alpha\dot\beta}-D_{\dot\alpha\dot\beta}), \quad \{\overline{\mathcal{Q}},\psi_\mu\}=2\sqrt{2}\nabla_\mu\phi,\\
&[\overline{\mathcal{Q}},D]=(2\nabla\psi)^+ + 2\sqrt{2}[\phi,\chi].
\end{aligned} \tag{3.7}$$

where $\psi_\mu=(\sigma_\mu)^{\alpha\dot\beta}\psi_{\alpha\dot\beta}$ and $F^+_{\dot\alpha\dot\beta}=(\sigma^{\mu\nu})_{\dot\alpha\dot\beta}F_{\mu\nu}$ is the self-dual part of $F_{\mu\nu}$ (see [10, Appendix A] for details). Observe that

$$\left[\overline{\mathcal{Q}},\left[\overline{\mathcal{Q}},A_\mu\right]\right]=2\sqrt{2}\nabla_\mu\phi \tag{3.8}$$

that is $\overline{\mathcal{Q}}^2$ is a gauge transformation. One can check this for other fields as well. This might seem to contradict $\overline{\mathcal{Q}}^2=0$ of (2.23) when central charge is 0. This is because the fields above are in the Wess-Zumino gauge and supersymmetry takes us out of the gauge giving us gauge equivalent field configurations. Thus $\overline{\mathcal{Q}}^2$ is closed only upto a gauge transformation. See Appendix A for details. Next important observation is that

$$S=\frac{1}{e^2}\{\overline{\mathcal{Q}},V\}-\frac{1}{2e^2}\int \mathrm{Tr}(F\wedge F) \tag{3.9}$$

where

$$V=\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{i}{4}\chi^{\dot\alpha\dot\beta}\left(F^+_{\dot\alpha\dot\beta}+D_{\dot\alpha\dot\beta}\right)-\frac{1}{2}\eta\left[\phi,\phi^\dagger\right]+\frac{1}{2\sqrt{2}}\psi_{\alpha\dot\alpha}\nabla^{\dot\alpha\alpha}\phi\right). \tag{3.10}$$

and the second term is topological, i,e only depends on the topology of X

$$-\frac{1}{8\pi^2}\int \mathrm{Tr}(F\wedge F)=k\in\mathbb{R} \tag{3.11}$$

The number k is called the *instanton number* (see Appendix C for details). This immediately implies that the stress tensor is $\overline{\mathcal{Q}}$-exact:

$$T_{\mu\nu}=\frac{\delta S}{\delta g^{\mu\nu}}=\left\{\overline{\mathcal{Q}},\frac{\delta V}{\delta g^{\mu\nu}}\right\}. \tag{3.12}$$

Thus if we fix the topology of X, then the extra piece is just a harmless constant and can be ignored in the calculation of the path integral. We will sum over all instanton numbers at the end to include all inequivalent gauge configurations. It turns out that there is one more quantity called the *second Stiefel-Whitney class* $w_2(X)$ (see Appendix C for definition) which characterises the gauge configurations completely. One can study spinors on X only if the second Stiefel-Whitney class is trivial. But since the twisted theory does not contain spinors, we should consider X with general nonzero second Stiefel-Whitney class. To emphasize this, we sometimes put a subscript w in the correlation functions. So for us, correlation functions[14] are

$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle_w = \int [DX] \mathcal{O}_1 \dots \mathcal{O}_n e^{-\frac{1}{e^2} S_{DW}} \tag{3.13}$$

where $S_{DW} = \{\overline{\mathcal{Q}}, V\}$ is the *Donaldson-Witten action*. Here $[DX]$ contains all fields appearing in $\mathcal{O}_1 \dots \mathcal{O}_n, S_{DW}$. The $\overline{\mathcal{Q}}$-exactness of S_{DW} has a very important implication for the exact calculation of path integrals: if $\mathcal{O}_1 \dots \mathcal{O}_n$ are $\overline{\mathcal{Q}}$-closed operators, then

$$\frac{\partial}{\partial e} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = 0 \tag{3.14}$$

Indeed

$$\begin{aligned} \frac{\partial}{\partial e} \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle &= \int [DX] \mathcal{O}_1 \dots \mathcal{O}_n \left(\frac{2}{e^3} S_{DW} \right) e^{-\frac{1}{e^2} S_{DW}} \\ &= \frac{2}{e^3} \langle \mathcal{O}_1 \dots \mathcal{O}_n S_{DW} \rangle = 0 \end{aligned} \tag{3.15}$$

since S_{DW} is $\overline{\mathcal{Q}}$-exact . Now since correlation functions do not depend on the coupling e, we can evaluate the path integral in the *saddle point approximation* $e \to 0$ and it will be *exact*! This will be done in Subsection 3.2. We now need to find the topological operators of the theory. But since

$$\overline{\mathcal{Q}}^2 \equiv \text{gauge transformation}, \tag{3.16}$$

we need gauge invariant topological operators. From (3.7), we see that ϕ is $\overline{\mathcal{Q}}$-closed but it is not gauge invariant since under gauge transformation

$$\phi \longrightarrow U \phi U^\dagger, \quad U \in G \tag{3.17}$$

when the gauge group G is a matrix Lie group. So gauge invariant operators can be constructed by taking trace. For $G = \mathrm{SU}(N)$, these operators are generated by

$$\mathcal{O}_n = \mathrm{Tr}\left(\phi^n\right), \quad n = 2, \dots, N \tag{3.18}$$

and then we can construct new topological operators using descent equation (2.12), (2.13). Note that $\mathcal{O}_1 = 0$ since the generators of $\mathrm{SU}(N)$ are traceless. We will take $G = \mathrm{SU}(2)$ in which case gauge invariant topological operators are generated by

$$\mathcal{O}^{(0)} \equiv \mathcal{O}_1 = \mathrm{Tr}\left(\phi^2\right). \tag{3.19}$$

[14]Note that we are not considering the factor of Z in the correlation function.

This operator can be interpreted as a map from a zero homology cycle $x \in H_0(X)$ to a topological operator $\mathcal{O}^{(0)}(x)$. We now use descent equation to construct other topological operators. We will restrict to the first two such operators. Recall that

$$G_\mu = \frac{i}{4} (\bar{\sigma}_\mu)^{\alpha\dot{\beta}} Q_{\alpha\dot{\beta}}. \tag{3.20}$$

We need the following commutators which can be checked using (A.27):

$$[G_\mu, \phi] = \frac{1}{2\sqrt{2}}\psi_\mu, \quad \{G_\mu, \psi_\nu\} = -\left(F^-_{\mu\nu} + D_{\mu\nu}\right) \tag{3.21}$$

where $F^-_{\mu\nu}$ is the anti-selfdual part of $F_{\mu\nu}$ and $D_{\mu\nu} = (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} D_{\dot{\alpha}\dot{\beta}}$. Then by descent equation (2.12), (2.13), we get two other topological operators:

$$I_1(\delta) = \int_\delta \mathcal{O}^{(1)}, \quad I_2(S) = \int_S \mathcal{O}^{(2)}, \tag{3.22}$$

where $\delta \in H_1(X), S \in H_2(X)$ and

$$\begin{aligned} \mathcal{O}^{(1)} &= \mathrm{Tr}\left(\frac{1}{\sqrt{2}}\phi\psi_\mu\right) dx^\mu, \\ \mathcal{O}^{(2)} &= -\frac{1}{2}\,\mathrm{Tr}\left(\frac{1}{\sqrt{2}}\phi\left(F^-_{\mu\nu} + D_{\mu\nu}\right) - \frac{1}{4}\psi_\mu\psi_\nu\right) dx^\mu \wedge dx^\nu. \end{aligned} \tag{3.23}$$

3.2 Donaldson invariants and Donaldson-Witten partition function

As we saw in the previous section, the path integral for the correlation functions of topological operators are independent of the coupling e and hence can be evaluated exactly in the saddle point approximation $e \to 0$. The saddle point approximation is described in detail in Appendix B. The main content of this approximation is that the path integral localises to the classical saddles of the action (see (B.8)). So we now need to find the classical saddles of the action. Let us first determine the bosonic saddles. A good way to do this is to determine the vacua since it minimises the action. The vacua is defined by vanishing vevs of fermionic fields (to preserve Lorentz invariance of the Lagrangian when expanded around the vev). But since the vacua must also be invariant under $\overline{\mathcal{Q}}$, we require $\overline{\mathcal{Q}}$(fermions) $= 0$. These are the so called *supersymmetric configurations*. From (3.7) we see that $\{\overline{\mathcal{Q}}, \chi_{\dot{\alpha}\dot{\beta}}\} = 0, \{\overline{\mathcal{Q}}, \psi_\mu\} = 0$ implies

$$F^+_{\mu\nu} = (\bar{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} D_{\dot{\alpha}\dot{\beta}}, \quad \nabla\phi = 0. \tag{3.24}$$

Here

$$F^+ \equiv \frac{F + *F}{2} = 0 \tag{3.25}$$

where

$$*F = \widetilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu, \quad \widetilde{F}_{\mu\nu} = \frac{\sqrt{g}}{2}\epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}. \tag{3.26}$$

But onshell $D = 0$ and hence the classical saddles for gauge fields are solutions to $F^+ = 0$. These are called *instanton configurations* or *anti-selfdual gauge fields.*[15] Moreover since the

[15]Because the selfdual part is zero.

