

Beauty And The Beast Part 2: Apprehending The Missing Supercurrent

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Based on arXiv: 2309.02382 with Gregory Moore

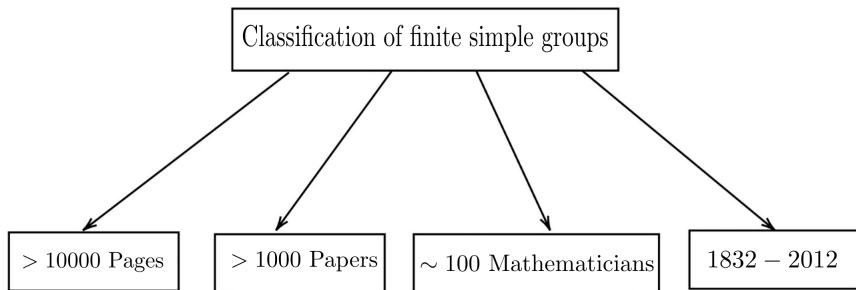
Outline

- 1 Monstrous Moonshine
- 2 Beauty and the Beast module
- 3 Construction of the Moonshine module
- 4 Gap in Beauty and the Beast
- 5 Solution

Monstrous Moonshine

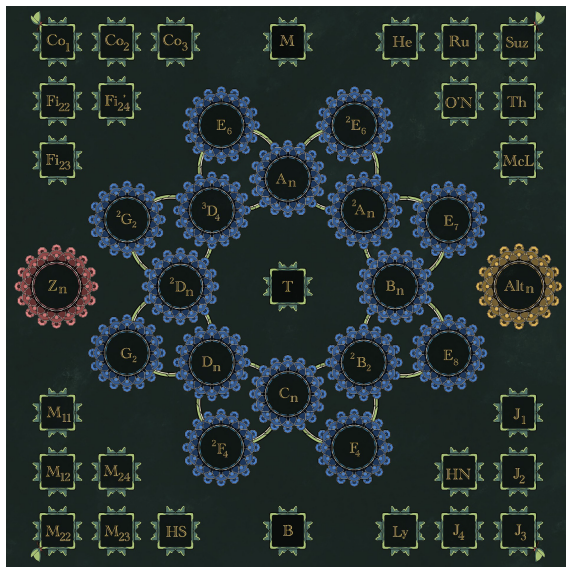
Finite simple groups

Finite groups with no nontrivial normal subgroups.

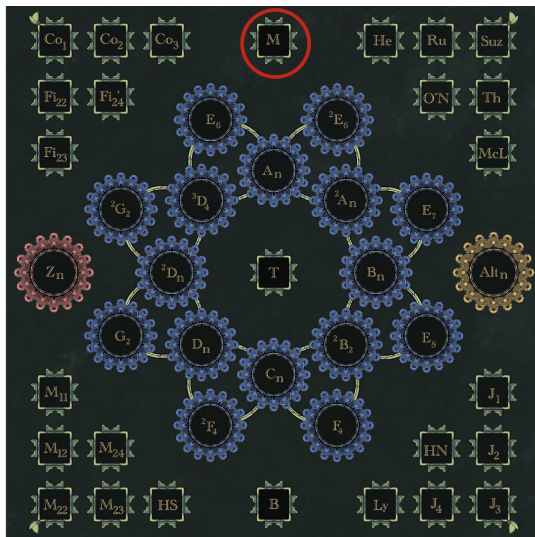


Galois, Cayley, Mathieu, ..., Conway, Fisher, Griess, ..., Gorenstein, Aschbacher, ..., Gonthier.

Finite simple groups



Monster group



M: Largest sporadic group of the order $\sim 8 \times 10^{53}$.

Monstrous Moonshine: $196884 = 1 + 196883$

$$J(q) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

Dimensions of irreps of $\mathbb{M}$: $r_n = 1, 196883, 21296876, 842609326, \dots$

$$1 = r_1$$

$$196884 = r_1 + r_2$$

$$21493760 = r_1 + r_2 + r_3$$

$$864299970 = 2r_1 + 2r_2 + r_3 + r_4$$

McKay-Thompson conjecture

There exists an infinite dimensional graded representation

$V^{\natural} = \bigoplus_{n=-1}^{\infty} V_n^{\natural}$ of $\mathbb{M}$ such that

$$\sum_{n=-1}^{\infty} \dim(V_n^{\natural}) q^n = J(q).$$

The Moonshine Module

- Frenkel-Lepowsky-Meurman constructed $V^{\natural}$ using the theory of vertex operator algebras.
- Physical interpretation: $\mathbb{Z}_2$ orbifold of a holomorphic CFT (the Leech torus theory) [Harvey (1985)]

