APPLICATIONS OF NUMBER THEORY IN STRING THEORY

A REPORT

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CERTIFICATE

This is to certify that **Ranveer Kumar Singh**, BS-MS (Dual Degree) student in department of Mathematics, has completed bonafide work on the thesis entitled **'Applications of number theory in string theory'** under my supervision and guidance.

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Ranveer Kumar Singh

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered. Somehow it should be possible to build them into a coherent group-theoretical structure, analogous to the structure of modular forms which Hecke built around the old theta-functions of Jacobi. This remains a challenge for the future. My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include mock theta-functions."

Freeman Dyson, 1988, [1].

ABSTRACT

Number theory finds wide ranging applications in mathematics. Surprisingly, it also has deep connections in theoretical physics. Specifically, it appears in string theory. Although, several objects from number theory appear in string theory we will study one particular application in this thesis. To be precise, a particular string theory is the Type IIB string theory compactified on $K3 \times T^2$. The set of orbits of black hole solutions in this theory under a certain equivalence relation forms a group which is isomorphic to the group of equivalence classes of positive definite binary quadratic forms, called class group. Several questions arise due to this identification which we try to answer in this thesis. We study a quantity called degeneracy of black hole microstates which is related to a partition function. The partition function in this theory is a Siegel modular form. This thesis mainly studies modular forms and class groups and tries to state the above mentioned connection of number theory with string theory with complete mathematical rigour.

CONTENTS

1. INTRODUCTION

1.1 Introduction for mathematicians

String theory in theoretical physics has rich mathematical content[11]. One particular aspect is the study of black holes in the theory. Black holes in string theory are characterised by two charge vectors Q and P called the electric charge vector and the magnetic charge vector respectively. There are three duality transformation on these black hole charges namely the S, T and $U-$duality transformation. In particular string theories, this corresponds to the $SL_2(\mathbb{Z})-$action on quadratic forms constructed out of the charge vectors. Under $U-$duality transformation, the set of orbits of black holes forms a group and is isomorphic to the class group of binary quadratic forms[14]. This raises several questions regarding the interpretation of the group operation on the black hole side[13]. In mathematics, there is a well known method to compose quadratic forms using Bhargava cubes[3]. We want to interpret this method on the black hole side physically. Moreover, we try to answer the questions using the degeneracy of black hole solution. Degeneracy is related to the partition function of the theory[12]. This partition function turns out to be a Siegel modular form. Degeneracy is equal to the Fourier coefficient of the Siegel modular form upto a sign[12]. Thus string theory is deeply connected to the theory of automorphic forms. Degeneracy by definition is a positive integer. Thus the study of the connection of string theory with number theory also predicts the sign changes of Fourier coefficients of the Siegel modular forms[19]. One particular type of black hole in string theory are single centered black holes. The degeneracy of single centered black holes are the Fourier coefficients of mock Jacobi forms[17]. Thus, there is a relation of mock modular forms and Jacobi forms with string theory. This thesis is mainly aimed at understanding these connections.

1.2 Introduction for Physicists

The theory of class numbers and class groups has been a venerable problem in number theory since Gauss. It is famous result that the number of orbits under the $SL_2(\mathbb{Z})$ action on the binary quadratic forms of fixed discriminant D are finite in number[10]. This finite set is called the class group. The $U-$duality equivalence class of black holes in Type IIB string theory is isomorphic to this group.

Another area in number theory is the study of modular forms which are holomorphic fucntions on the upper half space satisfying certain modularity conditions[4]. Generalisation of modular forms are the Siegel modular forms and the Jacobi forms[9, 8]. Relaxing the holomorphicity condition with an addition condition results in the theory of harmonic Maass forms and mock modular forms[2, 6, 7]. Mixing mock modular form and Jacobi form in a certain way gives the theory of mock Jacobi forms[17]. These function admits series expansion called the Fourier expansion. The partition function of the Type IIB string theory turns out to be a Siegel modular form and the degeneracy of black hole is the Fourier coefficient of this Siegel modular form. If we only consider single centered black hole, the degeneracy is the Fourier coefficient of certain mock Jacobi forms. This thesis first studies the mathematical topics in detail and then make this connection of number theory with string theory explicit.

1.3 Goals of the thesis

As mentioned in the introduction, this thesis is aimed at understanding the connections of black hole physics with number theory. To do so, we first systematically study automorphic forms.

1. We begin with the classical theory of modular forms for $SL_2(\mathbb{Z})$. We discuss modular forms of integral weight for $SL_2(\mathbb{Z})$ and its subgroups with examples in detail. We refer the reader to [4] for the proofs of the Theorems recorded without proof. Towards the end, we describe the need for defining modular forms of half integral weight by introducing some examples without going into the details which do not play any role in our investigation.

2. We then discuss the theory of harmonic Maass forms and mock modular forms in detail with examples.
3. We then discuss Jacobi forms in detail with a plethora of examples. We briefly discuss Siegel modular forms relevant to our analysis.
4. We then study the theory of class groups from the quadratic forms side, briefly discussing Dirichlet composition and Bhargava cubes.
5. Finally we examine the connections mentioned in the introduction. Explicitly we try to understand the following Theorem by G.W. Moore.

Theorem 1.1 (Moore)**.** *If $D < 0$ is a fundamental discriminant, then the U-duality equivalence classes of attractor black holes of entropy $S = \pi\sqrt{-D}$ admit a structure as a finite abelian group. Moreover, this group is isomorphic to the class group $C(D)$.*

This Theorem gives rise to several questions as to the interpretation of the group composition law on the black hole side. To be explicit, we list the questions posed in [13].

Questions 1.2. (i) Is there a natural physical interpretation of the group law described in Theorem 1.1 in terms of attractor black holes?

(ii) Is there a distinguished physical property of the identity class black hole, which corresponds to the class represented by the identity element 1_D?

(iii) Is there a physical interpretation of inverse black hole?

(iv) What is the physical interpretation of the order of a black hole corresponding to the order of an element in the class group?

1.4 Contributions of this thesis

In this thesis, apart from explicitly mentioning the mathematical content of black hole physics rigorously, we also provide proofs of some of the results which are know to physicists and mathematicians but have not been written down explicitly in the literature. To be precise, we provide the proof of the following Theorems in this thesis.

Theorem 1.3. *The Fourier expansion of* $Z(\Omega) = \frac{1}{\Phi_{10}(\Omega)}$ *is of the form*

$$Z(\Omega) = \sum_{\substack{m,n \geq -1 \\ r \in \mathbb{Z}}} g(m,r,n) q^n \zeta^r w^m \tag{1.1}$$

where Φ_{10} *is the Igusa cusp form.*

Theorem 1.4. $Z(\Omega)$ *has Fourier-Jacobi expansion of the form*

$$Z(\Omega) = \sum_{m \geq -1} \psi_m(\tau, z) w^m \tag{1.2}$$

where ψ_m *are meromorphic Jacobi forms of weight* -10 *and index* m*.*

Theorem 1.5. *The degeneracy of a black hole microstate remains invariant in its* $U-$*duality equivalence class.*

Theorem 1.6. *The degeneracy of a black hole* $U-$*duality equivalence class and its inverse class under Moore's identification are the same.*

Apart from these known results, we have made an attempt to prove a negative result. We have shown that degeneracy of black hole microstates is not a good quantity to study if we want to characterise $U-$duality equivalence classes. Precisely, we show that degeneracy does not recognize the class group composition of black hole classes. Towards the end, we attempt to give a definition of the identity black hole class using the asymptotic entropy upto first order linear correction. We also prove the compatibility of the definition with respect to the class group composition.

Definition 1.7. Let $D < 0$ be a fundamental discriminant. Consider the class group of $U-$duality equivalence classes of attractor black holes with leading entropy $S = \pi\sqrt{-D}$. The equivalence class with entropy upto leading correction given by

$$S_{id} = \pi\sqrt{-D} - \begin{cases} \ln\left(\left|g\left(\frac{2i}{\sqrt{-D}}\right)\right|^2 \frac{4^{12}}{D^6}\right) & \text{if } D \equiv 0 (\text{mod } 4) \\ \ln\left(\left|g\left(\frac{4i}{i+\sqrt{-D}}\right)\right|^2 \frac{(16D)^6}{(1-D)^{12}}\right) & \text{if } D \equiv 1 (\text{mod } 4) \end{cases}$$

is the identity class of the class group. Here $g(\tau) = \eta(\tau)^{24}$.

2. IMPORTANT THEOREMS AND DEFINITIONS

We write $z \in \mathbb{C}$ as $z = x + iy$ to denote a complex number with $x, y \in \mathbb{R}$. We denote by $\mathbb{H}$, the upper half plane which is the set

$$\mathbb{H} := \{z \in \mathbb{C} : y > 0\}.$$

We will often use the following Theorem to show holomorphicity of functions:

Theorem 2.1. *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of a domain Ω, then f is holomorphic in Ω.*

Theorem 2.2. *(Identity Theorem) Let f and g be two holomorphic functions on a domain Ω. Suppose $f = g$ on a subset $S \subset \Omega$ which has a limit point in Ω then $f = g$ on Ω.*

We will use the *Poisson summation formula.* To state the result, we first make some definitions.

Definition 2.3. Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a smooth function. We define the *Schwartz space* S as the set of smooth functions f such that for every pair of integers $m, n \geq 0$,

$$\sup_{x\in\mathbb{R}} |x^m f^{(n)}(x)| < \infty.$$

Definition 2.4. For $f \in S$, we define the *Fourier transform* $\widehat{f}$ by setting

$$\widehat{f} = \int_{-\infty}^{\infty} f(t) e^{-2\pi i t x} dt.$$

Theorem 2.5. *(Poisson summation formula) Let $f \in S$. Then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

Let us define the *Jacobi symbol* in terms of the usual *Legendre symbol.* Let d be an odd integer and c be any integer. Then the Jacobi symbol $\left(\frac{c}{d}\right)$ is defined as follows:

$$\left(\frac{c}{d}\right) = \begin{cases} \left(\frac{c}{d}\right) & \text{if } d \text{ is odd positive prime} \\ \prod_j \left(\frac{c}{p_j}\right) & \text{if } 0 < d = \prod_j p_j,\ p_j - \text{odd primes} \\ \left(\frac{c}{|d|}\right) & \text{if } d < 0, c > 0 \\ -\left(\frac{c}{|d|}\right) & \text{if } d < 0, c < 0 \\ 0 & \text{if } \gcd(c,d) > 1. \end{cases}$$

Definition 2.6. A *Dirichlet character modulo N* is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. This implies $\chi(1) = 1$.

Theorem 2.7. *The set of all Dirichlet characters modulo N forms a group of order $\phi(N)$ where ϕ is the* Euler's totient function.

3. CLASSICAL MODULAR FORMS

3.1 Modular forms of integral weight

Modular forms are holomorphic functions on the upper half plane satisfying a certain modularity condition with respect to some matrix group along with growth conditions at some specific points of the real axis of the complex plane. We will make this precise in later sections. We mainly follow [4] in this chapter.

3.1.1 The Setup

Let R be a commutative ring with identity. Define

$$GL_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R, ad - bc \in R^\times := R \setminus \{0\} \right\}$$

and

$$SL_2(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(R) : ad - bc = 1 \right\}.$$

It is easy to check that these are groups under matrix multiplication. We denote by $GL_2^+(\mathbb{R})$, the The group of invertible 2×2 matrices with entries in $\mathbb{R}$ with positive determinant. $SL_2(\mathbb{Z})$ is called the *full modular group*. The group $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ via the following action. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$, define

$$\gamma(z) := \frac{az + b}{cz + d}. \tag{3.1}$$

It is easy to check that

$$Im(\gamma(z)) = \frac{(ad-bc)Im(z)}{|cz+d|^2}. \tag{3.2}$$

Hence if $det(\gamma) = ad - bc > 0$ then γ maps $\mathbb{H}$ to $\mathbb{H}$. In particular, when $\gamma \in SL_2(\mathbb{Z})$ then

$$Im(\gamma(z)) = \frac{Im(z)}{|cz+d|^2}. \tag{3.3}$$

Remark 3.1. If $\gamma = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in GL_2^+(\mathbb{R})$ $(\lambda \neq 0)$ then $\gamma(z) = z$. Similarly $\pm I \in SL_2(\mathbb{Z})$ acts trivially on $\mathbb{H}$. Hence sometimes $PSL_2(\mathbb{Z}) \coloneqq SL_2(\mathbb{Z})/\{\pm I\}$ is called the *modular group*.

We now establish some properties of the full modular group.

Lemma 3.2. *$SL_2(\mathbb{Z})$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

Proof. First observe that $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $S^2 = -I$ for $n \in \mathbb{Z}$. Now for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$,

$$T^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+cn & b+dn \\ c & d \end{pmatrix} \quad \text{and} \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}.$$

We now break the analysis in two cases.

(i) *Case 1.* $c = 0$
Since $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we must have $ad = 1$ which gives $a = d = \pm 1$.
Thus $\gamma = \pm \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} = \pm T^{b'}$.

(ii) *Case 2.* $c \neq 0$
Without loss of generality, we can assume that $|a| \geq |c|$. If this is not the

case then apply S and exchange the row a negative sign. Then apply division algorithm,

$$a = qc + r, \qquad 0 \leq r < |c|.$$

Since $r < |c|$, we can apply S to maintain $|a'| \geq |c|$. Continuing in this way, we get to case 1.

$\square$

Remark 3.3. (i) $SL_2(\mathbb{Z})$ is also generated by finite order elements S and ST (ST has order 6).

(ii) $SL_2(\mathbb{Z})$ is also generated by infinite order elements T and $U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(iii) If $\varphi : SL_2(\mathbb{Z}) \to \mathbb{C}^\times$ is a homomorphism then it takes values in the subgroup of the 12^{th} roots of unity.

Definition 3.4. Let $N \geq 1$. Define

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

This is called the *principle congruence subgroup of level N.*

Note that $\Gamma(N)$ is a subgroup of $SL_2(\mathbb{Z})$ and $\Gamma(1) = SL_2(\mathbb{Z})$.

Theorem 3.5. *Consider the natural map $\pi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \pmod{N}$. Then π is a surjective homomorphism.*

It is easy to see that kernel of the map π in above fact is precisely $\Gamma(N)$. Thus $\Gamma(N) \triangleleft SL_2(\mathbb{Z})$.
Following cardinality relations follow from the first isomorphism Theorem.

(i) $|GL_2(\mathbb{Z}/p\mathbb{Z})| = (p^2 - 1)(p^2 - p)$

(ii) $|SL_2(\mathbb{Z}/p\mathbb{Z})| = p(p^2 - 1)$

(iii) $|SL_2(\mathbb{Z}/p^n\mathbb{Z})| = p^{3n-2}(p^2 - 1)$.

Here p is a prime.

Theorem 3.6. (Chinese remainder Theorem) Let $N = \prod_{p^\alpha \| N} p^\alpha$, then $\mathbb{Z}/N\mathbb{Z} \cong \otimes(\mathbb{Z}/p^\alpha\mathbb{Z})$.

Using this Theorem we can show that $|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$.

Definition 3.7. Let Γ be a subgroup of $SL_2(\mathbb{Z})$. Then Γ is called a *conguence subgroup* if $\Gamma(N) \subset \Gamma$ for some N.

Remark 3.8. (i) If $N' = kN$ for some $k \in \mathbb{Z}$ then $\Gamma(N') \subset \Gamma$. Thus the smallest such N is called the *level of* Γ.

(ii) Using first isomorphism Theorem,

$$\frac{SL_2(\mathbb{Z})}{\Gamma(N)} \cong SL_2(\mathbb{Z}/N\mathbb{Z})$$

Thus the index $[SL_2(\mathbb{Z}) : \Gamma(N)] = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) < \infty$. Thus every conguence subgroup of $SL_2(\mathbb{Z})$ is of finite index.

Define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

Clearly $\Gamma(N) \subset \Gamma_0(N)$. Thus $\Gamma_0(N)$ is a conguence subgroup.

Consider the map $\pi_0 : \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod N$. Then

$$Ker(\pi_0) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : {\scriptstyle \begin{matrix} c \equiv 0 (\mathrm{mod}\ N) \\ a \equiv d \equiv 1 (\mathrm{mod}\ N) \end{matrix}} \right\}$$

Define $\Gamma_1(N) := Ker(\pi_0)$. Then we have another map $\pi_1 : \Gamma_1(N) \to \mathbb{Z}/N\mathbb{Z}$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod N$.

Theorem 3.9. *The maps π_0 and π_1 are surjective homomorphisms.*

It is easy to observe that $Ker(\pi_1) = \Gamma(N)$. Thus we have the following containments.

$$\Gamma\underbrace{(N) \subset \Gamma_1(N}_{\text{index N}})\underbrace{\,\subset \Gamma_0(N) \subset S}_{\text{finite index}}L_2(\mathbb{Z})$$

The subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$ are called *Hecke subgroups* and they have finite index in $SL_2(\mathbb{Z})$.

Definition 3.10. Given $\Gamma \leq SL_2(\mathbb{Z})$, a subset $\mathcal{F} \subset \mathbb{H}$ is called a fundamental domain for Γ if $\mathcal{F}$ is closed with connected interior such that,

(i) Every point $z \in \mathbb{H}$ is $\Gamma-$equivalent to a point of $\mathcal{F}$ (*i.e.* $\exists\, \gamma \in \Gamma$ & $z' \in \mathcal{F}$ such that $\gamma(z) = z'$).

(ii) No two points in the interior of $\mathcal{F}$ are $\Gamma-$equivalent.

Let $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}) : n \in \mathbb{Z} \right\}$. A fundamental domain for Γ_∞ is given by

$$\mathcal{F} = \left\{ z \in \mathbb{H} : -\frac{1}{2} \leq Re(z) \leq \frac{1}{2} \right\}.$$

Theorem 3.11. *The set* $\mathcal{F} = \left\{ z \in \mathbb{H} : |Re(z)| \leq \frac{1}{2}, |z| \geq 1 \right\}$ *is a fundamental domain for* $\Gamma = SL_2(\mathbb{Z})$*. This is called the* standard fundamental domain*.*

We define the action of $GL_2(\mathbb{C})$ on $\widehat{\mathbb{C}} = \mathbb{C} \cup \{i\infty\}$ as follows. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$ define

$$\gamma(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \in \mathbb{C} \setminus \{-d/c\} \\ \infty & \text{if } z = -d/c \\ \frac{a}{c} & \text{if } z = i\infty \end{cases}$$

The action on $i\infty$ is defined in the limiting sense. More precisely,

$$\gamma(\infty) := \lim_{\omega \to 0} \frac{a(1/\omega) + b}{c(1/\omega) + d} = \frac{a}{c}$$

Definition 3.12. The set $\mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ is called the *extended upper half plane* and the set $\mathbb{Q} \cup \{i\infty\}$ is called the set of cusps of $SL_2(\mathbb{Z})$ and its congruence subgroups.

Lemma 3.13. Every cusp is $SL_2(\mathbb{Z})-$equivalent to $i\infty$.

In later sections, we will see that, we require functions to satisfy a *cusp condition* at every inequivalent cusp for them to be a modular form. In that case, we must make sure that the subgroup with respect to which we impose modularity condition must have finitely many inequivalent cusps. Next Lemma gives a condition on such subgroups.

Lemma 3.14. Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. Then the number of $\Gamma-$inequivalent cusps of Γ is atmost $[SL_2(\mathbb{Z}) : \Gamma]$.

Definition 3.15. (Slash/Stroke operator). Let $\mathcal{H}(\Omega)$ denote the space of holomorphic functions on some domain Ω. The Slash/Stroke operator of weight $k \in \frac{1}{2}\mathbb{Z}$ is defined as follows.

$$|_k : \mathcal{H}(\Omega) \times SL_2(\mathbb{Z}) \to \mathcal{H}(\Omega), \quad (f, \gamma) \mapsto (f|_k\gamma)$$
$$(f|_k\gamma)(z) = J(\gamma, z)f(\gamma(z))$$

where $J(\gamma, z)$ is some function called *automorphy factor.*

Proposition 3.16. If $|_k$ is a stroke operator and $J(\gamma, z)$ an automorphy factor then we have

$$(f|_k\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2 \text{ and } J(\gamma_1\gamma_2, z) = J(k, \gamma_1, \gamma_2 z)J(\gamma_2, z).$$

Example 3.17. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the function $J(k, \gamma, z) = (cz + d)^{-k}$ for $k \in \mathbb{Z}$ and $\gamma \in SL_2(\mathbb{Z})$, and $J(k, \gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d(\sqrt{cz + d})^{-2k}$ for $k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ and $\gamma \in \Gamma_0(4)$ are automorphy factors, where $\left(\frac{c}{d}\right)$ is the *Jacobi symbol* and

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv -1 \pmod 4 \end{cases}$$

.

Definition 3.18. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. A map $\nu : \Gamma \longrightarrow S^1$ is called a *multiplier system for* Γ *of weight* k if

(i) ν is a homomorphism.

(ii) If $-I \in \Gamma$ then $\nu(-I) = (-1)^k$.

Remark 3.19. The condition $f|_k\gamma = \nu(\gamma)f$ is called the *modularity condition*. In particular for $J(\gamma, z) = (cz+d)^{-k}$, $k \in \mathbb{Z}$, and trivial multiplier system, the modularity condition reduces to

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

3.1.2 Definition and Examples

We now define modular forms on $SL_2(\mathbb{Z})$ of integral weight.

Definition 3.20. A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$ on $SL_2(\mathbb{Z})$ if

(i) f is holomorphic.

(ii) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $z \in \mathbb{H}$.

(iii) $f(z)$ is bounded as $z \to i\infty$ (cusp condition).

Remark 3.21. The modularity condition (ii) in above definition is equivalent to checking the condition for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ as S and T generates $SL_2(\mathbb{Z})$ by Lemma 3.2. Thus we just have to check that

$$f(z+1) = f(z) \qquad f\left(-\frac{1}{z}\right) = z^k f(z).$$

The third condition along with the modularity condition $f(z+1) = f(z)$ in the definition can be used to prove the following Theorem.

Theorem 3.22. *If f is a modular form on $SL_2(\mathbb{Z})$, then f has an expansion of the form*

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

where $q = e^{2\pi i z}$ *and* $a_n \in \mathbb{C}\ \forall\ n \in \mathbb{N}$. *This is called the* Fourier expansion *of* f. *If* $a_0 = 0$, *then* f *is called a* cusp form. *In this case we say that* f *vanishes at* $i\infty$.

Notations 3.23. The space of all weight-k modular forms on $SL_2(\mathbb{Z})$ forms a vector space and is denoted by $M_k(SL_2(\mathbb{Z}))$. So does the space of weight k cusp forms and is denoted by $S_k(SL_2(\mathbb{Z}))$.

We now look at one particular example, the *Eisenstein series*.

Definition 3.24. Define

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^k} \qquad k \geq 4, even.$$

This formal sum is called the Eisenstein series of weight k.

Theorem 3.25. $G_k(z)$ *converges absolutely and uniformly on compact subsets of* $\mathbb{H}$ *and hence defines a holomorphic function on* $\mathbb{H}$. *Moreover*

$$G_k(z+1) = G_k(z) \qquad G_k\left(-\frac{1}{z}\right) = z^k G_k(z).$$

The next Theorem gives the Fourier expansion of $G_k(z)$.

Theorem 3.26. *For even* $k \geq 4$,

$$G_k(z) = 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) = \sum_{\substack{d|n \\ d\geq 1}} d^{k-1}$ *and* $\zeta(z)$ *is the Riemann zeta function.*

We can define the *normalised Eisenstein series* by making the constant term in the above Fourier expansion 1. We define

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)}.$$

Its Fourier expansion is

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where B_k are *Bernoulli numbers* defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Remark 3.27. Observe that for k odd, $E_k(z)$ vanishes identically since for every pair (m, n) there is a pair $(-m, -n)$. These two terms cancel because k is odd.

Put

$$E_2(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_1(n) q^n.$$

$E_2(z)$ does not satisfy the modularity condition. Rather it satisfies modularity with an additional term. This is an example of what are called *quasimodular forms.*

Theorem 3.28. *We have*

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{6c(c\tau + d)}{\pi i}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

The modularity condition puts a strong restriction on functions. In particular, a modular form is completely determined by the values it takes in the fundamental domain. As a result of this, the multiplicities of the zeros of a modular form in the fundamental domain satisfies a relation called the *valence formula.* We will not go into the details of the valence formula but just list the conclusions which are important.

Theorem 3.29. *(i)* $M_0(SL_2(\mathbb{Z})) = \mathbb{C}$.

(ii) $M_k(SL_2(\mathbb{Z})) = \{0\}$ *for* $k < 0$ *and* $k = 2$.

(iii) $E_4(z)$ *and* $E_6(z)$ *has a simple zero at* $z = e^{\frac{i\pi}{3}}$ *and* $z = i$ *respectively.*

(iv) $S_k(SL_2(\mathbb{Z})) = \{0\}$ *for* $k \leq 10$.

(v) $M_k(SL_2(\mathbb{Z})) = \mathbb{C}E_k$ *for* $4 \leq k \leq 10$.

It turns out that $M_k(SL_2(\mathbb{Z}))$ and $S_k(SL_2(\mathbb{Z}))$ are finite dimensional vector spaces. We have a nice dimension formula for these spaces. We can also construct basis for these spaces.

Theorem 3.30. *For $k \geq 0$,*

$$dim M_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2(mod\ 12) \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2(mod\ 12) \end{cases}$$

and for $k \geq 4$,

$$dim S_k(SL_2(\mathbb{Z})) = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1 & \text{if } k \equiv 2(mod\ 12) \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \not\equiv 2(mod\ 12). \end{cases}$$

Moreover the set $\{E_4^{\alpha} E_6^{\beta} : 4\alpha + 6\beta = k, \alpha, \beta \in \mathbb{Z}, \alpha, \beta \geq 0\}$ forms a basis for $M_k(SL_2(\mathbb{Z}))$.

3.1.3 The $j-$function

By Theorem 3.29 (i), there are no non-trivial modular forms of weight 0. But if we relax the holomorphicity condition, then we can construct modular forms of weight 0. Such functions will be called *modular functions*. Let us define it precisely.

Definition 3.31. A *modular function* with respect to Γ, a congruence subgroup of $SL_2(\mathbb{Z})$, is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = f(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and is meromorphic at all $\Gamma-$inequivalent cusps of Γ.

Given any modular form f, we can construct a modular function using next Theorem.

Theorem 3.32. *Let $f \in M_k(SL_2(\mathbb{Z}))$. Then $y^k|f(z)|^2$ is a modular function with respect to $SL_2(\mathbb{Z})$ where $z = x + iy$.*

We can characterise modular functions completely in terms of the $j-$function. Define the *j-function* by $j = E_4^3/\Delta$. Clearly

$$j\left(\frac{az+b}{cz+d}\right) = j(z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Moreover j has a simple pole at $i\infty$ since $\Delta(i\infty) = 0$. Thus j is a modular function with respect to $SL_2(\mathbb{Z})$. We have the following Theorem.

Theorem 3.33. *Let f be a meromorphic function on $\mathbb{H}$. The following are equivalent:*

1. *f is a modular function with respect to $SL_2(\mathbb{Z})$.*
2. *f is a quotient of two modular forms with respect to $SL_2(\mathbb{Z})$ of the same weight.*
3. *f is a rational function of j.*

3.1.4 Modular forms of higher level

We would now like to consider functions which satisfy modularity with respect to a congruence subgroup of $SL_2(\mathbb{Z})$.

Theorem 3.34. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ of level N. If f satisfies modularity with respect to Γ then f has a Fourier expansion of the form*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n q_h^n.$$

at $i\infty$ where $q_h = e^{2\pi iz/h}$ and h is an integer specific to Γ. This is called the Fourier expansion of f at $i\infty$.

Theorem 3.35. *If s is a cusp $\Gamma-$inequivalent to $i\infty$ then $(f|_k\gamma)(z)$ has Fourier expansion of the form*

$$(f|_k\gamma)(z) = \sum_{n=-\infty}^{\infty} b_n q_h^n$$

where $\gamma \in SL_2(\mathbb{Z})$ *is such that* $\gamma(i\infty) = s$. *This series is called the Fourier expansion of* f *at* s.

f is said to be holomorphic at a cusp $i\infty$ if the Fourier coefficient $a_n = 0$ for $n < 0$ in the above Fourier expansion. Holomorphicity at any other cusp is defined similarly.

Definition 3.36. A modular form of weight k for Γ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies modularity property for Γ and is holomorphic at every cusp. The space is denoted by $M_k(\Gamma)$. If f vanishes at all cusps then f is called a cusp form. The space of cusp forms is denoted by $S_k(\Gamma)$.

The first examples are again the Eisenstein series.

Definition 3.37. Let $k \geq 3$, $N \in \mathbb{N}$ and $g \in (\mathbb{Z}/N\mathbb{Z})^2$. Define the Eisenstein series by

$$G_{k,g}(z) = \sum_{\substack{(m,n)\equiv g (\text{mod } N)\\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}, \qquad z \in \mathbb{H}.$$

Again we can check all properties of modular forms for $\Gamma(N)$ and conclude $G_{k,g} \in M_k(\Gamma(N))$.

Theorem 3.38. *The Fourier expansion of* $G_{k,g}(z)$ *is given by* $G_{k,g}(z) = b_{k,g}(0) + \sum\limits_{n=1}^{\infty} b_{k,g}(n) q_N^n$ *where*

$$b_{k,g}(0) = \begin{cases} 0 & \text{if } a_1 \not\equiv 0 (\text{mod } N) \\ \sum\limits_{m\equiv a_2 (\text{mod } N)} m^{-k} & \text{if } a_1 \equiv 0 (\text{mod } N) \end{cases}$$

and

$$b_{k,g}(n) = \frac{(-2\pi i)^k}{N^k (k-1)!} \sum_{\substack{d\mid n\\ (n/d)\equiv a_1 (\text{mod } N)}} \frac{d^k}{|d|} e^{2\pi i a_2 d/N}$$

where summation is over all divisors (+ve as well as −ve) of n, *and* $g = (a_1, a_2)$.

We now have the following facts.

Theorem 3.39. *1. Let $f \in M_k(\Gamma)$ with Γ a congruence subgroup of $SL_2(\mathbb{Z})$. Let $\alpha \in GL_2^+(\mathbb{Q})$. Then $f|\alpha$ is holomorphic at every cusp $s \in \mathbb{Q} \cup \{i\infty\}$.*

2. Let $f \in S_k(\Gamma)$ with Γ a congruence subgroup of $SL_2(\mathbb{Z})$. Let $\alpha \in GL_2^+(\mathbb{Q})$. Then $f|\alpha$ vanishes at every cusp $s \in \mathbb{Q} \cup \{i\infty\}$.

Theorem 3.40. *Let $f(z) \in M_k(\Gamma_1(N))$ and suppose r is a positive integer. Then $f(rz) \in M_k(\Gamma_1(rN))$. Moreover if $f(z)$ is a cusp form, then so is $f(rz)$.*

Proof. Using the stroke operator for $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, we have $f(rz) = r^{-k/2} f(z) \Big| \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$. Now, the cusp condition follow from above two facts. To prove modularity, observe that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(rN)$,

$$\left(f(z) \Big| \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \right) \Big| \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(z) \Big| \begin{pmatrix} ra & rb \\ c & d \end{pmatrix} = \left(f(z) \Big| \begin{pmatrix} a & br \\ c/r & d \end{pmatrix} \right) \Big| \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$$

Since $rN|c$, we see that $\begin{pmatrix} a & br \\ c/r & d \end{pmatrix} \in \Gamma_1(N)$. Hence

$$\left(f(z) \Big| \begin{pmatrix} a & br \\ c/r & d \end{pmatrix} \right) \Big| \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = f(z) \Big| \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}.$$

We conclude that $f(rz) \in M_k(\Gamma_1(rN))$. □

Here also we have valence and dimension formula which uses Riemann surface theory for its proof but we will not discuss it here.

3.1.5 Modular forms with Nebentypus

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$ and χ be a dirichlet character modulo N. We have the following proposition.

Proposition 3.41. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, define $\psi(\gamma) = \chi(d)$. Then $\psi(\gamma_1\gamma_2) = \psi(\gamma_1)\psi(\gamma_2)$.

Thus we can modify the modularity condition in remark 3.19 by replacing the multiplier system with $\psi(\gamma)$. Thus we can define the space $M_k(\Gamma,\chi)$ of modular form with *Nebentypus* χ as follows:

$$M_k(\Gamma,\chi) := \left\{ f \in M_k(\Gamma) : f|_k\gamma = \chi(d)f,\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}.$$

Remark 3.42. If we allow f to have pole at cusps, then f is called a weakly holomorphic modular forms. The space of all weakly holomorphic modular forms of weight k with respect to a congruence subgroup Γ is denoted by $M_k^!(\Gamma)$.

3.2 Modular forms of half integral weight

Consider the second automorphy factor in Example 3.17. The automorphy factor indicates some kind of half integral transformation. We will now prove that the *classical theta function* is modular with that automorphy factor.