path integral only sums over gauge inequivalent gauge fields, we need to consider gauge inequivalent instantons. The set of all gauge inequivalent instanton configurations is called the *moduli space of instantons* and denoted by $\mathcal{M}_{\text{ASD}}$. See Appendix D for precise definitions and details about the moduli space of instantons. The classical saddles for ϕ are solutions to $\nabla_\mu \phi = 0$. To simplify matters, we assume that the instantons are *irreducible*[16] meaning that there are no non-trivial solutions[17] to $\nabla_\mu f = 0$ which implies that there are no non-trivial classical saddles for ϕ. Similarly there are no non-trivial classical saddles for $\phi^\dagger, \eta$. Let us now look at the classical saddles of fermions. It is shown in Appendix D that in the background of *irreducible* and *regular* instantons (see Appendix D for the precise definition),

$$N_\psi = \dim \mathcal{M}_{\text{ASD}} \tag{3.27}$$

where N_ψ is the number of independent classical saddles of ψ and χ has no non-trivial classical saddles. So the path integration measure $[DX]$ localises to a measure on $\mathcal{M}_{\text{ASD}}$ and Grassmann variables corresponding to the classical saddles of ψ:

$$[DX] = d\mu = da^1 \cdots da^n d\psi^1 \dots d\psi^n \tag{3.28}$$

where $n = \dim \mathcal{M}_{\text{ASD}}$. The direct consequence of this is that the partition function

$$Z_{\text{DW}} = 0 \tag{3.29}$$

since the action S_{DW} is a scalar and does not contain any Grassmann variables. So to get non-trivial topological observables we should consider

$$\langle \mathcal{O} \rangle = \int [DX] \mathcal{O} e^{-\frac{1}{e^2} S_{DW}} \tag{3.30}$$

where $\mathcal{O}$ contains all Grassmann variables $\psi^1 \dots \psi^n$ and is $\overline{\mathcal{Q}}$-closed. So after replacing all gauge fields in the topological operators by instanton configurations, a general operator whose correlation function is nonzero is of the form

$$\mathcal{O} = \Phi_{i_1 \dots i_n} \left(\{a_k\}, \phi\right) \psi^{i_1} \dots \psi^{i_n} \tag{3.31}$$

where $\Phi_{i_1 \dots i_n} \left(\{a_k\}, \phi\right)$ is a function on $\mathcal{M}_{\text{ASD}}$ and also depends on ϕ (cf. (3.19) and (3.23)). In other words, $\mathcal{O}$ must have ghost number $n = \dim \mathcal{M}_{\text{ASD}}$ (since ψ^i has ghost number 1, cf. (3.1)). Note that $\Phi_{i_1 \dots i_n} \left(\{a_k\}, \phi\right)$ is completely antisymmetric in $i_1, \cdots i_n$ and hence is an n-form on $\mathcal{M}_{\text{ASD}}$ once we get rid of ϕ dependence as we will explain below. In the saddle point approximation, the factor (see (B.5))

$$\exp\left(-\frac{1}{e^2} S_{DW}\left[\phi_{cl}\right]\right) = 1 \tag{3.32}$$

[16] The irreducibility condition is also required for $\mathcal{M}_{\text{ASD}}$ to be a well defined space. See Appendix D for more details.

[17] Recall that ∇ depends on the gauge connection

in our case. To see this, note that the classical saddles are $\phi = \phi^\dagger = \eta = \chi = 0$ and D is an auxiliary field and hence $D = 0$ on shell. Finally the kinetic term for the gauge field in S_{DW} is

$$-\frac{1}{4}\int_X \mathrm{Tr}(F^+ \wedge F^+), \tag{3.33}$$

so that the classical saddle $F^+ = 0$ gives $S_{DW}[\phi_{cl}] = 0$. Thus (see (B.9))

$$\langle \mathcal{O} \rangle = \int d\mu \left((-1)^{f(\{a_k\})} \sqrt{2\pi} \right) \Phi_{i_1 \ldots i_n}(a_k, \phi)\, \psi^{i_1} \ldots \psi^{i_n} \tag{3.34}$$

where $f(\{a_k\}) = 0, 1$ depending on a point $\{a_k\} \in \mathcal{M}_{\mathrm{ASD}}$. Assuming that $\mathcal{M}_{\mathrm{ASD}}$ is connected, one can show that the signs are all $+1$ (see Appendix B for some explanation). We thus have

$$\langle \mathcal{O} \rangle = \sqrt{2\pi} \int_{\mathcal{M}_{\mathrm{ASD}}} \Phi_{\mathcal{O}}(\phi, a) \tag{3.35}$$

where

$$\Phi_{\mathcal{O}}(\phi, a) = \Phi_{i_1 \ldots i_n}(a_k, \phi)\, da^{i_1} \wedge \cdots \wedge da^{i_n} \tag{3.36}$$

Finally we need to integrate out ϕ which does not have a classical saddle. To do this we replace ϕ by

$$\langle \phi \rangle = \sum_{a=1}^{3} \langle \phi^a(x) \rangle\, T^a \tag{3.37}$$

It has been calculated in detail in [8]. The result is

$$\langle \phi^a(x) \rangle = \frac{i}{\sqrt{2}} \int d^4y \sqrt{g}\, G^{ab}(x, y)\, [\psi_\mu(y), \psi^\mu(y)]^b \tag{3.38}$$

where

$$\nabla^2 G^{ab}(x, y) = \delta^{ab} \delta^4(x - y), \tag{3.39}$$

which gives us an expression for $\langle \phi^a(x) \rangle$ in terms of the classical saddles of ψ. We now replace ϕ in $\Phi_{\mathcal{O}}(\phi, a)$ by $\langle \phi(x) \rangle$ and normalise the integral by $\sqrt{2\pi}$ so that

$$\langle \mathcal{O} \rangle = \int_{\mathcal{M}_{\mathrm{ASD}}} \Phi_{\mathcal{O}} \tag{3.40}$$

Thus the path integral in Donaldson-Witten theory is basically integration of forms on $\mathcal{M}_{\mathrm{ASD}}$. Next, if we take a product of operators

$$\mathcal{O} = \mathcal{O}_1 \cdots \mathcal{O}_k \tag{3.41}$$

where $\mathcal{O}_i$ has ghost number n_i:

$$\mathcal{O}_i = \Phi^{(i)}_{r_1 \cdots r_{n_i}} \psi^{r_1} \ldots \psi^{r_{n_i}}, \tag{3.42}$$

then, for $\langle \mathcal{O} \rangle \neq 0$, we must have

$$\dim \mathcal{M}_{\mathrm{ASD}} = \sum_{i=1}^{k} n_i. \tag{3.43}$$

After replacing ϕ by $\langle\phi\rangle$ in $\mathcal{O}$, one can show that

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle = \int_{\mathcal{M}_{\mathrm{ASD}}} \Phi^{(1)} \wedge \ldots \wedge \Phi^{(k)} \tag{3.44}$$

In our theory we have the topological operators $\mathcal{O}^{(0)}(x), I_1(\delta), I_2(S), \ldots$ for $x \in H_0(X), \delta \in H_1(X), S \in H_2(X)$ and so on (see (3.19),(3.23) and (3.22)). Let $\Phi_x^{(0)} \in H^4(\mathcal{M}_{\mathrm{ASD}}), \Phi_\delta^{(1)} \in H^3(\mathcal{M}_{\mathrm{ASD}}), \Phi_S^{(2)} \in H^2(\mathcal{M}_{\mathrm{ASD}})$ be the forms associated to $\mathcal{O}^{(0)}(x), I_1(\delta), I_2(S)$. Then we have a map

$$H_*(X) \longrightarrow H^*(\mathcal{M}_{\mathrm{ASD}}). \tag{3.45}$$

This is the quantum field theoretic version of the Donaldson map μ_D. Thus a general correlation function has the form

$$\left\langle \mathcal{O}^\ell(x) I_1(\delta_{i_1}) \ldots I_1(\delta_{i_p}) I_2(S_{j_1}) \cdots I_2(S_{j_q}) \right\rangle = \int\limits_{\mathcal{M}_{\mathrm{ASD}}} \left(\wedge^\ell \Phi_x^{(0)}\right) \wedge \Phi_{i_1}^{(1)} \wedge \ldots \wedge \Phi_{i_p}^{(1)} \wedge \Phi_{j_1}^{(2)} \wedge \ldots \Phi_{j_q}^{(2)} \tag{3.46}$$

Note that since the ghost number of $\mathcal{O}^{(0)}, I_1(\delta), I_2(S)$ is 4,3 and 2 respectively, we need

$$4\ell + 3p + 2q = \dim \mathcal{M}_{\mathrm{ASD}}. \tag{3.47}$$

for the integral to be nontrivial. As noted in Subsection 3.1, we need to fix the second Stiefel-Whitney class of X[18] and sum over the instanton number of principal SU(2) bundles over X. The result is precisely the Donaldson invariants. In particular, the Donaldson-Witten partition function is given by

$$\begin{aligned} Z_{DW}(p, S) &= \left\langle e^{p\mathcal{O}(x) + I_2(S)} \right\rangle_w \\ &= \sum_{\ell, r \geq 0} \frac{p^\ell}{\ell! r!} \left\langle \mathcal{O}^\ell(x) I_2(S)^r \right\rangle_w \end{aligned} \tag{3.48}$$

where

$$\begin{aligned} P_D\left(x^\ell S^r\right) &\equiv \left\langle \mathcal{O}^\ell(x) I_2(S)^r \right\rangle_w \\ &= \int_{\mathcal{M}_{\mathrm{ASD}}(w)} \left(\wedge^\ell \Phi_x^{(0)}\right) \wedge \left(\wedge^r \Phi_S^{(2)}\right) \end{aligned} \tag{3.49}$$

are called the *Donaldson polynomials*. Again it is nonzero only when $4\ell + 2r = \dim \mathcal{M}_{\mathrm{ASD}}$. Thus we have given a quantum field theoretic interpretation of Donaldson invariants.

Acknowledgments

I would like to express my gratitude to my advisor Prof. Greg Moore for suggesting me this project and resolving numerous doubts. I would like to thank Vivek Saxena and Runkai Tao for various discussions which helped me understand the subject better. I would also like to thank Prof. Jan Manschot for useful discussions. Finally I would like to thank my colleagues Anindya Banerjee, Spencer Stubbs and Madhav Sinha for various discussions, comments and suggestions.

[18] We will indicate this with a subscript w

A Supersymmetric Gauge Theory

In this section we very briefly discuss supersymmetric gauge theories on Minskowski space $\mathbb{R}^4$. For a more detailed discussion and derivation of the results mentioned here, the reader is referred to. Throughout this section, we will use the notations and conventions of [10, Chapter 4, Appendix A]

A.1 Supersymmetry in Four Dimensions

Traditional quantum field theories have symmetry groups which are a direct product of the local Poincaré group and some internal symmetry. Supersymmetry extends the symmetry group by introducing fermionic generators in the algebra. These fermionic generators are called the supercharges. In addition to the Poincaré generators $P_\mu, M_{\mu\nu}$, the supersymmetry algebra includes $\mathcal{N}$ *supercharges* $Q_{\alpha I}, \overline{Q}^I_{\dot\alpha}$ with $I = 1, \ldots, \mathcal{N}$ which transform under the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ of the Lorentz group. They satisfy the The supersymmetry algebra involving the supercharges is given by

$$\begin{aligned}
&\left\{Q^I_\alpha, \overline{Q}_{\dot\beta J}\right\} = 2P_\mu \sigma^\mu_{\alpha\dot\beta}\delta^I_J, \quad \{Q^I_{\alpha,}, Q^J_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}Z_{IJ} \\
&\left\{\overline{Q}_{\dot\alpha I}, \overline{Q}_{\dot\beta J}\right\} = -2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\overline{Z}^{IJ}, \quad \left[Q^I_\alpha, P^\mu\right] = [\overline{Q}^I_{\dot\beta}, P^\mu] = 0 \\
&\left[Q^I_\alpha, M_{\mu\nu}\right] = (\sigma_{\mu\nu})_\alpha^{\ \beta} Q^I_\beta, \quad \left[\overline{Q}^{\dot\alpha}_I, M_{\mu\nu}\right] = (\sigma_{\mu\nu})^{\ \beta}_{\dot\alpha}\overline{Q}^\beta_I
\end{aligned} \tag{A.1}$$

