

ALGEBRAIC AND TOPOLOGICAL STRUCTURES FROM QUANTUM FIELD THEORIES

By

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ABSTRACT OF THE DISSERTATION

Algebraic and topological structures from quantum field theories

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In this thesis, we study algebraic and topological structures that emerge from quantum field theories in two and four dimensions. In four dimensions, we study the topological twisting of arbitrary 4d $\mathcal{N} = 2$ theories and show that the correlation functions of the twisted theory depend only on the smooth structure on the 4-manifold, characteristic classes for background gerbe connection and a generalized spin-c structure. We illustrate our proposal for topological twisting by working out some Lagrangian and non-Lagrangian class $\mathcal{S}$ theory examples. In two dimensions, we study a class of rational conformal field theories (CFT), called Wess-Zumino-Witten (WZW) models and their gaugings. Algebraically, 2d rational conformal field theories can be described using vertex operator algebras and their irreducible modules. We discover that this is not strictly true for gauged Wess-Zumino-Witten models, it is related to the Goddard-Kent-Olive coset CFT, which is the algebraic description of gauged WZW models, via the coupling of a 2d topological quantum field theory. We also study the application of vertex operator algebras to Monstrous Moonshine. We show that the spin lift of the Monster CFT, which is a $c = 24$ holomorphic CFT, has superconformal symmetry by explicitly constructing the superconformal current. We use classical and quantum error correcting codes for this construction.

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¹This paper is called “Arithmetic and Attractors” and it explores the connection between number theory and black holes [Moo98]

²I was rejected in 2019 application cycle and finally accepted in 2020.

almost broke me mentally.

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Dedication

To my father and my mother

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Chapter 1

Introduction

Physics is the study of patterns in the observations of the external world in such a way as to formulate a set of rules which could then be used to predict phenomena. For example, Newton’s second law, $F = ma$, captures all observations pertaining to the acceleration of an object when a force is applied on it, as well as it predicts the outcome of the experiment of applying a certain force on a body. This amounts to a theory in physics. A spectacular example of this endeavor is the standard model of particle physics. It explains observations of subatomic particles in the Large Hadron Collider as well as predicts the existence of new particles. It is based on the framework of quantum field theory (QFT). Unsurprisingly, QFT is one of the most important areas of theoretical physics. QFT is a culmination of efforts to unify quantum mechanics with special relativity and it must be admitted that it is still a work in progress.¹ As we keep uncovering new principles and new rules of QFT, we need new mathematics to describe it. Naturally, ever-increasing sophistication in mathematics finds a perfect spot in the description of QFTs. For a large part of the history of theoretical physics, this has been a one way street, where mathematics played a pivotal role in formulating physical theories. This started to change in the 1970s when some mathematicians, including Michael Atiyah, Vladimir Drinfeld, Nigel Hitchin, Yuri Manin, Isadore Singer started using physics motivated equations to prove new mathematical theorems [AHDM78, AW77]. This two way communication has become stronger since then. We will explore some of the aspects of this relationship in this thesis. This program of proving new results in mathematics using insights from physics has been called *Physical mathematics* [Moo14a, Moo14b]. This thesis is based on two such examples which we now describe.

The first example concerns the classification of smooth structures on 4-manifolds. Physics studies changes in systems, for example change in position, speed, acceleration and so on. These changes are measured by taking successive derivatives. Thus, the ability to take derivatives is of fundamental

¹In a lecture titled “What is a quantum field theory?” (can be accessed on the link: <https://www.youtube.com/watch?v=GZvs-ae4YRA>), Nathan Seiberg explains that the signature of a mature field is that the books and courses on the subject are more or less structured in a standard way. He concludes that “QFT is thus not only incomplete but also an immature field because every university and every book has its own way of teaching the subject”.