3.2.1 The classical theta function

Define the classical theta function by

$$\Theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2}; \quad z \in \mathbb{H},\ q = e^{2\pi i z}.$$

We will prove that the theta function satisfies the following transformation property:

$$\Theta(\gamma z) = J(\gamma, z)\Theta(z), \quad \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$
$$\text{where } J(\gamma,z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1}\sqrt{cz+d}.$$

We will follow [27] for the proof of this Theorem.

Theorem 3.43. *The theta function $\Theta(z)$ defines a holomorphic function on $\mathbb{H}$.*

Proof. Let Ω be any compact subset of $\mathbb{H}$. Then there exists some $T > 0$ such that $Im(z) < T$ for every $z \in \Omega$ for some T. Now for $z = x + iy$,

$$|q^{n^2}| = |e^{2\pi ixn^2}||e^{-2\pi yn^2}| \leq |e^{-2\pi yn}|.$$

Let $y_0 = \sup_{z\in\Omega} y$. Then this minimum exists and is positive since the minimum function is continuous and the domain is compact. Then we have that $|e^{-2\pi yn}| \leq |e^{-2\pi y_0 n}|$ with $|e^{-2\pi y_0}| < 1$. Put $M_n = |e^{-2\pi y_0 n}|$. Then the series $\sum_{n=0}^{\infty} M_n < \infty$ as it is just the geometric series. Now observe that

$$\Theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} = 2\sum_{n=0}^{\infty} q^{n^2} - 1.$$

Thus by *Weiratrass M test*, theta function converges uniformly and absolutely on compact subset Ω of $\mathbb{H}$ and hence defines a holomorphic function on $\mathbb{H}$. □

We will now prove the transformation of the theta function. The following Lemma will be used in the proof.

Lemma 3.44. Let $f(t) = e^{-(t+\alpha)^2\pi/y}$ where $\alpha \in \mathbb{R}, y > 0$. Then its Fourier transform $\widehat{f}(u) = \sqrt{y}e^{-\pi(u^2y-2i\alpha\sqrt{y}u)}$.

Then by Poisson summation formula we have

$$\sum_{n\in\mathbb{Z}} e^{-(n+\alpha)^2\pi/y} = \sqrt{y}\sum_{n\in\mathbb{Z}} e^{-\pi(n^2y-2i\alpha n)}. \tag{3.4}$$

Let us define $\widetilde{\Theta}(z) = \Theta(z/2) = \sum_{n\in\mathbb{Z}} e^{\pi in^2z}$. Then it is easy to see that

$$\widetilde{\Theta}(z+2) = \widetilde{\Theta}(z). \tag{3.5}$$

Moreover (3.4) for $\alpha = 0$ gives

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi/y} = \sqrt{y}\sum_{n\in\mathbb{Z}} e^{-\pi n^2 y}. \tag{3.6}$$

Then for $z = iy, y > 0$, (3.6) gives

$$\widetilde{\Theta}(-1/z) = \sqrt{-iz}\,\widetilde{\Theta}(z) \tag{3.7}$$

where $\sqrt{\ }$ is the usual branch positive on positive reals $\mathbb{R}^+$. Thus by Identity Theorem, (3.7) holds for every $z \in \mathbb{H}$. Equations (3.5) and (3.7) gives us the transformation of $\widetilde{\Theta}(z)$ under the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But we need a general transformation formula for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $a, d \equiv 0 (\mathrm{mod}\ 2)$. We will deal with $c > 0$. $c < 0$ is dealt with similarly. Observe that

$$\begin{aligned} \widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) &= \widetilde{\Theta}\left(\frac{a}{c} - \frac{1}{c(cz+d)}\right) \\ &= \sum_{n=-\infty}^{\infty} e^{\pi i n^2\left(\frac{a}{c} - \frac{1}{c(cz+d)}\right)}. \end{aligned}$$

We can break this sum modulo c by substituting $n = tc + m$ and summing t over $\mathbb{Z}$ and $0 \leq m \leq c - 1$. We get

$$\begin{aligned} \widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) &= \sum_{m=0}^{c-1} \sum_{t=-\infty}^{\infty} e^{\pi i (tc+m)^2 a/c} e^{-\pi i (tc+m)^2\left(\frac{1}{c(cz+d)}\right)} \\ &= \sum_{m=0}^{c-1} e^{\pi i m^2 a/c} \sum_{t=-\infty}^{\infty} e^{-\pi i \left(t + \frac{m}{c}\right)^2\left(\frac{c}{cz+d}\right)}. \end{aligned}$$

Using (3.4), we have that

$$\sum_{n\in\mathbb{Z}} e^{-i\pi(n+\alpha)^2/z} = \sqrt{z/i} \sum_{n\in\mathbb{Z}} e^{i\pi(n^2 z + 2\alpha n)}. \tag{3.8}$$

for $z = iy$. Thus by Identity Theorem (3.8) holds for every $z \in \mathbb{H}$. Using (3.8) for $\alpha = m/c$ and $z = (cz + d)/c$, we get

$$\widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) = (ic)^{-1/2}(cz+d)^{1/2} \sum_{m=0}^{c-1} e^{\pi i m^2 a/c} \sum_{\nu=-\infty}^{\infty} e^{2\pi i \frac{m\nu}{c} + i\nu^2\pi\left(z + \frac{d}{c}\right)}. \tag{3.9}$$

Put
$$\sum := \sum_{m=0}^{c-1} e^{i\pi m^2 a/c+2\pi i m\nu/c} = \sum_{m=0}^{c-1} e^{2\pi i\left(\frac{\alpha m^2+m\nu}{c}\right)}$$
where $a = 2\alpha$. Now since $ad - bc = 1$, thus $2\alpha d \equiv 1 (\text{mod } c)$ and c is odd. Thus if $\overline{\alpha}$ denotes the inverse of α in $\mathbb{Z}/c\mathbb{Z}$ that is $\alpha\overline{\alpha} \equiv 1(\text{mod } c)$, then we can substitute $m = r\overline{\alpha}$. This is because $\gcd(\alpha, c) = 1$ which implies $m\alpha$ would run over the same indices as m. Since $r \equiv m\alpha(\text{mod } c)$, thus r runs over same indices as m. We get

$$\sum = \sum_{r=0}^{c-1} e^{2\pi i\left(\frac{\overline{\alpha}(r^2+r\nu)}{c}\right)} = \sum_{r=0}^{c-1} e^{2\pi i\left(\frac{\overline{\alpha}(r+\overline{2}\nu)^2-\overline{\alpha}\overline{4}\nu^2}{c}\right)}$$

since $\overline{\alpha}(r+\overline{2}\nu)^2 - \overline{\alpha}\overline{4}\nu^2 = \overline{\alpha}(r^2+2\overline{2}\nu r) \equiv \overline{\alpha}(r^2+\nu r)(\text{mod } c)$. This gives

$$\begin{aligned}\sum = e^{2\pi i\left(\frac{-\overline{\alpha}\overline{4}\nu^2}{c}\right)} \sum_{r=0}^{c-1} e^{2\pi i\left(\frac{\overline{\alpha}(r+\overline{2}\nu)^2}{c}\right)} &= e^{2\pi i\left(\frac{-\overline{\alpha}\overline{4}\nu^2}{c}\right)} \sum_{r=0}^{c-1} e^{2\pi i\left(\frac{\overline{\alpha}r^2}{c}\right)} \\ &= e^{2\pi i\left(\frac{-\overline{\alpha}\overline{4}\nu^2}{c}\right)} \beta(\alpha, c)\end{aligned}$$

where $\beta(\alpha, c) = \sum_{r=0}^{c-1} e^{2\pi i\left(\frac{\overline{\alpha}r^2}{c}\right)}$ is a *Gauss sum*. Substituting in (3.9), we get

$$\begin{aligned}\widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) &= (ic)^{-1/2}(cz+d)^{1/2}\beta(\alpha,c) \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2 z} e^{2\pi i\left(-\frac{\nu^2}{c}\left(\overline{\alpha}\overline{4}-\frac{d}{2}\right)\right)} \\ &= (ic)^{-1/2}(cz+d)^{1/2}\beta(\alpha,c) \sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2 z} e^{2\pi i\left(\frac{\nu^2}{c}(\overline{2}d-\overline{\alpha}\overline{4})\right)}.\end{aligned}$$

But

$$\begin{aligned}2\alpha d \equiv 1(\text{mod } c) &\implies 2\alpha\overline{\alpha}d \equiv \overline{\alpha}(\text{mod } c) \\ &\implies \overline{4}2d \equiv \overline{4}\overline{\alpha}(\text{mod } c) \\ &\implies \overline{2}d \equiv \overline{4}\overline{\alpha}(\text{mod } c) \\ &\implies \overline{2}d - \overline{4}\overline{\alpha} \equiv 0(\text{mod } c).\end{aligned}$$

Thus we get

$$\widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) = (ic)^{-1/2}(cz+d)^{1/2}\beta(\alpha,c)\sum_{\nu=-\infty}^{\infty} e^{\pi i\nu^2 z}$$
$$= (ic)^{-1/2}(cz+d)^{1/2}\beta(\alpha,c)\widetilde{\Theta}(z).$$

We need to calculate $\beta(\alpha, c)$. Define

$$G(a,b,c) = \sum_{n=0}^{c-1} e^{2\pi i\left(\frac{an^2+bn}{c}\right)}.$$

Then $\beta(\alpha, c) = G(\overline{\alpha}, 0, c)$. We have the following Theorem (Theorem 1.5.2 of [21]):

Theorem 3.45. *For $gcd(a, c) = 1$, we have*

$$G(a,c) = G(a,0,c) = \begin{cases} 0 & \textit{if } c \equiv 2 (mod\ 4) \\ \varepsilon_c\sqrt{c}\left(\frac{a}{c}\right) & \textit{if } c \textit{ is odd} \\ (1+i)\varepsilon_a^{-1}\sqrt{c}\left(\frac{c}{a}\right) & a \textit{ is odd, } 4|c. \end{cases}$$

Since $\gcd(\alpha, c) = 1$, so $\gcd(\overline{\alpha}, c) = 1$ and c is odd, so $\beta(\alpha, c) = \varepsilon_c\sqrt{c}\left(\frac{\overline{\alpha}}{c}\right)$. But $\overline{\alpha} \equiv 2d (\mathrm{mod}\ c)$, so $\left(\frac{\overline{\alpha}}{c}\right) = \left(\frac{2d}{c}\right)$. Hence $\beta(\alpha, c) = \varepsilon_c\sqrt{c}\left(\frac{2d}{c}\right)$. So we get

$$\widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) = (ic)^{-1/2}(cz+d)^{1/2}\varepsilon_c\sqrt{c}\left(\frac{2d}{c}\right)\widetilde{\Theta}(z).$$

Making the transformation $z \mapsto -1/z$, we get

$$\widetilde{\Theta}\left(\frac{bz-a}{dz-c}\right) = (ic)^{-1/2}\frac{(dz-c)^{1/2}}{\sqrt{z}}\varepsilon_c\sqrt{c}\left(\frac{2d}{c}\right)\widetilde{\Theta}(-1/z)$$
$$= (ic)^{-1/2}\frac{(dz-c)^{1/2}}{\sqrt{z}}\varepsilon_c\sqrt{c}\left(\frac{2d}{c}\right)\sqrt{-iz}\widetilde{\Theta}(z)$$

where we used (3.7). Put $b = a', a = -b', c = -d'$ and $d = c'$. With this change we have

$$\widetilde{\Theta}\left(\frac{a'z+b'}{c'z+d'}\right) = (-id')^{-1/2}(c'z+d')^{1/2}\varepsilon_{-d'}\sqrt{-d'}\left(\frac{2c'}{-d'}\right)\sqrt{-i}\widetilde{\Theta}(z).$$

It can easily be checked that $\varepsilon_{-d} = i\varepsilon_d^{-1}$. This gives

$$\widetilde{\Theta}\left(\frac{a'z+b'}{c'z+d'}\right) = (c'z+d')^{1/2}\varepsilon_{d'}^{-1}\left(\frac{2c'}{d'}\right)\widetilde{\Theta}(z)$$

Thus we finally have

$$\widetilde{\Theta}\left(\frac{az+b}{cz+d}\right) = (cz+d)^{1/2}\varepsilon_{d}^{-1}\left(\frac{2c}{d}\right)\widetilde{\Theta}(z)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $b, c \equiv 0 (\text{mod } 2)$. Now since $\Theta(z) = \widetilde{\Theta}(2z)$, thus we get for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $b, c \equiv 0 (\text{mod } 2)$,

$$\begin{aligned}\Theta(\gamma z) = \widetilde{\Theta}(\gamma(2z)) = \widetilde{\Theta}\left(\frac{2az+b}{2cz+d}\right) &= (2cz+d)^{1/2}\varepsilon_{d}^{-1}\left(\frac{2c}{d}\right)\widetilde{\Theta}(2z) \\ &= (2cz+d)^{1/2}\varepsilon_{d}^{-1}\left(\frac{2c}{d}\right)\Theta(z).\end{aligned}$$

Thus for $\gamma = \begin{pmatrix} a & b/2 \\ 2c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $b, c \equiv 0 (\text{mod } 2)$, we get

$$\Theta(\gamma z) = (2cz+d)^{1/2}\varepsilon_{d}^{-1}\left(\frac{2c}{d}\right)\Theta(z).$$

Thus this gives

$$\Theta(\gamma z) = J(\gamma, z)\Theta(z), \quad \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$
$$\text{where } J(\gamma, z) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}\sqrt{cz+d}.$$

3.2.2 The Dedekind Eta function

Define the Dedekind eta function $\eta : \mathbb{H} \longrightarrow \mathbb{C}$ by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n); \quad q = e^{2\pi i \tau}.$$

We will first show that the infinite product on the right hand side converges absolutely and uniformly on compact subsets of $\mathbb{H}$. Thus Theorem 15.6 of [22] will imply that $\eta(\tau)$ defines a holomorphic function. Indeed for $f_n(\tau) = 1 - q^n$, we have that

$$\sum_{n=0}^{\infty} |1 - f_n| = \sum_{n=0}^{\infty} |q|^n.$$

Since $\tau \in \mathbb{H}$, thus $|q| = |e^{-2\pi y}| < 1$ where $\tau = x + iy$. Thus the series $\sum |q|^n$ converges being a geometric series. Thus Theorem 15.6 of [22] implies that $\eta(\tau)$ is holomorphic on $\mathbb{H}$.

Using Poisson summation formula, one can prove the following transformation property of the Dedekind eta function [4].

Theorem 3.46. *For $\tau \in \mathbb{H}$, we have*

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau) \text{ and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\,\eta(\tau)$$

where $\sqrt{\ }$ denotes the branch of square root in which $\sqrt{\tau} > 0$ if $\tau > 0$.

With this transformation property one can prove the following:

Theorem 3.47. *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \nu_\eta(\gamma)(c\tau + d)^{1/2}\eta(\tau)$$

where $\nu_\eta(\gamma)^{24} = 1$ and $\nu_\eta(\gamma_1\gamma_2) = \pm\nu_\eta(\gamma_1)\nu_\eta(\gamma_2)$. ν_η is a multiplier system for $SL_2(\mathbb{Z})$ and $\nu_\eta^2 : SL_2(\mathbb{Z}) \longrightarrow \mathbb{C}^\times$ is a character of order 12.

This Theorem shows that $\eta^{2k}(\tau) \in M_k(SL_2(\mathbb{Z}), \nu_\eta^{2k})$ if we could take care of the Fourier expansion (next Theorem). In particular we have

$$\eta(\tau)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \Delta(\tau) \in S_{12}(SL_2(\mathbb{Z})).$$

Theorem 3.48. *(Euler)*

$$\eta(\tau) = \sum_{n\geq 1}\left(\frac{-12}{n}\right)q^{n^2/24}; \quad \eta(\tau)^3 = \sum_{n\geq 1}\left(\frac{-4}{n}\right)nq^{n^2/8}$$

where

$$\left(\frac{-12}{n}\right) = \begin{cases} 1 & \textit{if } n \equiv \pm 1 \ (\textit{mod } 12) \\ -1 & \textit{if } n \equiv \pm 1 \ (\textit{mod } 12) \\ 0 & \textit{if } gcd(n,12) > 1 \end{cases}$$

$$\left(\frac{-4}{n}\right) = \begin{cases} \pm 1 & \textit{if } n \equiv \pm 1 \ (\textit{mod } 4) \\ 0 & \textit{if } n \equiv 0 \ (\textit{mod } 2). \end{cases}$$

These two examples give us automorphy factors to define modular forms of half integral weight. The precise theory of half integral modular forms is a bit complicated and we skip it for now. The reader is referred to [5] for details. We end this chapter with the a discussion of Eisenstein series. Put

$$\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} \in SL_2(\mathbb{Z}) : n \in \mathbb{Z}\right\}.$$

We say that γ_1 and γ_2 are related if $\gamma_1 = \gamma\gamma_2$ for some $\gamma \in \Gamma_\infty$. Denote the set of right cosets of Γ_∞ in $\Gamma_0(4)$ by $\Gamma_\infty \setminus \Gamma_0(4)$.

Definition 3.49. Let $k \geq 5$ be an odd integer. Put

$$E_{k/2}(z) = \sum_{\gamma\in\Gamma_\infty\setminus\Gamma_0(4)} J(\gamma, z)^{-k}, \quad F_{k/2} = (2z)^{k/2}E_{k/2}\left(-\frac{1}{4z}\right).$$

Then $E_{k/2}$ is called Eisenstein series and $F_{k/2}$ is called Eisenstein series associated to $E_{k/2}$.

It can be proved that $E_{k/2}, F_{k/2} \in M_{k/2}(\Gamma_0(4))$.

Theorem 3.50. *The Fourier expansion of $F_{k/2}$ has the form $F_{k/2}(z) = \sum\limits_{l=0}^{\infty} b_l q^l$ where*

$$b_l = \frac{\pi^{k/2}}{\Gamma(\frac{k}{2})e^{\pi i k/4}} l^{k/2-1} \sum_{n>0,odd} \varepsilon_n^k n^{-k/2} \sum_{0\leq j<n} \left(\frac{j}{n}\right) e^{-2\pi i l j/n}.$$

and $\Gamma(z)$ is the gamma function.

4. HARMONIC MAASS FORMS AND MOCK MODULAR FORMS

The forms which we have studied till now have been assumed to be holomorphic. We now further relax the holomorphicity condition on the functions and assume them to be smooth in real sense with some additional conditions. This leads to new kinds of modular forms. We follow [6] for this chapter.

4.1 Definition

Define the hyperbolic Laplacian. Define the weight-k ($\in \mathbb{R}$) hyperbolic Laplacian

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)+ikv\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)=-4v^2\frac{\partial}{\partial \tau}\frac{\partial}{\partial \bar{\tau}}+2ikv\frac{\partial}{\partial \bar{\tau}}.$$

where $\tau = u+iv$, $\frac{\partial}{\partial \tau}=\frac{1}{2}\left(\frac{\partial}{\partial u}+\frac{1}{i}\frac{\partial}{\partial v}\right)$ and $\frac{\partial}{\partial \bar{\tau}}=\frac{1}{2}\left(\frac{\partial}{\partial u}-\frac{1}{i}\frac{\partial}{\partial v}\right)$.

Harmonic Maass forms are smooth functions f on $\mathbb{H}$ which have transformion property similar to weight-k modular forms but with an additional condition that they are also annihilated by the weight-k hyperbolic Laplacian operator. In the upcoming discussion, we will see that weight-k weakly holomorphic modular forms are trivial examples of harmonic Maass forms. Thus, this theory can be considered as a natural generalization of the classical theory of modular forms.
We now state the precise definition following[2].

Definition 4.1. Let $k \in \frac{1}{2}\mathbb{Z}$. A smooth function (in real sense) $f : \mathbb{H} \to \mathbb{C}$ is called a weight-k harmonic Maass form on $\Gamma_0(N)$ ($4|N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$) if

(i) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$, we have

$$f\left(\frac{az+b}{cz+d}\right) = \begin{cases} (cz+d)^k f(\tau) & \text{if } k \in \mathbb{Z} \\ (\frac{c}{d})\varepsilon_d^{2k}(cz+d)^k f(\tau) & \text{if } k \in \frac{1}{2}+\mathbb{Z}. \end{cases}$$

(ii) $\Delta_k(f) = 0$.

(iii) There exists a polynomial $P_f(\tau) \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\epsilon v})$ as $v \to \infty$ for some $\epsilon > 0$. Similar conditions hold at other cusps.

If the third condition in the above definition is replaced by $f(\tau) = O(e^{\epsilon v})$, then f is said to be a harmonic Maass form of *manageable growth*. Space of harmonic Maass forms of weight k is denoted by $H_k(\Gamma)$ and that of harmonic Maass forms of manageable growth is denoted by $H_k^!(\Gamma)$.

We make the following remarks.

Remark 4.2. (i) Let s be a cusp $\Gamma_0(N)-$inequivalent to $i\infty$ and let $\gamma \in SL_2(\mathbb{Z})$ be such that $\gamma(i\infty) = s$. Then the cusp condition for f at s is given by the requirement that $f|_k\gamma$ rather than f satisfy Definition 4.1(iii).

(ii) For $k = 0$, weight-k harmonic Maass forms are called *harmonic Maass functions.*

(iii) The eigenfunctions of Δ_k which satisfy the modularity (Definition (4.1)(i)) and cusp condition (Definition (4.1)(ii)) in above Definition (4.1) are called *weak Maass forms.*

(iv) The polynomial $P_f(\tau) \in \mathbb{C}[q^{-1}]$ in Definition (4.1) (iii) is called the principal part of f at $i\infty$. Similar principal parts exist at all of the cusps.

(v) Since $P_f(\tau) = O(e^{\epsilon v})$ for some $\epsilon > 0$ as $v \to \infty$ for any $P_f(\tau) \in \mathbb{C}[q^{-1}]$. Thus if if $f \in H_k(\Gamma)$, then $f \in H_k^!(\Gamma)$. Thus $H_k(\Gamma) \subset H_k^!(\Gamma)$.

(vi) Harmonic Maass forms on $\Gamma_0(N)$ with Nebentypus χ, where χ is a Drichlet character modulo n, can also be defined in a similar way. The modularity

condition in Definition (4.1) is modified as follows:
For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$,

$$f\left(\frac{az+b}{cz+d}\right) = \begin{cases} \chi(d)(cz+d)^k f(\tau) & \text{if } k \in \mathbb{Z} \\ \chi(d)(\frac{c}{d})\varepsilon_d^{2k}(cz+d)^k f(\tau) & \text{if } k \in \frac{1}{2}+\mathbb{Z}. \end{cases}$$

We let $H_k(\Gamma_0(N), \chi)$ (respectively $H_k^!(\Gamma_0(N), \chi)$) denote the space of weight-k harmonic Maass forms (respectively with manageable growth) with Nebentypus χ.

(vii) The transformation laws in the definition can be framed conveniently in terms of the stroke operator. We define the weight-k stroke operator as follows:
For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$,

$$f|_k\gamma(\tau) = \begin{cases} (cz+d)^{-k} f(\gamma(\tau)) & \text{if } k \in \mathbb{Z} \\ (\frac{c}{d})\varepsilon_d^{2k}(cz+d)^{-k} f(\gamma(\tau)) & \text{if } k \in \frac{1}{2}+\mathbb{Z}. \end{cases}$$

Then the modularity condition can be succinctly written as

$$f|_k\gamma = f.$$

for all $\gamma \in \Gamma_0(N)$.

Let f be a weakly holomorphic weight-k modular form. Then it is holomorphic on $\mathbb{H}$ and may have poles at the cusps. Thus f is annihilated by the *anti holomorphic derivative* $\frac{\partial}{\partial \bar{\tau}}$. Thus $\Delta_k(f) = 0$. Moreover f satisfies modularity condition as required in Definition (4.1). To check the cusp condition, observe that f has a Fourier expansion of the form

$$f(z) = \sum_{n=n_0}^{\infty} a(n) q_h^n$$

at $i\infty$ where n_0 is some negative integer. The principal part $P_f(\tau)$ of f at $i\infty$ can simply be taken to be $f(z) = \sum_{n=n_0}^{0} a(n)q_h^n$. Thus we see that weakly homomorphic weight-k modular forms are trivial examples of harmonic Maass forms. Thus we have the following containments

$$M_k^!(\Gamma) \subset H_k(\Gamma) \subset H_k^!(\Gamma). \tag{4.1}$$

4.2 Fourier Expansion

The Fourier expansion of harmonic Maass forms involves the *incomplete gamma function.* The incomplete gamma function is defined as

$$\Gamma(s,z) := \int_z^\infty e^{-t} t^s \frac{dt}{t}.$$

This integral converges absolutely for $Re(s) > 0$ and $z \in \mathbb{C}$ (or any $s \in \mathbb{C}$ and $z \in \mathbb{H}$). It can be analytically continued in s using the functional equation in next Lemma.

Lemma 4.3. For $z \in \mathbb{C}$ and $Re(s) > 0$ we have,

$$\Gamma(s+1,z) = s\Gamma(s,z) + z^s e^{-z}.$$

Note that the functional equation gives analytic continuation of the incomplete gamma function in s to all of $\mathbb{C}$ except at the negative reals since $\Gamma(s,z)$ is not initially defined at $s = 0$. The next result resolves this issue by defining the incomplete gamma function at $s = 0$.

Lemma 4.4.

$$\Gamma(0,z) = Ein(z) - Log(z) - \gamma$$

where the entire function Ein, the complementary error integral is given by

$$Ein(z) := \int_0^z (1 - e^{-t})\frac{dt}{t}$$

and γ is the Euler-Mascheroni constant.

We also need the asymptotic behaviour of the incomplete gamma function.

Lemma 4.5. For $x \in \mathbb{R}$, we have that

$$\Gamma(s, x) \sim x^{s-1}e^{-x} \quad \text{as } |x| \to \infty.$$

We now establish the Fourier expansion for harmonic Maass forms[2].

Theorem 4.6. *Let $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ and $\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\}$. If $f \in H_k^!(\Gamma)$ then*

$$f(\tau) = f(u + iv) = \sum_{n>>-\infty} c_f^+(n)q^n + c_f^-(0)v^{1-k} + \sum_{\substack{n<<\infty \\ n\neq 0}} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n.$$

If $f \in H_k(\Gamma)$ then

$$f(\tau) = f(u + iv) = \sum_{n>>-\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n$$

at $i\infty$. Similar expressions hold at other cusps. The notation $\sum\limits_{n>>-\infty}$ means $\sum\limits_{n=\alpha_f}^{\infty}$ for some $\alpha_f \in \mathbb{Z}$. $\sum\limits_{n<<\infty}$ is defined similarly.

Proof. From Definition (4.1)(i), for $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we get $f(u + iv + 1) = f(u + iv)$. Thus if we consider f as a function of u only for a fixed v, then f has Fourier expansion (at $i\infty$) in the real sense, of the form

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_f(n, v)e^{2\pi inu}.$$

Now since $\Delta_k(f) = 0$, we get

$$\begin{aligned}
\Delta_k \left(a(v) + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} a(n,v) e^{2\pi i n u} \right) &= 0 \\
\implies \Delta_k(a(v)) + \Delta_k \left(\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} a(n,v) e^{2\pi i n u} \right) &= 0 \\
\implies \Delta_k(a(v)) + \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \Delta_k \left(a(n,v) e^{2\pi i n u} \right) &= 0.
\end{aligned}$$

where $a(0, v) = a(v)$ and we have interchanged the summation and Laplacian in view of the absolute and uniform convergence of the Fourier series. Comparing the coefficients of $e^{2\pi i n u}$ in above equation, we get $\Delta_k(a(v)) = 0$. This gives

$$\begin{aligned}
-v^2 \left(\frac{d^2 a(v)}{dv^2} \right) + ikv \left(i \frac{da(v)}{dv} \right) &= 0 \\
\implies v^2 a''(v) + kva'(v) &= 0
\end{aligned}$$

where $'$ is derivative with respect to v. This is Euler's second order homogeneous ordinary differential equation. To solve this, put $a(v) = v^r$, then the ODE transforms to

$$\begin{aligned}
v^2 r(r-1) v^{r-2} + kvrv^{r-1} &= 0 \\
\implies (r(r-1) + kr) v^r &= 0 \\
\implies r^2 - r + kr &= 0
\end{aligned}$$

which gives $r = 0, 1-k$. Thus

$$a(v) = c_f^+(0) + c_f^-(0) v^{1-k} \tag{4.2}$$

for arbitrary complex numbers $c_f^+(0)$ and $+c_f^-(0)$. Now we solve

$$\sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \Delta_k \left(a(n,v) e^{2\pi i n u} \right) = 0.$$

For $n \neq 0$, Put $a(n,v) = C(\omega)$ where $\omega = 2\pi n v$. Then by chain rule,

$$\begin{aligned}\frac{\partial C(\omega)}{\partial v} = \frac{\partial C(\omega)}{\partial \omega}\frac{\partial \omega}{\partial v} = 2\pi n \frac{\partial C(\omega)}{\partial \omega} \\ \frac{\partial^2 C(\omega)}{\partial v^2} = (2\pi n)^2 \frac{\partial^2 C(\omega)}{\partial \omega^2}.\end{aligned}$$

We now have

$$\begin{aligned}
&\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Delta_k\left(a(n,v)e^{2\pi i n u}\right) = 0 \\
&\Longrightarrow \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \Delta_k(a(n,v))e^{2\pi i n u} + a(n,v)\Delta_k\left(e^{2\pi i n u}\right) = 0 \\
&\Longrightarrow -v^2\left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} C(\omega)\frac{\partial^2 e^{2\pi i n u}}{\partial u^2} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\partial^2 C(\omega)}{\partial v^2}e^{2\pi i n u}\right) + \\
&ikv\left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} C(\omega)\frac{\partial e^{2\pi i n u}}{\partial u} + i\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\partial C(\omega)}{\partial v}e^{2\pi i n u}\right) = 0 \\
&\Longrightarrow -\left(\frac{\omega}{2\pi n}\right)^2\left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} C(\omega)\frac{\partial^2 e^{2\pi i n u}}{\partial u^2} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} (2\pi n)^2\frac{\partial^2 C(\omega)}{\partial \omega^2}e^{2\pi i n u}\right) + \\
&ik\frac{\omega}{2\pi n}\left(\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} C(\omega)(2\pi i n)e^{2\pi i n u} + i\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\partial C(\omega)}{\partial \omega}(2\pi n)e^{2\pi i n u}\right) = 0 \\
&\Longrightarrow -\omega^2\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(-C(\omega) - e^{2\pi i n u} + \frac{\partial^2 C(\omega)}{\partial \omega^2}e^{2\pi i n u}\right) - \\
&k\omega\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left(C(\omega)e^{2\pi i n u} + \frac{\partial C(\omega)}{\partial \omega}e^{2\pi i n u}\right) = 0 \\
&\Longrightarrow -\omega^2\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\left[\frac{\partial^2 C(\omega)}{\partial \omega^2} - C(\omega) + \frac{k}{\omega}\left(C(\omega) + \frac{\partial C(\omega)}{\partial \omega}\right)\right]e^{2\pi i n u} = 0
\end{aligned}$$

which finally gives

$$\frac{\partial^2 C(\omega)}{\partial \omega^2} - C(\omega) + \frac{k}{\omega}\left(C(\omega) + \frac{\partial C(\omega)}{\partial \omega}\right) = 0. \tag{4.3}$$

It is easily checked that $e^{-\omega}$ is a solution of (4.3). Another linearly independent solution to (4.3) is given by by $\Gamma(1-k, -2\omega)e^{-\omega}$ for $\omega \neq 0$ (we had $n \neq 0, \tau \in \mathbb{H}$ so $v \neq 0$.) where

$$\Gamma(1-k, -2\omega) = \int_{-2\omega}^{\infty} e^{-t} t^{-k} dt \qquad \text{(analytically continued)}$$

So, using (4.2) and the solutions to (4.3) we have the following series expansion for $f(\tau)$

$$f(\tau) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \left[c_f^+(n)e^{-\omega}e^{2\pi inu} + c_f^-(n)e^{-\omega}\Gamma(1-k, -2\omega)e^{2\pi inu}\right] + c_f^+(0) + c_f^-(0)v^{1-k}$$

which after putting the value of ω gives

$$f(\tau) = \sum_{n\in\mathbb{Z}} c_f^+(n)q^n + c_f^-(0)v^{1-k} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n. \tag{4.4}$$

Now we use Lemma (4.5) to obtain the restrictions on the sum. For large n, $\Gamma(1-k, -4\pi nv) \sim (-4\pi nv)^{-k}e^{4\pi nv}$. Thus the term of the second sum ar of the form $c_f^-(n)(-4\pi nv)^{-k}e^{2\pi nv}e^{2\pi inu} \neq O(e^{\epsilon v})$ for any $\epsilon > 0$ as n is unbounded. So the second sum must be bounded from above. Similarly, in the first sum, we have $c_f^+(n)q^n = c_f^+(n)e^{-2\pi nv}e^{2\pi inu} \neq O(e^{\epsilon v})$ for any $\epsilon > 0$ if n goes all the way to $-\infty$. Thus the first sum must be bounded from below. So we finally obtain

$$f(\tau) = \sum_{n>>-\infty} c_f^+(n)q^n + c_f^-(0)v^{1-k} + \sum_{\substack{n<<\infty\\ n\neq 0}} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n.$$

In particular, if $f \in H_k(\Gamma)$, then $\exists P_f(\tau) \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\epsilon v})$ for some $\epsilon > 0$ as $v \to \infty$. If in the second sum $n > 0$, then $-\pi nv < 0$ and as $v \to$

∞, $|-4\pi nv| \to \infty$ and hence $c_f^-(n)\Gamma(1-k,-4\pi nv)q^n \sim c_f^-(n)(-4\pi nv)^{-k}e^{2\pi nv}e^{2\pi inu} \neq O(e^{-\epsilon v})$ for any $\epsilon > 0$. Thus we must have $n < 0$ in second sum. Finally $c_f^-(0) = 0$ since $v^{1-k} \longrightarrow \infty$ as $v \longrightarrow \infty$. Thus we get for $f \in H_k(\Gamma)$

$$f(\tau) = \sum_{n>>-\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k,-4\pi nv)q^n.$$

$\square$

Remark 4.7. If $k = 1$, then v^{1-k} in the Fourier expansion is replaced by $\log v$.