The matrix Z_{IJ} is called *central charge* since one can show that it commutes with all the generators. We will be mostly interested in the case when $Z_{IJ} = 0$. In this case there is an internal U($\mathcal{N}$) symmetry which acts on the supercharges as

$$Q_{\alpha I} \to U_I^{\ J} Q_{\alpha J}, \quad \overline{Q}^I_{\dot\alpha} \to \overline{U}_J^{\ I}\overline{Q}^J_{\dot\alpha}. \quad U \in \mathrm{U}(\mathcal{N}). \tag{A.2}$$

This is called the *R-symmetry.* Let B^ℓ be the generators of R-symmetry. Then

$$[Q_{\alpha I}, B^\ell] = (b^\ell)_I^{\ J} Q_{\alpha J}, \quad [\overline{Q}^I_{\dot\alpha}, B^\ell] = -(b^\ell)_J^{\ I}\overline{Q}^J_{\dot\alpha} \tag{A.3}$$

where b^ℓ are the representation matrices. The supersymmetry algebra along with the commutators of the Poincaré algebra is called the *Super-Poincaré algebra.* The next step is to construct the irreducible representation of the Super-Poincaré algebra, called a *supermultiplet.* It turns out that the mass is constant in a supermultiplet but the spin can change resulting in a supermultiplet consisting of a bunch of particles with same mass but different spins. To construct these explicitly, for vanishing central charge we define the annihilation and creation operators as

$$a^I_\alpha = \begin{cases} \frac{1}{\sqrt{2m}} Q^I_\alpha & P^2 = m^2 > 0 \\ \frac{1}{\sqrt{2}} Q^I_\alpha & P^2 = 0 \end{cases}, \quad a^\dagger_{\dot\alpha I} = \begin{cases} \frac{1}{\sqrt{2m}} \overline{Q}_{\dot\alpha I} & P^2 = m^2 > 0 \\ \frac{1}{\sqrt{2}} \overline{Q}_{\dot\alpha I} & P^2 = 0. \end{cases} \tag{A.4}$$

One can show using the Super-Poincaré algebra that

$$\left\{a^I_\alpha, a^\dagger_{\dot\beta J}\right\} = \delta^I_J \delta_{\alpha\dot\beta}. \tag{A.5}$$

Then we get a supermultiplet by applying the creation operator on the *Clifford vacuum* $|\lambda_0\rangle$, with spin/helicity λ_0, defined as usual $a^I_\alpha|\lambda_0\rangle = 0$. One shows that the Clifford vacuum for massive states is characterised by mass $m > 0$ and spin j with a total of $2j+1$ degrees of freedom and transforms in the usual spin-j representation of the Lorentz algebra and the Clifford vacuum of massless states is characterised by the helicity λ with two degrees of freedom. The supersymmetry algebra also show that the creation operators raise the spin/helicity by 1/2 and the annihilation operator decrease it by 1/2. Thus a supermultiplet constructed on a Clifford vacuum $|\lambda_0\rangle$ has the spin/helicity content

$$|\lambda_0\rangle, \quad a^\dagger_{\dot{2}I}|\lambda_0\rangle \equiv |\lambda_0 + \frac{1}{2}\rangle_I, \quad a^\dagger_{\dot{2}I}a^\dagger_{\dot{2}J}|\lambda_0\rangle \equiv |\lambda_0 + 1\rangle_{IJ}, \quad \ldots, \quad a^\dagger_{\dot{2}1}\ldots a^\dagger_{\dot{2}\mathcal{N}}|\lambda_0\rangle \equiv |\lambda_0 + \frac{\mathcal{N}}{2}\rangle \tag{A.6}$$

This is not CPT invariant unless $\lambda_0 = -\mathcal{N}/4$ and hence in general we need to add the CPT conjugates. Some examples are shown in Table 1 and 2.

λ_0	Multiplet Name	Helicity Content
0	Chiral Multiplet	$(-1/2, 2\times(0), 1/2)$
$\frac{1}{2}$	Vector Multiplet	$(-1, -1/2, 1/2, 1)$
1	Gravitino Multiplet	$(-3/2, -1, 1, 3/2)$
$\frac{3}{2}$	Gravity Multiplet	$(-2, -3/2, 3/2, 2)$

Table 1. Massless $\mathcal{N} = 1$ supermultiplets.

λ_0	Multiplet Name	Helicity Content
0	Vector Multiplet	$(-1, 2(-1/2), 2(0), 2(1/2), 1)$
$-\frac{1}{2}$	Half-hypermultiplet	$(-1/2, 2(0), 1/2)$
$-\frac{1}{2}$	Hypermultiplet	$(2(-1/2), 4(0), 2(1/2))$

Table 2. Massless $\mathcal{N} = 2$ supermultiplets. The Hypermultiplet is obtained by adding the CPT conjugates to the half-hypermultiplets.

One can similarly construct higher $\mathcal{N}$ supermultiplets. Note that the helicity content of $\mathcal{N} = 2$ massless vector multplet is same as the helicity content of $\mathcal{N} = 1$ chiral and vector multiplet.This will be important later. To construct local fields and Lagrangains for supermultiplets, one introduces the *superspace* by adjoining Grassmann coordinates $\theta^\alpha, \overline{\theta}_{\dot{\alpha}}$ to the spacetime coordinates x^μ. On the superspace coordinates, the supercharges acts as

$$x^\mu \to x'^\mu + i\theta\sigma^\mu\overline{\xi} - i\xi\sigma^\mu\overline{\theta}, \quad \theta \to \theta' = \theta + \xi, \quad \overline{\theta} \to \overline{\theta}' = \overline{\theta} + \overline{\xi}. \tag{A.7}$$

To construct a field representation of the supermultiplets, we consider *superfields* which are simply functions $\mathcal{F}(x, \theta, \overline{\theta})$ on the superspace. We can Taylor expand the superfields[19] as

$$\begin{aligned}\mathcal{F}(x,\theta,\overline{\theta}) =& f(x) + \theta\phi(x) + \overline{\theta}\overline{\chi}(x) + (\theta\theta)m(x) + (\overline{\theta\theta})n(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) \\ &+ (\theta\theta)\overline{\theta}\overline{\lambda}(x) + (\overline{\theta\theta})\theta\psi(x) + (\theta\theta)(\overline{\theta\theta})d(x).\end{aligned} \tag{A.8}$$

[19]Note that the bracketed Grassmann coordinates or spinor fields are contracted in a Lorentz invariant way. See [10, Appendix A] for conventions on spinor contractions.

This is the most general function of Grassmann variables $\theta, \overline{\theta}$. The functions $f(x), \phi(x)$ and so on are called *component fields* and will represent various particles in a supermultiplet. Note that the coefficients of odd number of Grassmann coordinates are spinors. The $\mathcal{N} = 1$ chiral multiplet is represented by the chiral and antichiral superfield given by

$$\Phi(x,\theta) = \phi(x) + i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\nabla^2\phi(x) + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi(x)\sigma^\mu\overline{\theta} - \theta\theta F(x)$$
$$\bar{\Phi}(x,\bar{\theta}) = \bar{\phi}(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\bar{\phi} - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\nabla^2\bar{\phi} + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\bar{\theta}\bar{\theta}\theta\sigma^\mu\partial_\mu\bar{\psi} + \bar{\theta}\bar{\theta}\bar{F}(x).$$

Sometimes, we write $f^\dagger \equiv \bar{f}$ for any field f. Supersymmetry transformation of the component fields has the form

$$\delta\phi = \sqrt{2}\xi^\alpha\psi_\alpha, \quad \delta\psi_\alpha = \sqrt{2}\xi_\alpha F + i\sqrt{2}\bar{\xi}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi, \quad \delta F = i\sqrt{2}\bar{\xi}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^\alpha. \tag{A.9}$$

Here $\xi, \bar{\xi}$ are Grassmann parameters for supersymmetry transformation. These transformations reveal that supersymmetry indeed maps bosons to fermions and vice-versa. To construct action for superfields, we use the standard Berezin integrals. The kinetic term of the chiral multiplet is given by

$$\frac{1}{4}\int d^4x d^2\theta d^2\bar{\theta}\Phi^\dagger\Phi = \int d^4x \left(\phi^\dagger\nabla^2\phi - \frac{i}{2}\bar{\psi}^{\dot{\alpha}}\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\psi^\alpha - F^\dagger F\right) \tag{A.10}$$

Note that F turns out to be an auxiliary field since its equation of motion is $F = 0$. Thus the physical degrees of freedom of the chiral superfield matches that of chiral multiplet. In our discussion, we will only be interested in massless supermultiplets. One can write the most general $\mathcal{N} = 1$ Lagrangian for chiral superfield using what is called the *Kähler potential* and *superpotential* but we do not need it for the present discussion. Finally the $\mathrm{U}(1)_R$ symmetry acts on the chiral superfield as

$$R\Phi(x,\theta) = e^{2in\alpha}\Phi\left(x, e^{-i\alpha}\theta\right), \quad R\Phi^\dagger(x,\bar{\theta}) \; = e^{-2in\alpha}\Phi^\dagger\left(x, e^{i\alpha}\bar{\theta}\right). \tag{A.11}$$

Under this the component fields transform as

$$A \to e^{2in\alpha}A, \quad \psi \to e^{2i(n-1/2)\alpha}\psi, \quad F \to e^{2i(n-1)\alpha}F. \tag{A.12}$$

The $\mathcal{N} = 1$ vector multiplet is represented locally by a *real vector superfield* V satisfying $V = V^\dagger$. A general real superfield has expansion

$$\begin{aligned} V(x,\theta,\bar{\theta}) =& C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) - \theta\sigma^\mu\bar{\theta}A_\mu(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}\bar{M} + i\theta\theta\bar{\theta}\left(\bar{\lambda}(x) + \frac{1}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) \\ &- i\bar{\theta}\bar{\theta}\theta\left(\lambda(x) - \frac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left(D(x) - \frac{1}{2}\nabla^2C(x)\right) \end{aligned}$$

where $C^\dagger \equiv \bar{C} = C, \bar{\chi} = \chi^\dagger, A_\mu = A_\mu^\dagger$ and so on. To reduce the d.o.f, we impose *gauge symmetry* by requiring that V and V' given by $V \longmapsto V' = V + \Phi + \bar{\Phi}$ describe physically equivalent theories. This is the *abelian gauge transformation.* Here Φ and $\bar{\Phi}$ are the chiral and anti-chiral superfield which we identify as gauge parameters. If we take the component fields of Φ to be ϕ, ψ, F then it is easy to check that

$$C \to C + 2\,\mathrm{Re}(\phi), \quad \chi \to \chi + \sqrt{2}\psi, \quad M \to M - F, \quad D \to D, \quad \lambda \to \lambda, \quad A_\mu \to A_\mu + 2\partial_\mu\,\mathrm{Im}\,\phi. \tag{A.13}$$