importance in physics. This is of particular significance in four dimensions because we happen to inhabit a four-dimensional spacetime. In mathematics, the ability to take derivatives is studied in differential calculus and a space on which we can take derivatives is called a *smooth manifold* or simply a *manifold*. Specifying a way of taking derivatives is specifying the *smooth structure* on the manifold. Naturally, we want to consider spacetimes which are manifolds. Now the following problem arises: typically there are many (sometimes uncountably many) inequivalent smooth structures on a 4-manifold. In mathematics, it is not clear how to differentiate between these smooth structures. But in physics, particles moving on a spacetime may be able to distinguish two different smooth structures on the spacetime. This indeed turns out to be true. There are solutions to the *nonabelian Yang-Mills equations*, the same equations that govern the dynamics of particles in the standard model, which detect this difference. These solutions are called *instantons* [BPST75]. In 1982, Simon Donaldson defined the *Donaldson polynomial* [Don90] for a given 4-manifold $\mathbb{X}$, which is a two variable polynomial with rational coefficients. The coefficients are computed by integrating certain differential forms on the space of instantons (the so called *moduli space of instantons*) on $\mathbb{X}$. These polynomials are *invariants* of smooth structures, meaning that they only depend on the smooth structure of $\mathbb{X}$ and² can distinguish smooth structures on $\mathbb{X}$. This was a major breakthrough in mathematics. Many new results were proved using the Donaldson polynomials. At this stage, it was clear that physics had a role to play in classification of smooth structures. Michael Atiyah, speculating such a connection, asked if Donaldson invariants could be interpreted as physical quantities. Edward Witten answered the question in 1988 in a landmark paper [Wit88]. This paper initiated two fields, both led to significant developments in physical mathematics as we will explain below. First, he defined a *topological quantum field theory* (TQFT). Roughly speaking, a TQFT is a theory which does not depend on the metric on spacetime³. Atiyah then axiomatized the essential features of the definition given by Witten in [Ati89]. Getting back to the story, Witten also discovered *topological twisting*, a procedure of constructing a topological quantum field theory from certain *supersymmetric field theories*. He discovered that applying topological twisting to four dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theory with gauge group $SU(2)$, which can

²This is true only for some special 4-manifolds, see below (1.1.6) below.

³Technically speaking, Witten defined, what are called TQFTs of *cohomological type*. These theories admit a symmetry generated by a nilpotent scalar supercharge $\mathcal{Q}$ with the property that the stress tensor is $\mathcal{Q}$ -exact. The relevant observables of the theory are correlation functions of $\mathcal{Q}$ -closed operators. We give some more details in Section 1.1.3.

only be defined on a certain class of 4-manifolds called *spin* 4-manifolds,⁴ we get a theory (called twisted SYM theory) which can be defined on any oriented Riemannian 4-manifold. The remarkable achievement of [Wit88] was that it showed that Donaldson invariants are the correlation functions of certain operators in the twisted SYM theory. This equality motivated Witten to use equations much simpler than instanton equations, the so called *Seiberg-Witten equations* or *monopole equations*, to define new 4-manifold invariants [Wit94]. These new invariants were called *Seiberg-Witten invariants*. They can also be computed from twisted SYM theory with *matter* with gauge group $U(1)$. Computing Seiberg-Witten invariants is much easier than computing Donaldson invariants. Many new results were proved using Seiberg-Witten invariants.

In physics, the basic object for computing correlation functions is the *path integral*. Path integrals are usually not mathematically well defined but physicists have learned to work with it in a way which can be used to produce sensible results. Many physical principles, for example *dualities* and *renormalization group flow*, relate the path integrals of two completely different theories. This opens up the possibility of using physical ideas to compute the Donaldson invariants or even construct new invariants. Moore-Witten realized this dream in [MW97]. Under renormalization group flow, the SYM theory with gauge group $SU(2)$ is related to the SYM theory with gauge group $U(1)$. Physics then tells us that we can write Donaldson invariants in terms of Seiberg-Witten invariants and this was precisely what was shown in [MW97]. This physical result was then rigorously proven by mathematicians [FL01, FL02, FL15]. This was a major contribution of physical mathematics. We give some more details of this story in Section 1.1 below.

It turns out that there is an infinite class of 4d $\mathcal{N} = 2$ supersymmetric field theories and one can apply topological twisting to all of them. Potentially, they can lead to new 4-manifold invariants. In this thesis, we study the first step in this program. We study the topological twisting of an arbitrary 4d $\mathcal{N} = 2$ field theory and determine what extra structures are needed on the 4-manifold $\mathbb{X}$ to define the twisted theory.

We now discuss the second application of physical mathematics. Conformal field theories (CFTs) have been an immensely important area of physics owing to its application to *critical phenomena*. These theories are *scale-invariant*, meaning that the observables are invariant under scaling of lengths.

⁴The traditional viewpoint is that since 4d $\mathcal{N} = 2$ SYM has fermions, the theory can only be defined on 4-manifolds which admit *spin structure*, i.e., on *spin manifolds*. We also need a metric to write the action of the theory. As we argue in Chapter 4, this is not the correct viewpoint. We show that 4d $\mathcal{N} = 2$ theories can be defined on any oriented Riemannian 4-manifold by choosing appropriate background fields. Topological twisting then amounts to specifying the background field in a specific way.

Conformal field theories describe *phase transitions*, for example the water to vapor transition. In 1984, Belavin-Polyakov-Zamolodchikov (BPZ) discovered that in two-dimensional conformal field theories, the scale and Lorentz invariance symmetry actually enhances to an infinite-dimensional symmetry algebra, called the *Virasoro algebra* [BPZ84]. This led to a flurry of work which developed ways to solve the theory exactly. Deep algebraic structures, like *vertex operator algebras*, and number theoretic functions, like *modular forms*, were discovered in 2d CFTs.