Beauty and the Beast: Superconformal Symmetry in a Monster Module

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Abstract. Frenkel, Lepowsky, and Meurman have constructed a representation of the largest sporadic simple finite group, the Fischer–Griess monster, as the automorphism group of the operator product algebra of a conformal field theory with central charge $c = 24$. In string terminology, their construction corresponds to compactification on a $\mathbb{Z}_2$ asymmetric orbifold constructed from the torus $\mathbb{R}^{24}/\Lambda$, where Λ is the Leech lattice. In this note we point out that their construction naturally embodies as well a larger algebraic structure, namely a super-Virasoro algebra with central charge $\hat{c} = 16$, with the supersymmetry generator constructed in terms of bosonic twist fields.

$$V^{\natural} \xrightarrow{\text{non-local } \mathbb{Z}_2\text{-cover}} V^{\text{BB}}$$

$$\text{Virasoro} \longrightarrow \text{Super-Virasoro}$$

$$T_B(z) \longrightarrow T_B(z), T_F(z)$$

Construction of the Moonshine module

Leech torus theory

$$\Lambda_L := \left\{ \sum_{n=1}^{24} n_i \alpha_i : n_i \in \mathbb{Z} \right\} \subset \mathbb{R}^{24} \quad \cdot \quad \circ \quad \circ \quad \text{integral even selfdual } \Lambda_L(2) := \{ \lambda \in \Lambda_L : \lambda^2 = 2 \} = \emptyset$$

- Leech torus theory is a theory of 24 chiral bosons $X^i(z)$ with target $\mathbb{T}^{24} := \mathbb{R}^{24}/\Lambda_L$.

$$X^i(z) = q^i + ip^i \ln z + i \sum_{n \neq 0} \frac{a_n^j}{n} z^{-n}$$

$$[q^i, p^j] = i\delta^{ij}, \quad [a_n^i, a_m^j] = n\delta^{ij}\delta_{n+m,0}.$$

- Fock space: $a_n^i|\lambda\rangle = 0$ for $n > 0$ and $p^i|\lambda\rangle = \lambda^i|\lambda\rangle$. Hilbert space $\mathcal{H}(\Lambda_L)$.

Leech torus theory

$$\omega_L = \frac{1}{2}a_{-1} \cdot a_{-1}|0\rangle, \quad V(\omega_L, z) = T(z) = -\frac{1}{2} : \partial_z X \cdot \partial_z X : \quad c = 24$$

$$\mathcal{H}(\Lambda_L) \ni \psi = \left(\prod_{a=1}^M a_{-m_a}^{j_a} \right) |\lambda\rangle, \quad h_\psi = \frac{1}{2}\lambda^2 + \sum_{a=1}^M m_a.$$

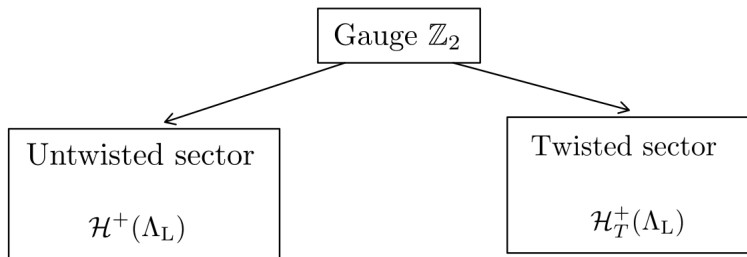
$$Z_{\Lambda_L}(q) = \text{Tr} q^{L_0 - c/24} = J(q) + 24, \quad T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

- $\mathbb{Z}_2$ symmetry:

$$\theta X^i \theta^{-1} = -X^i, \quad \theta a_n^i \theta^{-1} = -a_n^i, \quad \theta |\lambda\rangle = |-\lambda\rangle$$

$$\mathcal{H}(\Lambda_L) = \mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}^-(\Lambda_L)$$

$\mathbb{Z}_2$ orbifold



- Twisted sector fields $R^i(z)$

$$R^i(e^{2\pi i} z) = -R^i(z) \pmod{\Lambda_L}, \quad R^i(z) = \tilde{q}^i + i \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{c_k^i}{k} z^{-k}$$

- Translation symmetry $X \rightarrow X + \lambda$, $\lambda \in \Lambda_L$ breaks to $T_2(\Lambda_L) := \Lambda_L/2\Lambda_L$.