Definition 4.8. We call

$$f^+(\tau) = \sum_{n>>-\infty} c_f^+(n)q^n$$

the *holomorphic part* of f and

$$f^-(\tau) = c_f^-(0)v^{1-k} + \sum_{\substack{n<<\infty\\ n\neq 0}} c_f^-(n)\Gamma(1-k,-4\pi nv)q^n$$

the *nonholomorphic part* of f.

Remark 4.9. (i) If $f \in H_k^!(\Gamma)$ with $f^-(\tau) = 0$ then $f \in M_k^!(\Gamma)$.

(ii) Consider $f \in H_k(\Gamma)$ such that the Fourier expansion at the cusp $i\infty$ has the form

$$f(\tau) = \sum_{n=0}^{\infty} c_f^+(n)q^n + \sum_{\substack{n<<\infty\\ n\neq 0}} c_f^-(n)\Gamma(1-k,-4\pi nv)q^n.$$

Thus the holomorphic part is bounded near cusps and so the exponential growth at the cusps arises from the non holomorphic part. We denote by $H_k^{\#}(\Gamma)$, the space of $f \in H_k^!(\Gamma)$ with Fourier expansion as above.

4.3 Differential operators

In this section, we assume $k \in \frac{1}{2}\mathbb{Z}$. In particular if $k \in \mathbb{Z}$ then $\chi = 1$ and $k \in 2\mathbb{Z}$ since there are no odd integral weight modular forms. We also restrict to the group

$\Gamma_0(N)$. We would like to have maps between different spaces of harmonic Maass forms.

4.3.1 Maass operators and harmonic Maass forms

Define the following differential operator,

$$D = D_\tau = \frac{1}{2\pi i}\frac{\partial}{\partial\tau} \tag{4.5}$$

Lemma 4.10. Let $f : \mathbb{H} \to \mathbb{C}$ transform as

$$f(\gamma\tau) = (c\tau + d)^k f(\tau); \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}). \tag{4.6}$$

Then

$$D(f)(\gamma\tau) = (c\tau + d)^{k+2} D(f)(\tau) + \frac{k}{2\pi i} c(c\tau + d)^{k+1} f(\tau). \tag{4.7}$$

Proof. Differentiating Eq. (4.6) with respect to τ both sides we get

$$f'(\gamma\tau)\frac{d}{d\tau}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f'(\tau) + k(c\tau + d)^{k-1} c f(\tau). \tag{4.8}$$

Now

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{a}{c\tau + d} - c\frac{a\tau + b}{(c\tau + d)^2} &= \frac{a(c\tau + d) - c(a\tau + b)}{(c\tau + d)^2} \\ &= \frac{ad - bc}{(c\tau + d)^2} = (c\tau + d)^{-2}. \end{aligned}$$

Thus plugging this in to (4.8), we get

$$f'(\gamma\tau) = (c\tau + d)^{k+2} f'(\tau) + \frac{k}{2\pi i} c(c\tau + d)^{k+1} f(\tau).$$

Dividing by $2\pi i$ now gives the desired relation. □

Lemma (4.10) shows that for $k \neq 0$, the operator D does not preserve modularity.

But this can be rectified in a certain way. To this end, we define the *Maass Raising and Lowering operator.*

Definition 4.11. Put $\tau = u + iv \in \mathbb{H}$. Define the Maass raising operator R_k and Maass lowering operator L_k by

$$\begin{aligned} R_k &:= 2i\frac{\partial}{\partial \tau} + \frac{k}{v} = i\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right) + \frac{k}{v} \\ L_k &:= -2iv^2\frac{\partial}{\partial \overline{\tau}} = -iv^2\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right). \end{aligned} \tag{4.9}$$

Observe that $R_k = -4\pi D + \frac{k}{v}$. The reason for the names "raising" and "lowering" will become apparent in next Lemma.

Theorem 4.12. *Let f be a smooth function on $\mathbb{H}$, let $k \in \frac{1}{2}\mathbb{Z}$, and let $\gamma \in SL_2(\mathbb{Z})$ with $\gamma \in \Gamma_0(4)$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Then we have,*

(i) With respect to the slash operator, we have that

$$\begin{aligned} R_k(f|_k\gamma) &= R_k(f)|_{k+2}\gamma \\ L_k(f|_k\gamma) &= L_k(f)|_{k-2}\gamma. \end{aligned}$$

In particular, if f transforms as in Eq. (4.2) then we have,

$$R_k(f)|_{k+2}\gamma = R_k(f) \quad \text{and} \quad L_k(f)|_{k-2}\gamma = L_k(f).$$

(ii) The weight-k hyperbolic Laplacian may be expressed as

$$-\Delta_k = L_{k+2} \circ R_k + k = R_{k-2} \circ L_k. \tag{4.10}$$

(iii) If f is an eigenfunction of Δ_k with eigenvalue λ, then

$$\begin{aligned} \Delta_{k+2}(R_k(f)) &= (\lambda + k)R_k(f) \\ \Delta_{k-2}(L_k(f)) &= (\lambda - k + 2)L_k(f). \end{aligned}$$

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and for simplicity define

$$\rho_k(c,d) = \rho_k(\gamma) := \begin{cases} 1 & \text{if } k \in \mathbb{Z} \\ (\frac{c}{d})\varepsilon_d^{2k} & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

(i) $R_k(f) = 2if' + \frac{k}{v}f$. Thus

$$\left(R_k(f)|_{k+2}\gamma\right)(\tau) = 2i(f'|_{k+2}\gamma)(\tau) + \left(\left(\frac{k}{Im(\tau)}f\right)\bigg|_{k+2}\gamma\right)(\tau).$$

But

$$\begin{aligned}\left(\left(\frac{k}{Im(\tau)}f\right)\bigg|_{k+2}\gamma\right)(\tau) &= \frac{k}{Im(\gamma\tau)}f(\gamma\tau)\rho_{k+2}(\gamma)(c\tau+d)^{-(k+2)} \\ &= \frac{k}{v}|c\tau+d|^2 f(\gamma\tau)\rho_{k+2}(\gamma)(c\tau+d)^{-(k+2)}\end{aligned}$$

and

$$(f'|_{k+2}\gamma)(\tau) = (c\tau+d)^{-k}\rho_k(\gamma)f'(\gamma\tau)$$

by definition of the slash operator. Plugging these, we get

$$\left(R_k(f)|_{k+2}\gamma\right)(\tau) = \rho_{k+2}(\gamma)(c\tau+d)^{-(k+2)}\left(2if'(\gamma\tau) + \frac{k}{v}|c\tau+d|^2 f(\gamma\tau)\right).$$

On the other hand

$$\begin{aligned}&R_k(f|_k\gamma)(\tau) = R_k((c\tau+d)^{-k}\rho_k(\gamma)f(\gamma\tau)) \\ &= 2i\frac{\partial}{\partial\tau}((c\tau+d)^{-k}\rho_k(\gamma)f(\gamma\tau)) + \frac{k}{v}((c\tau+d)^{-k}\rho_k(\gamma)f(\gamma\tau)) \\ &= \rho_k(\gamma)\Bigg[2if'(\gamma\tau)\frac{\partial}{\partial\tau}\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-k} + 2if(\gamma\tau)\frac{\partial}{\partial\tau}(c\tau+d)^{-k} \\ &\qquad\qquad + \frac{k}{v}((c\tau+d)^{-k}(\gamma)f(\gamma\tau)\Bigg]\end{aligned}$$

where we used the chain rule. Now using $\frac{\partial}{\partial\tau}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{-2}$, we get

$$\begin{aligned}
R_k(f|_k\gamma)(\tau) &= \rho_k(\gamma)\Bigg[2if'(\gamma\tau)(c\tau+d)^{-k-2} + 2ikcf(\gamma\tau)(c\tau+d)^{-k-1} \\
&\qquad\qquad + \frac{k}{v}((c\tau+d)^{-k}f(\gamma\tau)\Bigg] \\
&= \rho_k(\gamma)(c\tau+d)^{-(k+2)}\left[2if'(\gamma\tau) - 2ikc(c\tau+d)f(\gamma\tau) + \frac{k}{v}(c\tau+d)^2 f(\gamma\tau)\right] \\
&= \rho_k(\gamma)(c\tau+d)^{-(k+2)}\left[2if'(\gamma\tau) + k(c\tau+d)f(\gamma\tau)\left(\frac{c\tau+d}{v} - 2ic\right)\right] \\
&= \rho_k(\gamma)(c\tau+d)^{-(k+2)}\left[2if'(\gamma\tau) + \frac{k}{v}(c\tau+d)f(\gamma\tau)\left(cu+civ+d-2icv\right)\right] \\
&= \rho_k(\gamma)(c\tau+d)^{-(k+2)}\left[2if'(\gamma\tau) + \frac{k}{v}(c\tau+d)f(\gamma\tau)(c\overline{\tau}+d)\right] \\
&= \rho_k(\gamma)(c\tau+d)^{-(k+2)}\left[2if'(\gamma\tau) + \frac{k}{v}|c\tau+d|^2 f(\gamma\tau)\right].
\end{aligned}$$

Now observe that

$$\begin{aligned}
\rho_{k+2}(\gamma) &:= \begin{cases} 1 & \text{if } k\in\mathbb{Z} \\ (\frac{c}{d})\varepsilon_d^{2(k+2)} & \text{if } k\in\frac{1}{2}+\mathbb{Z}. \end{cases} \\
&= \begin{cases} 1 & \text{if } k\in\mathbb{Z} \\ (\frac{c}{d})\varepsilon_d^{2k} & \text{if } k\in\frac{1}{2}+\mathbb{Z}. \end{cases} \\
&= \rho_k(\gamma).
\end{aligned}$$

Thus we see that $R_k(f|_k\gamma) = R_k(f)|_{k+2}\gamma$. For the lowering operator we need to extra care as f is also dependent on $\overline{\tau}$ and under slash operator, $\overline{\tau}\mapsto\gamma\overline{\tau}$. So with these points in mind, observe that

$$\begin{aligned}
L_k(f|_k\gamma)(\tau) &= -2iv^2\rho_k(\gamma)\frac{\partial}{\partial\overline{\tau}}(f(\gamma\tau,\gamma\overline{\tau}))(c\tau+d)^{-k} \\
&= -2iv^2\rho_k(\gamma)\left(\frac{\partial f}{\partial\overline{\tau}}\right)(\gamma\tau)(c\tau+d)^{-k}(c\overline{\tau}+d)^{-2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}(L_k(f)|_{k+2}\gamma)(\tau) &= \rho_{k-2}(\gamma)(c\tau+d)^{-(k-2)}\left(-\frac{2iv^2}{|c\tau+d|^4}\right)\left(\frac{\partial f}{\partial \overline{\tau}}\right)(\gamma\tau)\\ &= \rho_{k-2}(\gamma)(c\tau+d)^{-k}\left(-\frac{2iv^2}{(c\overline{\tau}+d)^2}\right)\left(\frac{\partial f}{\partial \overline{\tau}}\right)(\gamma\tau).\end{aligned}$$

Thus we have $L_k(f|_k\gamma) = L_k(f)|_{k-2}\gamma$ since $\rho_{k-2}(\gamma) = \rho_k(\gamma)$.

(ii) By definition, we have,

$$\begin{aligned}R_{k-2}\circ L_k &= \left(2i\frac{\partial}{\partial\tau}+\frac{k-2}{v}\right)\left(-2iv^2\frac{\partial}{\partial\overline{\tau}}\right)\\ &= 4\frac{\partial}{\partial\tau}\left(v^2\frac{\partial}{\partial\overline{\tau}}\right)-2iv(k-2)\frac{\partial}{\partial\overline{\tau}}\\ &= 4v^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}-4iv\frac{\partial}{\partial\overline{\tau}}-2iv(k-2)\frac{\partial}{\partial\overline{\tau}}\\ &= 4v^2\frac{\partial}{\partial\tau}\frac{\partial}{\partial\overline{\tau}}-2ivk\frac{\partial}{\partial\overline{\tau}}\\ &= -\Delta_k\end{aligned}$$

where we used the fact that

$$\frac{\partial}{\partial\tau}v^2 = \frac{1}{2}\left(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v}\right)(v^2) = -iv$$

For the second equality, we will act it on a test function f,

$$\begin{aligned}(L_{k+2}\circ R_k+k)f &= \left[\left(-2iv^2\frac{\partial}{\partial\overline{\tau}}\right)\left(2i\frac{\partial}{\partial\tau}+\frac{k}{v}\right)+k\right]f\\ &= 4v^2\left(\frac{\partial}{\partial\overline{\tau}}\frac{\partial}{\partial\tau}f\right)-2iv^2\frac{1}{2}\left(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v}\right)\left(\frac{k}{v}f\right)+kf\\ &= 4v^2\left(\frac{\partial}{\partial\overline{\tau}}\frac{\partial}{\partial\tau}f\right)-kf+kf-2ikv\frac{\partial}{\partial\overline{\tau}}\\ &= -\Delta_k f.\end{aligned}$$

Thus, we get $L_{k+2}\circ R_k + k = -\Delta_k$.

(iii) We have $\Delta_k(f) = \lambda f$. Then by (ii),

$$-(L_k \circ R_{k-2} \circ L_k)f = -L_k \circ ((R_{k-2} \circ L_k))f) = -L_k(-\lambda f) = \lambda L_k f$$

Thus $-(L_k \circ R_{k-2} \circ L_k)f - (k-2)L_k f = \lambda L_k f - (k-2)L_k f$. This gives $-(L_k \circ R_{k-2} - (k-2))L_k f = (\lambda - k + 2)L_k f$. Again by (ii), L.H.S of above equality is $\Delta_{k-2}L_k f$. Thus we finally get $\Delta_{k-2}L_k f = (\lambda - k + 2)L_k f$. To prove the other relation, we will use $\Delta_k = -(L_{k+2} \circ R_k + k)$. This gives

$$\begin{aligned}
&-(L_{k+2} \circ R_k + k)f = \lambda f \\
&\Rightarrow -(R_k \circ L_{k+2} \circ R_k + kR_k)f = \lambda R_k f \\
&\Rightarrow -(R_k \circ L_{k+2})(R_k f) - kR_k f = \lambda R_k f \\
&\Rightarrow \Delta_{k+2}(R_k f) = (\lambda + k)R_k f.
\end{aligned}$$

$\square$

Remark 4.13. (i) Raising and lowering operators preserve modularity up to a change in weight and preserve eigenfunctions of Δ_k up to a change in eigenvalue. Hence, these operators map weak Maass forms to weak Maass forms, up to a change in weight and eigenvalue.

(ii) R_k does not preserve meromorphicity. But for $k = 0$, we have $R_0 = -4\pi D$. Thus, in this case the raising operator preserves modularity as well as meomorphicity. For general integer k, it is possible to "compose" R_k with itself a special number of times to define maps between weakly holomorphic modular forms. We take up this issue in the next section.

4.3.2 Maps between spaces of weakly holomorphic modular forms

We first define the composition of raising operator as discussed in Remark 4.13. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, define

$$R_k^n := R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k \qquad \text{and} \qquad R_k^0 := 1 \tag{4.11}$$

This composition keeps track of the "correct" weights at each step. With this definition we have the following Theorem due to Zagier.

Theorem 4.14. *For all $k \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$, we have*

$$R_k^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (k+r)_{n-r} v^{r-n} (4\pi D)^r$$

where $(a)_n = a(a+1)\dots(a+n-1)$ is the rising factorial. We use the convention that $\binom{n}{-1} = 0$.

Proof. We use induction to prove the claim. For $n = 0$, the statement is trivial as both sides is 1. Suppose now that the identity is true for natural number n. We will show that the statement is true for $n+1$. First observe that

$$D(v) = \frac{1}{2\pi i}\frac{1}{2}\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) v = -\frac{1}{4\pi} = -(4\pi)^{-1}.$$

Now since

$$R_k^{n+1} = R_{k+2n} \circ R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k = R_{k+2n} \circ R_k^n,$$

we have

$$\begin{aligned}
&R_k^{n+1} = R_{k+2n} \circ R_k^n = -4\pi D \circ R_k^n + (k+2n)v^{-1} R_k^n \\
&= -4\pi \sum_{r=0}^{n} (-1)^r \binom{n}{r} (k+r)_{n-r} D\left(v^{r-n}(4\pi D)^r\right) + \\
&\sum_{r=0}^{n} (-1)^r \binom{n}{r} (k+r)_{n-r} (k+2n) v^{r-n-1} (4\pi D)^r \\
&= -4\pi \sum_{r=0}^{n} (-1)^r \binom{n}{r} (k+r)_{n-r} v^{r-n-1} (-(4\pi)^{-1})(4\pi D)^r + \\
&\sum_{r=0}^{n} (-1)^{r+1} \binom{n}{r} (k+r)_{n-r} v^{r-n} (4\pi D)^{r+1} + \\
&\sum_{r=0}^{n} (-1)^r \binom{n}{r} (k+r)_{n-r} (k+2n) v^{r-n-1} (4\pi D)^r
\end{aligned}$$

$$= \sum_{r=0}^{n}(-1)^r \binom{n}{r} (k+r)_{n-r} v^{r-n-1}(4\pi D)^r +$$
$$\sum_{r=0}^{n+1}(-1)^r \binom{n}{r-1} (k+r-1)_{n-r+1} v^{r-n-1}(4\pi D)^r +$$
$$\sum_{r=0}^{n}(-1)^r \binom{n}{r} (k+r)_{n-r}(k+2n) v^{r-n-1}(4\pi D)^r$$
$$= (-1)^{n+1} \binom{n}{n} (k+n)_0 v^0 (4\pi D)^{n+1} +$$
$$\sum_{r=0}^{n}(-1)^r \left[\binom{n}{r} (k+r)_{n-r}(r+k+n) + \binom{n}{r-1} (k+r-1)_{n-r+1}\right] v^{r-n-1}(4\pi D)^r$$
$$= (-1)^{n+1}(4\pi D)^{n+1} +$$
$$\sum_{r=0}^{n}(-1)^r \left[\binom{n}{r} (k+r)_{n-r}(r+k+n) + \binom{n}{r-1} (k+r-1)_{n-r+1}\right] v^{r-n-1}(4\pi D)^r.$$

But since $(k+r)_{n-r} = (k+r)(k+r+1)\ldots(k+r+n-r-1) = (k+r)(k+r+1)\ldots(k+n-1)$ and $(k+r-1)_{n-r+1} = (k+r-1)(k+r)\ldots(k+r-1+n-r) = (k+r-1)(k+r)\ldots(k+n-1)$, thus we get

$$R_k^{n+1} = (-1)^{n+1}(4\pi D)^{n+1} +$$
$$\sum_{r=0}^{n}(-1)^r (k+r)\ldots(k+n-1) \left[(r+k+n)\binom{n}{r} + (k+r-1)\binom{n}{r-1}\right] v^{r-n-1}(4\pi D)^r.$$

Now, note that

$$\left[(r+k+n)\binom{n}{r} + (k+r-1)\binom{n}{r-1}\right]$$
$$= \left[(r+k+n)\binom{n}{r} + (k+r-1+n-n)\binom{n}{r-1}\right]$$
$$= \left[(k+n)\left(\binom{n}{r} + \binom{n}{r-1}\right) + r\left(\binom{n}{r} + \binom{n}{r-1}\right) - (n+1)\binom{n}{r-1}\right]$$
$$= \left[(k+n)\binom{n+1}{r} + r\binom{n+1}{r} - (n+1)\binom{n}{r-1}\right]$$

It can easily be checked that $r\binom{n+1}{r} = (n+1)\binom{n}{r-1}$. Thus we have,

$$\begin{aligned}
&R_k^{n+1} = (-1)^{n+1}(4\pi D)^{n+1} + \\
&\sum_{r=0}^{n}(-1)^r(k+r)\dots(k+n-1)(k+n)\binom{n+1}{r} v^{r-n-1}(4\pi D)^r \\
&= (-1)^{n+1}(4\pi D)^{n+1} + \sum_{r=0}^{n}(-1)^r\binom{n+1}{r}(k+r)_{n+1-r}v^{r-n-1}(4\pi D)^r \\
&= \sum_{r=0}^{n+1}(-1)^r\binom{n+1}{r}(k+r)_{n+1-r}v^{r-n-1}(4\pi D)^r
\end{aligned}$$

as claimed. □

As discussed in previous section, we want to define map between spaces of weakly holomorphic modular forms. We will do this by relating the raising operator which preserves modularity and the differential operator D which preserves meromorphicity. This relation is called *Bol's identity.*

Theorem 4.15. *(Bol's identity) For $k \in \mathbb{N}$, we have*

$$D^{k-1} = \frac{1}{(-4\pi)^{k-1}} R_{2-k}^{k-1}.$$

In particular, we have that

$$D^{k-1} : M_{2-k}^{!}(\Gamma_0(N)) \longrightarrow M_k^{!}(\Gamma_0(N)).$$

Proof. We use Theorem 4.14 with n replaced by $k-1$ and k replaced by $2-k$. We get

$$\begin{aligned}
R_{2-k}^{k-1} &= \sum_{r=0}^{k-1}(-1)^r\binom{k-1}{r}(2-k+r)_{k+1-r}v^{r-k+1}(4\pi D)^r \\
&= (-1)^{k-1}v^0(4\pi D)^{k-1} + \sum_{r=0}^{k-2}(-1)^r\binom{k-1}{r}(2-k+r)_{k+1-r}v^{r-k+1}(4\pi D)^r.
\end{aligned}$$

But $(2-k+r)_{k+1-r} = 0$ for all $0 \le r \le k-2$ as $2-k+r \le 0 \le 2 = 2-k+r+k-r$

for all $0 \leq r \leq k-2$. Thus we get,

$$\begin{aligned} R_{2-k}^{k-1} &= (-1)^{k-1}(4\pi D)^{k-1} \\ \implies D^{k-1} &= \frac{1}{(-4\pi)^{k-1}} R_{2-k}^{k-1}. \end{aligned}$$

Now, using Theorem 4.12 (i), we see that if f satisfies modularity of weight $2-k$ then R_{2-k}^{k-1} satisfies modularity of weight $2-k+2(k-1)=k$. Meromorphicity at the cusps follows from the fact that D^{k-1} preserves meromorphicity. □

Theorem 4.15 shows that the *Bol operator* D^{k-1} maps weakly holomorphic modular forms of weight $2-k$ to weakly holomorphic modular forms of weight k. Infact, we have a much more general result concerning the Bol operator.

Theorem 4.16. *For $k \geq 2$ an integer, the following are true:*

(i) We have that

$$D^{k-1} : H_{2-k}(\Gamma_0(N)) \longrightarrow M_k^!(\Gamma_0(N)).$$

(ii) With the notations of Theorem 4.6, for $f \in H_{2-k}(\Gamma_0(N))$, we have

$$D^{k-1}(f(\tau)) = D^{k-1}(f^+(\tau)) = \sum_{n>>-\infty} c_f^+(n) n^{k-1} q^n.$$

(iii) We also have

$$D^{k-1} : H_{2-k}^!(\Gamma_0(N)) \twoheadrightarrow M_k^!(\Gamma_0(N)).$$

where the two headed arrow means the map is surjective.

(iv) With the notations of Theorem 4.6, for $f \in H_{2-k}^!(\Gamma_0(N))$, we have

$$D^{k-1}(f(\tau)) = D^{k-1}(f(\tau)) = (-4\pi)^{1-k}(k-1)! c_f^-(0) + \sum_{n>>-\infty} c_f^+(n) n^{k-1} q^n.$$

Proof. To prove (i), we claim that for $n \in \mathbb{N}$

$$D^{k-1}\left(\Gamma(k-1, 4\pi n v) q^{-n}\right) = 0.$$

To see this, first observe that

$$\Gamma(\alpha, \omega)e^{\omega} = \int_0^\infty (\omega + t)^{\alpha-1} e^{-t} dt.$$

For $\alpha = k-1, \omega = 4\pi nv$, we get

$$\Gamma(k-1, 4\pi nv)e^{4\pi nv} = \int_0^\infty (4\pi nv + t)^{k-2} e^{-t} dt$$

which gives

$$D^{k-1}\left(\Gamma(k-1, 4\pi nv)e^{4\pi nv}\right) = \left(\frac{1}{2\pi i}\right)^{k-1} \int_0^\infty \frac{\partial^{k-1}}{\partial \tau^{k-1}} (4\pi nv + t)^{k-2} e^{-t} dt = 0$$

because, $(4\pi nv + t)^{k-2}$ when differentiated $k-1$ times gives zero. Now

$$\Gamma(k-1, 4\pi nv)q^{-n} = \Gamma(k-1, 4\pi nv)e^{4\pi nv}e^{-2\pi nv}e^{-2\pi inu} = \Gamma(k-1, 4\pi nv)e^{4\pi nv}e^{-2\pi in\overline{\tau}}.$$

Thus

$$D^{k-1}\left(\Gamma(k-1, 4\pi nv)q^{-n}\right) = D^{k-1}\left(\Gamma(k-1, 4\pi nv)e^{4\pi nv}\right)e^{-2\pi in\overline{\tau}} = 0.$$

Now suppose $f \in H_{2-k}(\Gamma_0(N))$, then the non-holomorphic part is

$$f^{-}(\tau) = \sum_{n<0} c_f^{-}(n)\Gamma(k-1, -4\pi nv)q^n.$$

Thus by the calculation above, we see that $D^{k-1}(f^{-}(\tau)) = 0$. Thus $D^{k-1}(f(\tau)) = D^{k-1}(f^{+}(\tau))$ is holomorphic on $\mathbb{H}$. To prove that $D^{k-1}(f(\tau)) \in M_k^!(\Gamma_0(N))$, we must prove that it is meromorphic at the cusps of $\Gamma_0(N)$. It suffices to prove the Fourier expansion of $D^{k-1}(f(\tau))$ as in (ii) which will prove that $D^{k-1}(f(\tau))$ is meromorphic at $i\infty$. At other cusps, similar Fourier expansion for f exists. Thus meromorphicity at other cusps follows doing a similar calculation with the corresponding Fourier expansion of f. We now prove (ii). Observe that

$$D^{k-1}(q^n) = \left(\frac{1}{2\pi i}\right)^{k-1} \frac{\partial^{k-1}}{\partial\tau^{k-1}} e^{2\pi i n\tau} = n^{k-1}q^n.$$

Thus we get

$$D^{k-1}(f(\tau)) = D^{k-1}(f^+(\tau)) = \sum_{n>>-\infty} c_f^+(n)n^{k-1}q^n.$$

To prove (iii), observe that if $f \in H^!_{2-k}(\Gamma_0(N))$, then the non-holomorphic part is

$$f^-(\tau) = c_f^-(0)v^{k-1} + \sum_{n<<\infty} c_f^-(n)\Gamma(k-1, -4\pi nv)q^n.$$

Again $D^{k-1}(f(\tau)) = D^{k-1}(c_f^-(0)v^{k-1}) + D^{k-1}(f^+(\tau))$. But

$$\begin{aligned} D^{k-1}(c_f^-(0)v^{k-1}) &= c_f^-(0)\left(\frac{1}{2\pi i}\right)^{k-1} \frac{\partial^{k-1}}{\partial\tau^{k-1}} v^{k-1} \\ &= c_f^-(0)\left(\frac{1}{2\pi i}\right)^{k-1} \frac{1}{2^{k-1}}\left((-i)^{k-1}\frac{\partial^{k-1}}{\partial\tau^{k-1}} v^{k-1}\right) \\ &= c_f^-(0)(-4\pi)^{1-k}(k-1)! \end{aligned}$$

Thus we get the desired Fourier expansion of (iv). To prove the surjectivity of (iii), we will first need to define ξ operator and the *flipping operator*. □

Remark 4.17. (i) The image of f under the Bol operator is called the *Ghost* of f.

(ii) The map in Theorem 4.16 (i) is not surjective. Thus we can talk about the image of $H_{2-k}(\Gamma_0(N))$ under the Bol operator. The image can be characterised using the *regularised inner product* defined in terms of the usual *Petterson inner product*.

We constructed a map between space of harmonic Maass forms and space of weakly holomorphic modular forms using the Maass raising operator. A similar map can be defined using the Maass lowering operator.

Definition 4.18. For $k \in \mathbb{Z}$, define the ξ-*operator* as

$$\xi_k := 2iv^k \overline{\frac{\partial}{\partial \bar{\tau}}} = v^{k-2}\overline{L_k}.$$

We have the following Theorem.

Theorem 4.19. *Let $k \in \frac{1}{2}\mathbb{Z}$ and $N \in \mathbb{N}$. We have*

(i) $\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \twoheadrightarrow S_k(\Gamma_0(N))$.

(ii) For $f \in H_{2-k}(\Gamma_0(N))$, we have that

$$\xi_{2-k}(f(\tau)) = \xi_{2-k}(f^-(\tau)) = -(4\pi)^{k-1}\sum_{n=1}^{\infty}\overline{c_f^-(-n)}n^{k-1}q^n.$$

(iii) $\xi_{2-k} : H^!_{2-k}(\Gamma_0(N)) \twoheadrightarrow M^!_k(\Gamma_0(N))$.