A few years before the BPZ paper, mathematicians had completed the classification of finite simple groups. The conclusion was that there are 18 infinite families and 26 *sporadic groups* which do not fit into any of the infinite families. The largest of the sporadic group, which is called the *Monster group* [Gri82], exhibited some peculiar features [Tho79a]. The dimension of the irreducible representations of the Monster group was related to the coefficients of a certain modular form, called the *J*-invariant [Tho79a, Tho79b]. Since this observation seemed to be coincidental and an explanation seemed far-fetched, it was dubbed *Monstrous Moonshine* [CN79]. Due to conformal symmetry, the *torus partition function* of a 2d CFT is modular invariant [DFMS97]. Moreover, the torus partition function in a 2d CFT counts states. This means that the coefficients of the torus partition function of a 2d CFT are the dimensions of certain subspaces of the vector space of states of the theory. This indicates that 2d CFT might have a role to play in the explanation of Monstrous Moonshine. Indeed, it was shown by Frenkel-Lepowsky-Meurman [FLM88a] that there is an infinite-dimensional representation of the Monster group with the structure of a vertex operator algebra. The full Monstrous Moonshine conjecture was later proved by Borchers [Bor86a, Bor92a]. In physics, we interpret it as a 2d CFT with the Monster group as the symmetry group and whose *torus partition function* is the *J*-function. This CFT was called the Monster CFT. This gave a conceptual explanation for the Monstrous Moonshine phenomena. Some more details of this aspect of physical mathematics have been reviewed in Section 1.2. In 1988, Dixon-Ginsparg-Harvey [DGH88] observed, based on the physical description of the Monster CFT given by Harvey [Har85], that the conformal symmetry of the Monster CFT can in fact be enhanced to a larger symmetry, called *superconformal symmetry*. The essential aspect of this symmetry is the existence of a *superconformal current*. While they gave compelling evidence for the existence of a superconformal current in the Monster CFT, an explicit construction was missing. In this thesis, we will give an explicit construction of a superconformal current.

Another class of 2d CFTs studied in this thesis are the Wess-Zumino-Witten (WZW) models [Wit84]. These models are specified a compact semisimple Lie group G and a level k . They belong to a class of 2d CFTs which can be constructed completely using algebraic data of vertex operator algebras. Algebraically, in WZW models the Virasoro symmetry gets enhanced to the *affine*

Kac-Moody algebra. Now, given a subgroup H of G , one can construct another CFT from WZW models called *gauged WZW model*. These CFTs are denoted by $\text{WZW}(G/H)$. These models are important because they give the explicit construction of many of earlier CFTs, *minimal models* for example, constructed completely using the algebraic data of vertex operator algebras⁵. The algebraic construction of these vertex operator algebras goes by the name Goddard-Kent-Olive (GKO) coset construction [GKO85, GKO86]. It was believed that gauged WZW models are algebraically described by GKO coset construction [BRS88, ABR88, GK89]. In this thesis, we explore the relationship between the two in more detail. Topological quantum field theory makes an appearance in this relationship. We argue that the precise relationship between gauged WZW models and GKO coset construction is by coupling to a 2d TQFT.

In the rest of the chapter, we will give a more technical summary of the topics discussed above.

1.1 Topological twisting and 4-manifold invariants

1.1.1 Basics of 4-manifold topology

Recall that a *topological n -manifold* is a topological space⁶ M_n which is locally homeomorphic to $\mathbb{R}^n$, i.e., for every point $p \in M_n$, there is an open set $U \subset M_n$ with $p \in U$ and a homeomorphism⁷ $\varphi_U : U \rightarrow \varphi_U(U) \subset \mathbb{R}^n$. On overlapping charts (U_i, φ_{U_i}) and (U_j, φ_{U_j}) with $U_i \cap U_j \neq \emptyset$, the *transition map*

$$\varphi_{ij} := \varphi_{U_i} \circ \varphi_{U_j}^{-1} : \varphi_{U_j}(U_i \cap U_j) \longrightarrow \varphi_{U_i}(U_i \cap U_j) , \quad (1.1.1)$$

is required to be a homeomorphism. A pair (U, φ_U) is called a *chart* and a set of pairs $\{(U_i, \varphi_{U_i})\}$ such that $\cup_i U_i = M_n$ is called an *atlas*. We usually choose a *maximal atlas*, which is a maximal collection of charts such that the transition functions are homeomorphisms, and denote it by $\mathcal{A}$. Two topological manifolds are *equivalent* if they are homeomorphic as topological spaces. In many subfields of Physics, we are interested in smooth manifolds⁸.