Thus we can choose $\mathrm{Re}(\phi), \psi$ and F such that $C = \chi = M = 0$ and A_μ has the familiar gauge transformation with gauge parameter $2\,\mathrm{Im}(\phi)$. This is called the *Wess Zumino* (WZ) *gauge*

$$V_{WZ} = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D. \tag{A.14}$$

One can work out the supersymmetry transformation of the components of the vector superfield as before. Supersymmetry transformation takes out of WZ gauge because it introduces a ψ term. Thus to get the supersymmetry transformation of component fields and still remain in WZ gauge, we perform supersymmetry transformation as well as gauge transformation of V imposing the constraint that the final superfield is in WZ gauge. This gives

$$\begin{aligned} \delta A^\mu &= \partial^\mu\varphi - i\sigma^\mu_{\alpha\dot{\alpha}}\left(\xi^\alpha\bar{\lambda}^{\dot{\alpha}} - \bar{\xi}^{\dot{\alpha}}\lambda^\alpha\right) \\ \delta\lambda_\alpha &= i\xi_\alpha D - i\sigma^{\mu\nu\beta}_{\alpha}\xi_\beta F_{\mu\nu} \\ \delta D &= -\sigma^\mu_{\alpha\dot{\alpha}}\left(\xi^\alpha\partial_\mu\bar{\lambda}^{\dot{\alpha}} - \bar{\xi}^{\dot{\alpha}}\partial_\mu\lambda^\alpha\right) \end{aligned} \tag{A.15}$$

where $\varphi = 2\,\mathrm{Im}(\phi)$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The action for the vector multiplet takes the form

$$\frac{1}{e^2}\int d^4x\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} + \frac{1}{2}D^2\right) - \frac{\Theta}{64\pi^2}\int d^4x\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{A.16}$$

where e is the gauge coupling and Θ is the coupling for the *topological term*. This is the action for $\mathcal{N} = 1$ abelian Yang-Mills theory. Non-abelian vector superfield can be described in the usual way. Let G be a gauge group, usually a compact semisimple Lie group, with generators $T^a, a = 1, \ldots, \dim G$ and let $V^a, a = 1, \ldots, \dim G$ be real vector superfields. Then $V = V^aT^a$ is a Lie algebra valued superfield. The non-abelian gauge transformation of V is given by

$$e^{2V} \longrightarrow e^{-i\Phi}e^{2V}e^{i\Phi} \tag{A.17}$$

where[20] $\Phi = \Phi^aT^a, \bar{\Phi} = \bar{\Phi}^aT^a$ and $\Phi^a, \bar{\Phi}^a$ are chiral, antichiral superfields respectively. Again using gauge transformation one can set $C^a = M^a = \chi^a = 0$ for each real superfield V^a. As before, to find supersymmetry transformation of component fields of V^a, we have to make supersymmetry transformation of V^a in WZ gauge and then perform another gauge transformation to bring it back to WZ gauge. Without giving the details we simply list the transformation and refer to [11] for details:

$$\begin{aligned} \delta A^a_\mu &= \nabla_\mu\varphi^a - i\sigma^\mu_{\alpha\dot{\alpha}}\left(\xi^\alpha\bar{\lambda}^{\dot{\alpha}a} + \bar{\xi}^{\dot{\alpha}}\lambda^{\alpha a}\right) \\ \delta\lambda^a_\alpha &= f^{abc}\varphi^b\lambda^c_\alpha + i\xi_\alpha D^a - i\xi_\beta\left(\sigma^{\mu\nu}\right)^\beta_\alpha F^a_{\mu\nu}, \\ \delta D^a &= f^{abc}\varphi^bD^c - \sigma^\mu_{\alpha\dot{\alpha}}\left(\xi^\alpha\nabla_\mu\bar{\lambda}^{\dot{\alpha}a} - \bar{\xi}^{\dot{\alpha}}\nabla_\mu\lambda^{\alpha a}\right), \end{aligned} \tag{A.18}$$

[20] Wherever there is repeated Lie algebra indices $a, b, c, \ldots$, it is assumed to be summed over.

where

$$\begin{aligned}\nabla_\mu \varphi^a &= \partial_\mu \varphi^a + f^{abc} A^b_\mu \varphi^c \\ F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu\end{aligned} \tag{A.19}$$

with $\left[T^a, T^b\right] = i f^{abc} T^c$ and as before $\varphi^a = 2\,\mathrm{Im}\,(\phi^a)$. Here $(\phi^a, \lambda^a_\alpha)$ are the components of Φ^a. The action takes the form

$$S_{YM} = \frac{1}{e^2}\int d^4x \left(-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - i\lambda^{a\alpha}\sigma^\mu_{\alpha\dot\alpha}\nabla_\mu\bar\lambda^{a\dot\alpha} + \frac{1}{2}D^a D^a\right) - \frac{\Theta}{64\pi^2}\int d^4x \epsilon^{\mu\nu\rho\sigma}F^a_{\mu\nu}F^a_{\rho\sigma} \tag{A.20}$$

A.2 $\mathcal{N}=2$ Supersymmetric gauge theory

$\mathcal{N}=2$ theories are of immense importance for us since topological twisting of these theories as described in Section 2.2 gives us a topological quantum field theory of Witten type (see Section 2 for details). Recall that the $\mathcal{N}=2$ vector multiplet has same helicity content as that of an $\mathcal{N}=1$ chiral plus $\mathcal{N}=1$ vector multiplet. So to write an action for $\mathcal{N}=2$ vector multiplet with gauge group G, we take real vector superfields V^a and chiral superfields $\Phi^a, a = 1, \ldots, \dim G$ and construct

$$V = V^a T^a, \quad \Phi = \Phi^a T^a$$

with $(T^a)_{bc} = -i f^{abc}$ which says that V and Φ are in adjoint representation of G. The action then takes the form

$$S_{YM} + \int d^2\theta d^2\bar\theta d^4x \Phi^{\dagger a}\left(\mathrm{e}^{2V}\right)_{ab}\Phi^b. \tag{A.21}$$

In the WZ gauge

$$V = -\theta\sigma^\mu\bar\theta A_\mu - i\bar\theta\bar\theta\theta\lambda_2 + i\theta\theta\bar\theta\bar\lambda^2 + \frac{1}{2}\theta\theta\bar\theta\bar\theta D \tag{A.22}$$

and the chiral superfield takes the form

$$\Phi = \phi - \sqrt{2}\theta\lambda_1 + \theta\theta F, \quad \bar\Phi = \bar\phi - \sqrt{2}\bar\theta_1 + \bar\theta\bar\theta\bar F. \tag{A.23}$$

The action can then be written in component form as:

$$\begin{aligned}\frac{1}{e^2}\int d^4x\,\mathrm{Tr}\Big(&\nabla_\mu\phi^\dagger\nabla^\mu\phi - i\lambda_1\sigma^\mu\nabla_\mu\bar\lambda^1 - i\lambda_2\sigma^\mu\nabla_\mu\bar\lambda^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{2}D^2 + |F|^2 - \frac{1}{2}\left[\phi,\phi^\dagger\right]^2 - i\sqrt{2}\lambda_1^\alpha\left[\phi^\dagger, \lambda_{2\alpha}\right] + i\sqrt{2}\bar\lambda^1{}_\alpha\left[\bar\lambda^{2\dot\alpha}, \phi\right]\Big),\end{aligned} \tag{A.24}$$

where we rescaled $\Phi \longrightarrow \Phi/e$ and replaced $D \to D + [\phi, \phi^\dagger]$ in the action. We also chose the normalisation

$$\mathrm{Tr}\left(T^a T^b\right) = \delta^{ab}. \tag{A.25}$$

In addition to $\mathcal{N}=2$ supersymmetry there is $\mathrm{SU}(2)_R$ symmetry which is not manifest in the action above. To make it manifest we recognise ϕ^a, A^a_μ as $\mathrm{SU}(2)_R$ singlet, $\lambda^a_I, I = 1, 2$ as $\mathrm{SU}(2)_R$ doublet, similarly $\bar\lambda^{aI}$ as $\mathrm{SU}(2)_R$ doublet. Finally we organise

$$D^{IJ} = \begin{pmatrix}\sqrt{2}F & iD \\ iD & \sqrt{2}\bar F\end{pmatrix} \tag{A.26}$$

as $\mathrm{SU}(2)_R$ triplet. The indices are lowered and raised using ϵ_{IJ} and ϵ^{IJ} $\left(\epsilon_{12}=1, \epsilon_{IJ}\epsilon^{JK}=\delta_I^K\right)$. One can determine the supersymmetry transformation of the fields as before. The final transformation turns out to be

$$\begin{aligned}
&\delta\phi = \sqrt{2}\epsilon^{IJ}\xi_I\lambda_J, \\
&\delta A_\mu = i\xi_I\sigma_\mu\bar{\lambda}^I - i\lambda_I\sigma_\mu\bar{\xi}^I, \\
&\delta\lambda_{I\alpha} = D_I{}^J\xi_{J\alpha} - i\xi_{I\alpha}[\phi,\phi^\dagger] - i\sigma^{\mu\nu}{\alpha}^{\beta}_{\alpha}\xi_{I\beta}F_{\mu\nu} + i\sqrt{2}\epsilon_{IJ}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\xi}^{J\dot{\alpha}}\nabla_\mu\phi, \\
&\delta D^{IJ} = 2i\bar{\xi}^{(I}\bar{\sigma}^\mu\nabla_\mu\lambda^{J)} + 2i\nabla_\mu\bar{\lambda}^{(I}\bar{\sigma}^\mu\xi^{J)} + 2i\sqrt{2}\xi^{(I}[\lambda^{J)},\phi^\dagger] + 2i\sqrt{2}\xi^{(I}[\bar{\lambda}^{J)},\phi].
\end{aligned} \tag{A.27}$$