(iv) For $f \in H^!_{2-k}(\Gamma_0(N))$, we have that

$$\xi_{2-k}(f(\tau)) = \xi_{2-k}(f^-(\tau)) = (k-1)\overline{c_f^-(0)} - (4\pi)^{k-1}\sum_{n>>-\infty}\overline{c_f^-(-n)}n^{k-1}q^n.$$

Proof. To prove (i), we first prove that $\xi_{2-k}(f(\tau))$ satisfies modularity of weight k. Since $f \in H_{2-k}(\Gamma_0(N))$, thus

$$f|_{2-k}\gamma = f \qquad \forall \gamma \in \Gamma_0(N)$$

Thus, by Theorem 4.12 (i), we have

$$L_{2-k}(f)|_{-k}\gamma = L_{2-k}(f) \qquad \forall \;\; \gamma \in \Gamma_0(N). \tag{4.12}$$

Now for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$\left(\xi_{2-k}(f)|_k\gamma\right)(\tau) = \left([v^{-k}\overline{L_{2-k}(f)}]|_k\gamma\right)(\tau) = \frac{v^{-k}}{|c\tau+d|^{-2k}}\overline{L_{2-k}(f)(\gamma\tau)}\rho_k(\gamma)(c\tau+d)^{-k}.$$

By taking complex conjugation on both sides of Eq. (4.12), we get

$$\overline{L_{2-k}(f)(\gamma\tau)} = \frac{\overline{\left(L_{2-k}(f)\right)(\tau)(c\tau+d)^k}}{\overline{\rho_{-k}(\gamma)}}.$$

Thus we get

$$\begin{aligned}
\left(\xi_{2-k}(f)|_k\gamma\right)(\tau) &= \frac{v^{-k}}{(c\tau+d)^{-k}\overline{(c\tau+d)}^{-k}} \frac{\overline{\left(L_{2-k}(f)\right)(\tau)(c\tau+d)^k}}{\overline{\rho_{-k}(\gamma)}} \rho_k(\gamma)(c\tau+d)^{-k} \\
&= v^{-k}\left(L_{2-k}(f)\right)(\tau) \\
&= \left(\xi_{2-k}(f)\right)(\tau)
\end{aligned}$$

where the second last equality follows by noting that $\overline{\rho_{-k}(\gamma)} = \rho_k(\gamma)$. Next, observe that since $f^+(\tau)$ is holomorphic, thus $\frac{\partial f^+(\tau)}{\partial\overline{\tau}} = 0$. Hence $\xi_{2-k}(f^+) = 0$. We will now separately compute $\xi_{2-k}(f^-)$. We have

$$\begin{aligned}
\frac{\partial}{\partial\overline{\tau}}\Gamma(k-1,-4\pi nv) &= \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)\Gamma(k-1,-4\pi nv) = \frac{i}{2}\frac{\partial}{\partial v}\Gamma(k-1,-4\pi nv) \\
&= \frac{i}{2}\frac{\partial}{\partial(-4\pi nv)}\Gamma(k-1,-4\pi nv)\frac{d(-4\pi nv)}{dv} \\
&= \frac{-4\pi inv}{2}\frac{\partial}{\partial(-4\pi nv)}\int\limits_{-4\pi nv}^{\infty} e^{-t}t^{k-2}dt \\
&= \frac{-4\pi inv}{2}\frac{\partial}{\partial(-4\pi nv)}\left[\int\limits_0^{\infty} - \int\limits_0^{-4\pi nv} e^{-t}t^{k-2}dt\right] \\
&= \frac{4\pi inv}{2}(-4\pi nv)^{k-2}e^{4\pi nv} \\
&= -(4\pi n)^{k-1}iv^{k-2}\frac{(-1)^{k-1}}{2}e^{4\pi nv}.
\end{aligned}$$

where we used fundamental Theorem of calculus. Now,

$$\begin{aligned}
\xi_{2-k}(f^-(\tau)) &= 2iv^{2-k}\overline{\sum_{n<0} c_f^-(n)\frac{\partial}{\partial\overline{\tau}}\Gamma(k-1,-4\pi nv)q^n} \\
&= -2iv^{2-k}\overline{\sum_{n<0} c_f^-(n)(4\pi n)^{k-1}iv^{k-2}\frac{(-1)^{k-1}}{2}e^{4\pi nv}q^n} \\
&= -(4\pi)^{k-1}\sum_{n<0}\overline{c_f^-(n)}n^{k-1}(-1)^{k-1}e^{4\pi nv-2\pi in\overline{\tau}} \\
&= -(4\pi)^{k-1}\sum_{n<0}\overline{c_f^-(n)}(-n)^{k-1}e^{-2\pi in\tau} \\
&= -(4\pi)^{k-1}\sum_{n=1}^{\infty}\overline{c_f^-(-n)}n^{k-1}e^{2\pi in\tau} \\
&= -(4\pi)^{k-1}\sum_{n=1}^{\infty}\overline{c_f^-(-n)}n^{k-1}q^n.
\end{aligned}$$

Since at every other cusp, f has similar Fourier expansion, thus repeating above calculation at other cusps shows that $\xi_{2-k}(f)$ is indeed a cusp form. Above calculation gives (ii) as well. (iii) follows if we prove (iv). Now note that if $f \in H^!_{2-k}(\Gamma_0(N))$, then f^- has one more term namely $c_f^-(0)v^{k-1}$. Thus we just need to calculate $\xi_{2-k}(c_f^-(0)v^{k-1})$. We have

$$\begin{aligned}
\xi_{2-k}(c_f^-(0)v^{k-1}) &= 2iv^{2-k}\overline{\frac{\partial}{\partial\overline{\tau}}c_f^-(0)v^{k-1}} \\
&= 2iv^{2-k}\overline{c_f^-(0)}\,\overline{\frac{i}{2}\frac{\partial}{\partial v}v^{k-1}} \\
&= (k-1)v^{2-k}\overline{c_f^-(0)}v^{k-2} \\
&= (k-1)\overline{c_f^-(0)}.
\end{aligned}$$

Thus we get

$$\xi_{2-k}(f(\tau)) = \xi_{2-k}(f^-(\tau)) = (k-1)\overline{c_f^-(0)} - (4\pi)^{k-1}\sum_{n>>-\infty}\overline{c_f^-(-n)}n^{k-1}q^n.$$

Proving surjectivity of these maps requires the theory of Riemann surfaces and algebraic geometry. Thus we omit this part here. $\square$

Remark 4.20. The image of f under the ξ operator is called the *shadow* of f.

There is another important operator which "flips" the holomorphic and non-holomorphic parts. This operator is called the *flipping operator*.

Definition 4.21. For $f \in H_k^!(\Gamma_0(N))$ with $k \in -2\mathbb{N}_0$[1], define the *flip* of f by

$$\mathfrak{F}_k(f) := -\frac{v^{-k}}{(-k)!}\overline{R_k^{-k}(f)}.$$

With this definition, we have the following Theorem

Theorem 4.22. *Suppose that $f \in H_k^!(\Gamma_0(N))$ with $k \in -2\mathbb{N}_0$. Then the following are true:*

(i) The image $\mathfrak{F}_k(f)$ lies in $H_k^!(\Gamma_0(N))$.

(ii) The flipping operator is an involution, that is

$$\mathfrak{F}_k(\mathfrak{F}_k(f)).$$

(iii) The shadow of the flip of f is given by

$$\xi_k(\mathfrak{F}_k(f)) = \frac{(4\pi)^{1-k}}{(-k)!}D^{1-k}(f).$$

(iv) The image under the Bol operator of the flip of f is given by

$$D^{1-k}(\mathfrak{F}_k(f)) = \frac{(-k)!}{(4\pi)^{1-k}}\xi_k(f).$$

(v) Assuming the notation of Theorem 4.6 for the Fourier expansion of f, the Fourier expansion of the flip of f is given by

$$\mathfrak{F}_k(f(\tau)) = -\overline{c_f^-(0)}v^{1-k} - (-k)! \sum_{\substack{n>>-\infty \\ n\neq 0}} \overline{c_f^-(-n)}q^n$$
$$-\overline{c_f^+(0)} - \frac{1}{(-k)!}\sum_{\substack{n<<\infty \\ n\neq 0}} \overline{c_f^+(-n)}\Gamma(1-k, -4\pi n v)q^n.$$

[1] $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. To prove (i), we need to show modularity, annihilation by the hyperbolic Laplacian and appropriate growth condition. Let us first show modularity. Observe that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$\begin{aligned}\mathfrak{F}_k(f)(\gamma\tau)) &= -\frac{Im(\gamma\tau)^{-k}}{(-k)!}\overline{R_k^{-k}(f)(\gamma\tau)} \\ &= -\frac{v^{-k}}{(-k)!|c\tau+d|^{-2k}}\overline{R_k^{-k}(f)(\gamma\tau)}.\end{aligned}$$

But $R_k^{-k}(f)$ transforms as a modular form of weight $k + 2(-k) = -k$. So, using the slash operator for integer weight, we have

$$\overline{R_k^{-k}(f)(\gamma\tau)} = \overline{(c\tau+d)^{-k}R_k^{-k}(f(\tau))}.$$

This gives

$$\begin{aligned}\mathfrak{F}_k(f)(\gamma\tau)) &= -\frac{v^{-k}}{(-k)!|c\tau+d|^{-2k}}\overline{(c\tau+d)^{-k}R_k^{-k}(f(\tau))} \\ &= (c\tau+d)^k\left[-\frac{v^{-k}}{(-k)!}\overline{R_k^{-k}(f(\tau))}\right] \\ &= (c\tau+d)^k\mathfrak{F}_k(f(\tau)).\end{aligned}$$

Thus $\mathfrak{F}_k(f)$ transforms as a modular form of weight k. The growth condition of $\mathfrak{F}_k(f)$ is obvious from the growth condition of f since $\mathfrak{F}$ just contains a positive power of v and a k^{th} order derivative of f. To complete the proof of (i), we will assume and prove it next. First observe that for any test function f we have

$$\begin{aligned}(\xi_{2-k}\circ\xi)f &= \xi_{2-k}\left(2iv^k\overline{\frac{\partial f}{\partial\overline{\tau}}}\right) \\ &= 2iv^{2-k}\overline{\frac{\partial}{\partial\overline{\tau}}\left(2iv^k\overline{\frac{\partial f}{\partial\overline{\tau}}}\right)} \\ &= 2iv^{2-k}2i\frac{ik}{2}v^{k-1}\frac{\partial f}{\partial\overline{\tau}} - 2iv^{2-k}2iv^k\overline{\frac{\partial}{\partial\overline{\tau}}\overline{\frac{\partial f}{\partial\overline{\tau}}}} \\ &= -2ivk\frac{\partial f}{\partial\overline{\tau}} + 4v^2\frac{\partial}{\partial\tau}\frac{\partial f}{\partial\overline{\tau}}\end{aligned}$$

$$= -\left[-4v^2 \frac{\partial}{\partial \tau}\frac{\partial}{\partial \overline{\tau}} + 2ivk\frac{\partial}{\partial \overline{\tau}}\right] f$$
$$= -\Delta_k(f)$$

where we used the fact that

$$\frac{\partial v^k}{\partial \overline{\tau}} = \frac{ik}{2}v^{k-1} \qquad \text{and} \qquad \overline{\frac{\partial \overline{g}}{\partial \overline{\tau}}} = \frac{\partial g}{\partial \tau}.$$

These two facts follow by simply using the definition of the operators $\frac{\partial}{\partial \overline{\tau}}$ and $\frac{\partial}{\partial \tau}$. Thus we have $\Delta_k = -\xi_{2-k} \circ \xi_k$. Now using (iii), we have

$$\begin{aligned}\Delta_k(\mathfrak{F}_k(f)) &= -(\xi_{2-k} \circ \xi_k)f \\ &= -\frac{(4\pi)^{1-k}}{(-k)!}\xi_{2-k}(D^{1-k}(f)) = 0\end{aligned}$$

since by Theorem 4.16 (iii) $D^{1-k}(f) \in M^!_{2-k}(\Gamma_0(N))$ and by Theorem 4.19 (iv), $\xi_{2-k}(f) = 0$ if f is holomorphic. We now prove (iii). First observe that for any real analytic function $g : \mathbb{H} \to \mathbb{C}$, we have

$$\xi_k(v^{-k}\overline{g(\tau)}) = R_{-k}(g(\tau)). \tag{4.13}$$

To see this, observe that

$$\begin{aligned}\xi_k(v^{-k}\overline{g(\tau)}) &= 2iv^k\overline{\frac{\partial}{\partial \overline{\tau}}v^{-k}\overline{g(\tau)}} \\ &= 2iv^k\overline{\left(\frac{-ik}{2}v^{-k-1}\overline{g(\tau)}\right)} + 2iv^k v^{-k}\overline{\frac{\partial}{\partial \overline{\tau}}\overline{g(\tau)}} \\ &= -\frac{k}{v}g(\tau) + 2i\frac{\partial}{\partial \tau}g(\tau) \\ &= R_{-k}(g(\tau)).\end{aligned}$$

Now for $g = R_k^{-k}(f)$, we get

$$\begin{aligned}\xi_k(v^{-k}\overline{R_k^{-k}(f)}) = R_{-k}(R_k^{-k}(f)) &= (R_{k+2(-k)} \circ R_k^{-k})(f) = R_k^{1-k}(f) \\ &= (-4\pi)^{1-k}D^{1-k}(f) \\ &= -(4\pi)^{1-k}D^{1-k}(f).\end{aligned}$$

where we used the Bol's identity to get the last equality and the fact that $k \in -2\mathbb{N}_0$. On dividing both sides by $(-k)!$ in previous equation we get (iii). To prove (ii), we will first show by induction that for $0 \le l \le -k$

$$\mathfrak{F}_k(\mathfrak{F}_k(f)) = \frac{l!}{(-k)!(-k-l)!} v^{-k} \overline{R_{k+2l}^{-k-l}\left(v^{-k-2l}\overline{R_k^{-k-l}(f)}\right)}. \tag{4.14}$$

<u>Base case:</u> $l = 0$

$$\begin{aligned} R.H.S. &= \frac{1}{((-k)!)^2} v^{-k} \overline{R_k^{-k}\left(v^{-k}\overline{R_k^{-k}(f)}\right)} \\ &= \frac{-v^{-k}}{(-k)!} \overline{R_k^{-k}\left(\frac{-v^{-k}}{(-k)!}\overline{R_k^{-k}(f)}\right)} \\ &= \mathfrak{F}_k(\mathfrak{F}_k(f)) = L.H.S. \end{aligned}$$

<u>Inductive step:</u> Suppose (4.14) is true for $0 \le l \le -k-1$, then we prove it for $l+1$. Observe that

$$\begin{aligned} \mathfrak{F}_k(\mathfrak{F}_k(f)) &= \frac{l!}{(-k)!(-k-l)!} v^{-k} \overline{R_{k+2l}^{-k-l}\left(v^{-k-2l}\overline{R_{k-2(-k-l+1)}\left(R_k^{-k-l-1}(f)\right)}\right)} \\ &= \frac{l!}{(-k)!(-k-l)!} v^{-k} \overline{R_{k+2l}^{-k-l}\left(v^{-k-2l}\overline{R_{k+2(-k-l-1)}\left(R_k^{-k-l-1}(f)\right)}\right)} \\ &= \frac{l!}{(-k)!(-k-l)!} v^{-k} \overline{R_{k+2l}^{-k-l-1} \circ R_{k+2l}\left(v^{-k-2l}\overline{R_{-k-2l-2}\left(R_k^{-k-l-1}(f)\right)}\right)}. \end{aligned}$$

We would like to calculate $R_{k+2l}\left(v^{-k-2l}\overline{R_{-k-2l-2}\left(R_k^{-k-l-1}(f)\right)}\right)$. To do this observe that for any real analytic function g, we have

$$\begin{aligned} R_k\left(v^{-k}\overline{R_{-k-2}(g(\tau))}\right) &= 2i\frac{\partial}{\partial\tau}\left(v^{-k}\overline{R_{-k-2}(g(\tau))}\right) + \frac{k}{v}\left(v^{-k}\overline{R_{-k-2}(g(\tau))}\right) \\ &= 2i\left(\overline{R_{-k-2}(g(\tau))}\right)\frac{\partial v^{-k}}{\partial\tau} + 2iv^{-k}\frac{\partial}{\partial\tau}\left(\overline{R_{-k-2}(g(\tau))}\right) \\ &+ \frac{k}{v}\left(v^{-k}\overline{R_{-k-2}(g(\tau))}\right) \\ &= -kv^{-k-1}\left(\overline{R_{-k-2}(g(\tau))}\right) + 2iv^{-k}\frac{\partial}{\partial\tau}\left(\overline{R_{-k-2}(g(\tau))}\right) \\ &+ \frac{k}{v}\left(v^{-k}\overline{R_{-k-2}(g(\tau))}\right) \end{aligned}$$

$$
\begin{aligned}
&= v^{-k-2} 2iv^2 \frac{\partial}{\partial \tau} \left(\overline{R_{-k-2}(g(\tau))} \right) \\
&= v^{-k-2} \overline{\left(-2iv^2 \frac{\partial}{\partial \overline{\tau}} R_{-k-2}(g(\tau)) \right)} \\
&= v^{-k-2} \overline{(L_{-k} \circ R_{-k-2}(g(\tau)))} \\
&= v^{-k-2} \overline{(-\Delta_{-k-2} + k + 2) g(\tau)}
\end{aligned}
$$

where we used Theorem 4.12 (iii) for the last equality. In particular if $\Delta_{-k-2} g(\tau) = \lambda g(\tau)$, then

$$
R_k \left(v^{-k} \overline{R_{-k-2}(g(\tau))} \right) = v^{-k-2} (-\overline{\lambda} + k + 2) \overline{g(\tau)}.
$$

Replacing k by $k + 2l$ and g by $R_k^{-k-l-1}(f)$, we get

$$
R_{k+2l} \left(v^{-k-2l} \overline{R_{-k-2l-2}(R_k^{-k-l-1}(f))} \right) = v^{-k-2l-2} \overline{(-\Delta_{-k-2l-2} + k + 2l + 2) R_k^{-k-l-1}(f)}.
$$

We now need to calculate $\Delta_{-k-2l-2} R_k^{-k-l-1}(f)$ knowing the fact that $\Delta_k f = 0$. To do this we will again prove a general result concerning eigenvalue of f.
Let $\Delta_k f = \lambda f$ then for $m \in \mathbb{N}_0$, we have

$$
\Delta_{k+2m}(R_k^m(f)) = (\lambda + m(k + m - 1)) R_k^m(f).
$$

Again we will prove this by induction. For $m = 0$, the relation holds identically. For $m = 1$, we have $\Delta_{k+2}(R_k(f)) = (\lambda + k) R_k(f)$ by Theorem 4.12. Suppose the statement is true for m, then for $m + 1$ we get

$$
\Delta_{k+2(m+1)}(R_k^{m+1}(f)) = \Delta_{k+2m+2}(R_{k+2m}(R_k^m(f))).
$$

By induction hypothesis $\Delta_{k+2m}(R_k^m(f)) = (\lambda + m(k + m - 1)) R_k^m(f)$. Then by Theorem 4.12 (iii), we have

$$
\begin{aligned}
\Delta_{k+2(m+1)}(R_k^{m+1}(f)) &= \Delta_{k+2m+2}(R_{k+2m}(R_k^m(f))) \\
&= (\lambda + m(m + k - 1) + k + 2m) R_k^m(f) \\
&= (\lambda + mk + m^2 - m + k + 2m) R_k^m(f) \\
&= (\lambda + (m + 1)(k + m)) R_k^m(f).
\end{aligned}
$$

Using this result for $m = -k-l-1$, we have

$$\Delta_{-k-2l-2}(R_k^{-k-l-1}(f)) = (-k-l-1)(k-k-l-1-1)R_k^m(f) = (k+l+1)(l+2)R_k^m(f).$$

Thus we get

$$\begin{aligned} &R_{k+2l}\left(v^{-k-2l}\overline{R_{-k-2l-2}(R_k^{-k-l-1}(f))}\right) \\ &= v^{-k-2l-2}(-(k+l+1)(l+2)+k+2l+2)\overline{R_k^{-k-l-1}(f)} \\ &= -(k+l)(l+1)v^{-k-2l-2}\overline{R_k^{-k-l-1}(f)}. \end{aligned}$$

Thus we get

$$\begin{aligned} \mathfrak{F}_k(\mathfrak{F}_k(f)) &= \frac{l!}{(-k)!(-k-l)!}v^{-k}\overline{R_{k+2l}^{-k-l-1}\left(-(k+l)(l+1)v^{-k-2l-2}\overline{R_k^{-k-l-1}(f)}\right)} \\ &= \frac{l!(-k-l)(l+1)}{(-k)!(-k-l)!}v^{-k}\overline{R_{k+2l}^{-k-l-1}\left(v^{-k-2l-2}\overline{R_k^{-k-l-1}(f)}\right)} \\ &= \frac{(l+1)!}{(-k)!(-k-l-1)!}v^{-k}\overline{R_{k+2l}^{-k-l-1}\left(-(k+l)(l+1)v^{-k-2l-2}\overline{R_k^{-k-l-1}(f)}\right)}. \end{aligned}$$

Thus by induction we have proved (4.14). For $l=-k$, (4.14) gives

$$\mathfrak{F}_k(\mathfrak{F}_k(f)) = \frac{(-k)!}{(-k)!}v^{-k}(v^k f) = f.$$

We now prove (iv). First observe that $R_k^{1-k}(\mathfrak{F}_k(f)) = R_{-k}\left(R_k^{-k}(\mathfrak{F}_k(f))\right)$. Using (4.13) for $g = R_k^{-k}(\mathfrak{F}_k(f))$, we get

$$\begin{aligned} R_k^{1-k}(\mathfrak{F}_k(f)) = \xi_k\left(v^{-k}\overline{R_k^{-k}(\mathfrak{F}_k(f))}\right) &= -(-k)!\xi_k\left(\mathfrak{F}_k(\mathfrak{F}_k(f))\right) \\ &= -(-k)!\xi_k(f). \end{aligned}$$

Now by Bol's identity we have

$$D^{1-k} = \frac{1}{(-4\pi)^{1-k}}R_k^{1-k} = -\frac{1}{(4\pi)^{1-k}}R_k^{1-k}$$

since $k \in -2\mathbb{N}_0$. Thus we finally get

$$D^{1-k}(\mathfrak{F}_k(f)) = \frac{(-k)!}{(4\pi)^{1-k}}\xi_k(f).$$

To prove (iv), observe that since $\mathfrak{F}_k(f) \in H_k^!(\Gamma_0(N))$, thus it has a Fourier expansion of the form

$$\mathfrak{F}_k(f(\tau)) = \sum_{n>>-\infty} d_f^+(n)q^n + d_f^-(0)v^{1-k} + \sum_{\substack{n<<\infty \\ n\neq 0}} d_f^-(n)\Gamma(1-k,-4\pi nv)q^n.$$

By Theorem 4.19 (iv) for appropriate weight, we have

$$\xi_k\left(\mathfrak{F}_k(f)\right) = (k-1)\overline{d_f^-(0)} - (4\pi)^{k-1}\sum_{\substack{n>>-\infty \\ n\neq 0}} \frac{\overline{d_f^-(-n)}}{n^{k-1}}q^n. \tag{4.15}$$

By Theorem 4.16 (iv) with appropriate weight and using (iii) above, we have

$$\xi_k\left(\mathfrak{F}_k(f)\right) = \frac{(4\pi)^{1-k}}{(-k)!}D^{1-k}(\mathfrak{F}_k(f)) = -(k-1)c_f^-(0) + \frac{(4\pi)^{k-1}}{(-k)!}\sum_{\substack{n>>-\infty \\ n\neq 0}} \frac{c_f^-(-n)}{n^{k-1}}q^n. \tag{4.16}$$

Comparing coefficients in (4.15) and (4.16), we get

$$\text{for} \quad n \neq 0 \quad d_f^-(n) = -\frac{\overline{c_f^+(n)}}{(-k)!} \quad \text{and} \quad d_f^-(0) = -\overline{c_f^-(0)}. \tag{4.17}$$

Again using Theorem 4.16 (iv) for appropriate weight, we get

$$D^{1-k}(\mathfrak{F}_k(f)) = -\frac{(1-k)!}{(4\pi)^{1-k}}d_f^-(0) + \sum_{\substack{n>>-\infty \\ n\neq 0}} \frac{d_f^+(n)}{n^{k-1}}q^n. \tag{4.18}$$

By Theorem 4.19 (iv) with appropriate weight and using (iv) above, we have

$$D^{1-k}(\mathfrak{F}_k(f)) = \frac{(-k)!}{(4\pi)^{1-k}}\xi_k(f) = \frac{(1-k)!}{(4\pi)^{1-k}}\overline{c_f^-(0)} - (-k)!\sum_{\substack{n>>-\infty \\ n\neq 0}} \frac{\overline{c_f^+(-n)}}{n^{k-1}}q^n. \tag{4.19}$$

Comparing coefficients in (4.18) and (4.19), we get

$$\text{for} \quad n \neq 0 \quad d_f^+(n) = -(-k)!\overline{c_f^+(n)}. \tag{4.20}$$

Finally using Theorem 4.14 we see that

$$\begin{aligned}\mathfrak{F}_k(c_f^+(0)) &= -\frac{v^{-k}}{(-k)!}\overline{R_k^{-k}(c_f^+(0))} = -\frac{v^{-k}}{(-k)!}\frac{(k)_{-k}}{v^{-k}}\overline{c_f^+(0)} \\ &= -\frac{k(k+1)\dots(k-k-1)}{(-k)!}\overline{c_f^+(0)}\end{aligned}$$

$$= -\frac{(-1)^k(-k)(-k-1)\dots(-1)}{(-k)!}\overline{c_f^+(0)} = -\overline{c_f^+(0)}.$$

Thus we have

$$d_f^+(0) = -\overline{c_f^+(0)}. \tag{4.21}$$

Using (4.17), (4.20) and (4.21), we get

$$\mathfrak{F}_k(f(\tau)) = -\overline{c_f^-(0)}v^{1-k} - (-k)! \sum_{\substack{n>>-\infty \\ n\neq 0}} \overline{c_f^-(-n)}q^n$$

$$-\overline{c_f^+(0)} - \frac{1}{(-k)!}\sum_{\substack{n<<\infty \\ n\neq 0}} \overline{c_f^+(-n)}\Gamma(1-k,-4\pi nv)q^n.$$

□

We are now ready to prove the surjectivity of the Bol operator in Theorem 4.16 (iv). Let $f \in M_k^!(\Gamma_0(N))$. Then since $\xi_{2-k} : H_{2-k}^!(\Gamma_0(N)) \twoheadrightarrow M_k^!(\Gamma_0(N))$ is surjective by Theorem 4.19 (iii), $\exists \widetilde{f} \in H_{2-k}^!(\Gamma_0(N))$ such that $\xi_{2-k}(\widetilde{f}) = f$. Since $k \geq 2$ in Theorem 4.19, thus $2-k \leq 0$ and $2-k \in -2\mathbb{N}_0$ as $-I \in \Gamma_0(N)$ which implies there are no harmonic Maass forms of odd weight due to modularity. Thus $g = \mathfrak{F}_{2-k}(\widetilde{f}) \in H_{2-k}^!(\Gamma_0(N))$. By Theorem 4.22 (iv)

$$D^{k-1}(g) = \frac{(k-2)!}{(4\pi)^{k-1}}f.$$

Put

$$\widetilde{g} = \frac{(4\pi)^{k-1}}{(k-2)!} g.$$

Then $D^{k-1}(\widetilde{g}) = f$.

4.4 Mock-modular forms and shadows

Definition 4.23. (i) A *mock modular form* of weight $2-k$ is the holomorphic part f^+ of a harmonic Maass form of weight $2-k$ for which f^- is non trivial.

(ii) If $f \in H_{2-k}(\Gamma_0(N))$, we refer to the cusp form

$$\xi_{2-k}(f(\tau)) = -(4\pi)^{k-1} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{k-1} q^n$$

as the *shadow* of the mock modular form f^+.

(iii) Given a mock modular form f, the harmonic Maass form $\widehat{f}$ of which f is the holomorphic part, is called the *completion* of f.

(iv) We refer to a mock modular form of weight $1/2$ or $3/2$ whose shadow is a linear combination of unary theta functions as a *mock theta function.*

Remark 4.24. If $f \in H^!_{2-k}(\Gamma_0(N))$ then we refer to the modular form $\xi_{2-k}(f)$ as the *shadow* of f.

Given a mock modular form and its shadow, the corresponding nonholomorphic part can be determined using next Theorem.

Theorem 4.25. *Let $f \in H_{2-k}(\Gamma_0(N))$ and suppose the mock modular form f^+ has shadow $g(\tau) = \sum_{n=1}^{\infty} c_g(n) q^n \in S_k(\Gamma_0(N))$. Then the nonholomorphic part f^- satisfies*

$$f^-(\tau) = 2^{1-k} i \int_{-\overline{\tau}}^{i\infty} \frac{g^c(\tau)}{(-i(\omega+\tau))^{2-k}} d\omega$$

where $g^c(\tau) := \overline{g(-\overline{\tau})} = \sum_{n=1}^{\infty} \overline{c_g(n)} q^n$.

Proof. First observe that

$$i(2\pi n)^{1-k}\Gamma(k-1,4\pi nv)q^{-n} = i(2\pi n)^{1-k}\left(\int_{4\pi nv}^{\infty} e^{-t}t^{k-2}dt\right)e^{-2\pi in\tau}.$$

Substitute $\omega = \frac{it}{2\pi n}$ then $d\omega = \frac{i}{2\pi n}dt$. We get

$$\begin{aligned}
i(2\pi n)^{1-k}\Gamma(k-1,4\pi nv)q^{-n} &= i(2\pi n)^{1-k}\left[\int_{2iv}^{i\infty} e^{2\pi in\omega}\left(\frac{2\pi n}{i}\omega\right)^{k-2}\left(\frac{2\pi n}{i}\right)d\omega\right]e^{-2\pi in\tau}\\
&= \frac{1}{i^{k-2}}\int_{2iv}^{i\infty} e^{2\pi in(\omega-\tau)}\omega^{k-2}d\omega\\
&= \int_{2iv}^{i\infty}\frac{e^{2\pi in(\omega-\tau)}}{(-i\omega)^{2-k}}d\omega\\
&= \int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi in\omega}}{(-i(\omega+\tau))^{2-k}}d\omega
\end{aligned}$$

where the last equality comes by replacing $\omega-\tau$ by ω. Now we have

$$\begin{aligned}
2^{1-k}i\int_{-\overline{\tau}}^{i\infty}\frac{g^c(\tau)}{(-i(\omega+\tau))^{2-k}}d\omega &= 2^{1-k}i\int_{-\overline{\tau}}^{i\infty}\frac{\sum_{n=1}^{\infty}\overline{c_g(n)}e^{2\pi in\omega}}{(-i(\omega+\tau))^{2-k}}d\omega\\
&= 2^{1-k}i\sum_{n=1}^{\infty}\overline{c_g(n)}\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi in\omega}}{(-i(\omega+\tau))^{2-k}}d\omega\\
&= 2^{1-k}i\sum_{n=1}^{\infty}\overline{c_g(n)}i(2\pi n)^{1-k}\Gamma(k-1,4\pi nv)q^{-n}.
\end{aligned}$$

By Theorem 4.19 (ii), we have $c_g(n) = -(4\pi)^{k-1}\overline{c_f^-(-n)}n^{k-1}$. Thus we have

$$2^{1-k} i \int_{-\overline{\tau}}^{i\infty} \frac{g^c(\tau)}{(-i(\omega+\tau))^{2-k}} d\omega = \sum_{n=1}^{\infty} c_f^-(-n)\Gamma(k-1, 4\pi n v)q^{-n}$$
$$= \sum_{n<0} c_f^-(n)\Gamma(k-1, -4\pi n v)q^n = f^-(\tau).$$

□

4.5 Examples

We will discuss two examples of harmonic Maass forms.

The Eisenstein series $E_2^*(\tau)$

We have the Eisenstein series

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n.$$

Definition 4.26. (The Eisenstein series of weight 2) For $\tau \in \mathbb{H}$, the non-holomorphic weight-2 Eisenstein series $E_2^*(\tau)$ is defined by

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi v} \quad \text{where} \quad \tau = u + iv.$$

Theorem 4.27. *The non-holomorphic weight-2 Eisenstein series $E_2^*(\tau)$ is a harmonic Maass form of weight 2 for the group $SL_2(\mathbb{Z})$. Moreover the mock modular form $E_2(\tau)$ has shadow $3/\pi$.*

Proof. Let us first show modularity. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then by Theorem 3.28 we have

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{6c(c\tau+d)}{\pi i}.$$

Using this transformation rule, we have

$$E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = E_2\left(\frac{a\tau+b}{c\tau+d}\right) - \frac{3}{\pi Im(\gamma\tau)}$$
$$= (c\tau+d)^2E_2(\tau) + \frac{6c(c\tau+d)}{\pi i} - \frac{3|c\tau+d|^2}{\pi v}$$

$$= (c\tau+d)^2\left[E_2(\tau) - \frac{6ic}{\pi(c\tau+d)} - \frac{3(c\overline{\tau}+d)}{\pi v(c\tau+d)}\right]$$
$$= (c\tau+d)^2\left[E_2(\tau) - \frac{1}{(c\tau+d)}\left(\frac{6icv+3cu-3civ+3d}{\pi v}\right)\right]$$
$$= (c\tau+d)^2\left[E_2(\tau) - \frac{3}{\pi v}\right] = (c\tau+d)^2E_2^*(\tau).$$

Next, observe that

$$\Delta_2(E_2^*(\tau)) = \Delta_2(E_2(\tau)) - \Delta_2(\frac{3}{\pi v}) = 0 + v^2\frac{\partial^2}{\partial v^2}\frac{3}{\pi v} + -2ivi\frac{\partial}{\partial v}\frac{3}{\pi v}$$
$$= 2v^2\frac{3}{\pi v^3} - 2v\frac{3}{\pi v^2} = 0.$$

since $E_2(\tau)$ is holomorphic on $\mathbb{H}$. Moreover observe that $\lim_{v\to\infty} E_2^*(\tau) = 1 = O(e^{\epsilon v})$ for any $\epsilon > 0$. Thus $E_2^*(\tau) \in H_2^!(SL_2(\mathbb{Z}))$. Lastly we need to check that $\xi_2(E_2^*(\tau)) = \frac{3}{\pi}$. Indeed

$$\xi_2(E_2^*(\tau)) = \xi_2\left(-\frac{3}{\pi v}\right) = -2iv^2\overline{\frac{\partial}{\partial\overline{\tau}}\frac{3}{\pi v}} = iv^2i\frac{\partial}{\partial v}\frac{3}{\pi v} = \frac{3}{\pi}.$$

$\square$

Zagier's 3/2−weight nonholomorphic Eisenstein series

Let us first define the Hurwitz class number[26].

Definition 4.28. (Hurwitz class number) For integer $N \geq 0$, the Hurwitz class number is defined as follows : $H(0) = -\frac{1}{12}$. If $N \equiv 1$ or $2 (\mathrm{mod} 4)$ then $H(N) = 0$. Otherwise $H(N)$ is the number of classes of binary quadratic forms of discriminant $-N$, except that those classes which have a representative which is a multiple of $x^2 + y^2$ should be counted with weight 1/2 and those which have a representative which is a multiple of x^2+xy+y^2 should be counted with weight 1/3.