A *smooth n -manifold* or simply an n -manifold is a topological n -manifold such that all transition functions are smooth i.e., infinitely differentiable. The choice of a maximal atlas with smooth transition functions is called a *smooth structure* on M_n . A smooth structure allows us to differentiate

⁵Up to a caveat that we will see below.

⁶There are some technical requirements, namely M_n must be paracompact and Hausdorff [Mun00].

⁷A map between topological spaces is called a homeomorphism if it is bijective, continuous with a continuous inverse.

⁸For example, Kinematics which studies the motion of objects. Many quantities quantifying motion of object, for example velocity, acceleration, jerk, is measured by derivatives. But there are also quantities which are not measured by derivatives, for example static and kinetic friction.

functions on a manifold or study the notion of smooth maps between two manifolds. Two n -manifolds are said to be *equivalent* if there exists a diffeomorphism between them, i.e., a smooth bijective homeomorphism such that the inverse is also smooth. Note that two manifolds may be equivalent as topological manifolds but not as smooth manifolds. We now ask the question

Question 1. *Given a topological manifold, can it be endowed with a smooth structure? If yes, how many inequivalent smooth structures are there?*

In dimensions other than four, any compact topological manifold has a finite number of inequivalent smooth structures [KS77, Sco05]. For example, there are 28 inequivalent smooth structures on the 7-sphere $\mathbb{S}^7$ [Mil56, KM63]. These 7-spheres equipped with various smooth structures are called *exotic spheres*. In contrast, any compact 4-manifold can have countably infinite number of inequivalent smooth structures. For noncompact 4-manifolds, the situation is hopeless. A typical example of noncompact case is $\mathbb{R}^4$ itself which admits uncountably many inequivalent smooth structures⁹ [Gom85, Tau87]. We will thus restrict to compact 4-manifolds. In addition, we will assume that it is closed, oriented and without boundary and denote it by $\mathbb{X}$.

Simon Donaldson in 1983 [Don83b] used gauge theory motivated equations to define new invariants of 4-manifolds, now called *Donaldson invariants* [Don90]. These are polynomials with rational coefficients defined using the homology cycles of $\mathbb{X}$. The significance of these invariants is that it can distinguish between smooth structures on a 4-manifold: if the Donaldson invariants computed using two smooth structures are different, then the two smooth structures cannot be equivalent. In the next section, we give an overview of the construction of Donaldson invariants and its relation to physics.

1.1.2 Donaldson invariants

In 1975, Belavin-Polyakov-Schwartz-Tyupkin [BPST75] found a localized, finite-energy solution to Yang-Mills equation on $\mathbb{R}^4$ and called them *instantons*. Their realization was that if the field strength F of a gauge field A is (anti) selfdual, i.e., $(F^+ = 0) \ F^- = 0$, where

$$F^\pm = \frac{1}{2}(F \pm \star F) , \quad (1.1.2)$$

where $\star F$ is the Hodge dual of F , then F automatically satisfies the Yang-Mills equation. Consequently,

$$F^+ = 0 , \quad (1.1.3)$$

⁹On $\mathbb{R}^n$ with $n \neq 4$, there is a unique smooth structure up to equivalence [Sta62].

is called the *anti selfdual instanton equation*. Mathematicians then started studying the instanton equation [AW77, AHDM78, AHS77]. The space of solutions to the instanton equation modulo gauge transformations is called the *moduli space of instantons*, and denoted by $\mathcal{M}_{G,k,\mu}$. The notation involves dependence on the gauge group G , which we take to be a compact connected semisimple Lie group, $k, \mu(b)$, called the *instanton number* and *'t Hooft flux*¹⁰ respectively, are topological invariants of the *principal G -bundle* on which the gauge fields and field strength are defined. In general, the moduli space of instantons is noncompact but it was shown that it can be compactified and made into a smooth manifold [AHS78]. Moreover, its dimension can be calculated using index theory [AS68a, AS68b, AS68c, APS75].

Donaldson invariants are defined as integrals of certain cohomology classes on the (compactified) moduli space of instantons. The cohomology classes are defined by the *Donaldson map*

$$\mu_D : H_n(\mathbb{X}) \longrightarrow H^{4-n}(\mathcal{M}_{G,k,\mu}) . \quad (1.1.4)$$

On simply connected 4-manifolds, we only need to consider $H_0(\mathbb{X}), H_2(\mathbb{X})$. For $p^{(0)} \in H_0(\mathbb{X}), S^{(2)} \in H_2(\mathbb{X})$, we define the Donaldson invariants by

$$\mathfrak{P}_D^{\ell,r} := \int_{\mathcal{M}_{G,k,\mu}} \mu_D(p^{(0)})^\ell \mu_D(S^{(2)})^r . \quad (1.1.5)$$