Here (IJ) indicates symmetrization of the indices, the ξ_I^α and $\bar{\xi}^I_{\dot{\alpha}}$ are Grassmann parameters. With this the action takes the form

$$\begin{aligned}
\frac{1}{e^2}\int d^4x\,\mathrm{Tr}\Bigg(&\nabla_\mu\phi^\dagger\nabla^\mu\phi - i\lambda_I\sigma^\mu\nabla_\mu\bar{\lambda}^I - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{4}D_{IJ}D^{IJ} \\
&-\frac{1}{2}\left[\phi,\phi^\dagger\right]^2 - \frac{i}{\sqrt{2}}\epsilon^{IJ}\lambda_I{}^\alpha\left[\phi^\dagger,\lambda_{J\alpha}\right] - \frac{i}{\sqrt{2}}\epsilon_{IJ}\bar{\lambda}^I{}_\alpha\left[\bar{\lambda}^{J\dot{\alpha}},\phi^\dagger\right]\Bigg)
\end{aligned} \tag{A.28}$$

Now the $\mathrm{SU}(2)_R$ symmetry is manifest in the action as well as the supersymmetry transformations. Since the full R-symmetry is $\mathrm{U}(2)_R$ but at the level of algebra $\mathrm{U}(2)$ is same as[21] $\mathrm{SU}(2)\times\mathrm{U}(1)$, thus we need to specify how the fields transform under $\mathrm{U}(1)_R$. On the fields, $\mathrm{U}(1)_R$ acts with charge q_R given by

$$\begin{aligned}
&A_\mu \to A_\mu, \quad q_R=0; \quad D_{IJ}\to D_{IJ}, \quad q_R=0; \\
&\lambda_{I\alpha}\to e^{i\varphi}\lambda_{I\alpha}, \quad q_R=1; \quad \phi\to e^{2i\varphi}\phi, \quad q_R=2; \\
&\bar{\lambda}^I_{\dot{\alpha}}\to e^{-i\varphi}\bar{\lambda}^I_{\dot{\alpha}}, \quad q_R=-1; \quad \phi^\dagger\to e^{-2i\varphi}\phi^\dagger, \quad q_R=-2.
\end{aligned} \tag{A.29}$$

The $\mathrm{U}(1)_R$ charges q_R are also called *ghost numbers*. $\mathrm{U}(1)_R$ acts on the superspace coordinates $\theta,\bar{\theta}$ with charge $q_R=-1,1$ respectively. It is easy to see that under $\mathrm{U}(1)_R$

$$W_\alpha \to \mathrm{e}^{-i\varphi}W_\alpha\left(e^{i\varphi}\theta\right), \quad \Phi\to e^{-2i\varphi}\Phi\left(\mathrm{e}^{i\varphi}\theta\right). \tag{A.30}$$

and hence the action is clearly invariant under $\mathrm{U}(1)_R$.

B Saddle Point Approximation

Recall that for a function $f:\mathbb{R}\longrightarrow\mathbb{R}$, $x_\star$ is called a *saddle point* if

$$f'(x_\star)=0, \quad f''(x_\star)>0. \tag{B.1}$$

For a multivariable function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$, saddle point is defined as

$$\left.\frac{\partial f}{\partial x_i}\right|_{x=x_\star}=0, \quad \det\left(\left.\frac{\partial^2 f}{\partial x_i\partial x_j}\right|_{x=x_\star}\right)>0.$$

[21]More precisely $\mathrm{U}(2)\cong(\mathrm{SU}(2)\times\mathrm{U}(1))/\mathbb{Z}_2$.

In analogy, saddle point for functionals can be defined as

$$\left.\frac{\delta S[\phi]}{\delta \phi}\right|_{\phi=\phi_\star}=0, \quad \det\left(\left.\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}\right|_{\phi=\phi_\star}\right)>0. \tag{B.2}$$

Consider a path integral of the form

$$Z[\mathcal{O}]=\int[Dphi]e^{-\frac{1}{e^2}S[\phi]}\mathcal{O}(\phi)$$

where DX contains path integral over all fields appearing in $S[\phi]$, $\mathcal{O}(\phi)$ is a function of fields in $S[\phi]$ and g is a coupling parameter. We want to compute the path integral when $g\to 0$. In this limit, field configurations at which $S[\phi]$ attains a minima dominates the path integral since other field configurations are exponentially suppressed. Such field configurations are precisely the saddle points, which we call classical saddles. Suppose ϕ_{cl} be one such *classical saddles* of a bosonic field ϕ. We expand $S[\phi]$ and $\mathcal{O}(\phi)$ around ϕ_{cl} : write $\phi=\phi_{cl}+\delta\phi$ and expand

$$\begin{aligned} S[\phi] &= S\left[\phi_{cl}\right]+\left(\left.\frac{\delta S}{\delta\phi}\right|_{\phi=\phi_{cl}}\right)\delta\phi+\frac{1}{2}\delta\phi\left(\left.\frac{\delta^2 S}{\delta\phi\delta\phi}\right|_{\phi=\phi_{cl}}\right)\delta\phi+O(\delta\phi)^3 \\ &= S\left[\phi_{cl}\right]+\frac{1}{2}\delta\phi\left(\left.\frac{\delta^2 S}{\delta\phi\delta\phi}\right|_{\phi=\phi_{cl}}\right)\delta\phi+O(\delta\phi)^3 \end{aligned} \tag{B.3}$$

and similarly

$$\mathcal{O}(\phi)=\mathcal{O}\left(\phi_{c1}\right)+O(\delta\phi) \tag{B.4}$$

Plugging this in the action, we get

$$Z[\mathcal{O}]=e^{-\frac{1}{e^2}S[\phi_{cl}]}\int[D\delta\phi]e^{-\frac{1}{2e^2}\delta\phi\frac{\delta^2 S}{\delta\phi\delta\phi}\delta\phi+O\left((\delta\phi)^3\right)}\left(\mathcal{O}\left(\phi_{cl}\right)+O(\delta\phi)\right).$$

Changing $\delta\phi\to e\delta\phi$ we get

$$Z[\mathcal{O}]=e^{-\frac{1}{e^2}[\phi_{c1}]}e\int[D\delta\phi]e^{-\frac{1}{2}\delta\phi\frac{\delta^2 S}{\delta\phi\delta\phi}\delta\phi+O\left(e^3(\delta\phi)^3\right)}\left(\mathcal{O}\left(\phi_{cl}\right)+O(e\delta\phi)\right)$$

which in $e\to 0$ can be written as

$$\begin{aligned} Z[\mathcal{O}] &= e^{-\frac{1}{e^2}S[\phi_{cl}]}e\int[D\delta\phi]e^{-\frac{1}{2}\delta\phi\frac{\delta^2 S}{\delta\phi\delta\phi}\delta\phi}\mathcal{O}\left(\phi_{cl}\right)(1+O(e)) \\ &= e^{-\frac{1}{e^2}S[\phi_{cl}]}\mathcal{O}\left(\phi_{cl}\right)\sqrt{\frac{2\pi e^2}{\det\Delta_B}}(1+O(e)) \end{aligned} \tag{B.5}$$

where

$$\Delta_B=\left.\frac{\delta^2 S}{\delta\phi\delta\phi}\right|_{\phi=\phi_{cl}} \tag{B.6}$$

is a differential operator. We used the Gaussian path integral for bosonic coordinates. If the field $\phi=\psi$ was fermionic, then we would use the Grassmann Gaussian integral to get

$$Z[\mathcal{O}]=e^{-\frac{1}{e^2}S[\psi_{cl}]}\mathcal{O}\left(\psi_{cl}\right)\operatorname{Pf}\left(\Delta_F\right)e(1+O(e))$$

where

$$i\Delta_F = \left.\frac{\delta^2 S}{\delta\psi\delta\psi}\right|_{\psi=\psi_{cl}} \tag{B.7}$$

If there are more than one classical saddles then we have to integrate over all of them. That is

$$Z[\mathcal{O}] = \int_{\text{classical saddles}} d\phi_{cl} e^{-\frac{1}{e^2}S[\phi_{cl}]}\mathcal{O}\left(\phi_{cl}\right)\sqrt{\frac{2\pi e^2}{\det\Delta_B}}(1+O(e)). \tag{B.8}$$

Note that in $e \to 0$, this approximation becomes exact since the corrections are all of linear or higher order in the coupling. In a supersymmetric theory, one can show that there are equal number of fermionic and bosonic classical saddles [8]. Moreover for every eigenvalue of Δ_F

$$i\Delta_F\Psi = \lambda\Psi, \quad \lambda \neq 0$$

there is a corresponding eigenvalue of Δ_B:

$$\Delta_B\Phi = \lambda^2\Phi.$$

This means that in saddle point approximation of supersymmetric path integral we will have

$$\begin{aligned} Z[\mathcal{O}] &= \sum_{\{\phi_{cl},\psi_{cl}\}} e^{-\frac{1}{e^2}S[\phi_{cl}]}e^{\frac{1}{e^2}S[\psi_{cl}]}\int [D\delta\phi D\delta\psi]e^{-\frac{1}{2}\delta\phi\Delta_B\delta\phi - i\frac{1}{2}\delta\psi_F\delta\psi}\mathcal{O}\left(\phi_{cl},\psi_{cl}\right)(1+O(e)) \\ &= \sum_{\{\phi_{cl},\psi_{cl}\}} e^{-\frac{1}{e^2}S[\phi_{cl}]}e^{-\frac{1}{e^2}S[\psi_{cl}]}(\pm\sqrt{2\pi})\mathcal{O}\left(\phi_{cl},\psi_{c1}\right)(1+O(e)) \end{aligned} \tag{B.9}$$

where $\pm$ is undetermined and stems from the fact that

$$f(\psi_{cl},\phi_{cl}) = \frac{\mathrm{Pf}(\Delta_F)}{\sqrt{\det(\Delta_B)}} = \pm 1. \tag{B.10}$$

This ambiguity can be resolved in certain cases. For example when the set of classical saddles is a connected manifold, which is usually the case in applications, then one can declare [8] that $f(\psi_{cl},\phi_{cl}) = 1$. This is consistent since the function $f(\psi_{cl},\phi_{cl})$ is continuous and there are no zero eigenvalues for Δ_F (which means $f(\psi_{cl},\phi_{cl}) \neq 0$ everywhere).

C Bundle Theory and Characteristic Classes

In this appendix, we will recall the basic definition and properties of characteristic classes. We refer the reader to [6] for details.

C.1 Principal bundles and connection 1-forms

A fibre bundle is a tuple (E,π,X,F,G) usually denoted as $E \xrightarrow{\pi} X$ with E,X,F smooth manifolds called the *total space, base space and the typical fibre* respectively, $\pi : E \longrightarrow X$ a surjective smooth map called the *projection*, G a Lie group satisfying the following properties:

(i) For every $x \in X$, $F_x := \pi^{-1}(x)$ called the *fibre above* x is diffeomorphic to F.

(ii) G acts on F on the left.