$\mathbb{Z}_2$ orbifold

Quantum theory realises broken symmetry via a central extension of $T_2(\Lambda_L)$:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{H}(T_2(\Lambda_L)) \xrightarrow{-} T_2(\Lambda_L) \longrightarrow 0$$

$$\mathcal{H}(T_2(\Lambda_L)) = \{\pm e_\lambda : \lambda \in T_2(\Lambda_L)\}$$

$$e_\lambda e_\mu = (-1)^{\lambda \cdot \mu} e_\mu e_\lambda = \varepsilon(\lambda, \mu) e_{\lambda+\mu}, \quad e_\lambda^2 = (-1)^{\frac{1}{2}\lambda^2}.$$

Unique unitary 2^{12} -dimensional irrep $\mathcal{S}$ of $\mathcal{H}(T_2(\Lambda_L))$.

$$\mathcal{H}_T(\Lambda_L) = \text{Span}_{\mathbb{C}}\{\mathcal{F} \otimes \mathcal{S}\}$$

$$\chi = \left(\prod_{a=1}^N c_{-k_a}^{j_a} \right) |0\rangle_T \otimes \chi_0, \quad h_\chi = \frac{3}{2} + \sum_{a=1}^M k_a$$

$$\theta \text{ action: } \theta c_k^i \theta^{-1} = -c_k^i, \quad \theta \chi_0 = -\chi_0, \quad \chi_0 \in \mathcal{S}.$$

$$V^\natural := \mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}_T^+(\Lambda_L)$$

Superconformal symmetry

Conformal symmetry: stress tensor T_B with OPE

$$T_B(z)T_B(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T_B(w) + \frac{1}{(z-w)}\partial_w T_B(w)$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}.$$

Superconformal symmetry

Conformal symmetry: stress tensor T_B with OPE

$$T_B(z)T_B(w) \sim \frac{3\hat{c}/4}{(z-w)^4} + \frac{2}{(z-w)^2}T_B(w) + \frac{1}{(z-w)}\partial_w T_B(w)$$
$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\hat{c}}{8}m(m^2-1)\delta_{m,-n}$$

Superconformal symmetry: supercurrent $T_F(z)$ of dimension $\frac{3}{2}$ with OPE:

$$T_B(z)T_F(w) \sim \frac{3/2}{(z-w)^2}T_F(w) + \frac{1}{(z-w)}\partial_w T_F(w)$$
$$T_F(z)T_F(w) \sim \frac{\hat{c}/4}{(z-w)^3} + \frac{1/2}{(z-w)}T_B(w)$$

Gap in the Beauty and the Beast

Beauty and the Beast again

- $V^{\text{BB}} = \mathcal{H}(\Lambda_L) \oplus \mathcal{H}_T(\Lambda_L)$ [Dixon-Ginsparg-Harvey (1988)]
- “non-local” because OPE of fields in $\mathcal{H}^-(\Lambda_L)$ with fields in $\mathcal{H}_T^-(\Lambda_L)$ has square root singularity.
- $\mathcal{S} \subset \mathcal{H}_T^-(\Lambda_L)$ is a 2^{12} dimensional subspace of weight $3/2$ vectors. DGH claimed that one of them is the supercurrent.



- $\mathbb{Z}_2$ -cover?

Question: $\mathbb{Z}_2$ -cover?

Solution: Spin lift

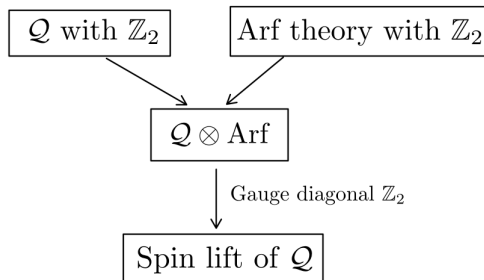
Spin lift of a QFT

Given a bosonic 2d QFT $\mathcal{Q}$ with nonanomalous $\mathbb{Z}_2$ symmetry, one can construct a spin QFT via spin lift.