The Donaldson polynomials are then generating function of the Donaldson invariants [Don90]:

$$Z_D[g, p, S] := \sum_{\ell, r \geq 0} \frac{p^\ell}{\ell!} \frac{S^r}{r!} \mathfrak{P}_D^{\ell,r} , \quad \mathfrak{P}_D^{\ell,r} \in \mathbb{Q} , \quad (1.1.6)$$

where g is the metric on $\mathbb{X}$, p, S are formal variables corresponding to $p^{(0)} \in H_0(\mathbb{X}), S^{(2)} \in H_2(\mathbb{X})$ respectively. The rational numbers $\mathfrak{P}_D^{\ell,r}$ are independent of the metric on $\mathbb{X}$ as long as the dimension $b_2^+(\mathbb{X})$ of the space of selfdual 2-forms on $\mathbb{X}$ is greater than 1.

1.1.3 Topological twisting and Seiberg-Witten invariants

Physics made a reentry into the scene in 1988 when Edward Witten showed that Donaldson invariants can be understood as correlation function of certain operators of a *topologically twisted* theory [Wit88]. More precisely, Witten showed that topological twisting of the 4d $\mathcal{N} = 2$ supersymmetric Yang-Mills (SYM) theory with gauge group G leads to a topological field theory of cohomological type. What this means is that after performing topological twist, the theory admits a scalar supercharge $\mathcal{Q}$ which is nilpotent

$$\mathcal{Q}^2 = 0 , \quad (1.1.7)$$

¹⁰In Chapter 4, we will see that the 't Hooft flux can be thought of as background field for “1-form symmetry” corresponding to a subgroup of the center of G .

and the stress tensor of the theory, which is the response of the theory to variation of the metric on $\mathbb{X}$, is $\mathcal{Q}$ -exact, i.e.,

$$T_{\mu\nu} = \{\mathcal{Q}, \Lambda_{\mu\nu}\} . \quad (1.1.8)$$

The resulting theory is called *Donaldson-Witten theory*. This has profound consequences:

1. The partition function of the theory is independent of the metric. Let S_{SYM} be the action of the twisted 4d $\mathcal{N} = 2$ SYM theory. Then

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}} Z_W &= \int [\mathcal{D}\text{VM}] e^{-S_{\text{SYM}}} \\ &= - \int [\mathcal{D}\text{VM}] \frac{\delta S_{\text{SYM}}}{\delta g_{\mu\nu}} e^{-S_{\text{SYM}}} \\ &\propto \int [\mathcal{D}\text{VM}] T_{\mu\nu} e^{-S_{\text{SYM}}} \\ &= \int [\mathcal{D}\text{VM}] \mathcal{Q} \cdot \Lambda_{\mu\nu} e^{-S_{\text{SYM}}} . \end{aligned} \quad (1.1.9)$$

Here VM is the notation for fields in the vector multiplet¹¹. The construction of $\mathcal{Q}$ comes from the supercharges and it acts as a differential operator on the field space. Moreover, since the action S_{SYM} is invariant under $\mathcal{Q}$, we can write

$$\begin{aligned} \frac{\delta}{\delta g_{\mu\nu}} Z_W &\propto \int [\mathcal{D}\text{VM}] \mathcal{Q} \cdot (\Lambda_{\mu\nu} e^{-S_{\text{SYM}}}) \\ &= \int [\mathcal{D}\text{VM}] (\text{Total derivative}) = 0 , \end{aligned} \quad (1.1.10)$$

assuming that fields vanish on the boundary of the field space¹².

2. The same argument shows that the correlation function of $\mathcal{Q}$ -exact observables are identically zero.

This means that the correlation function of observables in the $\mathcal{Q}$ -cohomology:

$$\mathcal{H}_{\mathcal{Q}} := \frac{\text{Ker } \mathcal{Q}}{\text{Im } \mathcal{Q}} , \quad (1.1.11)$$

are metric independent and hence potential invariants of smooth manifolds. The theory admits a $\mathcal{Q}$ -closed point observable $\mathcal{O}^{(0)}(p^{(0)}), p^{(0)} \in \mathbb{X}$. Witten constructed $\mathcal{Q}$ -closed surface observable $\mathcal{O}^{(2)}(S^{(2)})$ for $S^{(2)} \subset \mathbb{X}$ a closed surface in $\mathbb{X}$, using the so called *descent equations*. These observables only depend on the homology class of $p^{(0)} \in H_0(\mathbb{X})$ and $S^{(2)} \in H_2(\mathbb{X})$ up to $\mathcal{Q}$ -exact terms which

¹¹Vector multiplet is a supermultiplet of the 4d $\mathcal{N} = 2$ super Poincaré algebra. It contains a gauge field, a complex scalar and two Weyl fermions.