(iii) There is an open covering $\{U_i\}$ of X and diffeomorphisms

$$\phi_i : U_i \times F \longrightarrow \pi^{-1}(U_i)$$

called a *local trivialization* such that $\pi \circ \phi_i = \mathbb{1}_{U_i}$.

(iv) The map $\phi_i(x, \cdot) =: \phi_{i,x} : F \longrightarrow \pi^{-1}(x)$ is a diffeomorphism. On $U_i \cap U_j$, $\phi_{i,x}^{-1} \circ \phi_{j,x} : F \to F$ is an element of G, *i.e.*, $G \ni t_{ij}(x) := \phi_{i,x}^{-1} \circ \phi_{j,x}$ acts on F as a diffeomorphism. That is $\phi_{i,x}$ and $\phi_{j,x}$ are related by $t_{ij}(x)$:

$$\phi_{j,x} = \phi_{i,x} \circ t_{ij}.$$

The functions t_{ij} are called *transition functions* and they satisfy the consistency conditions:

$$\begin{aligned} & t_{ii}(p) = \mathbb{1}_G, \quad p \in U_i \\ & t_{ij}(p) = t_{ji}(p)^{-1}, \quad p \in U_i \cap U_j \\ & t_{ij}(p) t_{jk}(p) t_{ki}(p) = \mathbb{1}_G, \quad p \in U_i \cap U_j \cap U_k. \end{aligned} \tag{C.1}$$

If the fiber F is a k-dimensional complex (real) vector space then the fiber bundle is called a complex (real) *vector bundle of rank* k. A rank 1 vector bundle is called a line bundle. The typical examples are the tangent bundle, the cotangent bundle and the bundle of forms on the manifold. We refer to [6] for precise definitions.

A principal G-bundle over a manifold X is a fiber bundle with typical fiber G and is denoted by $P \xrightarrow{\pi} X$. G acts on itself on the left as usual by left multiplication. It also acts on P on right as follows: let $p \in P$ and U_i be a local trivialization around $\pi(p) = x \in U_i$. Let $\phi_i : U_i \times G \longrightarrow \pi^{-1}(U_i)$ be a diffeomorphism and suppose $\phi_i^{-1}(p) = (x, g_i)$ for some $g_i \in G$. Then for any $a \in G$, we define pa as

$$pa = \phi_i(x, g_i a).$$

A typical example is the frame bundle of a manifold X.

Suppose (ρ, V) be a representation of G, then we can construct the *associated vector bundle* $E = P \times_\rho V$ by defining

$$E = P \times_\rho V = (P \times V)/\sim$$

where $\sim$ is

$$(p, v) \sim (p', u) \Leftrightarrow p = p'g \text{ and } v = \rho(g)^{-1} u$$

for some $g \in G$. The projection operator $\pi_E : E \longrightarrow X$ is defined as

$$\pi_E([p, u]) = \pi(p).$$

This is well-defined since $\pi(p) = \pi(pg)$ for all $g \in G$. The transition functions on E are $\rho \circ t_{ij}$. The structure group is clearly $\mathrm{GL}(V)$. Given an vector bundle $E \xrightarrow{\pi} X$ with structure group G, it can be considered as the associated vector bundle to a principle G-bundle with the same transition function. For explicit construction see [6, Page 370]. An important example of an associated vector bundle is the *adjoint bundle* $\mathfrak{g}_P \equiv P \times_{\text{adj}} \mathfrak{g}$ obtained by taking V to be $\mathfrak{g}$- the Lie algebra of G and $\rho : G \longrightarrow \mathfrak{gl}(\mathfrak{g})$ be the adjoint action which is the differential of conjugation map on G.

A choice of connection on $P \xrightarrow{\pi} X$ is a choice of a 1-form called *connection 1-form* which takes values in $\mathfrak{g}$ - the Lie algebra of G and satisfies certain properties. That is a connection 1-form $\mathcal{A}$ is a map

$$\mathcal{A} : TP \longrightarrow \mathfrak{g}$$

with certain properties. Given a local trivialisation $\{U_i\}$ and local sections[22] $\sigma_i : U_i \longrightarrow P$, we can construct local expressions for the connection by pullback:

$$\mathcal{A}_i := \sigma_i^* \mathcal{A}, \quad \text{so that} \quad \mathcal{A}_i : TU_i \longrightarrow \mathfrak{g}. \tag{C.2}$$

1-forms $\mathcal{A}_i$ have to satisfy *gauge covariance*: on $U_i \cap U_j$,

$$\mathcal{A}_j = t_{ij}^{-1} \mathcal{A}_i t_{ij} + i t_{ij}^{-1} dt_{ij}.$$

The local connection 1-form is called *Yang-Mills potential.* The *curvature* or *field strength* of a connection $\mathcal{A}$ is the *exterior covariant derivative* of $\mathcal{A}$:

$$\mathcal{F} = D\mathcal{A} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}. \tag{C.3}$$

which is a $\mathfrak{g}$-valued 2-form. Locally it is again given by

$$\mathcal{F}_i = \sigma_i^* \mathcal{F} \text{ on } U_i$$

Suppose $\{U_i, x^\mu\}$ is a coordinate chart and we write

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu, \quad \mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} dx^\mu \otimes dx^\nu$$

then

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu].$$

Note that $\mathcal{A}_\mu, \mathcal{F}_{\mu\nu} : U_i \longrightarrow \mathfrak{g}$. Recall that a representation $\rho : G \longrightarrow \mathrm{GL}(V)$ descends to a representation $\rho_* : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ given by the derivative as[23]

$$\rho_* (V_1) = \frac{d}{dt} \rho \left(e^{itV_1} \right) \Big|_{t=0}.$$

[22] A local section is simply a map $\sigma_i : U_i \longrightarrow P$ satisfying $\pi \circ \sigma_i = \mathbb{1}_{U_i}$.
[23] We follow physicists convention to define Lie algebra, that is $e^{itX} \in G$ for every $t \in \mathbb{R}$ and $X \in \mathfrak{g}$

Let $\{T^a\}_{a=1}^{\mathrm{dimG}}$ be the generators of G and denote by $T_R^a = \rho_*(T^a)$. Let $s \in \Gamma(E,U)$ be a local section of E on $U \subset X$ and $Y \in \Gamma(TX,X)$ a vector field. The connection $\mathcal{A}$ induces a connection $\mathcal{A}^E$ on E which locally takes the form

$$\mathcal{A}^E = \mathcal{A}_\mu^a T_R^a dx^\mu \quad \text{where} \quad \mathcal{A} = \mathcal{A}_\mu^a T^a dx^\mu \tag{C.4}$$

is the local expression of the connection $\mathcal{A}$ on P. We can use this connection to define covariant derivative on E. We have

$$\nabla_Y^{\mathcal{A}} s = ds(Y) + (\rho_*(\mathcal{A})(Y)s) = (ds)(Y) + \mathcal{A}^E(Y)s.$$

Pointwise, this becomes

$$\left(\nabla_Y^{\mathcal{A}} s\right)(x) = (ds)(Y_x) + \mathcal{A}_\mu^a T_R^a Y^\mu(x) s(x)$$

where $Y = Y^\mu \frac{\partial}{\partial x^\mu}$. In particular for $Y = \partial_\mu$, we get

$$\left(\nabla_\mu^{\mathcal{A}} s\right)(x) = \partial_\mu s + \mathcal{A}_\mu^a T_R^a s(x).$$

One can show that the sum of two local connection 1-forms another local connection 1-form. This means that we can think of the space $\mathcal{A}$ of all connections as an affine space whose tangent space at $\mathcal{A}$ is given by

$$T_{\mathcal{A}}\mathcal{A} \cong \Omega^1(X, \mathfrak{g}_P). \tag{C.5}$$

All this can be transferred to any associated vector bundle $E = P \times_\rho V$. We can again construct the associated adjoint bundle

$$\mathfrak{g}_E = P \times_{\mathrm{adj}} \rho_*(\mathfrak{g})$$

where G acts on $\rho_*(\mathfrak{g})$ via ρ :

$$\begin{aligned} G \times \rho_*(\mathfrak{g}) &\longrightarrow \rho_*(\mathfrak{g}) \\ g \cdot \rho_*(T) &= \rho_*(\mathrm{ad}_g(T)). \end{aligned}$$

Then the space of all induced connection $\mathcal{A}^E$ on E is an affine space with tangent space

$$T_{\mathcal{A}}\mathcal{A}^E \cong \Omega^1(X, \mathfrak{g}_E).$$

A gauge transformation is described by a local function $u \in C^\infty(U,G) =: \mathcal{G}$. If G is a matrix Lie group, then the local connection $\mathcal{A}^U$ on U transforms as

$$\mathcal{A}^U \longrightarrow u^*\left(\mathcal{A}^U\right) = u(x)\mathcal{A}^U u(x)^{-1} + i du(x) u^{-1}(x)$$

and the curvature transforms as

$$\mathcal{F}^U \longrightarrow u^*\left(\mathcal{F}^U\right) = u(x)\mathcal{F}^U u(x)^{-1}. \tag{C.6}$$

In particular, if we take $E \xrightarrow{\pi} X$ to be the adjoint bundle, then

$$u^*\left(\mathcal{A}^U\right) = \mathcal{A}^U + i\left(\nabla^{\mathcal{A}} u\right) u^{-1}. \tag{C.7}$$

C.2 Characteristic classes

Characteristic classes are certain cohomology classes on a manifold X. Many of them are defined using curvature on a vector bundle on X and are elements of de Rham cohomology group. These classes are invariants of the bundle structure, that is, two equivalent bundles determine the same characteristic classes. The essential ingredient to construction of characteristic classes is the *Chern-Weil homomorphism* which determines a closed form on X from an *invariant polynomial*[24] and a curvature 2-form on the vector bundle[25]. The characteristic class does not depend on the choice of connection. Without going into details, here we will list some characteristic classes and their properties, refering to [6, Chapter 11] for proofs, which will be required for the discussion in Section **??**.

Let $E \xrightarrow{\pi} X$ be a complex vector bundle of rank r with a curvature 2-form $\mathcal{F}$. The *total Chern class* is defined by

$$\begin{aligned} c(E) &:= \det\left(\mathbb{1} + \frac{i\mathcal{F}}{2\pi}\right) \\ &=: 1 + c_1(E) + c_2(E) + \cdots + c_k(E) \end{aligned} \tag{C.8}$$

where we have expanded the cohomology class into homogenous forms. Note that since $\mathcal{F}$ is degree 2, $c_j(E) \in H^{2j}(X)$ which implies that $c_j(E) = 0$ for $2j > \mathrm{dim}X$. The Chern class satisfies [26]

$$c(E \oplus F) = c(E) \wedge c(F), \quad c(E \otimes F) = c(E) + c(F). \tag{C.9}$$

For a complex line bundle[27], $c_1(L)$ is the only nontrivial class and is a *complete invariant*, meaning that two complex line bundles are equivalent if and only if their first Chern class is the same. Rank 2 vector bundles with structure group SU(2) are completely classified by the second Chern class.