Arf theory: On a Riemann surface Σ_ρ with spin structure ρ

$$Z_{\text{Arf}}[\Sigma_\rho] := e^{i\pi \text{Arf}[\rho]}, \quad \text{Arf}[\rho] = \text{ind}_2 D_\rho = \text{Dim}(\text{Ker}(D_\rho) \cap (\gamma = +1)) \bmod 2.$$

$$\text{Arf}(S_{\text{NS}}^1) \cong \mathbb{C}^{1|0}, \quad \text{Arf}(S_{\text{R}}^1) \cong \mathbb{C}^{0|1}.$$



Spin lift of Moonshine module

$$Z^{\text{SL}}[\Sigma_\rho] = \frac{1}{2} \sum_{t \in H^1(\Sigma, \mathbb{Z}_2)} Z_{\mathcal{Q}}[P_t] \exp(i\pi \text{Arf}[\rho + t]).$$

$$\mathcal{H}^{\text{SL}}(S_{\text{NS}}^1) = \mathcal{H}_{\mathcal{Q}}^+ \oplus \mathcal{H}_T^-, \quad \mathcal{H}^{\text{SL}}(S_{\text{R}}^1) = \mathcal{H}_{\mathcal{Q}}^- \oplus \mathcal{H}_T^+.$$

When $\mathcal{Q}$ is a CFT

$$\text{Tr}_{\mathcal{H}_T} q^{L_0 - \frac{c}{24}} = Z_{\mathcal{H}_{\mathcal{Q}}}^{\iota}(-1/\tau), \quad Z_{\mathcal{H}_{\mathcal{Q}}}^{\iota}(\tau) = \text{Tr}_{\mathcal{H}_{\mathcal{Q}}} \iota q^{L_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}.$$

- $V^{\natural}$ has a $\mathbb{Z}_2$ symmetry

$$\begin{aligned} \mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}_T^+(\Lambda_L) \\ \iota = +1, \quad \iota = -1 \end{aligned}$$

- Spin lift associated to this $\mathbb{Z}_2$ is $V^{\text{BB}} = \mathcal{H}(\Lambda_L) \oplus \mathcal{H}_T(\Lambda_L)$
[Lin-Shao (2021)]

Question: Where is the supercurrent?

Solution: Beauty and the Beast Part 2

Beauty and the Beast Part 2

Recall: $h_{\chi_0} = \frac{3}{2}$ for all $\chi_0 \in \mathcal{S}$.

Goal: find $\chi_s \in \mathcal{S}$ such that $V(\chi_s, z)$ is a supercurrent. OPE of $V(\chi_s, z)$ with itself.

THE CONFORMAL FIELD THEORY OF ORBIFOLDS

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Conformal Field Theories, Representations and Lattice Constructions

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CONFORMAL FIELD THEORY OF TWISTED VERTEX OPERATORS

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Beauty and the Beast Part 2

Dolan-Goddard-Montague formalism: physical restatement of VOA and its modules.

$\mathcal{H}(\Lambda_L)$ is a VOA

$\mathcal{H}_T(\Lambda_L)$ is a module of $\mathcal{H}(\Lambda_L)$

Vertex operators for $\phi \in \mathcal{H}(\Lambda_L)$

$$V(\phi, z) : \mathcal{H}(\Lambda_L) \rightarrow \mathcal{H}(\Lambda_L), \quad V_T(\phi, z) : \mathcal{H}_T(\Lambda_L) \rightarrow \mathcal{H}_T(\Lambda_L).$$

Intertwiners of the module: for $\psi \in \mathcal{H}_T(\Lambda_L)$

$$W(\psi, z) : \mathcal{H}(\Lambda_L) \rightarrow \mathcal{H}_T(\Lambda_L), \quad \overline{W}(\psi, z) : \mathcal{H}_T(\Lambda_L) \rightarrow \mathcal{H}(\Lambda_L)$$

$$W(\psi, z)\phi := e^{zL-1}V_T(\phi, -z)\psi, \quad \overline{W}(\overline{\chi}, z) = z^{-2h_\chi}W(e^{z^*L_1}\chi, 1/z^*)^\dagger.$$

$V^{\text{BB}} = \mathcal{H}(\Lambda_L) \oplus \mathcal{H}_T(\Lambda_L)$ with vertex operators

$$\tilde{V}(\phi, z) = \begin{pmatrix} V(\phi, z) & 0 \\ 0 & V_T(\phi, z) \end{pmatrix}, \quad \tilde{V}(\psi, z) = \begin{pmatrix} 0 & \overline{W}(\psi, z) \\ W(\psi, z) & 0 \end{pmatrix}.$$

Beauty and the Beast Part 2

$$T_B(z) = \tilde{V}(\omega_L, z), \quad \omega_L = \frac{1}{2}a_{-1} \cdot a_{-1}|0\rangle.$$

DGM gives the general form of OPE of $V(\chi_s, z)$ with itself.

$$\tilde{V}(\chi, z)\tilde{V}(\chi, w) \sim \frac{\alpha(\chi)}{(z-w)^3} + \frac{\alpha(\chi)/8}{(z-w)}T_B(w) + \sum_{\lambda^2=4} \frac{\kappa_\lambda(\chi)}{z-w}\tilde{V}(|\lambda\rangle, z),$$