¹²Sometimes this is not true and the partition function depends on the metric. This is the reason for metric dependence of Donaldson invariants for $b_2^+(\mathbb{X}) \leq 1$.

decouple from the theory [KMMN19]. On simply connected 4-manifolds, these are the only nontrivial observables in $\mathcal{H}_Q$. We can then formally write the path integral for the generating function of the correlation function for these observables as

$$\begin{aligned} Z_W[g, p, S] &:= \int [\mathcal{D}V M] \exp \left(p \mathcal{O}^{(0)}(p^{(0)}) + S \mathcal{O}^{(2)}(S^{(2)}) \right) e^{-S_{\text{DW}}} \\ &= \sum_{\ell, r \geq 0} \frac{p^\ell}{\ell!} \frac{S^r}{r!} \left(\mathcal{O}^{(0)}(p^{(0)}) \right)^\ell \left(\mathcal{O}^{(2)}(S^{(2)}) \right)^r . \end{aligned} \quad (1.1.12)$$

The remarkable result of [Wit88] is that up to a constant¹³

$$Z_D[g, p, S] = Z_W[g, p, S] . \quad (1.1.13)$$

This story gets even better. In 1994, Seiberg and Witten found the exact low energy effective theory (LEET) of the 4d $\mathcal{N} = 2$ SYM with gauge group $SU(2)$ [SW94]. This UV theory admits a moduli space of vacua labeled by a complex parameter u . The LEET at a generic point u is a $U(1)$ SYM theory completely determined by a *prepotential*. Seiberg and Witten determined the exact prepotential using an auxiliary Riemann surface called the *Seiberg-Witten curve*. There are some singular points on the moduli space of vacua at which there are additional massless degrees of freedom. More precisely, there are singularities at $u = \pm \Lambda_0$, where Λ_0 is the dynamically generated scale of the theory, called the *monopole* and *dyon singularities* and there is another singularity at $u = \infty$. At the monopole and dyon points, there is a $U(1)$ SYM theory with a massless *hypermultiplet*¹⁴ coupled to the $U(1)$ gauge group with fixed electromagnetic charge. Looking ahead, Witten generalized the topological twisting to the $U(1)$ SYM with a massless hypermultiplet coupled to it [Wit94]. The twisted scalar field in the hypermultiplet was called the *monopole field*. The equations of motion of this abelian theory is called *Seiberg-Witten equations* or *monopole equations*. Witten then used the moduli space of solutions to the Seiberg-Witten equations to construct *new* 4-manifold invariants [Wit94]. These new invariants were called the *Seiberg-Witten invariants*. Since these invariants were constructed using the theory on the moduli space of vacua of $SU(2)$ SYM theory, Witten also conjectured that Donaldson invariants can be written in terms of Seiberg-Witten invariants. This was a remarkable statement because Seiberg-Witten invariants are easier to compute owing to the fact that the gauge group is abelian. Later in [MW97], Moore and Witten gave a more precise relation between Donaldson and Seiberg-Witten invariants using renormalization group flow. The precise form of the relation is

$$Z_D = Z_u + \sum_{u=\pm\Lambda_0, \infty} Z_{\text{SW}}^u , \quad (1.1.14)$$

¹³See [Sin22] for a review.

¹⁴Hypermultiplet is another supermultiplet of 4d $\mathcal{N} = 2$ super Poincaré algebra. It contains a complex scalar and two Weyl fermions.

where Z_u is the contribution from the generic points on the moduli space given by an integration over the u -plane away from the singularities, and is called the *u-plane integral*, and Z_{SW}^u is the *Seiberg-Witten contributions* from the singularities given in terms of Seiberg-Witten invariants and some universal couplings obtained from the Seiberg-Witten curve [MW97]. Later, mathematicians gave a rigorous proof of the Witten conjecture relating Donaldson and Seiberg-Witten invariants [FL01, FL02, FL15, FL18, FL19, GNY11]. This is one of the greatest successes of the program physical mathematics. Many old known results in 4-manifold theory were given new simpler proofs using Seiberg-Witten invariants [Don96, Moo17].

1.1.4 New 4-manifold invariants from physics?

The physical theory used in the above discussion is the 4d $\mathcal{N} = 2$ SYM theory with gauge group $\text{SU}(2)$. But the process of topological twisting described in [Wit88] only uses the features of 4d $\mathcal{N} = 2$ supersymmetry algebra and there is an infinite class of such theories. Many of them are Lagrangian theories but a large class of them are non-Lagrangian, the so called theories of class $\mathcal{S}$ [Wit97, GMN09, Gai09]. The natural question is

Do we get new 4-manifold invariants from any 4d $\mathcal{N} = 2$ theory?