For real vector bundles of rank r with structure group[28] $\mathrm{O}(r)$, one defines the *total Pontrjagin class* by

$$\begin{aligned} p(E) &:= \det\left(\mathbb{1} + \frac{\mathcal{F}}{2\pi}\right) \\ &=: 1 + p_1(E) + p_2(E) + \cdots + p_{[r/2]}(E). \end{aligned} \tag{C.10}$$

The jth Pontrjagin class $p_j(E) \in H^{4j}(X, \mathbb{R})$ and vanishes if $4j > \mathrm{dim}X$. One has the relation

$$p_j(E) = (-1)^j c_{2j}(E_{\mathbb{C}}) \tag{C.11}$$

[24] For a matrix Lie group G with Lie algebra $\mathfrak{g}$, a polynomial $f : \mathfrak{g} \longrightarrow \mathbb{C}$ is called an invariant polynomial if $f(gTg^{-1}) = f(T)$ for every $g \in G, T \in \mathfrak{g}$.

[25] Note that a vector bundle can be thought of as an associated vector bundle to a principal bundle and hence we can define connection 1-form and a curvature on any vector bundle.

[26] The tensor product and direct sum of bundles is defined by letting the fiber to be the tensor product and direct sum of the individual fibers and changing transition function accordingly.

[27] We usually denote a complex line bundle by L.

[28] If the fibers have an inner product structure, then one can require the transition function to preserve the inner product and hence reduce the structure group to $\mathrm{O}(r)$.

where $E_{\mathbb{C}}$ is the *complexified vector bundle* obtained by complexifying the fibers and the transition functions. If X is a 4-manifold then $p_1(E)$ is the only nontrivial class. But since it is a top form, its integral over X is an invariant of the bundle structure:

$$\int_X p_1(E) = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}(\mathcal{F}\wedge\mathcal{F}) = k \in \mathbb{R}. \tag{C.12}$$

k is called the *instanton number*. This means that isomorphic G-bundles have same instanton number. In particular, for an SU(2) bundle, $k \in \mathbb{Z}$.

C.3 Reduction (lifting) of structure groups

Suppose we have a principal G_1-bundle P_1 over X with local trivialising open sets $\{U_i\}$ on X and transition functions[29] $t_{ij} : U_{ij} \longrightarrow G_1$. Suppose we want to know when can the structure group (and hence the fiber) of P_1 be changed to another Lie group G_2 given a Lie group homomorphism $\phi : G_1 \longrightarrow G_2$ (See [7, Appendix A.1]). Ofcourse we would have new transition functions $\widetilde{t}_{ij} : U_{ij} \longrightarrow G_2$ with $\widetilde{t}_{ij}(x)^{-1} = \widetilde{t}_{ji}(x), \widetilde{t}_{ii} = \mathbb{1}_{G_2}$. Let us denote the new bundle by P_2. For consistency, we assume that $\{U_i\}$ is a *simple* cover, meaning that any intersection of U_i is either empty or simply connected. A consistent construction requires the condition that $\phi \circ \widetilde{t}_{ij}(x) = t_{ij}(x)$ for all $x \in U_{ij}$ and for all U_{ij}. Moreover the obstruction to such a construction is the cocycle condition (cf. (C.1))

$$\zeta_{ijk}(x) \equiv \widetilde{t}_{ij}(x)\widetilde{t}_{jk}(x)\widetilde{t}_{ki}(x) = \mathbb{1}_{G_2}. \tag{C.13}$$

In general $\zeta_{ijk} \neq \mathbb{1}_{G_2}$ in which case changing the structure group is not possible. Note that since $\phi \circ \widetilde{t}_{ij}(x) = t_{ij}(x)$, we have $\zeta_{ijk} : U_{ijk} \longrightarrow \mathrm{Ker}(\phi)$. Infact this function defines a Čech 2-cocyle and hence a cohomology class $f_2(P_1, G) \in H^2(X, \mathrm{Ker}(\phi))$ in the Čech cohomology group[30] of X with coefficients in $\mathrm{Ker}(\phi)$. Thus the obstruction to changing the structure group is measured by $f_2(P_1, G)$. When the homomorphism ϕ is injective (surjective), we call this construction of P_2 as the reduction (lifting) of the structure group. We now describe some relevant examples.

C.3.1 Spin, Spin^c-structure and Stiefel-Whitney class

Let X be an oriented Riemannian n-manifold. Consider the orthonormal frame bundle FX of X. The structure group of FX is SO(n). We want to lift the structure group to Spin(n) via the covering map $\varphi : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$. Since $\mathrm{Ker}(\varphi) = \mathbb{Z}_2$, $w_2(FX) \in H^2(X, \mathbb{Z}_2)$. We usually just write $w_2(X)$ for $w_2(FX)$ and call it the *second Stiefel-Whitney class*. If $w_2(X) = 0$ then X is called a *spin manifold* and is said to admit a *spin structure* and the bundle $S(X)$ with structure group Spin(n) and transition functions $\widetilde{t}_{ij}$ is called the *spinor bundle*. Spinors[31] on a general Riemannian manifold are sections of the spinor bundle and

[29] We use the symbol $U_{ijk\ldots l} \equiv U_i \cap U_j \cap U_k \cap \cdots \cap U_l$.

[30] Čech cohomology is defined in terms of symmetric functions on $U_{ijk\ldots l}$. For a definition of $H^r(X, \mathbb{Z}_2)$ see [6, Page 449]. The generalisation to $H^r(X, \mathbb{Z}_n)$ is strightforward, which is what we will need.

[31] Physically, we are more concerned with Lorentzian signature and hence the structure group of the spinor bundle must be Spin$(1, n-1)$. But for the present discussion, we work with Euclidean quantum field thoery and hence our spacetime manifold is Riemannian.

hence do not exist if the manifold is not spin. In case of a 4-manifold, the (double cover of) Lorentz group is

$$\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2) \tag{C.14}$$

Thus the transition functions are of the form $\widetilde{t}_{ij} = (\widetilde{t}^{+}_{ij}, \widetilde{t}^{-}_{ij}) : U_{ij} \longrightarrow \mathrm{SU}(2) \times \mathrm{SU}(2)$. The transition functions $\widetilde{t}^{\pm}_{ij}$ determine complex rank 2 associated vector bundles which we denote by $S^{\pm}(X)$. Weyl spinors are sections of $S^{\pm}(X)$ and Dirac spinors are sections of $S^{+}(X) \oplus S^{-}(X)$. Most of the 4-manifolds are not spin and hence spinors do not really make sense on a general Riemannian 4-manifold. But one can still put spinors on any orientable 4-manifold by defining what is called a Spin^c-structure. The idea is basically to lift the structure group from SO(4) to Spin^c(4) defined as

$$\mathrm{Spin}^c(4) := \{(u_1, u_2) : u_1, u_2 \in \mathrm{U}(2), \det(u_1) = \det(u_2)\} \subset \mathrm{U}(2) \times \mathrm{U}(2). \tag{C.15}$$

Again the two factors give two complex rank 2 associated vector bundles which we denote by $W^{\pm}$ and call them *chiral spinor bundles*. Spinors can then be defined to be sections of the chiral spinor bundles. The crucial result is that any compact orientable 4-manifold admits a Spin^c-structure. A useful way to think about Spin^c-structure is the following: suppose the manifold is not spin, then $w_2(X) \neq 0$. Suppose $w_2(X)$ is the mod 2 reduction of $\bar{w}_2(X) \in H^2(X, \mathbb{Z})$, that is $w_2(X)$ admits an *integral lift* $\bar{w}_2(X)$. Since line bundles are completely classified by first Chern class, we have a line bundle L with $c_1(L) = \bar{w}_2(X)$. The square root[32] $L^{1/2}$ of the line bundle does not exist globally and the obstruction[33] is again measured by the Stiefel-Whitney class $w_2(X)$. We can then define $S_L = S \otimes L$ and S_L now exists. Clearly the transition functions of S_L give us a Spin^c-bundle.

As discussed above, spinors are sections of (chiral) spinor bundles, but very often spinors are also charged under the gauge group or there is a nontrivial flavour group[34]. In such cases, spinors are sections of the tensor product of the (chiral) spinor bundle with the appropriate vector bundle associated to the gauge bundle and a bundle for the flavor group.

C.3.2 't Hooft flux

Let G be a Lie group and $P \overset{\pi}{\longrightarrow} X$ be a principal $G/Z(G)$-bundle. Suppose we want to lift the structure group to G via the projection map $\pi_{Z(G)} : G \longrightarrow G/Z(G)$. From the discussion in Appendix C.3, the obstruction to such a lifting is a cohomology class $f_2(P, G) \in H^2(X, Z(G))$. This class is called the 't Hooft flux.

D Moduli Space of Instantons

Let X be a smooth, compact oriented Riemannian 4-manifold with metric $g_{\mu\nu}$. Let $\mathcal{A}$ be a connection on the principal fibre bundle $P \overset{\pi}{\longrightarrow} X$ with structure group G and let $\mathcal{F}$ be the

[32] The square root $L^{1/2}$ of a line bundle L is again a line bundle with transition function which square to the transition functions of L.

[33] One can check the cocycle condition for $L^{1/2}$ fails by a number which represents $w_2(X)$.