Theorem 4.29. *(Zagier) Let $H(n)$ be the Hurwitz class number. Then the function*

$$\mathcal{H}(\tau) \coloneqq -\frac{1}{12} + \sum_{n=1}^{\infty} H(n)q^n + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 v\right) q^{-n^2} + \frac{1}{8\pi\sqrt{v}}$$

where $\tau = u + iv$ is a $3/2-$weight harmonic Maass form of manageable growth on $\Gamma_0(4)$. Moreover we have $\xi_{3/2}(\mathcal{H}) = -\frac{1}{16}\Theta$ where $\Theta(\tau) \coloneqq \sum\limits_{n\in\mathbb{Z}} q^{n^2}$ is the classical theta function.

5. JACOBI FORMS AND SIEGEL MODULAR FORMS

Jacobi forms are holomorphic functions in two variables with some modularity conditions and elliptic property with respect to some modular group and some lattice respectively. We will discuss these concepts more precisely in a while. The motivation to study Jacobi forms comes from two different considerations. The first is a systematic study of Siegel modular forms which we will discuss in sections to come. Jacobi forms appear naturally in Siegel modular forms. Secondly, the study of the generating functions for the representation of numbers by quadratic forms motivates the idea of Jacobi forms. These generating functions were first studied by Jacobi and hence the name of the subject is derived. Most of the material in this chapter can be found in [8, 9].

5.1 Jacobi theta function

We will briefly discuss Jacobi theta function which will motivate the first definition of Jacobi forms. The proofs of the results discussed in this section can be found in section 6 of Chapter VII of [25].

Definition 5.1. (i) (Lattice) For n $\mathbb{R}-$linearly independent complex numbers $\omega_1, \omega_2, \ldots, \omega_n$, the additive group

$$L = L(\omega_1, \omega_2, \ldots, \omega_n) := \left\{ \sum_{i=1}^{n} x_i \omega_i : \; x_i \in \mathbb{Z} \right\}$$

is called the *lattice* of rank n generated by $\omega_1, \omega_2, \ldots, \omega_n$.

(ii) (Fundamental Parallelepiped) Let L be a lattice of rank n. Then the set

$$\mathcal{P}(L) := \left\{ \sum_{i=1}^{n} x_i \omega_i : \ 0 \leq x_i < 1 \right\}$$

is called the *fundamental parallelepiped*.

(iii) (Determinant of lattice) The *determinant* $\det(L)$ of the lattice L is defined as the volume of the fundamental parallelepiped $\mathcal{P}(L)$. L is called *unimodular* if $\det(L) = 1$.

(iv) (Dual lattice) Given a lattice L and a quadratic form $(.,.) : L \times L \longrightarrow \mathbb{Z}$, its *dual lattice* L^* is defined as

$$L^* := \{\omega \in L : (\omega, v) \in \mathbb{Z} \ \forall \ v \in L\}.$$

L is called *self-dual* if $L^* = L$.

Theorem 5.2. *Given a lattice L, it is self-dual if and only if it is unimodular if and only if its rank is divisible by* 8.

Definition 5.3. A lattice L is called an integral, even, positive-definite quadratic lattice if

(i) $L \cong \mathbb{Z}^n; \ n > 0$.

(ii) For a quadratic form $(.,.) : L \times L \longrightarrow \mathbb{Z}$, $(v, v) \in 2\mathbb{Z} \ \forall \ v \in L$.

(iii) $(v, v) > 0 \ \forall \ v \in L \setminus \{0\}$.

Put

$$r_L(2n) = \#\{v \in L : (v, v) = 2n\}.$$

Then for every $n \in \mathbb{Z}$, $r_L(2n) < \infty$.

Definition 5.4. Given an integral, positive-definite, even quadratic lattice L and $\tau \in \mathbb{H}$, define the *theta function* for L as

$$\vartheta_L(q) := \sum_{n \geq 0} r_L(2n) q^n; \quad q = e^{2\pi i n \tau}. \tag{5.1}$$

With this definition we have

$$\vartheta_L(q) = \sum_{n\geq 0} e^{\pi i(v,v)\tau} = \vartheta_L(\tau).$$

Theorem 5.5. *The theta function $\vartheta_L(\tau)$ defines a holomorphic function on $\mathbb{H}$. Moreover it satisfies the following transformation property:*

(i) $\vartheta_L(\tau+1) = \vartheta_L(\tau)$.

(ii) $\vartheta_L\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{n/2}(\det(L))^{-1/2}\vartheta_{L^*}(\tau)$.

Theorem 5.5 along with (5.1) shows that $\vartheta_L(\tau)$ is a modular form for $SL_2(\mathbb{Z})$ of weight $n/2$ if L is self-dual.
For $m \in \mathbb{Z},\ m > 0$, let $u \in L$ be such that $(u,u) = 2m$. For $n, l \in \mathbb{Z}$, put

$$r_{L,u}(n,l) := \#\{v \in L : (v,v) = 2n;\ (v,u) = l\}.$$

Observe that $r_{L,u}(n,l) = 0 \Leftrightarrow (4nm - l^2) < 0$. To see this let $v \in L$ be such that $(v,v) = 2n$ and $(v,u) = l$. Consider the restriction of the quadratic form $(.,.)|_{L'} : L' \longrightarrow L'$ where $L' :=$Span$\{u,v\}$. Then the matrix of the quadratic form $(.,.)|_{L'}$ is

$$\begin{pmatrix} (v,v) & (v,u) \\ (u,v) & (u,u) \end{pmatrix} = \begin{pmatrix} 2n & l \\ l & 2m \end{pmatrix}.$$

Since the quadratic form is positive-definite, its determinant $(4nm - l^2) \geq 0$. So if $(4nm - l^2) < 0$ then $r_{L,u}(n,l) = 0$.

Definition 5.6. Given an integral, positive-definite, even quadratic lattice $L, \tau \in \mathbb{H}, z \in \mathbb{C}, m \in \mathbb{Z},\ m > 0$ and $u \in L$ such that $(u,u) = 2m$, define the *Jacobi theta function* for L as

$$\vartheta^m_{L,u}(\tau,z) := \sum_{v\in L} e^{\pi i((v,v)\tau + 2(v,u)z)} = \sum_{\substack{n,l\in\mathbb{Z} \\ (4nm-l^2)\geq 0}} r_{L,u}(n,l)e^{2\pi i(n\tau + lz)}.$$

The following result is Theorem 7.1 of [8].

Theorem 5.7. *The Jacobi theta function $\vartheta^m_{L,u}(\tau, z)$ defines a holomorphic function on $\mathbb{H} \times \mathbb{C}$. Moreover if L is self-dual, even rank lattice then $\vartheta^m_{L,u}(\tau, z)$ satisfies the following transformation property:*

(i) $\vartheta^m_{L,u}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^{n/2} e^{2\pi i m \frac{cz^2}{c\tau+d}} \vartheta^m_{L,u}(\tau, z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $\tau \in \mathbb{H}, z \in \mathbb{C}$.

(ii) $\vartheta^m_{L,u}(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \vartheta^m_{L,u}(\tau, z), \quad \forall (\lambda, \mu) \in \mathbb{Z}^2$.

This kind of a transformation property is the essence of Jacobi forms. We now define Jacobi forms precisely.

5.2 First definition of Jacobi forms

Definition 5.8. Let $k, m \in \mathbb{Z},\ m \geq 0$. Let $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function which satisfies the following properties:

(i)

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e\left(\frac{mcz^2}{c\tau+d}\right)\varphi(\tau, z) \ \ \forall \tau \in \mathbb{H}, z \in \mathbb{C}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{5.2}$$

where $e(t) = e^{2\pi i t}$. This is called *modular invariance.*

(ii)

$$\varphi(\tau, z + \lambda\tau + \mu) = e(-m(\lambda^2\tau + 2\lambda z))\varphi(\tau, z) \ \ \forall \lambda, \mu \in \mathbb{Z} \tag{5.3}$$

This is called *Elliptic invariance.*

(iii) Using $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in (i) gives $\varphi(\tau+1, z) = \varphi(\tau, z)$ and Using $(\lambda, \mu) = (0, 1)$ in (ii) gives $\varphi(\tau, z+1) = \varphi(\tau, z)$. Thus we get a Fourier expansion of the form

$$\varphi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n \zeta^r$$

where $q = e^{2\pi i\tau}, \zeta = e^{2\pi iz}$. φ is called a holomorphic (cusp or weak) Jacobi form of weight k and index m if

$$\begin{aligned} c(n,r) = 0 \quad \text{unless} \quad & (4nm - r^2) \geq 0 && \text{(holomorphic)} \\ & (4nm - r^2) > 0 && \text{(cusp)} \\ & n \geq 0 && \text{(weak)} \\ & n \geq n_0 \text{ (possibly negative)} && \text{(weakly holomorphic).} \end{aligned} \tag{5.4}$$

Notations 5.9. We denote by $\mathbb{J}_{k,m}, \mathbb{J}^{\text{cusp}}_{k,m}, \mathbb{J}^{\text{w}}_{k,m}$ and $\mathbb{J}^{!\text{w}}_{k,m}$ the space of holomorphic, cusp, weak and weakly holomorphic Jacobi forms of weight k and index m respectively. If we allow φ to have poles in $\mathbb{H} \times \mathbb{C}$ and at $i\infty$ then we call φ as *meromorphic Jacobi form.*

Corollary 5.10. For a self-dual lattice L, the Jacobi theta function in definition 5.6 is a holomorphic Jacobi form of weight $n/2$ and index m.

Let $\varphi(\tau, z) \in \mathbb{J}_{k,m}$ and put $z = 0$. By Definition 5.6 we get

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, 0\right) = (c\tau + d)^k \varphi(\tau, 0) \;\; \forall \tau \in \mathbb{H}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$$\varphi(\tau, 0) = \sum_{n \in \mathbb{Z}} \left(\sum_{(4nm - r^2) \geq 0} c(n,r) \right) q^n$$

Since $m > 0$ thus $(4nm - r^2) \geq 0$ implies $n \geq 0$ ($n < 0 \implies (4nm - r^2) \leq 0$). Thus

$$\varphi(\tau, 0) = \sum_{n \geq 0} c(n) q^n; \quad c(n) = \sum_{\substack{n \in \mathbb{Z} \\ (4nm - r^2) \geq 0}} c(n,r)$$

Thus $\varphi(\tau, 0) \in M_k(SL_2(\mathbb{Z}))$. Conversely we ask the following question

Questions 5.11. Given $f \in M_k(SL_2(\mathbb{Z}))$, does there exist $\varphi(\tau, z) \in \mathbb{J}_{k,m}$ such that $\varphi(\tau, 0) = f$?

This question has a positive answer and we will deal with this question later. More generally, we ask the following question.

Questions 5.12. Given $\varphi(\tau, z) \in \mathbb{J}_{k,m}$, what modular properties does $\varphi\left(\tau, \frac{1}{2}\right)$, $\varphi\left(\tau, \frac{z+1}{2}\right), \varphi(\tau, q\tau + p)$ satisfy where $(q,p) \in \mathbb{Q}^2$?

To answer this question we need a more refined definition of Jacobi forms. Specifically we need to define the transformation conditions in terms of the stroke operator with respect to the *Jacobi modular group*. This is the content of the next section.

5.3 Second defintion of Jacobi forms

The definition using stroke operator is closely related to the theory of *Siegel modular forms*. We will study Siegel modular forms systematically in later sections. For now we will just need some terminology. Let us first define the *Siegel modular group* of *degree n*.

$$Sp_n(\mathbb{Z}) := \{g \in GL_{2n}(\mathbb{Z}) : gJg^t = J\}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ with I_n the $n \times n$ identity matrix. Define the following subgroup of $Sp_2(\mathbb{Z})$

$$\Gamma^J := \left\{ \begin{pmatrix} * & 0 & * & * \\ * & 1 & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_2(\mathbb{Z}) \right\}.$$

It can easily be checked that Γ^J is a subgroup of $Sp_2(\mathbb{Z})$. The group Γ^J is called the Jacobi modular group. $SL_2(\mathbb{Z})$ can be embedded in Γ^J as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^J.$$

Moreover

$$M = \begin{pmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^J \implies \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Define another subgroup of Γ^J, the *Heisenberg group* as follows:

$$H(\mathbb{Z}) := \left\{ \left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right] := \begin{pmatrix} 1 & 0 & 0 & p \\ -q & 1 & p & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^J \right\} < \Gamma^J.$$

Remark 5.13. Let

$$M = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^J \quad \text{then} \quad \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^{-1} M = \begin{pmatrix} 1 & 0 & 0 & p \\ -q & 1 & p & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H(\mathbb{Z}).$$

Thus $\Gamma^J = SL_2(\mathbb{Z}) \cdot H(\mathbb{Z})$.

Properties of the Heisenberg group

1. $\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right]\left[\begin{pmatrix} p' \\ q' \end{pmatrix}, r'\right] = \left[\begin{pmatrix} p+p' \\ q+q' \end{pmatrix}, r + r' + \det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\right]$. Thus $H(\mathbb{Z})$ is not commutative.
2. $\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right]^{-1} = \left[\begin{pmatrix} -p \\ -q \end{pmatrix}, -r\right]$.

3. Let $h = \left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right]$ and $h' = \left[\begin{pmatrix} p' \\ q' \end{pmatrix}, r'\right]$, then $hh'h^{-1}h'^{-1} = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\right]$. Thus the first commutator subgroup of the Heisenberg group is $[H(\mathbb{Z}), H(\mathbb{Z})] = \left\{\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 2r\right] : r \in \mathbb{Z}\right\}$.

4. The center of $H(\mathbb{Z})$ is $Z\left(H(\mathbb{Z})\right) = \left\{\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, r\right] : r \in \mathbb{Z}\right\}$.

5. $H(\mathbb{Z})$ is the central extension of $\mathbb{Z} \times \mathbb{Z}$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} H(\mathbb{Z}) \xrightarrow{g} \mathbb{Z} \times \mathbb{Z} \longrightarrow 0$$

where $f(r) = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, r\right]$ and $g\left(\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right]\right) = (p, q)$.

6. $SL_2(\mathbb{Z})$ acts on $H(\mathbb{Z})$ as follows:

$$M \in SL_2(\mathbb{Z}) : [M]\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right][M] = \left[M\begin{pmatrix} p \\ q \end{pmatrix}, r\right]$$

7. $H(\mathbb{Z}) \trianglelefteq \Gamma^J$ and $\Gamma^J = SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$.

8. The map $\nu_H : H(\mathbb{Z}) \longrightarrow \{\pm 1\}$ given by $\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right] \mapsto (-1)^{p+q+pq+r}$ is a character of the Heisenberg group. This is called the *binary character*. We have $\nu_H(ghg^{-1}) = \nu_H(h)\ \forall\ g \in SL_2(\mathbb{Z})$.

Let us now define the *Siegel upper half space.*

Definition 5.14. For $n \in \mathbb{Z}, n \geq 1$, define the Siegel upper half space of *genus n*

$$\mathbb{H}_n := \{Z = X + iY \in M_n(\mathbb{C}) : Z^t = Z, Y > 0\}$$

where $Y > 0$ means that the matrix Y is positive definite. In particular $\mathbb{H}_2 = \left\{\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} : \tau, \sigma \in \mathbb{H}; z \in \mathbb{C}, det(Im(\Omega)) > 0\right\}$.

We now define the action of the $Sp_n(\mathbb{Z})$ on $\mathbb{H}_n$. Before that, we need a Lemma, the proof of which is straightforward computation. Hence we will skip the proof.

Lemma 5.15. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_{2n}(\mathbb{Z})$ with $A, B, C, D \in M_n(\mathbb{Z})$. Then

(i) $M \in Sp_n(\mathbb{Z})$ if and only if $A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = I_n$ if and only if $AB^t = BA^t, CD^t = DC^t, AD^t - BC^t = I_n$.

(ii) $M^t \in Sp_n(\mathbb{Z})$ if $M \in Sp_n(\mathbb{Z})$.

(iii) $M^{-1} = \begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix} \in Sp_n(\mathbb{Z})$ if $M \in Sp_n(\mathbb{Z})$.

Using Lemma 5.15 we can prove the following Theorem:

Theorem 5.16. *For* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$ *and* $Z \in \mathbb{H}_n$*, the map* $\varphi : Sp_n(\mathbb{Z}) \times \mathbb{H}_n \longrightarrow \mathbb{H}_n$ *defined by*

$$(M, Z) \mapsto M \circ Z := (AZ + B)(CZ + D)^{-1}$$

defines a group action of $Sp_n(\mathbb{Z})$ *onto* $\mathbb{H}_n$*.*

Let us define the stroke operator for functions on $\mathbb{H}_n$. Let $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ be a function and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$. Define the weight-$k$ stroke operator as:

$$(f|_k M) := det(CZ + D)^{-k} f(M \circ Z). \tag{5.5}$$

It can be checked that this definition of stroke operator satisfies $(f|_k M_1 M_2) = (f|_k M_1)|_k M_2$. We would now like to study the action of Γ^J on $\mathbb{H}_2$ and the corresponding stroke operator. Let $Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$ and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Then we have

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \circ Z = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \begin{pmatrix} a\tau + b & az \\ z & \omega \end{pmatrix} \begin{pmatrix} c\tau + d & cz \\ 0 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} a\tau + b & az \\ z & \omega \end{pmatrix} \begin{pmatrix} \frac{1}{c\tau+d} & \frac{-cz}{c\tau+d} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{a\tau+b}{c\tau+d} & \frac{z}{c\tau+d} \\ \frac{z}{c\tau+d} & \omega - \frac{cz^2}{c\tau+d} \end{pmatrix}$$

For $h = \left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right] \in H(\mathbb{Z})$ we have

$$h \circ Z = \begin{pmatrix} 1 & 0 & 0 & p \\ -q & 1 & p & r \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \begin{pmatrix} \tau & \tau + p \\ q\tau + z + p & qz + \omega + r \end{pmatrix} \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \tau & z + q\tau + p \\ z + q\tau + p & q^2\tau + 2qz + pq + \omega + r \end{pmatrix}$$

Let $\varphi(\tau, z)$ be a function on $\mathbb{H} \times \mathbb{C}$ then $\exists\, \omega \in \mathbb{H}$ such that $\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$. Thus $\varphi(\tau, z)e^{2\pi i\omega m}$ is a function on $\mathbb{H}_2$. Applying the stroke operator of (5.5), we get

$$\left(\varphi(\tau, z)e^{2\pi i\omega m}\right)\big|_k[M] = (c\tau + d)^{-k} e^{-2\pi i m\frac{cz^2}{c\tau+d}} \varphi\left(\frac{a\tau + d}{c\tau + d}, \frac{z}{c\tau + d}\right) e^{2\pi i\omega m} \tag{5.6}$$

and

$$\left(\varphi(\tau, z)e^{2\pi i\omega m}\right)\big|_k h = e^{2\pi i m(q^2\tau + 2qz + pq + r)} \varphi(\tau, z + q\tau + p) e^{2\pi i\omega m}. \tag{5.7}$$

Using (5.6) and (5.7) we will now define the stroke operator for Jacobi forms.

Definition 5.17. Let $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}; k, m \in \mathbb{Z}, m \geq 0$, For $(\tau, z) \in \mathbb{H} \times \mathbb{C}\ \exists!\, \omega \in \mathbb{H}$ such that $\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$. For $g \in \Gamma^J$ define

$$\varphi|_{k,m} g := \left(\left(\varphi(\tau, z)e^{2\pi i\omega m}\right)\big|_k g\right) e^{-2\pi i\omega m}.$$

The stroke operator $|_{k,m}$ satisfies the following properties:

(i) $\varphi|_{k,m}g$ depends only on τ and z.

(ii) $(\varphi|_{k,m}g_1g_2) = (\varphi|_{k,m}g_1)|_{k,m}g_2$.

We will now give the second definition of Jacobi forms. First observe that the transformation (i) in Definition 5.8 is equivalent to

$$\varphi|_{k,m}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \varphi(\tau, z) \;\forall\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

and the transformation (ii) in Definition 5.8 is equivalent to

$$\varphi|_{k,m}\left[\begin{pmatrix} p \\ q \end{pmatrix}, r\right] = \varphi(\tau, z) \;\forall\; (p, q) \in \mathbb{Z}^2.$$

Since $\Gamma^J = SL_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$, thus these transformations are equivalent to $\varphi|_{k,m}g = \varphi \;\forall\; g \in \Gamma^J$. Further, we can translate the Fourier expansion for holomorphic Jacobi form in (5.4) in terms of the variable $Z \in \mathbb{H}_2$ as follows:

$$\varphi(\tau, z)e^{2\pi i m\omega} = \sum_{T \geq 0} a_f(T)e^{2\pi i \;\mathrm{trace}(TZ)} \tag{5.8}$$

where the sum runs over all positive semi-definite matrices of the form $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$; $n, r \in \mathbb{Z}$. To see this observe that for $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$, we have $e^{2\pi i \;\mathrm{trace}(TZ)} = e^{2\pi i(n\tau + rz + m\omega)} = q^n\zeta^r e^{2\pi i m\omega}$ and $T \geq 0 \Leftrightarrow n, m \geq 0$ and $nm - r^2/4 \geq 0$. Thus using (5.4) for holomorphic Jacobi form, we get (5.8) with $a_f(T) = c(n, r)$.

Definition 5.18. A holomorphic function $\varphi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a holomorphic Jacobi form of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}, m \geq 0$ if $\varphi|_{k,m}g = \varphi \;\forall\; g \in \Gamma^J$ and φ has a Fourier expansion of the form (5.8).

The proof of the following Theorem is a trivial exercise using elliptic invariance of φ.

Theorem 5.19. *Let φ be an Jacobi (holomorphic, weak or cusp) form of index m and weight-k with Fourier expansion $\varphi(\tau, z) = \sum_{n,r} c(n, r)q^n\zeta^r$. Then $c(n, r)$ depends* *only on $4mn - r^2$ and $r(\text{mod } 2m)$. If k is even and $m = 1$ or m is prime, then $c(n, r)$ depends only on $4mn - r^2$. If $m = 1$ and k odd, then ϕ is identically zero.*

Remark 5.20. Theorem 5.19 shows that $c(n + mk^2 + k, r + 2mk) = c(n, r)$ for every $k \in \mathbb{Z}$. In particular, $c(n, r) = c(4mn - r^2, r)$.

5.4 Special values of Jacobi forms

We will answer Question 5.12 in this section. Put $X = \begin{pmatrix} p \\ -q \end{pmatrix} \in \mathbb{Q}^2$ then

$$\varphi(\tau, z)|_{k,m}[X, 0] = e^{2\pi i m(q^2\tau + 2qz + pq)}\varphi(\tau, z + q\tau + p).$$

Put $z = 0$ then

$$\varphi_X(\tau) := (\varphi(\tau, z)|_{k,m}[X, 0])\,\big|_{z=0} = e^{2\pi i m(q^2\tau + pq)}\varphi(\tau, q\tau + p).$$

Put

$$\Gamma_X := \left\{M \in SL_2(\mathbb{Z}) : \; MX \equiv X(\text{mod } \mathbb{Z}^2)\right\}$$

where for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $MX \equiv X(\text{mod } \mathbb{Z}^2)$ means $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p \\ -q \end{pmatrix} = \begin{pmatrix} ap - bq \\ cp - dq \end{pmatrix} \equiv \begin{pmatrix} p \\ -q \end{pmatrix}(\text{mod } \mathbb{Z}^2) \implies ap - bq - p, cp - dq + q \in \mathbb{Z}$. It can be checked that Γ_X is a congruence subgroup of $SL_2(\mathbb{Z})$ with level N =minimum positive integer such that $NX \in \mathbb{Z}^2$.

Theorem 5.21. *Let $\varphi \in \mathbb{J}_{k,m}, X = \begin{pmatrix} p \\ -q \end{pmatrix} \in \mathbb{Q}^2$ then the function $\varphi_X(\tau) = e^{2\pi i m(q^2\tau + pq)}\varphi(\tau, q\tau + p)$ is a modular form of weight k with respect to Γ_X with a character $\chi_X(M) = e^{2\pi i \ \det(MX, X)}$ where MX and X are the columns of the matrix (MX, X).*

Proof. We analyse the modular behaviour of $(\varphi|_{k,m}[X, 0])\,\big|_{k,m}[M]$ for $M \in SL_2(\mathbb{Z})$.

Observe that

$$
\begin{aligned}
&[X,0][M] = [M]\left([M]^{-1}[X,0][M]\right) \quad \text{and} \quad \varphi|_{k,m}[M] = \varphi \\
&\implies (\varphi|_{k,m}[X,0])\,\big|_{k,m}[M] = (\varphi|_{k,m}[M])\,\big|_{k,m}\left([M]^{-1}[X,0][M]\right) \\
&= \varphi\big|_{k,m}\left([M^{-1}X,0]\right) \quad \forall\ M \in SL_2(\mathbb{Z}).
\end{aligned}
$$

where we used Property 6 of the Heisenberg group. Now, for $M \in \Gamma_X, MX - X \in \mathbb{Z}^2$, thus
$\varphi|_{k,m}[M^{-1}X - X, 0] = \varphi$. Observe that

$$
\begin{aligned}
[M^{-1}X,0] &= M^{-1}X,0][X,0]^{-1}[X,0] = [M^{-1}X - X, -det(M^{-1}X,X)][X,0] \\
&= [M^{-1}X - X, 0][0, -det(M^{-1}X,X)][X,0].
\end{aligned}
$$

Thus

$$\varphi_{k,m}[M^{-1}X,0] = e^{2\pi i det(MX,X)}\varphi|_{k,m}[X,0].$$

Thus we get

$$(\varphi|_{k,m}[X,0])\,\big|_{k,m}[M] = \chi_X(M)\varphi|_{k,m}[X,0] \ \ \forall\ M \in \Gamma_X \tag{5.9}$$

where $\chi_X(M) = e^{2\pi i det(MX,X)}$ is a character of the group Γ_X. The modular equation (5.9) for $z = 0$ gives

$$\varphi_X(\tau)|_k M = \chi_X(M)\varphi_X \ \ \forall\ M \in \Gamma_X.$$

We now analyse the Fourier expansion of φ_X.

$$
\begin{aligned}
\varphi_X(\tau) &= e^{2\pi i m(q^2\tau + pq)}\varphi(\tau, q\tau + p) \\
&= e^{2\pi i m(q^2\tau + pq)} \sum_{(4nm - r^2) \geq 0} c(n,r) e^{2\pi i(n\tau + r(q\tau + p))} \\
&= e^{2\pi i mpq} \sum_{(4nm - r^2) \geq 0} c(n,r) e^{2\pi i rp} e^{2\pi i(mq^2 + rq + n)\tau}
\end{aligned}
$$

but for $Q(x) = mx^2 + rx + n$, the discriminant is $D = r^2 - 4mn \leq 0$. Thus $Q(x) \geq 0$ as $m \geq 0$. Thus $mq^2 + rq + n \geq 0$. Thus φ_X is holomorphic at $i\infty$. We further need to analyse the Fourier expansion of φ_X at other cusps of Γ_X. Observe that for $M \in SL_2(\mathbb{Z})$,

$$\varphi_X|_k M = (\varphi|_{k,m}[X,0])\,\big|_{k,m}[M]\big|_{z=0} = \left(\varphi_{k,m}[M^{-1}X,0]\right)\big|_{z=0}.$$

Put $X' = M^{-1}X = \begin{pmatrix} p' \\ q' \end{pmatrix} \in \mathbb{Q}^2$. Then we just need to analyse Fourier expansion of $\varphi_{X'}$ at $i\infty$ and similar calculation shows that

$$(\varphi_X|_k M)(\tau) = \sum_{n\geq 0} a(n)q^n.$$

Thus φ_X is holomorphic at all cusps of Γ_X. □

Remark 5.22. The analysis of the Fourier expansion of φ_X in above Theorem shows that $c(n,r) = 0$ unless $(4nm - r^2) \geq 0 \Leftrightarrow \forall\ X \in \mathbb{Q}^2, \varphi_X(\tau)$ is holomorphic at $i\infty$.

5.5 The zeros of elliptic functions

In this section we will quantify the zeros of functions which satisfy elliptic invariance. This in turn will help in proving the finite dimensionality of the space of holomorphic Jacobi forms. For $\tau \in \mathbb{C}$, let $\mathbb{Z}\tau + \mathbb{Z}$ denote the lattice generated by 1 and τ.

Theorem 5.23. *Let $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. For $m \in \mathbb{Z}$ and $\tau \in \mathbb{H}$, suppose $\varphi(z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\varphi(z)\ \ \forall \lambda,\mu \in \mathbb{Z}$. If $\varphi \not\equiv 0$ then $\varphi(z)$ has exactly $2m$ zeros in any fundamental parallelepiped of $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$.*

Proof. Choose the fundamental parallelepiped $D_{z_0} := \{z_0 + x + iy :\ x, y \in [0,1)\}$ such that φ does not vanish on ∂D_{z_0}. Then by argument principle we get

$$\frac{1}{2\pi i}\int\limits_{\partial D_{z_0}} \frac{\varphi'(z)}{\varphi(z)}dz = \text{the number of zeros of } \varphi(z).$$

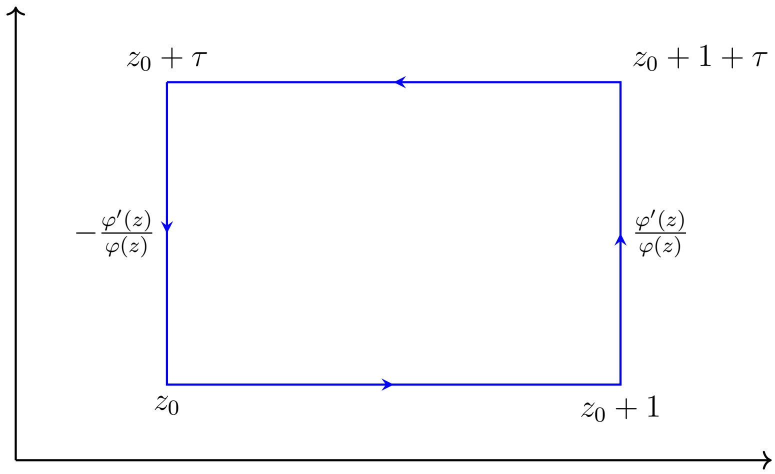


Fig. 5.1: Contour for integral

By evaluating the integral along each side of the contour we get

$$\frac{1}{2\pi i}\int\limits_{\partial D_{z_0}} \frac{\varphi'(z)}{\varphi(z)}dz = 0 - \frac{(-4\pi i m)}{2\pi i} = 2m.$$

□

Remark 5.24. (i) If φ is meromorphic then $2m$ = number of zeros – the number of poles.

(ii) If φ is holopmorphic then $2m \geq 0$. So $m \geq 0$. This is why we define Jacobi forms only for non-negative index.

(iii) For $\varphi \in \mathbb{J}^{\mathrm{w}}_{k,m}$, the map $z \mapsto \varphi(\tau, z)$ satisfies the hypothesises of Theorem 5.23. Thus $\varphi(\tau, z)$ has exactly $2m$ zeros in any fundamental parallelepiped of $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$.

Proposition 5.25. Let φ be a function as in Theorem 5.23. Then φ is determined upto a constant by its zeros.