These computations have been done for $\text{SU}(N)$ SYM theory [MM98], the minimal Argyres-Douglas theory [MN17] but have not led to new 4-manifold invariants. The computation has also been done for Lagrangian theories with matter with gauge group $\text{SU}(2)$, again without new invariants [LMn95, HPP95, AFM22, AFM23, MM21]. But not all hope is lost because it is not yet known if class $\mathcal{S}$ theories also lead to old invariants. The first step to answer this question is to understand the topological twisting of a general 4d $\mathcal{N} = 2$ field theory.

Let us first discuss the original notion of topological twisting in context of 4d $\mathcal{N} = 2$ SYM theory. The symmetry group of the theory is

$$\Gamma = G \times \text{SU}(2)_- \times \text{SU}(2)_+ \times \text{SU}(2)_R, \quad (1.1.15)$$

where G is the gauge group, $\text{Spin}(4) = \text{SU}(2)_- \times \text{SU}(2)_+$ is the Lorentz group (in Euclidean signature) and $\text{SU}(2)_R$ is the R-symmetry group. The basic idea of topological twisting in [Wit88] is to replace the symmetry group by

$$\Gamma' = G \times \text{SU}(2)_- \times \text{SU}(2)', \quad (1.1.16)$$

where $\text{SU}(2)' = \text{diag}(\text{SU}(2)_+ \times \text{SU}(2)_R)$. Then fields of the vector multiplet are then considered as representation of Γ' . It turns out that as representation of Γ' , all fields of the vector multiplet

are bosonic (see (D.4.95) for details). The theory can thus be defined on an arbitrary oriented Riemannian 4-manifold $\mathbb{X}$. The twisted action on an arbitrary oriented Riemannian 4-manifold $\mathbb{X}$ can be obtained by minimally coupling the SYM action in terms of twisted fields to gravity (see Chapter 4 for the precise description). This gives us the twisted theory. If we now follow the same procedure to a theory with hypermultiplets, then we run into an issue: the fields in the twisted hypermultiplet (see (D.4.98) for details) are all fermionic as representation of Γ' . Thus, it seems that the twisted theory with hypermultiplets can only be defined on spin 4-manifolds. It was realized that the twisted theory with hypermultiplets can be defined on arbitrary 4-manifolds by turning on background fields [HPP95, LMn97, Wit94, MW97]. This procedure requires a $\text{Spin}^c(4)$ -structure on $\mathbb{X}$ which exists on any oriented 4-manifold $\mathbb{X}$. In this view of topological twisting, we can twist any Lagrangian 4d $\mathcal{N} = 2$ theory by introducing background fields and choosing appropriate topological structure on $\mathbb{X}$.

1.1.5 Contributions of this thesis

The view of topological twisting described above is not appropriate for non-Lagrangian theories. In [MSS24], we developed a more geometric view of twisting using the functorial definition of quantum field theory. We then applied the procedure to some non-Lagrangian class $\mathcal{S}$ theories and determined the necessary topological structure needed on $\mathbb{X}$ to define the twisted theory. A result of this analysis is the distinction between the physical theory and the twisted theory. We explain that any 4d $\mathcal{N} = 2$ field theory can be defined on an oriented Riemannian compact 4-manifold by appropriately choosing background fields. Then the topological twisting of the theory amounts to changing the symmetry group of the theory by using the notion of *transfer of structure group*. For Lagrangian theories, by using the Q -symmetry of the path integral, we show that formally the correlation functions of the twisted theory are independent of the metric and background fields and only depend on the orientation and smooth structure on $\mathbb{X}$.

1.2 Moonshine, 2d conformal field theory and 2d topological quantum field theories

1.2.1 Moonshine and 2d conformal field theories

Moonshine phenomena have played an important role in the history of string theory. The explanation of *Monstrous Moonshine* by Borcherds, Frenkel, Lepowsky, and Meurman [Bor86b, Bor92b, Bor97, FLM88a] played an important role in the development of vertex operator algebra theory and string compactification on orbifolds [DFMS87, DHVW85, DHVW86]. For a nice recent review see [HHP22].

We now briefly review the first example of Moonshine phenomena, called Monstrous moonshine.