Real payoff of DGM formalism: choose an orthonormal basis of $\mathcal{S}$

$$\alpha(\chi)^* = -is^\dagger s^*, \quad \boxed{\kappa_\lambda(\chi) = -16is^\dagger \gamma_\lambda s}, \quad \gamma_\lambda = \text{matrix of } e_\lambda.$$

Goal: choose $\chi_s \in \mathcal{S}$ such that $\alpha(\chi_s) = 1$ and $\kappa_\lambda(\chi_s) = 0$ for all $\lambda^2 = 4$.
Then

$$\boxed{T_F(z) = 2\tilde{V}(\chi_s, z) \text{ with } \hat{c} = 16}$$

First requirement is easy. Take $\chi_1 = e^{-i\pi/4}\chi$ for any unit vector $\chi \in \mathcal{S}$.

Quantum error correction codes

Brute force is hopeless: there are 196560 vectors with $\lambda^2 = 4$.

Moonshine, Superconformal Symmetry, and Quantum Error Correction

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A quantum code on n qubits for a set of errors $\{E_i\}$ is a certain subspace $C_q \subset \mathbb{F}^{2^n}$.

$$E_i C_q \subset C_q^\perp.$$

Recall $\kappa_\lambda(\chi) = -16is^\dagger \gamma_\lambda s$. Want to find a QEC for the set of errors $\{\gamma_\lambda : \lambda^2 = 4\}$. Then take $\chi_s \in C_q$.

Superconformal sublattices $\rightarrow$ codes

$$2\Lambda_L \subset \Lambda_{\text{SC}} \subset \Lambda_L$$

$2^{12} \qquad 2^{12}$

$$2|\lambda_1 \cdot \lambda_2|, 4|\lambda|^2, \lambda^2 > 4, \quad \lambda, \lambda_1, \lambda_2 \in \Lambda_{\text{SC}}$$

Choose a rep of $\mathcal{H}_2(T_2(\Lambda_L))$ using $2^{12} \times 2^{12}$ Dirac gamma matrices γ_i in 24 dimensions.

$$w \in \mathbb{F}_2^{24}, \quad \gamma_w := \prod_{i=1}^{24} \gamma_i^{w^i}, \quad \gamma_w \gamma_{w'} = \varepsilon(w, w') \gamma_{w+w'}, \quad \boxed{\text{Tr}(\gamma_w) = 0, \quad w \neq 0}$$

$$C_{\text{cl}} = \{w \in \mathbb{F}_2^{24} : \gamma_\lambda = \gamma_w, \lambda \in \Lambda_{\text{SC}}\}, \quad |C_{\text{cl}}| = 2^{12}$$

is a classical code. Choose $b : C_{\text{cl}} \rightarrow U(1)$ such that $\frac{b(w+w')}{b(w)b(w')} = \varepsilon(w, w')$ [Freed-Moore-Segel (2007)] . Then

$$P = \frac{1}{2^{12}} \sum_{w \in C_{\text{cl}}} \tilde{\gamma}_w, \quad \tilde{\gamma}_w = b(w) \gamma_w, \quad P^2 = P, \quad \text{rank}(P) = 1.$$

- $C_q = \text{Im} P$ is a quantum code for error operations $\{\gamma_\lambda : \lambda^2 = 4\}$.

Examples of Λ_{SC}

Unit vector $\chi_s \in \text{Im}P \implies T_F(z) = 2\tilde{V}(e^{-i\pi/4}\chi_s, z)$ is a supercurrent



$$\sqrt{2}\Lambda_L \xrightarrow{\text{isometry}} \Lambda_L \quad [\text{Dong-Li-Mason-Nortan (1996)}]$$

- One can construct explicit embeddings using $\mathbb{Z}_4$ codes [Harada-Kitazume (1999)]

Image of $\sqrt{2}\Lambda_L$ is an example of Λ_{SC} .

- We constructed two distinct embeddings using pseudo Golay code and Quaternary Golay code.

Atleast two distinct supercurrents in V^{BB}

Final Comments

- No $N = 2$ SUSY because all dimension 1 currents are non-local with respect to supercurrents.
- The subspace $\mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}_T^-(\Lambda_L)$ is a superconformal VOA [G.W. Moore, [R.K.S](#) (2023)]. What is its Automorphism group? Possibly larger than $2^{1+24} \cdot \text{Co}_0$.
- Genus zero property of Monster? Recent progress [Duncan-Frenkel (2009), Paquette-Persson-Volpato (2017)]

Thank You