Proof. Let φ and ψ have the same zeros. That is $\varphi(z_1) = \psi(z_1) = 0 \ldots \varphi(z_{2m}) = \psi(z_{2m}) = 0$. Then $\forall\ z_0 \notin \{z_1 \ldots z_{2m}\}$, $\varphi(z_0), \psi(z_0) \neq 0$. Consider the function $\varrho(z) = \psi(z_0)\varphi(z) - \varphi(z_0)\psi(z)$. Then ϱ again satisfies the hypothesises of Theorem

5.23 but has at least $2m+1$ zeros (including z_0). Then by Theorem 5.23 $\varrho \equiv 0$. Thus we get

$$\psi(z_0)\varphi(z) - \varphi(z_0)\psi(z) = 0 \implies \psi(z) = \frac{\psi(z_0)}{\varphi(z_0)}\varphi(z).$$

Thus any elliptic function of index m is determined by its zeros upto a constant. $\square$

Remark 5.26. For $\varphi \in \mathbb{J}^{\mathrm{w}}_{k,0}$, then $\varphi(\tau,z) - \varphi(\tau,0)$ has a zero at $z=0$. Thus by Theorem 5.23, $\varphi(\tau,z) = \varphi(\tau,0)$. But $\varphi(\tau,0) \in M_k$. Thus we proved that

$$\mathbb{J}_{k,0} \subset \mathbb{J}^{\mathrm{w}}_{k,0} = M_k \subset \mathbb{J}_{k,0}.$$

Theorem 5.27. *$\mathbb{J}_{k,m}$ is a finite dimensional vector space.*

Proof. It is easy to check that $\mathbb{J}^{\mathrm{w}}_{k,m}$ is a vector space. For the finite dimensionality part, choose distinct $X_i = \begin{pmatrix} p_i \\ -q_i \end{pmatrix} \in \mathbb{Q}^2$, $i = 1, 2, \ldots, 2m+1$ such that $X_i \not\equiv X_j (\mathrm{mod}\ \mathbb{Z}^2)$. By Theorem 5.21, the map

$$\Sigma : \varphi \in \mathbb{J}_{k,m} \longrightarrow \bigoplus_{i=1}^{2m+1} M_k(\Gamma_{X_i}, \chi_{X_i}); \quad \Sigma(\varphi(\tau,z)) = (\varphi_{X_i}(\tau))_{i=1}^{2m+1}$$

is well defined. Moreover we claim that this map is injective. To see this, suppose $\Sigma(\varphi_1(\tau,z)) = \Sigma(\varphi_2(\tau,z))$ then $\varphi_{1X_i}(\tau) = \varphi_{2X_i}(\tau)$. We claim that $\varphi_1(\tau,z) = \varphi_2(\tau,z)$. Indeed for a fixed $\tau \in \mathbb{H}$, there are $2m+1$ distinct zeros $z = p_i\tau + q_i$ of the elliptic function $\psi := \varphi_1(\tau,z) - \varphi_2(\tau,z)$. Thus by Theorem 5.23, we have that $\psi \equiv 0$ which implies $\varphi_1(\tau,z) = \varphi_2(\tau,z)$. Thus φ is determined uniquely by $(\varphi_{X_i}(\tau))_{i=1}^{2m+1}$. Thus

$$\dim \mathbb{J}_{k,m} \leq \sum_{i=1}^{2m+1} \dim M_k(\Gamma_{X_i}, \chi_{X_i}) < \infty$$

since $M_k(\Gamma_{X_i}, \chi_{X_i})$ is finite dimensional. $\square$

5.6 Taylor expansion of Jacobi forms

Let $\varphi \in \mathbb{J}_{k,m}$ then it has a Fourier expansion of the form

$$\varphi(\tau, z) = \sum_{\substack{n\in\mathbb{Z} \\ (4nm-r^2)\geq 0}} c(n,r)q^n\zeta^r.$$

We want to study the Taylor expansion in z:

$$\varphi(\tau, z) = \sum_{d\geq d_0\geq 0} f_d(\tau)z^d$$

where $\mathrm{ord}_{z=0}\varphi(\tau, z) = d_0$ is the order of the zero of φ at $z = 0$. We have

$$f_0(\tau) = \varphi(\tau, 0) \in M_k(SL_2(\mathbb{Z})).$$

In general, we have $f_d(\tau + 1) = f_d(\tau)$ using $\varphi(\tau + 1, z) = \varphi(\tau, z)$. This gives the Fourier expansion

$$f_d(\tau) = \sum_{n\geq 0} c(n)q^n.$$

In fact, we have a more specific Fourier expansion as in next Lemma.

Lemma 5.28. If $d > 0$ then $f_d(\tau) = \sum\limits_{n>0} c(n)q^n$ for some coefficients $c(n)$.

Proof. If $n = 0$ then $c(n, r) = 0 \;\forall\; r \neq 0$ since $4mn - r^2 < 0$ in this case. Thus any power of z^d; $d > 0$ in the Fourier expansion of φ coming from $\zeta^r = e^{2\pi irz}$ comes with a positive power of q. □

f_d is not a modular form in general for $d > 0$.

Proposition 5.29. $\varphi \in \mathbb{J}_{k,m}$ is determined uniquely by its first $2m + 1$ Taylor coefficients $f_0(\tau), \ldots, f_{2m}(\tau)$.

Proof. Suppose φ and ψ have the same first $2m + 1$ Taylor coefficients. Then $\mathrm{ord}_{z=0}(\varphi - \psi) \geq 2m + 1$ and index of $\varphi - \psi$ is m. Thus by Proposition 5.25, $\varphi - \psi$ has atmost $2m$ zeros. But

$$\varphi(\tau, z) - \psi(\tau, z) = \sum_{d\geq 2m+1} (f_d^{\varphi} - f_d^{\psi})(\tau)z^d = z^{2m+1}\sum_{d\geq 0}(f_{d+2m+1}^{\varphi} - f_{d+2m+1}^{\psi})(\tau)z^d.$$

Thus the zero $z = 0$ has multiplicity atleast $2m+1$. Thus we must have $\varphi(\tau, z) = \psi(\tau, z)$. □

Proposition 5.30. Let $\varphi \in \mathbb{J}_{k,m}$ then it has Taylor expansion of the shape

$$\varphi(\tau, z) = \sum_{\substack{d \equiv k (\text{mod } 2) \\ d \geq 0}} f_d(\tau) z^d.$$

Proof. For $-I \in SL_2(\mathbb{Z})$, the modularity gives $\varphi(\tau, -z) = (-1)^k \varphi(\tau, z)$. The Taylor expansion of $\varphi(\tau, z)$ has form

$$\varphi(\tau, z) = \sum_{d \geq 0} f_d(\tau) z^d.$$

This gives

$$\varphi(\tau, -z) = \sum_{d \geq 0} (-1)^d f_d(\tau) z^d.$$

Thus using $\varphi(\tau, -z) = (-1)^k \varphi(\tau, z)$, we get

$$\begin{aligned} &\sum_{d \geq 0} (-1)^d f_d(\tau) z^d = (-1)^k \sum_{d \geq 0} f_d(\tau) z^d \\ &\implies \sum_{d \geq 0} [(-1)^{d-k} - 1] f_d(\tau) z^d = 0 \\ &\implies d - k \in 2\mathbb{Z}. \end{aligned}$$

Thus we get the Taylor expansion of φ as

$$\varphi(\tau, z) = \sum_{\substack{d \equiv k (\text{mod } 2) \\ d \geq 0}} f_d(\tau) z^d.$$

□

Proposition 5.31. Let $\varphi \in \mathbb{J}_{k,m}$. Then $f_0(\tau) = \varphi(\tau, 0) \in M_k(SL_2(\mathbb{Z}))$. If $\varphi(\tau, 0) \equiv 0$ then $f_{d_0}(\tau) \in S_{k+d_0}(SL_2(\mathbb{Z}))$ where $\text{ord}_{z=0} \varphi(\tau, z) = d_0$.

Proof. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$\varphi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{mcz^2}{c\tau+d}} \varphi(\tau, z)$$

We substitute the Taylor expansion

$$\sum_{l\geq 0} f_l(M\tau)\frac{z^l}{(c\tau+d)^l} = (c\tau+d)^k e^{2\pi i m \frac{mcz^2}{c\tau+d}} \sum_{l\geq 0} f_l(\tau) z^l.$$

Comparing the d_0^{th} coefficient, we get $f_{d_0}(M\tau) = (c\tau+d)^{k+d_0} f_{d_0}(\tau)$. Thus $f_{d_0}(\tau) \in M_{k+d_0}(SL_2(\mathbb{Z}))$. Since $d_0 \neq 0$, thus by Lemma 5.28 we see that f_{d_0} is indeed a cusp form. □

Theorem 5.32. $\mathbb{J}_{2,1} = \{0\}$ *and* $\dim \mathbb{J}_{4,1}$, $\dim \mathbb{J}_{6,1}$, $\dim \mathbb{J}_{8,1} \leq 1$. *Moreover* $\dim \mathbb{J}_{6,3}^{cusp}$, $\dim \mathbb{J}_{10,1}^{cusp}$, $\dim \mathbb{J}_{8,2}^{cusp} \leq 1$.

Proof. Let $\varphi(\tau, z) \in \mathbb{J}_{2k,1}$ then using Proposition 5.30 we have

$$\varphi(\tau, z) = f_0(\tau) + f_2(\tau)z^2 + f_4(\tau)z^4 + \dots$$

where $f_0(\tau) \in M_{2k} = \mathbb{C}E_{2k}; 2k = 4, 6, 8, 10$. If $f_0(\tau) = 0$, then $f_2(\tau) \in S_{2k+2} = \{0\}; 2k+2 < 12$ as $S_{12} \neq \{0\}$. For weight $2k = 4, 6, 8$, $\varphi(\tau, z) \in \mathbb{J}_{2k,1}$ is determined by $f_0(\tau)$ by Proposition 5.29 and for all cases $f_0(\tau)$ is a multiple of E_{2k}. Thus $\dim \mathbb{J}_{2k,1} \leq 1$ for $2k = 4, 6, 8$. If $2k = 2$ then $M_{2k} = \{0\}$. Thus $f_0(\tau) = 0$ which means $f_2(\tau) \in S_4 = \{0\} \implies f_2(\tau) = 0$. Thus $\mathrm{ord}_{z=0}(\varphi) \geq 4$. Thus by Theorem 5.23 $\varphi \equiv 0$. Thus $\mathbb{J}_{2,1} = \{0\}$. Finally suppose $\varphi_{6,3} \in \mathbb{J}_{6,3}^{\mathrm{cusp}}$. Then

$$\varphi_{6,3}(\tau, z) = f_0(\tau) + f_2(\tau)z^2 + f_4(\tau)z^4 + f_6(\tau)z^6 + \dots$$

Then $f_0 \in S_6 = \{0\} \implies f_2 \in S_8 = \{0\} \implies f_4 \in S_{10} = \{0\}$. We know that $S_{12} = \mathbb{C}\Delta$ where Δ is the Ramanujan's cusp form. Moreover $\varphi_{6,3}$ is determined upto a constant by $f_6(\tau) = c\Delta(\tau)$. Thus $\dim \mathbb{J}_{6,3}^{\mathrm{cusp}} \leq 1$. Similar argument shows that $\dim \mathbb{J}_{10,1}^{\mathrm{cusp}}$, $\dim \mathbb{J}_{8,2}^{\mathrm{cusp}} \leq 1$. □

In later sections we will prove that $\dim \mathbb{J}_{6,3}^{\mathrm{cusp}} = \dim \mathbb{J}_{10,1}^{\mathrm{cusp}} = \dim \mathbb{J}_{8,2}^{\mathrm{cusp}} = 1$ by explicitly constructing examples.

5.7 Jacobi-ϑ series and examples of Jacobi forms

Let us first define the Jacobi theta series.

Definition 5.33. For $\tau \in \mathbb{H}, z \in \mathbb{C}$, the series

$$\vartheta(\tau, z) := \sum_{\substack{n \equiv 1 \pmod 2 \\ n \in \mathbb{Z}}} (-1)^{\frac{n-1}{2}} e^{\pi i \left(\frac{n^2}{4}\tau + nz\right)} = \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8} \zeta^{n/2}$$

where $q = e^{2\pi i \tau}, \zeta = e^{2\pi i z}$.

Using the Jacobi triple product formula

$$\prod_{n=0}^{\infty} (1 - q^{2n+2})(1 + q^{2n+1}x)\left(1 + \frac{q^{2n+1}}{x}\right) = \sum_{n \in \mathbb{Z}} q^{n^2} x^n$$

one can prove that

$$\vartheta(\tau, z) = -q^{1/8}\zeta^{-1/2} \prod_{n=1}^{\infty} (1 - q^{n-1}\zeta)(1 + q^n \zeta^{-1})(1 - q^n).$$

The next Theorem shows that the Jacobi theta function defines a holomorphic function on $\mathbb{H} \times \mathbb{C}$.

Theorem 5.34. *The Jacobi theta function $\vartheta(\tau, z)$ converges absolutely and uniformly on compact subsets of $\mathbb{H} \times \mathbb{C}$.*

Proof. Let $|Im(z)| < c$ and $Im(\tau) > \varepsilon$ then we have

$$\left| e^{\pi i \left(\frac{n^2}{4}\tau + nz\right)} \right| < e^{\pi n c - \pi \varepsilon \frac{n^2}{4}} = e^{\pi n \left(c - \varepsilon \frac{n_0}{4}\right)} e^{\pi \varepsilon n \frac{(n-n_0)}{4}} < \left(e^{-\pi\varepsilon}\right)^{\frac{n(n-n_0)}{4}}$$

where $n_0 > 4c/\varepsilon$. Now absolute and uniform convergence follows by Weirstrass M test. $\square$

Lemma 5.35. $\vartheta(\tau, 0) = 0$ for any $\tau \in \mathbb{H}$.

Proof. First observe that

$$\begin{aligned}\vartheta(\tau, -z) = \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8} \zeta^{-n/2} &= \sum_{n \in \mathbb{Z}} \left(\frac{-4}{-n}\right) q^{n^2/8} \zeta^{n/2} \\ &= -\sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) q^{n^2/8} \zeta^{n/2} = -\vartheta(\tau, z)\end{aligned}$$

since $\left(\frac{-4}{-n}\right) = -\left(\frac{-4}{-n}\right)$. Putting $z = 0$, we get $\vartheta(\tau, 0) = -\vartheta(\tau, 0) \implies \vartheta(\tau, 0) = 0$. $\square$

Theorem 5.36. *For $\lambda, \mu \in \mathbb{Z}$, we have*

$$\vartheta(\tau, z + \lambda\tau + \mu) = (-1)^{\lambda+\mu} e^{-\pi i(\lambda^2\tau + 2\lambda z)} \vartheta(\tau, z).$$

This is called quasi-periodicity of the ϑ function.

Proof. We directly substitute series expansion on L.H.S. We get

$$\sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) e^{\pi i\left(\frac{n^2}{4}\tau + n(z + \lambda\tau + \mu)\right)}.$$

Observe that $\left(\frac{n^2}{4}\tau + n(z + \lambda\tau + \mu)\right) = \left(\frac{n}{2} + \lambda\right)^2 \tau - \lambda^2\tau + z(n + 2\lambda) - 2\lambda z$. Put $\left(\frac{n}{2} + \lambda\right) = \frac{N}{2}$. Then we get

$$\vartheta(\tau, z + \lambda\tau + \mu) = (-1)^{\mu} e^{-\pi i(\lambda^2\tau + \mu)} \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) e^{\pi i\left(\frac{N^2}{4}\tau + Nz\right)}.$$

But we also have that $\left(\frac{-4}{n}\right) = (-1)^{\lambda} \left(\frac{-4}{n+2\lambda}\right) = (-1)^{\lambda} \left(\frac{-4}{N}\right)$ and moreover N varies over all $\mathbb{Z}$. So we get

$$\begin{aligned}\vartheta(\tau, z + \lambda\tau + \mu) &= (-1)^{\lambda+\mu} e^{-\pi i(\lambda^2\tau + 2\lambda z)} \sum_{N \in \mathbb{Z}} \left(\frac{-4}{N}\right) e^{\pi i\left(\frac{N^2}{4}\tau + Nz\right)} \\ &= (-1)^{\lambda+\mu} e^{-\pi i(\lambda^2\tau + 2\lambda z)} \vartheta(\tau, z).\end{aligned}$$

$\square$

Proposition 5.37. $\vartheta(\tau, z) = 0$ if and only if $z \in \mathbb{Z}\tau + \mathbb{Z}$. Moreover the order of this zero is 1.

Proof. We will apply Theorem 5.23 to ϑ since it is an elliptic function of index $m = 1/2$. The additional factor in elliptic invariance does not bother us as in the proof of Theorem 5.23, we need the following equalities

$$\frac{\vartheta(\tau, z+1)}{\vartheta(\tau, z+1)} = \frac{-\vartheta_z(\tau, z)}{-\vartheta(\tau, z)} = \frac{\vartheta_z(\tau, z)}{\vartheta(\tau, z)},$$

$$\frac{\vartheta_z(\tau, z+\tau)}{\vartheta(\tau, z+\tau)} = \frac{-(\vartheta_z(\tau, z) - 2\pi i \vartheta(\tau, z))e^{-\pi i(\tau+z)}}{-\vartheta(\tau, z)e^{-\pi i(\tau+z)}} = \frac{\vartheta_z(\tau, z)}{\vartheta(\tau, z)} - 2\pi i.$$

Thus in any fundamental parallelepiped of $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$, there is only one zero. Thus order of any zero is one. Moreover by Lemma 5.35, $\vartheta(\tau, 0) = 0$. Thus we get our result. □

Proposition 5.38. $\frac{\partial \vartheta(\tau,z)}{\partial z}\Big|_{z=0} = 2\pi i \eta(\tau)^3$.

Proof. The result follows by directly using the definition of ϑ function and Euler's Theorem 3.48. □

We would now like to analyse the quasi-periodicity in terms of the Jacobi modular group. Since the index of the theta function is $1/2$, we will analyse $\vartheta(\tau, z)e^{\pi i \omega}$ where $\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2$. Observe that

$$\begin{aligned}\left(\vartheta(\tau, z)e^{\pi i \omega}\big|_k\right)\left[\begin{pmatrix} \mu \\ \lambda \end{pmatrix}, 0\right] &= e^{\pi i(\lambda^2\tau + 2\lambda z + \lambda\mu)}\vartheta(\tau, z + \lambda\tau + \mu)e^{\pi i \omega} \\ &= (-1)^{\lambda+\mu+\lambda\mu}\vartheta(\tau, z)e^{\pi i \omega}\end{aligned}$$

where we used Theorem 5.36. Thus we see that the initial factor $(-1)^{\lambda+\mu}$ is actually the binary character ν_H of the Heisenberg group.

Theorem 5.39. *For $\tau \in \mathbb{H}$ and $z \in \mathbb{C}$ we have*

$$\vartheta(\tau+1, z) = e^{\frac{\pi i}{4}}\vartheta(\tau, z) \text{ and } \vartheta\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -\sqrt{\frac{\tau}{i}}e^{\pi i \frac{z^2}{\tau}}\vartheta(\tau, z).$$

Proof. To prove the first relation, we just substitute the definition of theta function.

$$\vartheta(\tau+1,z) = \sum_{\substack{n\equiv 1 (\text{mod } 2)\\ n\in\mathbb{Z}}} (-1)^{\frac{n-1}{2}} e^{\pi i\left(\frac{n^2}{4}(\tau+1)+nz\right)} = e^{\frac{\pi i}{4}} \sum_{\substack{n\equiv 1 (\text{mod } 2)\\ n\in\mathbb{Z}}} (-1)^{\frac{n-1}{2}} e^{\pi i\left(\frac{n^2}{4}\tau+nz\right)}$$
$$= e^{\frac{\pi i}{4}}\vartheta(\tau,z).$$

We will prove the second relation without using Poisson summation formula. We will prove that

$$\vartheta\left(-\frac{1}{\tau},z\right) = -\sqrt{\frac{\tau}{i}}e^{\pi i z^2\tau}\vartheta(\tau,z\tau)$$

which for$z \mapsto \frac{z}{\tau}$ will give us the desired property. Put $\xi_\tau(z) = e^{\pi i z^2\tau}\vartheta(\tau,z\tau)$. Observe that

$$\xi_\tau(z+1) = e^{\pi i(z+1)^2\tau}\vartheta(\tau,z\tau+\tau) = (-1)e^{\pi i(z^2+2z+1)\tau}e^{-\pi i(2\tau z+\tau)}\vartheta(\tau,z\tau)$$
$$= -e^{\pi i z^2\tau}\vartheta(\tau,z\tau) = -\xi_\tau(z)$$

and

$$\xi_\tau\left(z-\frac{1}{\tau}\right) = e^{\pi i\left(z-\frac{1}{\tau}\right)^2\tau}\vartheta(\tau,z\tau-1) = (-1)e^{\pi i\left(z^2-2\frac{z}{\tau}+\frac{1}{\tau^2}\right)\tau}\vartheta(\tau,z\tau)$$
$$= -e^{\pi i z^2\tau}\vartheta(\tau,z\tau)e^{-\pi i\left(2z-\frac{1}{\tau}\right)} = -e^{-\pi i\left(2z-\frac{1}{\tau}\right)}\xi_\tau(z)$$

where we used the quasi-periodicity of the theta function twice. Hence $\xi_\tau(z)$ is quasi periodic function for the lattice $-\frac{1}{\tau}\mathbb{Z}+\mathbb{Z}$ with index $1/2$. Also $\vartheta\left(-\frac{1}{\tau},z\right)$ is quasi periodic for the lattice $-\frac{1}{\tau}\mathbb{Z}+\mathbb{Z}$ with index $1/2$. Moreover $\vartheta\left(-\frac{1}{\tau},z\right)$ and $\xi_\tau(z)$ have the same zero (at $z=0$). Thus by Proposition 5.25 we have

$$\vartheta\left(-\frac{1}{\tau},z\right) = c(\tau)\xi_\tau(z).$$

We now prove that $c(\tau) = -\sqrt{\frac{\tau}{i}}$. To do this we would like to differentiate the equation $\vartheta\left(-\frac{1}{\tau},z\right) = c(\tau)\xi_\tau(z)$ both sides with respect to z and evaluate it at

$z = 0$. We get

$$\begin{aligned}
&\frac{\partial}{\partial z}\vartheta\left(-\frac{1}{\tau}, z\right)\Big|_{z=0} = c(\tau)\frac{\partial}{\partial z}e^{\pi i z^2\tau}\vartheta(\tau, z\tau)\Big|_{z=0}\\
\implies &2\pi i\eta\left(-\frac{1}{\tau}\right)^3 = c(\tau)\left(2\pi i\tau e^{\pi i z^2\tau}\vartheta(\tau, z\tau)|_{z=0} + 2\pi i\tau e^{\pi i z^2\tau}\eta(\tau)^3\right)\\
\implies &\left(\sqrt{\frac{\tau}{i}}\right)^3 \eta(\tau)^3 = \tau c(\tau)\eta(\tau)^3\\
\implies &c(\tau) = -\sqrt{\frac{\tau}{i}}.
\end{aligned}$$

where we used Theorem 3.46, Lemma 5.35 and Proposition 5.38. □

Noting that $\nu_\eta\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)^3 = e^{\frac{\pi i}{4}}$ and $\nu_\eta\left(\begin{pmatrix}0 & -1\\ 0 & 1\end{pmatrix}\right)^3 = -1$ and using Theorem 5.39, following Corollary follows.

Corollary 5.40.

$$\vartheta\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \nu_\eta(M)^3(c\tau+d)^{1/2}e^{\pi i\left(\frac{cz^2}{c\tau+d}\right)}\vartheta(\tau, z) \;\; \forall \tau \in \mathbb{H}, z \in \mathbb{C},$$
$$M = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$$

and

$$\vartheta(\tau, z+\lambda\tau+\mu) = \nu_H(h)e^{-\pi i(\lambda^2\tau+2\lambda z)}\vartheta(\tau, z) \;\; \forall\; h = \left[\begin{pmatrix}\mu\\ \lambda\end{pmatrix}, r\right] H(\mathbb{Z}).$$

Finally we analyse the Fourier expansion of the Jacobi theta series. We have

$$\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}}\left(\frac{-4}{n}\right)q^{n^2/8}\zeta^{n/2}.$$

Note that $c(n, r) = c\left(\frac{n^2}{8}, \frac{n}{2}\right) = \left(\frac{-4}{n}\right)$. Thus $4mn - r^2$ for these Fourier coefficients is $4\left(\frac{n^2}{8}\right)\left(\frac{1}{2}\right) - \frac{n^2}{4} = 0$. Thus the Fourier expansion satisfies the holomorphicity condition. Thus using Property 8 of the Heisenberg group, we have the following Theorem.

Theorem 5.41. *The Jacobi theta series is a Jacobi modular form of weight* $1/2$ *and index* $1/2$ *with respect to the Jacobi modular group with charater* ν_H *and multiplier system* ν_η^3*.*

Using the Jacobi theta series, we can construct several examples of Jacobi forms.

Example 5.42. (i) $\varphi_{-2,1}(\tau,z) = \frac{\vartheta^2(\tau,z)}{\eta^6(\tau)} \in \mathbb{J}^{\text{weak}}_{-2,1}$

To see this, first observe that weight of $\varphi_{-2,1}(\tau,z)$ is $2\times\frac{1}{2} - 6\times\frac{1}{2} = -2$. Moreover $\eta(\tau)\neq 0$, thus $\varphi_{-2,1}(\tau,z)$ is holomorphic on $\mathbb{H}\times\mathbb{C}$. Moreover, the Fourier expansion of $\varphi_{-2,1}(\tau,z)$ is of the form

$$\varphi_{-2,1}(\tau,z) = (\zeta - 2 + \zeta^{-1}) + q(\dots) + \dots$$

Thus $\varphi_{-2,1}$ is a weak Jacobi form. In fact $\varphi_{-2,1}(\tau,z)$ generates $\mathbb{J}^{\text{weak}}_{-2,1}$.

(ii) $\varphi_{-1,\frac{1}{2}}(\tau,z) = \frac{\vartheta(\tau,z)}{\eta^3(\tau)}$ is a weak Jacobi form of index $1/2$ and weight -1 with a character ν_H. It has a Fourier expansion of the form

$$\varphi_{-1,\frac{1}{2}}(\tau,z) = (\zeta^{1/2} - \zeta^{-1/2}) + q(\dots) + \dots$$

(iii) $\varphi_{10,1}(\tau,z) = \eta^{18}(\tau)\vartheta^2(\tau,z) \in \mathbb{J}^{\text{cusp}}_{10,1}$

Checking weight and index is trivial. Next thing to check is that the character and multiplier system are trivial. Note that $\vartheta^2(\tau,z)$ comes with trivial character and multiplier system ν_η^6 and $\eta(\tau)^{18}$ comes with multiplier system ν_η^6. Since $\nu_\eta^{24} = 1$, $\varphi_{10,1}$ has trivial character. To check the cusp condition, note that $\varphi_{10,1}(\tau,z) = \Delta(\tau)\varphi_{-2,1}$. Since $\Delta(\tau)$ is a cusp form, thus we have $\varphi_{10,1}(\tau,z) \in \mathbb{J}^{\text{cusp}}_{10,1}$. By Theorem 5.32, we have $\dim \mathbb{J}^{\text{cusp}}_{10,1} \leq 1$. Thus $\mathbb{J}^{\text{cusp}}_{10,1} = \mathbb{C}\varphi_{10,1}$. We will give an alternative proof of this fact.

(iv) $\varphi_{8,2}(\tau,z) = \eta^{12}(\tau)\vartheta^4(\tau,z) \in \mathbb{J}^{\text{cusp}}_{10,1}$. Thus $\mathbb{J}^{\text{cusp}}_{8,2} = \mathbb{C}\varphi_{8,2}$ by Theorem 5.32.

(v) $\varphi_{6,3}(\tau,z) = (\eta(\tau)\vartheta(\tau,z))^6 \in \mathbb{J}^{\text{cusp}}_{6,3}$. Thus $\mathbb{J}^{\text{cusp}}_{6,3} = \mathbb{C}\varphi_{6,3}$ by Theorem 5.32.

(vi) $\varphi_{4,4}(\tau,z) = \vartheta^8(\tau,z) \in \mathbb{J}_{4,4}$.

Remark 5.43. In general we have the Fourier expansion of the above Jacobi forms as

$$\varphi_{k,1} = \sum_{n=0}^{\infty} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \leq 4mn}} C_k(4n - r^2) q^n \zeta^r, \qquad k = -2, 0, 10, 12. \tag{5.10}$$

for some coefficients $C_k(n)$. We have that $\varphi_{k,1}$ is a weak Jacobi form of weight k for $k = -2, 0$ and Jacobi forms for $k = 10, 12$ of index 1. In particular,

$$\varphi_{0,1} = \frac{\zeta^2 + 10\zeta + 1}{\zeta} + 2\frac{(\zeta - 1)^2(5\zeta^2 - 22\zeta + 5)}{\zeta^2} q + \ldots$$

Lemma 5.44. $\mathbb{J}^{\text{weak}}_{-2,1} = \mathbb{C}\varphi_{-2,1}$.

Proof. Let $\psi_{-2,1} \in \mathbb{J}^{\text{weak}}_{-2,1}$. Consider the Taylor expansion of $\psi_{-2,1}$

$$\psi_{-2,1}(\tau, z) = f_{-2}(\tau) + f_0(\tau) z^2 + \ldots$$

In this case too, we have that $f_{-2}(\tau) = \varphi_{-2,1}(\tau, 0) \in M_{-2} = \{0\}$. Thus by modifying Proposition 5.31 suitably to take care of the fact that $\varphi_{-2,1}$ is a weak Jacobi form, we get that $f_0(\tau) \in M_0(SL_2(\mathbb{Z})) = \mathbb{C}$. If $f_0(\tau) = 0$ then $\text{ord}_{z=0}\psi_{-2,1} \geq 4$. Thus by Theorem 5.23, $\psi_{-2,1} = 0$. We know that $\varphi_{-2,1} \neq 0$, thus the only zero of $\varphi_{-2,1}$ is at $z = 0$ of order 2. Now if $f_0(\tau) \neq 0$ then by Proposition 5.25, $\psi_{-2,1} = c\varphi_{-2,1}$ for some $c \in \mathbb{C}$. Thus we have that $\mathbb{J}^{\text{weak}}_{-2,1} = \mathbb{C}\varphi_{-2,1}$. $\square$

Lemma 5.45. $\mathbb{J}^{\text{cusp}}_{10,1} = \mathbb{C}\varphi_{10,1}$.

Proof. Let $\psi^{\text{cusp}}_{10,1} \in \mathbb{J}^{\text{cusp}}_{10,1}$. Consider the Taylor expansion of $\psi_{-2,1}$

$$\psi^{\text{cusp}}_{-2,1}(\tau, z) = f^{\text{cusp}}_{10}(\tau) + f^{\text{cusp}}_{12}(\tau) z^2 + \ldots$$

In this case, we have that $f^{\text{cusp}}_{10}(\tau) = \varphi_{10,1}(\tau, 0) \in S_{10} = \{0\}$. Thus $f^{\text{cusp}}_{12} \in S_{10} = \mathbb{C}\Delta$. Now if $f^{\text{cusp}}_{12} = 0$ then $\psi^{\text{cusp}}_{10,1} = 0$ by exactly same argument as in Lemma 5.44. Proceeding as in Lemma 5.44, we get $\mathbb{J}^{\text{cusp}}_{10,1} = \mathbb{C}\varphi_{10,1}$. $\square$

All the examples of Jacobi forms that we have constructed till now are of even weight. We would now like to construct examples of Jacobi forms of odd weight. The next proposition deals with this construction.

Theorem 5.46. *For $t \in \mathbb{N}$, define $\vartheta_t(\tau, z) := \vartheta(\tau, tz)$. Then we have the following:*

(i) $\vartheta_t(\tau, z) \in \mathbb{J}_{\frac{1}{2}, \frac{t^2}{2}}$ *with character* ν_H^t *and multiplier system* ν_η^3.

(ii) $\forall\ \varphi \in \mathbb{J}_{k,m}$, $\varphi(\tau, tz) \in \mathbb{J}_{k,t^2 m}$.

Proof. Using Corollary 5.40, we have for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\vartheta_t\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = \nu_\eta(M)^3(c\tau + d)^{1/2} e^{2\pi i \frac{t^2}{2}\left(\frac{cz^2}{c\tau+d}\right)} \vartheta_t(\tau, z)$$

and for $\mu, \lambda \in \mathbb{Z}$

$$\begin{aligned} \vartheta_t(\tau, z + \lambda\tau + \mu) = \vartheta(\tau, t(z + \lambda\tau + \mu)) &= (-1)^{t\lambda + t\mu} e^{-\pi i(t^2\lambda^2\tau + 2t^2\lambda z)} \vartheta_t(\tau, z) \\ &= \nu_H^t e^{-\pi i(t^2\lambda^2\tau + 2t^2\lambda z)} \vartheta_t(\tau, z). \end{aligned}$$

Thus $\vartheta_t(\tau, z) \in \mathbb{J}_{\frac{1}{2}, \frac{t^2}{2}}$ with character ν_H^t and multiplier system ν_η^3. (ii) follows similarly. $\square$

Example 5.47. $\varphi_{11,2} = \eta(\tau)^{21}\vartheta(\tau, 2z) \in \mathbb{J}_{11,2}^{\text{cusp}}$ by Proposition 5.46.

5.8 Theta Series expansion

Let us first define a variant of the theta series defined in Section 5.7. For $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}$, define

$$\vartheta_{m,\ell}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \ell \pmod{2m}}} q^{r^2/4m} \zeta^r = \sum_{n \in \mathbb{Z}} q^{(\ell + 2mn)^2/4m} \zeta^{\ell + 2mn}.$$

We will record a property of this theta series without proof.

Theorem 5.48. *The theta series $\vartheta_{m,\ell}$ is a Jacobi form of index m and weight $1/2$. In particular we have that*

$$\begin{aligned} \vartheta_{m,\ell}(\tau + 1, z) &= e^{2\pi i \ell^2/4m} \vartheta_{m,\ell}(\tau, z) \\ \vartheta_{m,\ell}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sqrt{\frac{\tau}{2mi}\frac{mz^2}{\tau}} \sum_{r \pmod{2m}} e^{-2\pi i r\ell/2m} \vartheta_{m,r}(\tau, z) \end{aligned}$$

We can now prove the following Theorem.

Theorem 5.49. *Let φ be a Jacobi form of index m and weight k. Then*

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\vartheta_{m,\ell}(\tau, z) \tag{5.11}$$

where h_ℓ transforms like vector valued modular form of weight $k - 1/2$ with respect to $SL_2(\mathbb{Z})$.