[34] For example, if we have N spinors, then they can rotate into each other giving us a flavor group U(N).

associated curvature. The Yang-Mills action for the gauge field is given by

$$S_{YM} = \frac{1}{2}\mathrm{Tr}\left(\int_X \mathcal{F} \wedge *\mathcal{F}\right) = \frac{1}{4}\mathrm{Tr}\left(\int_X d^4x\sqrt{g}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right) \tag{D.1}$$

where $*\mathcal{F}$ is the Hodge dual of $\mathcal{F}$ and we have chosen a local section (U, s) of the bundle to write the action in terms of the local gauge field $\mathcal{A}_\mu$ and its field strength $\mathcal{F}_{\mu\nu}$. Now we have

$$\mathcal{F} \wedge *\mathcal{F} = \left(\mathcal{F}^+ + \mathcal{F}^-\right) \wedge \left(*\mathcal{F}^+ + *\mathcal{F}^-\right) = \left(\mathcal{F}^+ \wedge *\mathcal{F}^+\right) + \left(\mathcal{F}^- \wedge *\mathcal{F}^-\right), \tag{D.2}$$

where we used the fact that

$$\omega \wedge \eta = (-1)^{\deg(\omega)\deg(\eta)}\eta \wedge \omega \tag{D.3}$$

Next, using the fact that $*\mathcal{F}^\pm = \pm\mathcal{F}^\pm$ we get

$$\mathcal{F} \wedge \mathcal{F} = \left(\mathcal{F}^+ \wedge *\mathcal{F}^+\right) - \left(\mathcal{F}^- \wedge *\mathcal{F}^-\right). \tag{D.4}$$

This implies that

$$\mathcal{F} \wedge *\mathcal{F} = 2\left(\mathcal{F}^\pm \wedge *\mathcal{F}^\pm\right) \mp \mathcal{F} \wedge \mathcal{F}. \tag{D.5}$$

This gives us

$$\begin{aligned} S_{YM} &= \mathrm{Tr}\left(\int_X \mathcal{F}^\pm \wedge *\mathcal{F}^\pm\right) \mp \frac{1}{2}\mathrm{Tr}\left(\int_X \mathcal{F} \wedge \mathcal{F}\right) \\ &= \mathrm{Tr}\left(\int_X \mathcal{F}^\pm \wedge *\mathcal{F}^\pm\right) \pm 4\pi^2 k \end{aligned} \tag{D.6}$$

where k is the instanton number. Since the first term is positive definite, the action is bounded below[35] by $4\pi^2 k$ and hence the action is minimised when

$$\mathcal{F}^\pm = 0. \tag{D.7}$$

Since supersymmetric configurations restrict us to $\mathcal{F}^+ = 0$, we only consider this case. A connection $\mathcal{A}$ such that $\mathcal{F}^+ = 0$ is called an *anti-selfdual connection* or an *instanton*. Thus instantons minimise the Yang-Mills action. Let $\mathcal{A}$ be the space of all connections on $P \xrightarrow{\pi} X$. The group gauge transformations $\mathcal{G}$ acts on $\mathcal{A}$. Moreover (C.6) implies that the anti-selfduality condition is gauge invariant. To define the moduli space of instantons, we get rid of the gauge redundancy[36]. This leads us to define

$$\mathcal{M}_{\mathrm{ASD}} := \{[\mathcal{A}] \in \mathcal{A}/\mathcal{G} : \mathcal{F}(\mathcal{A})^+ = 0\}. \tag{D.8}$$

Note that this definition is well defined since the anti-selfduality condition is independent of the choice of representative of $[\mathcal{A}]$. We also fix the topology class of the bundle in defining the moduli space of instantons. So if $G = \mathrm{SO}(3)$, then we need to fix the instanton number

[35] Note that since $\int_X \omega \wedge *\omega$ is positive definite for any 2-form ω, k is positive (negative) when $\mathcal{F}^+ = 0(\mathcal{F}^- = 0)$

[36] This is particularly relevant for physical purposes since in path integrals, we only integrate over gauge inequivalent fields once a gauge has been fixed.

and the second Steifel-Whitney class (see Appendix C for details). To make sense of $\mathcal{M}_{\rm ASD}$, we first need to make sense of $\mathcal{A}/\mathcal{G}$. Here $\mathcal{G}$ can be thought of an infinite dimensional Lie group which acts on another infinite dimensional space of fields $\mathcal{A}$. So one can hope to make sense of the quotient space $\mathcal{A}/\mathcal{G}$ if $\mathcal{G}$ acts freely, that is the isotropy group[37] $\Gamma_{\mathcal{A}}$ of a nonzero connection is the center $Z(\mathcal{G})$ of $\mathcal{G}$. If the isotropy group of $\mathcal{A}$ is equal to the center of $\mathcal{G}$ then $\mathcal{A}$ is called an *irreducible connection*. For the adjoint bundle, which is what we are concerned with, from (C.7) and (C.5), we see that $\mathcal{A}$ is irreducible if $\mathrm{Ker}(\nabla^{\mathcal{A}}) = 0$ where $\nabla^{\mathcal{A}} : \Omega^0(X, \mathfrak{g}_E) \longrightarrow \Omega^1(X, \mathfrak{g}_E)$. Let $\mathcal{A}^*$ be the set of irreducible connections and $\widehat{\mathcal{G}} = \mathcal{G}/Z(\mathcal{G})$, then $\mathcal{A}^*/\widehat{\mathcal{G}}$ is now well defined. We now want to see when infinitesimal gauge deformation of an instanton remains an instanton. Indeed if we consider an infinitesimal deformation of an instanton $\mathcal{A}$ by $a \in \Omega^1(X, \mathfrak{g}_E)$ then we want

$$\mathcal{F}(\mathcal{A} + a)^+ = 0. \tag{D.9}$$

From (C.3), we get

$$\mathrm{SD}(a) := p^+(\nabla^{\mathcal{A}} a + a \wedge a) = 0, \tag{D.10}$$

where $p^+ : \Omega^2(X, \mathfrak{g}_E) \longrightarrow \Omega^{2,+}(X, \mathfrak{g}_E)$ is the projection to the subspace of selfdual 2-forms. Thus to linear order in a, the deformation remains an instanton if

$$p^+(\nabla^{\mathcal{A}} a) = 0. \tag{D.11}$$

All this can be packed into a complex:

$$0 \longrightarrow \Omega^0(X, \mathfrak{g}_E) \xrightarrow{\nabla^{\mathcal{A}}} \Omega^1(X, \mathfrak{g}_E) \xrightarrow{\mathrm{SD}} \Omega^{2,+}(X, \mathfrak{g}_E) \longrightarrow 0 \tag{D.12}$$

This is called the *Atiyah-Hitchin-Singer complex* or the *instanton deformation complex* [12]. Note that this complex is defined for each $\mathcal{A} \in \mathcal{A}$. It is *not* an exact sequence. The *index* of this complex, which is also called the *virtual dimension* of $\mathcal{M}_{\rm ASD}$, is defined as

$$\mathrm{vdim}\mathcal{M}_{\rm ASD} := \mathrm{ind} := \mathrm{dim}H^1_{\mathcal{A}} - \mathrm{dim}H^0_{\mathcal{A}} - \mathrm{dim}H^2_{\mathcal{A}} \tag{D.13}$$

where

$$H^0_{\mathcal{A}} := \mathrm{Ker}(\nabla^{\mathcal{A}}), \quad H^1_{\mathcal{A}} := \frac{\mathrm{Ker}(p^+\nabla^{\mathcal{A}})}{\mathrm{Im}(\nabla^{\mathcal{A}})}, \quad H^2_{\mathcal{A}} := \mathrm{Coker}(p^+\nabla^{\mathcal{A}}) := \frac{\Omega^{2,+}(X, \mathfrak{g}_E)}{\mathrm{Im}(\mathrm{SD})} \tag{D.14}$$

The result of this analysis is that the *Atiyah-Singer index theorem* gives the index of the AHS complex in terms of the topological data of the bundle. For an SU(2) bundle, the result is [7]

$$\mathrm{vdim}\mathcal{M}_{\rm ASD} = 8k - 3(b_2^+ - b_1 + 1), \tag{D.15}$$

where k is the instanton number, b_1 is the dimension of the first homology group $H_1(X)$ and $b_2^+ = \mathrm{dim}H^{2,+}(X)$ where $H^2(X) \cong H^{2,+}(X) \oplus H^{2,-}(X)$ is the decomposition under the Hodge operator $\omega \to *\omega$ since $*^2 = 1$ for Riemannian 4-manifolds. We now claim that for any $\mathcal{A}$, we have

$$T_{[\mathcal{A}]}\mathcal{M}_{\rm ASD} = H^1_{\mathcal{A}}. \tag{D.16}$$

[37]The isotropy group of a connection $\mathcal{A}$ is defined by $\Gamma_{\mathcal{A}} := \{u \in \mathcal{G} : u^*(\mathcal{A}) = \mathcal{A}\}$.

Indeed, $\mathrm{Ker}(p^+\nabla^{\mathcal{A}})$ contains all infinitesimal deformations of $\mathcal{A}$ which preserve the anti-selfduality condition and $\mathrm{Im}(\nabla^{\mathcal{A}})$ determines all instantons gauge equivalent to $\mathcal{A}$. If $\mathcal{A} \in \mathcal{A}^*$ then $H^0_{\mathcal{A}} = 0$. If in addition $H^2_{\mathcal{A}} = 0$ then the connection is called *regular*. Thus the dimension of the moduli space of regular, irreducible instantons is equal to the virtual dimension given in (D.15). We end this section by describing how the AHS complex can be used to prove (3.27). Indeed, note that the classical saddles of ψ come from the term $\chi_{\dot\alpha\dot\beta}\overline{\sigma}^{\mu\dot\alpha\alpha}\nabla_\mu\psi^{\dot\beta}_\alpha$ and $\eta\nabla^{\dot\alpha\alpha}\psi_{\alpha\dot\alpha}$. In the coordinate basis

$$\chi_{\dot\alpha\dot\beta}\overline{\sigma}^{\mu\dot\alpha\alpha}\nabla_\mu\psi^{\dot\beta}_\alpha \longrightarrow \chi^{\mu\nu}\nabla_\mu\psi_\nu. \tag{D.17}$$

Here $\chi_{\mu\nu} = (\bar\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\chi_{\dot\alpha\dot\beta}$ is a selfdual 2-form. This term in the action can be written as

$$\int_X d^4x\sqrt{g}\chi^{\mu\nu}\nabla_\mu\psi_\nu = -\int_X (\nabla\psi)\wedge *\chi = -\int_X (\nabla\psi)\wedge\chi = -\int_X p^+(\nabla\psi)\wedge\chi \tag{D.18}$$

where we used the fact[38] that $\omega\wedge\omega' = p^+(\omega)\wedge\omega'$ for a selfdual 2-form ω'. Thus the classical saddles for ψ are solutions to

$$p^+(\nabla^{\mathcal{A}}\psi) = 0. \tag{D.19}$$

In addition to this, the equation of motion for η gives another constraint that the classical saddles for ψ satisfy:

$$\nabla_\mu\psi^\mu = 0. \tag{D.20}$$

Note that solutions to (D.19) gives us elements of $\mathrm{Ker}(p^+(\nabla^{\mathcal{A}}))$ and (D.20) is the familiar gauge fixing condition. Since we only care about gauge-inequivalent classical saddles, thus the dimension of the space of gauge-nequivalalent classical saddles for ψ is $\mathrm{dim}H^1_{\mathcal{A}}$. Similarly, since χ is selfdual 2-form, the dimension of the space of gauge-inequivalent classical saddles for χ is $\mathrm{dim}H^2_{\mathcal{A}}$. Thus for irreducible, regular connections, index theorem immediately proves (3.27).

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