Remark 5.50. When we say that $h_\ell(\tau)$ is a vector valued modular form, we mean that the vector $\vec{h}(\tau) = (h_\ell)_{\ell \pmod{2m}}$ transforms as follows:

$$\vec{h}(\gamma\tau) = (c\tau + d)^{k-1/2} U(\gamma)\vec{h}(\tau), \quad \forall\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

where $U(\gamma)$ is some $2m \times 2m$ matrix.

Proof. Using Theorem 5.19, we have the Fourier expansion of φ as

$$\varphi(\tau, z) = \sum_{\substack{n \in \mathbb{Z} \\ (4nm - r^2) \geq 0}} c(4nm - r^2, r) q^n \zeta^r.$$

We can rearrange the sum by breaking it mod $2m$. We get

$$\varphi(\tau, z) = \sum_{\ell=0}^{2m-1} \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \ell \pmod{2m}}} \sum_{n \geq r^2/4m} c(4nm - r^2, r) q^n \zeta^r.$$

Using Theorem 5.19 to write $c(4nm - r^2, r) = c_\ell(4nm - r^2)$, we get

$$\varphi(\tau, z) = \sum_{\ell \pmod{2m}} \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \ell \pmod{2m}}} \sum_{n \geq r^2/4m} c_\ell(4nm - r^2) q^n \zeta^r.$$

Put $N = (4nm - r^2)$, we get

$$\varphi(\tau, z) = \sum_{\ell (\bmod\ 2m)} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\ell(\bmod\ 2m)}} \sum_{N\geq 0} c_\ell(4nm - r^2) q^{\frac{N+r^2}{4m}} \zeta^r.$$

Rearranging terms we get

$$\begin{aligned}
\varphi(\tau, z) &= \sum_{\ell (\bmod\ 2m)} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\ell(\bmod\ 2m)}} \left(\sum_{N\geq 0} c_\ell(4nm - r^2) q^{\frac{N}{4m}} \right) q^{\frac{r^2}{4m}} \zeta^r \\
&= \sum_{\ell (\bmod\ 2m)} \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\ell(\bmod\ 2m)}} h_\ell(\tau) q^{\frac{r^2}{4m}} \zeta^r \\
&= \sum_{\ell (\bmod\ 2m)} h_\ell(\tau) \sum_{\substack{r\in\mathbb{Z} \\ r\equiv\ell(\bmod\ 2m)}} q^{\frac{r^2}{4m}} \zeta^r \\
&= \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau) \vartheta_{m,\ell}(\tau, z).
\end{aligned}$$

where $h_\ell(\tau) = \sum\limits_{N\geq 0} c_\ell(4nm - r^2) q^{\frac{N}{4m}}$. Now using the transformation of φ for $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and using Theorem 5.48, we get

$$\begin{aligned}
h_\ell(\tau + 1) &= e^{-2\pi i \ell^2/4m} h_\ell(\tau) \\
h_\ell\left(-\frac{1}{\tau}\right) &= \frac{\tau^k}{\sqrt{2m\tau/i}} \sum_{r\ (\bmod\ 2m)} e^{-2\pi i r\ell/2m} h_r(\tau).
\end{aligned}$$

We can absorb the extra factor $1/\sqrt{2m/i}$ in $e^{-2\pi i r\ell/2m}$ to get the $2m \times 2m$ matrix $U(M)$. $\square$

Remark 5.51. The series in Theorem 5.49 is called *Theta series decomposition*.

5.9 Siegel modular forms

We will now systematically study Siegel modular forms. Throughout this section, we will assume the notation of section 5.3. Let us first define these modular forms.

We will mainly follow [9] for this section.

5.9.1 Definition and Fourier expansion

Definition 5.52. Let $k, n \in \mathbb{Z}, n \geq 1$. A function $f : \mathbb{H}_n \to \mathbb{C}$ is called a Siegel modular form of weight k and degree n with respect to $Sp_n(\mathbb{Z})$ if

(i) f is holomorphic.

(ii) $f\left((AZ+B)(CZ+D)^{-1}\right) = det(CZ+D)^k f(Z) \ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{Z})$, $Z \in \mathbb{H}_n$.

Note that there is no condition on the Fourier expansion of f. The Fourier expansion here is automatic for $n > 1$ if f is assumed to be holomorphic. This is called the *Koecher principle*. The proof of this Theorem can be found in [9].

Theorem 5.53. *Let f be a Sigel modular form of weight k and degree n. Then f has a Fourier expansion of the form*

$$f(Z) = \sum_{T \geq 0} a_f(T) e^{2\pi i \ trace(TZ)} \tag{5.12}$$

where the sum runs over all half-integral[1]*, positive semi-definite matrices of size n.*

Remark 5.54. f is called a *Siegel cusp form* if in the Fourier expansion (5.12), the sum runs only over positive definite matrices T.

5.9.2 Fourier Jacobi expansion

We are particularly interested in Siegel modular forms of degree 2. We have the following Lemma.

[1] Matrices whose non-diagonal entries are from $\frac{\mathbb{Z}}{2}$.

Lemma 5.55. For $n = 2$, the Fourier expansion (5.12) takes a simple form

$$f(\Omega) = \sum_{\substack{n,r,m\in\mathbb{Z} \\ n,m,4mn-r^2\geq 0}} a_f(n,r,m)q^n\zeta^r w^m$$

where $q = e^{2\pi i\tau}, \zeta = e^{2\pi iz}$ and $w = e^{2\pi i\sigma}$.

Proof. For $\Omega \in \mathbb{H}_2$, we have $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$ where $\tau, \sigma \in \mathbb{H}$ and $z \in \mathbb{C}$. Also any 2×2 half-integral matrix is of the form $T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$ where $n, r, m \in \mathbb{Z}$. For T to be positive semi definite, we must have both eigenvalues positive which is equivalent to having $nm - r^2/4 \geq 0$ and $n, m \geq 0$. Moreover we have $\mathrm{trace}(TZ) = n\tau + rz + m\sigma$. Thus using (5.12), we get

$$\begin{aligned} f(\Omega) &= \sum_{\substack{n,r,m\in\mathbb{Z} \\ n,m,4mn-r^2\geq 0}} a_f(n,r,m)e^{2\pi i(n\tau+rz+m\sigma)} \\ &= \sum_{\substack{n,r,m\in\mathbb{Z} \\ n,m,4mn-r^2\geq 0}} a_f(n,r,m)q^n\zeta^r w^m. \end{aligned}$$

where $a_f(T) = a_f(n,r,m)$. $\square$

We will now investigate the claimed connection between Jacobi forms and Siegel modular forms.

Theorem 5.56. *(Fourier-Jacobi expansion) Let f be a Siegel modular form of weight k and degree 2. Then f has an expansion of the form*

$$f(\Omega) = \sum_{m=0}^{\infty} \varphi_m(\tau, z)w^m$$

where each φ_m is a Jacobi form of weight k and index m.

Proof. Using Lemma 5.55, we can rearrange terms and write

$$\begin{aligned} f(\Omega) &= \sum_{m=0}^{\infty} \left(\sum_{\substack{n,r\in\mathbb{Z} \\ 4mn-r^2\geq 0}} a_f(n,r,m) q^n \zeta^r \right) w^m \\ &= \sum_{m=0}^{\infty} \varphi_m(\tau,z) w^m. \end{aligned}$$

where $\varphi_m(\tau,z) = \sum\limits_{\substack{n,r\in\mathbb{Z} \\ 4mn-r^2\geq 0}} a_f(n,r,m) q^n \zeta^r$. To show that φ is a Jacobi form, we need to check transformation properties and the Fourier expansion. But Fourier expansion is obvious from the definition of φ_m. We now check modularity and elliptic invariance. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ we have $f|_k[M] = f$ where we used the notation of section 5.3. If we write $f(\Omega) = f(\tau,z,\sigma)$, then this gives

$$f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, \sigma - \frac{cz^2}{c\tau+d}\right) = (c\tau+d)^k f(\tau,z,\sigma).$$

Plugging in the Fourier-Jacobi expansion both sides gives

$$\sum_{m=0}^{\infty} (c\tau+d)^k \varphi_m(\tau,z) w^m = \sum_{m=0}^{\infty} \varphi_m\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) e^{-2\pi i m \frac{cz^2}{c\tau+d}} w^m.$$

Comparing coefficients of w^m on both sides we get the modularity:

$$\varphi_m\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} \varphi_m(\tau,z).$$

Similarly for $\lambda,\mu \in \mathbb{Z}$, using modularity for f with respect to $\left[\begin{pmatrix} \mu \\ \lambda \end{pmatrix}, 0\right]$, we get elliptic invariance. □

Remark 5.57. We can ask a converse question. Given a sequence of Jacobi forms $\{\varphi_m\}_{m=0}^{\infty}$, does the formal sum $f(\tau,z,\sigma) = \sum\limits_{m=0}^{\infty} \varphi_m(\tau,z) e^{2\pi i m\sigma}$ define a Siegel modular form? The answer to this question is positive but under some conditions on the sequence of Jacobi forms. This has been investigated in a paper by Raum

[20].

5.9.3 Igusa cusp form

We will discuss (without proof) an example of a Siegel cusp form which is of interest in later chapters in our discussion. Define

$$\Phi_{10} = q\zeta w \prod_{(m,r,n)>0} (1 - q^n \zeta^r w^m)^{2C_0(4mn-r^2)} \tag{5.13}$$

where $(m,r,n) > 0$ means that $m,r,n \in \mathbb{Z}$ with $m > 0, n \geq 0$ or $m \geq 0, n > 0$ or $m = n = 0, r < 0$. The exponents $C_0(4mn - r^2)$ are the coefficients in the Fourier expansion in (5.10). Φ_{10} is called the Igusa cusp form of weight 10 and degree 2.

Theorem 5.58. *Φ_{10} is a Siegel cusp form of weight 10 and degree 2 with a zero at $z = 0$. Moreover it has Fourier expansion of the form*

$$\sum_{\substack{n,r,m \in \mathbb{Z} \\ 4mn - r^2 > 0}} a_{10}(n,r,m) q^n \zeta^r w^m,$$

where

$$a_{10}(n,r,m) = \sum_{d | gcd(n,r,m), d>0} d^{k-1} C_{10}\left(\frac{4mn - r^2}{d^2}\right)$$

with $C_{10}(d)$ as in (5.10).

The function of interest to physics is the inverse of the Igusa cusp form. We will analyse it in now. Put

$$Z(\Omega) = \frac{1}{\Phi_{10}(\Omega)}.$$

$Z(\Omega)$ has pole at $z = 0$. Thus it is a *meromorphic* Siegel modular form of weight -10 and degree 2. We will now anlayse its Fourier expansion.

Theorem 5.59. *The Fourier expansion of $Z(\Omega)$ is of the form*

$$Z(\Omega) = \sum_{\substack{m,n \geq -1 \\ r \in \mathbb{Z}}} g(m,r,n) q^n \zeta^r w^m. \tag{5.14}$$

Proof. We will prove that $Z(\Omega)$ has simple poles at $q, w = 0$. First observe that the product representation of $Z(\Omega)$ is

$$Z(\Omega) = q^{-1}\zeta^{-1}w^{-1} \prod_{(m,r,n)>0} (1 - q^n\zeta^r w^m)^{-2C_0(4mn-r^2)}$$

and note that the exponents of q and w is always non negative. Thus we have the following limits.

$$\lim_{q\to 0} q^{l+1}\frac{1}{\Phi_{10}} = \lim_{q\to 0} q^{l+1}q^{-1}\zeta^{-1}w^{-1} \prod_{(m,r,n)>0} (1 - q^n\zeta^r w^m)^{-2C_0(4mn-r^2)} = 0 \quad \forall l > 0$$

and that

$$\lim_{q\to 0} q\frac{1}{\Phi_{10}} = \lim_{q\to 0} \zeta^{-1}w^{-1} \prod_{(m,r,n)>0} (1 - q^n\zeta^r w^m)^{-2C_0(4mn-r^2)} \neq 0$$

Similar limits hold for w as well. This shows that $Z(\Omega)$ has a simple pole at $q = 0$ and $w = 0$. Thus the Fourier expansion of $Z(\Omega)$ has the desired form. □

Theorem 5.60. *$Z(\Omega)$ has Fourier-Jacobi expansion of the form*

$$Z(\Omega) = \sum_{m\geq -1} \psi_m(\tau, z)w^m \tag{5.15}$$

where ψ_m are meromorphic Jacobi forms of weight -10 and index m.

Proof. We can rearrange th series in (5.14) as follows:

$$Z(\Omega) = \sum_{m\geq -1} \left(\sum_{\substack{n\geq -1 \\ r\in\mathbb{Z}}} g(m,r,n)q^n\zeta^r \right) w^m.$$

Put $\psi_m(\tau, z) = \sum_{\substack{n\geq -1 \\ r\in\mathbb{Z}}} g(m,r,n)q^n\zeta^r$. Then checking the transformation properties of ψ_m is routine and follows exactly as done in Theorem 5.56. The Fourier expansion of ψ_m corresponds to weakly holomorphic Jacobi forms by (5.4). But ψ_m has a pole at $z = 0$. Thus each ψ_m is a meromorphic Jacobi form of weight -10 and index m. $\square$

6. CLASS GROUPS

The subject of binary quadratic forms and class groups has its roots back to seventeenth and eighteenth century. It was Gauss who first introduced the notion of $SL_2(\mathbb{Z})$-equivalence classes of binary quadratic forms and provided a composition law for this set of equivalence classes which made it into a group called class group[24]. Dirichlet in 1938 studied the ideal classes of ring of integers of quadratic number fields and gave an equivalent definition of class group. Dirichlet's analysis gave an easier way to compose pairs of binary quadratic forms. Finally Manjul Bhargava in 2000 gave an easy way to look at these compositions and showed that this composition law is one of atleast fourteen composition laws of this type[3]. The results contained in this chapter can be found in [10].

6.1 Definition

For $m, r, n \in \mathbb{Z}$, we first define a binary quadratic form.

Definition 6.1. A *binary quadratic form* is a function $q : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $q(x, y) = mx^2 + rxy + ny^2$ where $m, r, n \in \mathbb{Z}$. The integer $D = r^2 - 4mn$ is called the *discriminant* of $q(x, y)$. The binary quadratic form $q(x, y)$ is called

(i) *primitive* if $\gcd(m, r, n) = 1$.

(ii) *positive definite* if $D < 0$ and $m > 0$.

(iii) *reduced* if $|r| \leq |m| \leq |n|$ and $r \geq 0$ when $|m| = |r|$ or $|m| = |n|$.

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$, the change of variables

$$q(x, y) \mapsto Aq(x, y) = (ax + by, cx + dy)$$

is called *linear change of variables.* If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $q(x,y)$ and $q(ax+cy, bx+dy)$ are said to be *properly equivalent.* Any quadratic form is equivalent to a primitive form and it can be obtained by a series of linear change of variables of the form $T^k q$ and Sq where $T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Theorem 6.2. *For any quadratic form, there is a sequence of linear change of variables, each of type T^k or S such that the transformed quadratic form is reduced.*

The proof of this Theorem gives a recipe to reduce any quadratic form. We will describe the algorithm now.

Corollary 6.3. (Definite reduction algorithm)

1. Given coefficients $a, b, c \in \mathbb{Z}$, put $q(x,y) = ax^2 + bxy + cy^2$.
2. By division algorithm, obtain $k, r \in \mathbb{Z}$ with $b = k(2a) + r$ and $|r| \leq |a|$. The change of variables $q \mapsto T^k q$ results in a quadratic form with $|b| \leq |a|$.
3. If $|a| \leq |c|$, go to Step 5.
4. Put $q = Sq$ to produce a quadratic form with $|a| < |c|$ and go back to Step 2.
5. If $b = -|a|$, put $q := T^{sgn(a)} q$ to replace $ax^2 - |a|xy + cy^2$ with $ax^2 + (-|a| + 2|a|)xy + cy^2 = ax^2 + |a|xy + cy^2$ where $sgn(a) = \frac{a}{|a|}$.
6. If $b < 0$ and $a = |c|$, put $q := Sq$ to replace $ax^2 + bxycy^2$ with $cx^2 - bxy + ay^2$.

Observe that given any quadratic form $q(x,y) = mx^2 + rxy + ny^2$, we can associate a symmetric matrix to it in the following fashion

$$q(x,y) = mx^2 + rxy + ny^2 \mapsto \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix}$$

and the quadratic form itself can be written as

$$q(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} m & r/2 \\ r/2 & n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We will denote the matrix associated to $q(x, y)$ by Q_q. Consider the set of all primitive, positive definite binary quadratic forms with fixed discriminant D and denote it by $\mathcal{Q}_D$. The next Theorem gives an action of $SL_2(\mathbb{Z})$ on $\mathcal{Q}_D$.

Theorem 6.4. *For* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *and* $q(x, y) \in \mathcal{Q}_D$*, the map* $SL_2(\mathbb{Z}) \times \mathcal{Q}_D \longrightarrow \mathcal{Q}_D$ *given by*

$$(M, q) \mapsto (M \circ q)(x, y) = q(ax + cy, bx + dy) = \begin{pmatrix} x \\ y \end{pmatrix}^t MQ_qM^t \begin{pmatrix} x \\ y \end{pmatrix}.$$

is a group action of $SL_2(\mathbb{Z})$ *on the set* $\mathcal{Q}_D$*.*

The above action can be viewed as an action on the matrix Q_q given by $(M \circ Q_q) = MQ_qM^t$. Denote the set of orbits under this action by $\mathcal{Q}_D/SL_2(\mathbb{Z})$.

Definition 6.5. The set $C(D) = \mathcal{Q}_D/SL_2(\mathbb{Z})$ is called the *class group* and the cardinality $|C(D)|$ is called the *class number*.

Theorem 6.6. *The cardinality* $|C(D)|$ *of the class group is finite.*

Notation. Given a quadratic form $q(x, y) = mx^2 + rxy + ny^2$, we will denote by $[m, r, n]$ the class in $C(D)$ represented by q.

6.2 Group structure

$C(D)$ admits a group structure with respect to some composition which was first introduced by Gauss (1801). Let us first recall the following identity attributed to 7th century Indian mathematician Brahmagupta.

Proposition 6.7. For any integers x_1, y_1, x_2, y_2, D, we have

$$(x_1^2 + Dy_1^2)(x_2^2 + Dy_2^2) = (x_1x_2 - Dy_2y_2)^2 + D(x_1y_2 + Dx_2y_1)^2.$$

We can summarize the previous identity by saying that the numbers of the form x^2+Dy^2 are closed under multiplication. In 1801, Gauss asked in his *Disquisitiones Arithmeticae* whether it was possible to generalize this to numbers of a more general form, namely $ax^2 + bxy + cy^2$. He comes up with the answer: Yes!

Theorem 6.8. *Let $a_1x_1^2+b_1x_1y_1+c_1y_1^2$ and $a_2x_2^2+b_2x^2y^2+c_2y_2^2$ be binary quadratic forms of discriminant D. Then, there exists an (explicit) transformation (change of variables):*

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_1 & y_1 \\ y_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$$

and integers A, B, C such that $(a_1x_1^2 + b_1x_1y_1 + c_1y_1^2)(a_2x_2^2 + b_2x^2y^2 + c_2y_2^2) = AX^2 + BXY + CY^2$. Moreover $B^2 - 4AC = D$.

Theorem 6.9. *The set $C(D)$ forms a finite abelian group.*

It is worth noting that the modern notion of a group did not exist when Gauss wrote his *Disquisitiones*. However, it is clear that, without using our modern terms, this is really what he was after. The issue with Gauss composition is that it is highly non-trivial to compute the composition of two classes. This task was taken up by one of Gauss's students: *Peter Gustav Jejune Dirichlet*. Dirichlet went on to investigate the ideals of the ring of integers of quadratic number fields $\mathbb{Q}(\sqrt{D})$ and came up with an equivalent formulation of class groups with this perspective and also showed that the two class groups are isomorphic. We will not go into details of this formulation but simply state the results[23].

Theorem 6.10. *(Dirichlet's united form) Let $[m_1, r, n_1]$ and $[m_2, r, n_2]$ be two members of class group $C(D)$ such that $gcd(m_1, m_2) = 1$ (Two such quadratic forms are called united). Put $n = gcd(n_1, n_2)$. The class group composition of the two classes is given by*

$$[m_1, r, n_1] * [m_2, r, n_2] = [m_1m_2, r, n] \tag{6.1}$$

Theorem 6.11. *Let $[m_1, r_1, n_1]$ and $[m_2, r_2, n_2]$ be two members of class group $C(D)$. Let $e = gcd\left(n_1, n_2, \frac{r_1+r_2}{2}\right)$. Then there is a unique integer R modulo $2n_2n_2/e^2$ such that:*

$$R \equiv r_1 \ mod \ \frac{2n_1}{e}, \quad R \equiv r_2 \ mod \ \frac{2n_2}{e}, \quad R^2 \equiv D \ mod \ \frac{4n_1n_2}{e^2}$$

Moreover, we have the following class group composition

$$[m_1, r_1, n_1] * [m_2, r_2, n_2] = \left[\frac{e^2(R^2 - D)}{4n_1n_2}, R, \frac{n_1n_2}{e^2}\right]. \tag{6.2}$$

The proof of this Theorem gives us the recipe to calculate R. We will not go into the proof but we will simply give the formula for R.

Corollary 6.12. With the assumptions and notation of Theorem (4.12), we have that

$$R = \frac{1}{e}\left(\alpha n_1 r_2 + \beta n_2 r_1 + \gamma \frac{r_1 r_2 + D}{2}\right) \tag{6.3}$$

where α, β and γ are such that $\alpha n_1 + \beta n_2 + \gamma \frac{r_1 + r_2}{2} = e$.

Corollary 6.13. Let $\left[1, \frac{r_1 + i\sqrt{-D}}{2n_1}\right]$ and $\left[1, \frac{r_2 + i\sqrt{-D}}{2n_2}\right]$ be two vectors associated to class group elements $[m_1, r_1, n_1]$ and $[m_2, r_2, n_2]$ with discriminant $D < 0$. Then with the notations of Theorem (4.12), we have the following composition

$$\left[1, \frac{r_1 + i\sqrt{-D}}{2n_1}\right] * \left[1, \frac{r_2 + i\sqrt{-D}}{2n_2}\right] = \left[1, \frac{e^2(R + i\sqrt{-D})}{4n_1n_2}\right] \tag{6.4}$$

$$R = \frac{1}{e}\left(\alpha n_1 r_2 + \beta n_2 r_1 + \gamma \frac{r_1 r_2 + D}{2}\right) \tag{6.5}$$

where α, β and γ are such that $\alpha n_1 + \beta n_2 + \gamma \frac{r_1 + r_2}{2} = e$.

Theorem 6.14. *Given a class $[m, r, n]$, its inverse class is given by*

$$[m, r, n]^{-1} = [m, -r, n] = [n, r, m].$$

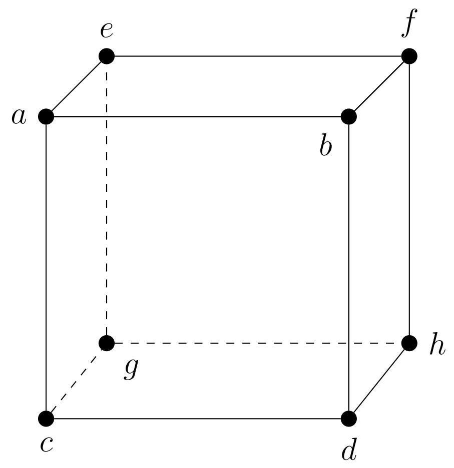
Theorem 6.15. *The identity class 1_D in $C(D)$ is given by*

$$1_D = \begin{cases} \left[1, 0, -\frac{D}{4}\right] & \text{if } D \equiv 0 (mod\ 4) \\ \left[1, 1, \frac{1-D}{4}\right] & \text{if } D \equiv 1 (mod\ 4). \end{cases}$$

6.3 Bhargava's composition

In 2000, Manjul Bhargava came up with an elegant and beautiful perspective on these class group composition using $2 \times 2 \times 2$ cubes.

Definition 6.16. (Bhargava Cube): A Bhargava cube $\mathcal{C}_{abcdefgh}$ is a $2 \times 2 \times 2$ array of elements in $\mathbb{Z}$:



Given a Bhargava Cube, we can construct three pairs of 2×2 matrices and corresponding to each pair we can construct a quadratic form. We index the three pairs by F, L and T and the two matrices in each pair by M_i and N_i for $i \in \{F, L, T\}$ The matrices are listed below:

- $i = F$ $\quad M_F = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad N_F = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$
- $i = L$ $\quad M_L = \begin{bmatrix} a & c \\ e & g \end{bmatrix} \quad N_L = \begin{bmatrix} b & d \\ f & h \end{bmatrix}$
- $i = T$ $\quad M_T = \begin{bmatrix} a & e \\ b & f \end{bmatrix} \quad N_T = \begin{bmatrix} c & g \\ d & h \end{bmatrix}$

The recipe to construct a quadratic form from these pairs is the following:
For $i \in \{F, L, T\}$, put

$$q_i(x, y) = -det(xM_i - yN_i).$$

Definition 6.17. A Bhargava cube is called *primitive* if all its associated quadratic forms are primitive.

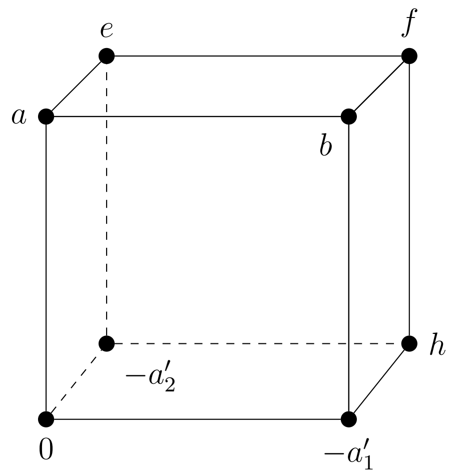
We have the following Theorems due to Bhargava:

Theorem 6.18. *(Bhargava) The three forms associated with any primitive Bhargava cube $\mathcal{C}$ of discriminant D satisfy $[q_F][q_L][q_T] = 1_D$.*

Proposition 6.19. Let q_1 and q_2 be two quadratic forms of the same discriminant. There exists a Bhargava cube for which $q_F = q_1$ and $q_L = q_2$.

Thus, given two classes $[q_1]$ and $[q_2]$, we have got a recipe to compute the composition of these classes using the Theorem above. We first need to construct a Bhargava cube for which $q_F = q_1$ and $q_L = q_2$. The existence of such a cube is guaranteed by the proposition above. Now by Theorem above we have $[q_1][q_2][q_T] = 1$. Thus the composition $[q_1][q_2] = [q_T]^{-1}$. Thus the problem of class composition reduces to constructing a suitable Bhargava cube. In fact the proof of Proposition 6.19 gives us a way to construct such a cube. The steps to do so are outlined below.

Let $q_i(x, y) = a_i x^2 + b_i x + c_i$; $i = 1, 2$ be two quadratic forms. The cube which we get from Proposition 6.19 is of the form shown below.



We calculate various vertices in the following way:

$a = gcd(a_1, a_2, (b_1 + b_2)/2)$ $\quad a'_i = a_i/a$; $i = 1, 2$ $\quad h = (b_1 + b_2)/2a$

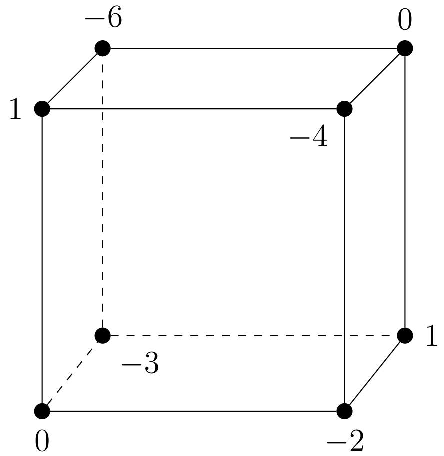
To calculate e, f, g define the following other quantities. $d_i = gcd(a'_i, h)$; $a''_i = a'_i/d_i$; $h_i = h/d_i$; $c'_1 = c_1/d_2$; $c'_2 = c_2/d_1$. In terms of these quantities consider the congruences

$$f \equiv -a''_2 c'_1 \ (mod\ h_2) \qquad f \equiv -a''_1 c'_2 \ (mod\ h_1)$$

Solving this congruence (solution exists according to the construction), we find f and then if $h \neq 0$ then $e = -(c_1 + a'_2 f)/h$ and $b = -(c_2 + a'_1 f)/h$. If $h = 0$ then $f = -c_1/a'_2 = -c_2/a'_1$ and for e and b we take any solution of $-a'_1 e + a'_2 b = b_1$ (again solution exists by construction).

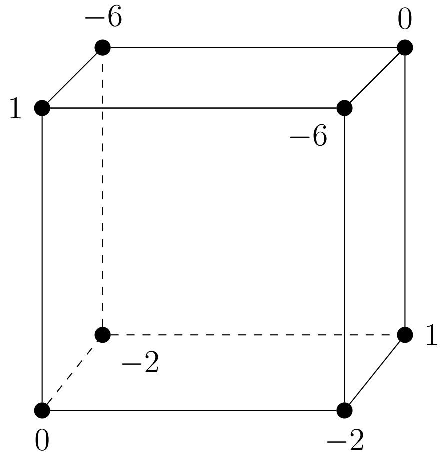
Example 6.20. Now let us apply the above theory to compute some of the compositions. Consider $D = -47$. It is known that the class number $h(D) = 5$ and the $SL_2(\mathbb{Z})$-equivalence classes are [1,1,12], [2,±1,6] and [3,±1,4]. We would like to compose two classes from this list and get another class in this list.

- $[2, 1, 6] * [3, 1, 4]$
  Using above procedure we get the values of the vertices of Bhargava cube. It is shown below.



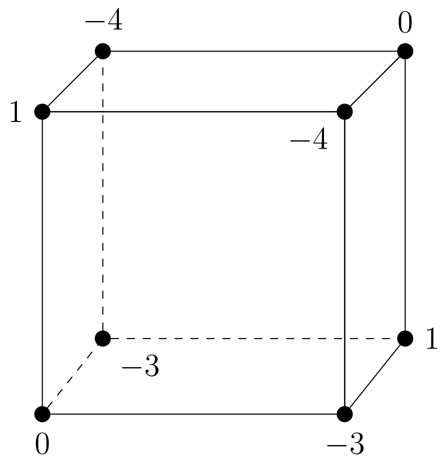
  The associated forms are $q_F(x, y) = 2x^2 + xy + 6y^2 = q_1(x, y)$ $q_L(x, y) = 3x^2 + xy + 4y^2 = q_2(x, y)$ and finally $q_T(x, y) = 24x^2 - 23xy + 6y^2$. We reduce q_T using the definite reduction algorithm in Corollary 6.3 to get $2x^2 + xy + 6y^2$. Thus we have $[2, 1, 6] * [3, 1, 4] = [2, 1, 6]^{-1} = [2, -1, 6]$

- $[2, 1, 6] * [2, 1, 6]$
  Using above procedure we get the values of the vertices of Bhargava cube. It is shown below.



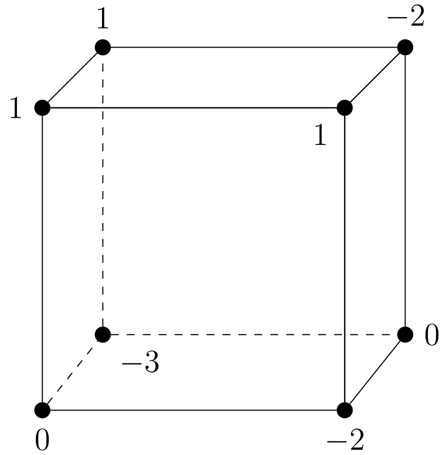
The associated forms are $q_F(x, y) = 2x^2 + xy + 6y^2 = q_1(x, y)$ $q_L(x, y) = 2x^2 + xy + 6y^2 = q_2(x, y)$ and finally $q_T(x, y) = 36x^2 - 23xy + 4y^2$. Reducing we get $3x^2 + xy + 4y^2$. Thus we have $[2, 1, 6] * [2, 1, 6] = [3, 1, 4]^{-1} = [3, -1, 4]$

- $[3, 1, 4] * [3, 1, 4]$
  Using above procedure we get the values of the vertices of Bhargava cube. It is shown below.



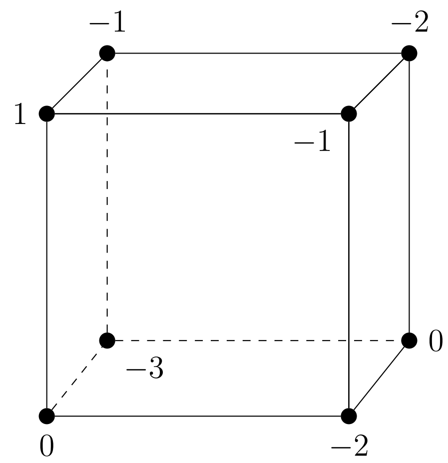
The associated forms are $q_F(x,y) = 3x^2+xy+4y^2 = q_1(x,y)$ $q_L(x,y) = 3x^2+xy+4y^2 = q_2(x,y)$ and finally $q_T(x,y) = 16x^2-23xy+9y^2$. Again reducing, we get $2x^2-xy+6y^2$. Thus we have $[3,1,4]*[3,1,4] = [2,-1,6]^{-1} = [2,1,6]$

- $[2,1,6]*[3,-1,4]$
  Using above procedure we get the values of the vertices of Bhargava cube. It is shown below.



The associated forms are $q_F(x,y) = 2x^2+xy+6y^2 = q_1(x,y)$ $q_L(x,y) = 3x^2-xy+4y^2 = q_2(x,y)$ and finally $q_T(x,y) = 3x^2+5xy+6y^2$. Legendre reduction gives $3x^2-xy+4y^2$. Thus we have $[2,1,6]*[3,-1,4] = [3,-1,4]^{-1} = [3,1,4]$

- $[2,-1,6]*[3,1,4]$
  Using above procedure we get the values of the vertices of Bhargava cube. It is shown below.



The associated forms are $q_F(x,y) = 2x^2 - xy + 6y^2 = q_1(x,y)$ $q_L(x,y) = 3x^2 + xy + 4y^2 = q_2(x,y)$ and finally $q_T(x,y) = 3x^2 - 5xy + 6y^2$. Reduction gives $3x^2+xy+4y^2$. Thus we have $[2,-1,6]*[3,1,4] = [3,1,4]^{-1} = [3,-1,4]$.

Thus we have the following list of compositions:

$[2,1,6] * [3,1,4] = [2,-1,6]$

$[2,1,6] * [2,1,6] = [3,-1,4]$

$[3,1,4] * [3,1,4] = [2,1,6]$

$[2,1,6] * [3,-1,4] = [3,1,4]$

$[2,-1,6] * [3,1,4] = [3,-1,4]$

7. BLACK HOLES AND CLASS GROUPS

7.1 Introduction

We will first state the connection between black holes and class group concretely in the form of a Theorem due to G.W. Moore. The specific black hole solutions which admit such a correspondence are those appearing in type IIB string theory compactified on $K3 \times T^2$. These are the set of supersymmetric BPS black holes. Given a black holes solution we can associate two physical quantities P and Q called *magnetic* and *electric* charge vector respectively. These charge vectors are elements of $\mathbb{Z}^{28}$ and have integer coefficients. Given a black hole solution, we can define a transformation on this solution to get another solution. This is called U-duality transformation. To characterise this transformation, consider a black hole solution with charge vectors P and Q and consider the following matrix

$$Q_{Q,P} = \begin{pmatrix} P^2 & P \cdot Q \\ P \cdot Q & Q^2 \end{pmatrix} = 2 \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$$

where $m, r, n \in \mathbb{Z}$, $(Q^2, P^2, Q \cdot P)$ is the combinations $(Q^T LQ, P^T LP, Q^T LP)$ where L is the $SO(6,22)$ invariant metric for the dot product. Then U-duality transformation is exactly the $SL_2(\mathbb{Z})-$action on the quadratic form determined by $Q_{Q,P}$. Through out this chapter, by fundamental discriminant we will mean that $D \equiv 0, 1 (\mathrm{mod}\ 4)$ and that D is square-free when $D \equiv 1 (\mathrm{mod}\ 4)$ and $D/4$ is a square-free number congruent to 2 or 3 modulo 4 when $D \equiv 0 (\mathrm{mod}\ 4)$. We can now state the Theorem.

Theorem 7.1 (Moore [14, 13]). *If $D < 0$ is a fundamental discriminant, then the*

U-duality equivalence classes of attractor black holes of entropy $S = \pi\sqrt{-D}$ admit a structure as a finite abelian group. Moreover, this group is isomorphic to the class group $C(D)$.

Following questions arise naturally in this context.

Questions 7.2. (i) Is there a natural physical interpretation of the group law described in Theorem 4.1 in terms of attractor black holes?

(ii) Is there a distinguished physical property of the identity class black hole, which corresponds to the class represented by the identity element 1_D?

(iii) Is there a physical interpretation of inverse black hole?

(iv) What is the physical interpretation of the order of a black hole corresponding to the order of an element in the class group?

There are other questions that one might ask. It is known, for example, that there are finitely many values of fundamental discriminant D for which the class group is trivial. This was originally conjectured by Gauss in 1801[24]. On the black hole side, this means that there are finitely many values of entropy for which there is a unique $U-$duality class. What is the physical interpretation of this fact? We will try to answer the third question using the notion of *degeneracy*.

Definition 7.3. Let P and Q be charge vectors. The number of underlying microstates $d(Q, P)$ with charge vectors P and Q is called *degeneracy*.

7.2 Degeneracy of black hole class

Degeneracy of black hole microstate is related to something called *partition function*. Knowing the partition function for a given solution in string theory, one can obtain the degeneracy for a particular configuration.

Theorem 7.4. *[17] The partition function associated with $\frac{1}{4}$BPS black hole solution of type IIB string theory compactified on $K3 \times T^2$ is given by*

$$Z(\Omega) = \frac{1}{\Phi_{10}(\Omega)} \tag{7.1}$$

where $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$; $\tau, \sigma \in \mathbb{H}$; $z \in \mathbb{C}$ *and*

$$\Phi_{10}(\Omega) = q\zeta w \prod_{(m,r,n)>0} (1 - q^n \zeta^r w^m)^{2C_0(4mn-r^2)}$$

is the Igusa cusp form as given in (5.13).

The partition function is a meromorphic Siegel modular form of weight -10 and degree 2 with double poles at $z = 0$. We now state the relation between degeneracy and partition function. There are two perspective for this relation. In the first case, we express the degeneracy using an integral called the *Fourier integral* involving the partition function and in the second perspective, we observe that the degeneracy is related to the Fourier coefficients in the Fourier expansion of the partition function. We will discuss both perspective and observe their relation.

Theorem 7.5. *[18] Let P and Q be charge vectors. Then the degeneracy $d(Q, P)$ is given by*

$$d(Q,P) = (-1)^{Q\cdot P+1} \int\limits_{\mathcal{C}} d\tau d\sigma dz \; e^{-\pi i(\sigma Q^2 + 2zQ\cdot P + \tau P^2)}, \tag{7.2}$$

where $\mathcal{C}$ is a three real dimensional subspace of the three complex dimensional space labelled by

$$\begin{aligned} (\sigma, \tau, z) &\equiv (\sigma_1 + i\sigma_2, \tau_1 + i\tau_2, z_1 + iz_2), \\ \sigma_2 = M_1, \tau_2 = M_2, z_2 &= M_3 \quad \text{and} \quad 0 \le \tau_1, \sigma_1, z_1 \le 1, \end{aligned} \tag{7.3}$$

where

$$\begin{aligned} M_1 &= \Lambda\left(\frac{|\lambda|^2}{\lambda_2} + \frac{Q_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right) \\ M_2 &= \Lambda\left(\frac{1}{\lambda_2} + \frac{P_R^2}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right) \\ M_3 &= -\Lambda\left(\frac{\lambda_1}{\lambda_2} + \frac{Q_R\cdot P_R}{\sqrt{Q_R^2 P_R^2 - (Q_R\cdot P_R)^2}}\right), \end{aligned} \tag{7.4}$$

where Λ is a large positive number and

$$Q_R^2 = Q^T(M+L)Q, P_R^2 = P^T(M+L)P, Q_R \cdot P_R = Q^T(M+L)P,$$

$\lambda = \lambda_1 + i\lambda_2$ denotes the asymptotic value of the axion-dilaton moduli which belong to the gravity multiplet and M is the asymptotic value of the 28×28 symmetric matrix valued moduli field of the matter multiplet satisfying $MLM^T = L$.

We can now invert the Fourier integral (7.2) to define the degeneracy in the second perspective.

Theorem 7.6. *[12] Let P and Q be charge vectors. Then the degeneracy $d(Q,P)$ is given by*

$$d(Q,P) = (-1)^{Q\cdot P+1} g\left(\frac{Q^2}{2}, Q\cdot P, \frac{P^2}{2}\right) \tag{7.5}$$

where $g(m,r,n)$ are the coefficients of Fourier expansion of $Z(\Omega)$,

$$Z(\Omega) = \sum_{m,r,n} g(m,r,n) e^{2\pi i(m\sigma + rz + n\tau)}. \tag{7.6}$$

The relation between the two perspective is the following. Different choices of (M_1, M_2, M_3) in (7.4) mean that we expand $Z(\Omega)$ in different ways to get different values of $g(m,r,n)$ in the expansion (7.6). Conversely, if we define $d(Q,P)$ in a given domain of the asymptotic moduli space by (7.5), then the then the choice of (M_1, M_2, M_3) is determined by requiring that the series (7.6) be convergent for $(\sigma_2, \tau_2, z_2) = (M_1, M_2, M_3)$.
To analyse the posed question, we first need to extract the attractor black holes degeneracy. To do this, we need to choose the following special values of the parameters,

$$Q_R^2 = 2Q^2, P_R^2 = 2P^2, \lambda_2 = \frac{\sqrt{Q^2P^2 - (Q\cdot P)^2}}{P^2}, \lambda_1 = \frac{Q\cdot P}{P^2}$$

Substituting this in Eq. (7.4) gives,

$$\begin{aligned} M_1 = 2\Lambda\frac{Q^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, M_2 = 2\Lambda\frac{P^2}{\sqrt{Q^2P^2-(Q\cdot P)^2}}, \\ M_3 = -2\Lambda\frac{Q\cdot P}{\sqrt{Q^2P^2-(Q\cdot P)^2}} \end{aligned} \tag{7.7}$$

Thus, given $T-$duality invariants $(Q^2/2, Q\cdot P, P^2/2) \equiv (m,r,n)$, the attractor black holes degeneracy is given by $d(Q,P) = (-1)^{r+1}g(m,r,n)$ if the series in (7.6) converges for $(\sigma_2, \tau_2, z_2) = (M_1, M_2, M_3)$ where (M_1, M_2, M_3) is given by (7.4). We will follow the following strategy. We will first assign degeneracy to each $U-$duality class. Then we will try to relate the degeneracy of different classes. This in turn will give us some insight into the interpretation of different classes. But now there is one issue that we need to fix. We need to prove that degeneracy does not vary in a given $U-$duality class if we are to assign degeneracies to $U-$duality classes. We will first prove this and then finally prove that degeneracy of a black hole class and its inverse class is the same.
Let P, Q be charge vectors. The $U-$duality class represented by this charge vector will be denoted by $[m,r,n]$ where $Q^2/2 = m, P^2/2 = n$ and $P\cdot Q = r$. To show that degeneracy does not vary in a given $U-$duality class, we need to show that if $Q_{Q',P'} = MQ_{Q,P}M^t$ for some $M \in SL_2(\mathbb{Z})$, then $d(Q,P) = d(Q',P')$. We also need to make sure that the the series (7.6) for the transformed partition function induced by the embedding of $SL_2(\mathbb{Z})$ into $Sp_2(\mathbb{Z})$ converges for $(\tilde{\tau}_2, \tilde{\sigma}_2, \tilde{z}_2) = (M'_1, M'_2, M'_3)$.
To be precise, let

$$\Omega' = M\Omega M^t \quad \text{for} \quad M \in SL_2(\mathbb{Z})$$

and (M'_1, M'_2, M'_3) be as in (7.4) for transformed charge vectors Q', P'. Then, we need to show that the series (7.6) for $Z(\Omega')$ converges for $(\sigma'_2, \tau'_2, z'_2) = (M'_1, M'_2, M'_3)$.
Let us first prove the first part.

Theorem 7.7. *If $Q_{Q',P'} = MQ_{Q,P}M^t$ for some $M \in SL_2(\mathbb{Z})$, then $d(Q,P) = d(Q',P')$.*

Proof. First observe that $(-1)^{P\cdot Q} = (-1)^{P'\cdot Q'}$. To see this let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then if

$$Q_{Q,P} = \begin{pmatrix} P^2 & P \cdot Q \\ P \cdot Q & Q^2 \end{pmatrix} = \begin{pmatrix} 2n & r \\ r & 2m \end{pmatrix}$$

then

$$Q_{Q',P'} = \begin{pmatrix} P'^2 & P' \cdot Q' \\ P' \cdot Q' & Q'^2 \end{pmatrix} =$$
$$\begin{pmatrix} * & 2can + r(ad+bc) + 2mbd \\ 2can + r(ad+bc) + 2mbd & * \end{pmatrix}.$$

But $ad - bc = 1$, so $Q' \cdot P' = 2(can + mbd + rbc) + r = 2(can + mbd + rbc) + P \cdot Q$. Thus $(-1)^{P \cdot Q} = (-1)^{P' \cdot Q'}$. We now need to show that

$$g\left(\tfrac{Q^2}{2}, Q \cdot P, \tfrac{P^2}{2}\right) = g\left(\tfrac{Q'^2}{2}, Q' \cdot P', \tfrac{P'^2}{2}\right).$$

It suffices to prove this for $M = S, T$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ since S, T generate $SL_2(\mathbb{Z})$. We first consider T. From Eq. (5.14) and Eq. (5.15)

$$\psi_m(\tau, z) = \sum_{\substack{n \geq -1 \\ r \in \mathbb{Z}}} g(m, r, n) q^n \zeta^r.$$

Now observe that for

$$Q_{Q,P} = 2 \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}, \; Q_{Q',P'} = T Q_{Q,P} T^t = 2 \begin{pmatrix} n+m+r & r/2+m \\ r/2+m & m \end{pmatrix}.$$

The corresponding degeneracies involve $g(m, r, n)$ and $g(m, r+2m, n+m+r)$ which are same as a result of remark 5.20 for $k = 1$. We now consider S. Next, observe that

$$S Q_{Q,P} S^t = \begin{pmatrix} m & -r/2 \\ -r/2 & n \end{pmatrix}.$$

Thus we need to show that $g(m, r, n) = g(n, -r, m)$. Now since $Z(\Omega)$ is a Siegel modular form and $S \in GL_2(\mathbb{Z})$, thus $\begin{pmatrix} U & 0 \\ 0 & (U^t)^{-1} \end{pmatrix} \in Sp_2(\mathbb{Z})$. Hence

$$Z(S \Omega S^t) = Z(\Omega)$$

Moreover for

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}, \qquad S\Omega S^t = \begin{pmatrix} \sigma & -z \\ -z & \tau \end{pmatrix}.$$

We now insert the expansion from Eq. (7.6) on both sides to get

$$\sum_{\substack{m,n\geq -1\\ r\in\mathbb{Z}}} g(m,r,n)q^m\zeta^{-r}w^n = \sum_{\substack{m,n\geq -1\\ r\in\mathbb{Z}}} g(m,r,n)q^n\zeta^r w^m.$$

Renaming the variables $n \leftrightarrow m, r \mapsto -r$, we see that the sum ranges over the same terms and hence we get

$$\sum_{\substack{m,n\geq -1\\ r\in\mathbb{Z}}} g(n,-r,m)q^n\zeta^r w^m = \sum_{\substack{m,n\geq -1\\ r\in\mathbb{Z}}} g(m,r,n)q^n\zeta^r w^m.$$

which finally gives $g(m,r,n) = g(n,-r,m)$. $\square$

We now show that $Z(\Omega')$ indeed converges for $(\sigma_2', \tau_2', z_2') = (M_1', M_2', M_3')$. But observe that the domain also transforms under $U-$duality transformation exactly in the same way. That is, Ω' and $Q_{Q',P'}$ are related to Ω and $Q_{Q,P}$ respectively in exactly the same way under $U-$duality transformation. Hence the series converges in the new domain as well.
Now, to prove that the degeneracy of a $U-$duality class and its inverse class is the same, we need to show that

$$(-1)^{Q\cdot P+1}g\left(\tfrac{Q^2}{2}, Q\cdot P, \tfrac{P^2}{2}\right) = (-1)^{-Q\cdot P+1}g\left(\tfrac{Q^2}{2}, -Q\cdot P, \tfrac{P^2}{2}\right).$$

since $[m,r,n]^{-1} = [m,-r,n] = [n,r,m]$. We further need to justify the convergence of the series for the transformed domain. First observe that $(-1)^{Q\cdot P+1} = (-1)^{-Q\cdot P+1}$. Secondly, we justify the convergence. Observe that under the class group action on the (σ,τ,z) domain we have, $(\sigma',\tau',z') = (\sigma,\tau,-z)$ as explained in the proof of the next Theorem. At the same time, it is easily seen that $(M_1', M_2', M_3') = (M_1, M_2, -M_3)$. We conclude that under these transformations, the series remains invariant and hence converges. We conclude the proof by proving the next Theorem.

Theorem 7.8. *We have $g(m,r,n) = g(m,-r,n)$ where g is the coefficients in the Fourier expansion in Eq. (7.6).*

Proof. Let $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(\mathbb{Z})$. Then

$$Z(U\Omega U^t) = (-1)^{-10} Z(\Omega) = Z(\Omega).$$

Also note that

$$U\Omega U^t = \begin{pmatrix} \tau & -z \\ -z & \sigma \end{pmatrix}.$$

Plugging the Fourier expansion of Eq. 7.6, we get

$$\sum_{\substack{m,n \geq -1 \\ r \in \mathbb{Z}}} g(m,r,n) q^n \zeta^{-r} w^m = \sum_{\substack{m,n \geq -1 \\ r \in \mathbb{Z}}} g(m,r,n) q^n \zeta^r w^m.$$

Again renaming the indices $r \mapsto -r$, we get

$$\sum_{\substack{m,n \geq -1 \\ r \in \mathbb{Z}}} g(m,-r,n) q^n \zeta^r w^m = \sum_{\substack{m,n \geq -1 \\ r \in \mathbb{Z}}} c(m,r,n) q^n \zeta^r w^m$$

which gives $g(m,r,n) = g(m,-r,n)$. □

7.3 Entropy of a black hole class

Now that we have one connection between a $U-$duality class and its inverse class, we may now try to connect other $U-$duality classes using degeneracy. The integral or series representation does not help much because of the complicated nature of class group operation. Hence, we now probe the asymptotic entropy of attractor black holes upto linear correction in charges which is calculated using the *Entropy function approach*[12]. We first discuss the result and then try to understand the class group operations.

Define

$$\tau_2 = \frac{\sqrt{Q^2P^2 - (Q \cdot P)^2}}{P^2}, \tau_1 = \frac{Q \cdot P}{P^2} \quad \text{and} \quad \tau = \tau_1 + i\tau_2 \tag{7.8}$$

The entropy function analysis for these black holes gives the statistical entropy of black holes with charge vectors Q, P to be

$$S_{stat} \simeq -\Gamma_B(\tau) \text{ at } \frac{\partial \Gamma_B(\tau)}{\partial \tau} = 0 \tag{7.9}$$

where

$$-\Gamma_B(\tau) = \frac{\pi}{2\tau_2}|Q - \tau P|^2 - \ln g(\tau) - \ln g(-\overline{\tau}) - (k+2)\ln 2\tau_2 + \ln 4\pi K + \mathcal{O}(Q^{-2}) \tag{7.10}$$

where $g(\tau) = \eta(\tau)^{24}$ and K is a constant and $k = 10$ for our case. Now since, $S_{stat} = \ln d(Q, P)$, we can study the relation between degeneracies using the statistical entropy. Since τ depends on the charges, thus the class group composition also acts on τ. Thus we first want to relate $\tau' s$ corresponding to different $U-$duality classes. As a first example, consider the *Dirichlet composition* of *united forms* of Theorem 6.10. Suppose, τ, τ' and $\widehat{\tau}$ are the quantities defined in (7.8) corresponding to the classes $[m_1, r, n_1]$, $[m_2, r, n_2]$ and $[m_1 m_2, r, n]$ respectively satisfying the hypothesis of Theorem 6.10 (and hence the relation (6.1)), then it can easily be seen that

$$\hat{\tau} = k_1 \tau = k_2 \tau' \text{ where } k_1 = n_1/n \text{ and } k_2 = n_2/n$$

Here as usual we have for the class $[m, r, n]$, $Q^2/2 = m, P^2/2 = n$ and $Q \cdot P = r$. Thus at least for this special case, the problem of relating degeneracies reduces to relating the quantities $g(\tau)$ and $g(k\tau)$ for some constant integer k. Furthermore, if we impose the condition that $n_1 | n_2$ (or $n_2 | n_1$), then $n = n_1$ (or $n = n_2$) which implies $k_1 = 1$ (or $k_2 = 1$). Thus we get $g(\widehat{\tau}) = g(\tau)$ (or $g(\widehat{\tau}) = g(\tau'))$ which in turn means $d(\widehat{Q}, \widehat{P}) = d(Q, P)$ (or $d(\widehat{Q}, \widehat{P}) = d(Q', P')$) using relation (7.10). For general elements of the class group, we need to use general Dirichlet composition as stated in the Theorem 6.11. Now observe that with the notations introduced in (7.8), the composition in (6.4) is exactly

$$[1, \tau] * [1, \tau'] = [1, \widehat{\tau}]$$

The next step is to find $\widehat{\tau}$ in terms of τ and τ'. The relation is obvious from (6.5) and the result is

$$\widehat{\tau} = \alpha\tau' + \beta\tau + \gamma\tau\tau' \tag{7.11}$$

Now we would like to relate $g(\widehat{\tau}), g(\tau')$ and $g(\tau)$. The first question we should ask is whether such a relation is possible. To answer this question, first consider the simplest possible composition of classes as in Theorem (6.10). In this case, we need to relate $g(\tau)$ and $g(k\tau)$ for some $k \in \mathbb{Z}$. We first prove a technical Lemma which shows that we can take k to be a positive integer.

Lemma 7.9. If $[m_i, r_i, n_i], \quad i = 1, 2$ are two united forms with discriminant $D < 0$ then we can assume n_i to be positive for $i = 1, 2$.

Proof. First observe that $[m, r, n]$=$[n, -r, m]$ as the representatives are related by the matrix S. Moreover $m_i > 0$ for each i since these quadratic forms are positive definite and $D < 0$. Now suppose any one or both of n_i is negative. We will follow the following steps to find representatives of these two united forms with positive coefficients of y^2 which are still united :

1. Apply S to both classes to get new representatives $[n_i, -r_i, m_i], \quad i = 1, 2$.
2. Apply T^{k_i} on $[n_i, -r_i, m_i]$ to get representatives $[n_i - k_i r + m_i k_i^2, -r_i + 2k_i m_i, m_i], \quad i = 1, 2$. We will determine k_i later.
3. Apply S on the two classes to get representatives $[m_i, r_i - 2k_i m_i, n_i - k_i r + m_i k_i^2], \quad i = 1, 2$.

Now, to get k_i, we will use the fact that $[m_i, r_i - 2k_i m_i, n_i - k_i r + m_i k_i^2], \quad i = 1, 2$ are united forms and $n_i - k_i r + m_i k_i^2 > 0$ for each i. The first constraint is $r_1 - 2k_1 m_1 = r_2 - 2k_2 m_2$. This is satisfied if we choose $k_1 = cm_2, k_2 = cm_1$ for any $c \in \mathbb{Z}$. Now we can choose c large enough such that $n_i - k_i r + m_i k_i^2 > 0$. This completes the proof. □

The proof of Theorem 3.40 shows that $f(rz) \in M_k(\Gamma_1(rN)) \setminus M_k(\Gamma_1(N))$. We apply this Theorem to $g(k\tau)$ for positive k. We get that $g(k\tau) \in S_{12}(\Gamma_1(k))$ since $g(\tau) \in S_{12}(\Gamma_1(1))$ and $\Gamma_1(1) = SL_2(\mathbb{Z})$. Moreover since $\Gamma_1(k) \subset \Gamma_1(1)$, thus $S_{12}(\Gamma_1(1)) \subset S_{12}(\Gamma_1(k)$. Thus $S_{12}(\Gamma_1(1)$ is a one dimensional subspace of $S_{12}(\Gamma_1(k)$ and $g(k\tau) \in S_{12}(\Gamma_1(k)) \setminus S_{12}(\Gamma_1(1))$. Thus $g(k\tau)$ and $g(\tau)$ are linearly independent. Now observe that if $g(k\tau) = F(g(\tau))$, then F must be linear in $g(\tau)$ since higher powers will increase weight of $F(g(\tau)$ and other type of functions (exponential, logarithm and so on) will destroy modularity. Now suppose $g(k\tau) = F_1(\tau)g(\tau) + F_2(\tau)$ assuming that F_2 does not contain $g(\tau)$ and $g(k\tau)$, then F_1 must be modular with weight 0 but only weight zero modular forms are constants. So $F_1 = c \in \mathbb{C}$. Moreover, $F_2(\tau) = g(k\tau) - cg(\tau)$ implies that $F_2 \in S_{12}(\Gamma_1(k))$. Suppose $\dim(S_{12}(\Gamma_1(k))) = n$. Then there exists a basis $\{f_1, f_2, ..., f_{n-1}, f_n\}$ such that $f_1 = g(\tau)$ and $f_2 = g(k\tau)$. Then we have

$$F_2(\tau) = \sum_{i=3}^{n} c_i f_i(\tau)$$

This relation gives

$$-cf_1(\tau) - f_2(\tau) + \sum_{i=3}^{n} c_i f_i(\tau) = 0$$

This contradicts the fact that $\{f_1, f_2, ..., f_{n-1}, f_n\}$ is a basis. Thus there cannot be any relation between $g(\tau)$ and $g(k\tau)$. Now, Lemma (7.9) says that we may always assume k to be positive so that the discussion above says that there cannot be any relation between the degeneracies corresponding to united forms classes of black holes. There can be one other way in which we can relate the degeneracies. We can study the quantity $g(\tau)g(-\overline{\tau})(2\tau_2)^{12}$ since

$$\begin{aligned}\ln\left(d(Q,P)\right) = S_{stat} = -\Gamma_B(\tau) = \frac{\pi}{2\tau_2}|Q - \tau P|^2 - \ln\left[g(\tau)g(-\overline{\tau})(2\tau_2)^{12}\right] \\ + \ln 4\pi K + O(Q^{-2})\end{aligned}$$

We will again show that such a relation does not exist in this case. Theorem 3.32 implies that $g(\tau)g(-\overline{\tau})(2\tau_2)^{12}$ is a modular function. Thus by Theorem 3.33, we have that

$$g(\tau)g(-\overline{\tau})(2\tau_2)^{12} = \frac{F_1(\tau)}{F_2(\tau)}$$

where F_1 and F_2 are modular forms with respect to $SL_2(\mathbb{Z})$ of the same weight.

Also observe that if

$$g(\tau)g(-\overline{\tau})(2\tau_2)^{12} = G(g(k\tau)g(-k\overline{\tau})(2k\tau_2)^{12})$$

for some function G then

$$F_1(\tau) = G(g(k\tau)g(-k\overline{\tau})(2k\tau_2)^{12})F_2(\tau) \tag{7.12}$$

But since $g(k\tau)g(-k\overline{\tau})(2k\tau_2)^{12}$ is a modular function for $\Gamma_1(k)$, so is $G(g(k\tau)g(-k\overline{\tau})(2k\tau_2)^{12})$ since G must be sufficiently "nice" as $g(\tau)g(-\overline{\tau})(2\tau_2)^{12}$ is holomorphic on $\mathbb{H}$. But now observe that (7.12) cannot hold since L.H.S satisfies modularity with respect to $SL_2(\mathbb{Z})$ but R.H.S does not satisfy modularity with respect to the full modular group. This is because $G(g(k\tau)g(-k\overline{\tau})(2k\tau_2)^{12})$ is a modular function for $\Gamma_1(k)$ and $\Gamma_1(k) \neq SL_2(\mathbb{Z})$ unless $k = 1$.

7.4 Identity black hole

Let us look at the identity black hole class. By theorem 6.15, the identity class in the class group is given by

$$1_D = \begin{cases} \left[1, 0, -\frac{D}{4}\right] & \text{if } D \equiv 0 (\text{mod } 4) \\ \left[1, 1, \frac{1-D}{4}\right] & \text{if } D \equiv 1 (\text{mod } 4). \end{cases}$$

By (7.4), the corresponding attractor point in the moduli space is

$$\tau = \begin{cases} \frac{2i}{\sqrt{-D}} & \text{if } D \equiv 0 (\text{mod } 4) \\ \frac{4i}{i+\sqrt{-D}} & \text{if } D \equiv 1 (\text{mod } 4). \end{cases}$$

Thus, observe that for a fixed D, the entropy upto linear correction remains constant in the identity class. Explicitly, the entropy upto linear correction is

$$S_{id} = \pi\sqrt{-D} - \begin{cases} \ln\left(\left|g\left(\frac{2i}{\sqrt{-D}}\right)\right|^2 \frac{4^{12}}{D^6}\right) & \text{if } D \equiv 0 (\text{mod } 4) \\ \ln\left(\left|g\left(\frac{4i}{i+\sqrt{-D}}\right)\right|^2 \frac{(16D)^6}{(1-D)^{12}}\right) & \text{if } D \equiv 1 (\text{mod } 4) \end{cases}$$

This physical property may be used to define the identity class. Let us make this definition.

Definition 7.10. Let $D < 0$ be a fundamental discriminant. Consider the class group of $U-$duality equivalence classes of attractor black holes with leading entropy $S = \pi\sqrt{-D}$. The equivalence class with entropy upto leading correction given by

$$S_{id} = \pi\sqrt{-D} - \begin{cases} \ln\left(\left|g\left(\frac{2i}{\sqrt{-D}}\right)\right|^2 \frac{4^{12}}{D^6}\right) & \text{if } D \equiv 0 (\text{mod } 4) \\ \ln\left(\left|g\left(\frac{4i}{i+\sqrt{-D}}\right)\right|^2 \frac{(16D)^6}{(1-D)^{12}}\right) & \text{if } D \equiv 1 (\text{mod } 4). \end{cases}$$

is called the identity class of the class group.

Let us check that this definition is compatible with the class group operation. Let $[m, r, n]$ be one class. We have that $\left[1, 0, -\frac{D}{4}\right] * [m, r, n] = [m, r, n]$. Let τ, τ' and $\widehat{\tau}$ be the corresponding attractor points in the moduli space. Then we show that $\widehat{\tau} = \tau'$. Since $\left[1, 0, -\frac{D}{4}\right] = \left[-\frac{D}{4}, 0, 1\right]$, thus $\alpha = 1, \beta = \gamma = 0$ using Corollary 6.12. Thus by Eq. (7.11), we get $\widehat{\tau} = \tau'$.

8. DIRECTIONS FOR FURTHER WORK

Although much of the theory of automorphic forms appears in string theory, one part which was discovered as recently as 2012 by Dabholkar, Murthy and Zagier (DMZ) is most relevant to the materials discussed in this thesis in previous chapters. In [17], DMZ begin by proving a structure theorem for Jacobi forms. They prove that every Jacobi form φ can be decomposed into a *finite part* φ^F and a *polar part* φ^P. They go on proving that the finite part φ^F is a mock Jacobi form which are a mix of Jacobi forms and mock modular forms. Then they analyse its relevance in string theory relating the mock Jacobi form to the degeneracy of single centered black holes in Type IIB string theory compactified on $K3 \times T^2$. The proof of the structure theorem mentioned above requires the theory of Apell Lerch sums and has rich mathematical content. We will now outline the result precisely in terms of the physical quantities discussed in the thesis.
Recall that the partition function for the theory is given by

$$\frac{1}{\Phi_{10}(\Omega)} = \sum_{m \geq -1} \psi_m(\tau, z) w^m. \tag{8.1}$$

We now define the polar part of ψ_m as follows,

$$\psi_m^P(\tau, z) := \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \sum_{s \in \mathbb{Z}} \frac{q^{ms^2+s}\zeta^{2ms+1}}{(1-\zeta q^s)^2} \tag{8.2}$$

where $q = e^{2\pi i \tau}$ and $\zeta = e^{2\pi i z}$. $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind eta function. Also $p_{24}(n)$ counts the number of partitions of an integer n into 24 parts. Define the finite or Fourier part of ψ_m as

$$\psi_m^F \coloneqq \psi_m - \psi_m^P. \tag{8.3}$$

It turns out that ψ_m^F is a *mock Jacobi form* of weight -10 and index m. The indexed degeneracies of the single-centered black hole of magnetic charge invariant $Q^2/2 = m$ which is obtained from the attractor mechanism, are the Fourier coefficients of the function ψ_m^F. To be precise, if we have the Fourier expansion of ψ_m^F as follows,

$$\psi_m^F = \sum_{n,r} c(n,r) q^n \zeta^r \tag{8.4}$$

then the indexed degeneracy $d_{micro}(n,m,r)$ corresponding to the single-centered black holes are given by

$$d_{micro}(n,m,r) = (-1)^{r+1} c(n,r). \tag{8.5}$$

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