The classification of finite simple groups¹⁵ in pure mathematics was completed in a long series of papers over several decades by several mathematicians¹⁶. There are 18 infinite families of finite groups and 26 sporadic groups which do not fit into any of the 18 infinite families. Out of the 26 sporadic groups, the largest finite simple group is called the *Monster group*, denoted by $\mathbb{M}$. The number of elements in $\mathbb{M}$ is of the order of $\sim 8.08 \times 10^{53}$. The existence of the Monster group was predicted by Bernd Fischer (unpublished) and Griess [Gri76] and later proved by Robert Griess in 1982 [Gri82]. Griess gave the explicit construction of $\mathbb{M}$ as the automorphism group of a 196883-dimensional commutative nonassociative algebra [Gri82]. The character table of $\mathbb{M}$, which is a 194×194 array was calculated by Fischer and Livingston in 1979. The smallest dimensional nontrivial complex irreducible representation of $\mathbb{M}$ is a 196883-dimensional representation. In 1979, John McKay, in a letter to John G. Thompson [Tho79a, Tho79b], made the following intriguing observation:

$$196884 = 1 + 196883 . \quad (1.2.1)$$

This might seem a triviality, but the crucial observation is the following: 196884 is the first nontrivial coefficient of the Fourier expansion of the J -function, which is the unique *modular-invariant* function on $\mathbb{H}$ with zero constant coefficient:

$$J(\tau) = \frac{1}{q} + 196884q + 21493760q^2 + 864299970q^3 + \dots , \quad q := e^{2\pi i\tau}, \quad \tau \in \mathbb{H} := \{x + iy : y > 0\} . \quad (1.2.2)$$

Modular invariance means that

$$J\left(\frac{a\tau + b}{c\tau + d}\right) = J(\tau) , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) , \quad (1.2.3)$$

where $\mathrm{SL}(2, \mathbb{Z})$ is the group of 2×2 integer matrices with determinant 1. The systematic observation is the following: let r_n denote the dimension of the n^{th} (in the increasing order of dimension) irreducible complex representation of $\mathbb{M}$ and c_n be the coefficient of q^n of $J(\tau)$. Then

$$\begin{aligned} c_{-1} &= r_1 \\ c_1 &= r_1 + r_2 \\ c_2 &= r_1 + r_2 + r_3 \\ c_3 &= 2r_1 + 2r_2 + r_3 + r_3 \\ &\dots \end{aligned} \quad (1.2.4)$$

¹⁵A finite group is called simple if the only normal subgroups are either trivial or the entire group.

¹⁶The Wikipedia page (https://en.wikipedia.org/wiki/Classification_of_finite_simple_groups) on classification of finite simple groups is a very good summary of the subject along with the history.

Later, this observation was made precise into a conjecture¹⁷ by Conway and Norton in 1979 [CN79], what is called the *Monstrous-Moonshine conjecture*, which is now a theorem as we will explain below.

Theorem 1.2.1. [Conway-Norton Monstrous Moonshine, 1979] *There exists a natural infinite-dimensional graded representation of $\mathbb{M}$*

$$V^{\natural} = \bigoplus_{n=0}^{\infty} V_{(n)}^{\natural} = V_{(0)}^{\natural} \oplus V_{(1)}^{\natural} \oplus V_{(2)}^{\natural} \oplus \dots, \quad (1.2.5)$$

such that each $V_{(n)}^{\natural}$ is finite dimensional and for each conjugacy class $[g]$ of the Monster $\mathbb{M}$, the McKay-Thompson series $T_g(\tau)$, defined by

$$T_g(\tau) := \text{Tr}_{V^{\natural}} g q^{L_0-1}, \quad (1.2.6)$$

where L_0 is the grading operator, $L_0|_{V_{(n)}^{\natural}} = n$, is a normalized Hauptmodul for a subgroup $\Gamma_g \subset \text{SL}(2, \mathbb{R})$ commensurable with $\text{SL}(2, \mathbb{Z})$.

Some terms require clarification. The normalized Hauptmodul for a subgroup $\Gamma \subset \text{SL}(2, \mathbb{R})$ is the unique generator of the function field on the *modular curve* $Y(\Gamma) := \mathbb{H}/\Gamma$ with no constant term and first coefficient in q -expansion equal to 1. This requires the Riemann surface $Y(\Gamma)$ to be genus zero. In other words, the Hauptmodul for Γ is the unique modular invariant function for Γ , i.e., satisfies (1.2.3) for Γ , with no constant term and first coefficient 1. Two subgroups G, H of $\text{SL}(2, \mathbb{R})$ are said to be commensurable to each other if the group $G \cap H$ is finite index in both G and H . To connect to the observation of McKay and Thompson, the conjecture states that the *graded dimension* of the representation

$$T_e(\tau) := \text{Tr}_{V^{\natural}} q^{L_0-1} = \sum_{n=0}^{\infty} \left(\dim V_{(n)}^{\natural} \right) q^{n-1} = J(\tau), \quad (1.2.7)$$

or more succinctly, $\Gamma_e = \text{SL}(2, \mathbb{Z})$.

The first major achievement towards the proof of the Monstrous Moonshine conjecture was announced in the celebrated work of Frenkel, Lepowsky and Meurman (FLM) [FLM85, FLM89] in which they introduced the notion of *vertex operator algebras*¹⁸ (VOAs). The notion of vertex operators was motivated from conformal field theory and string theory, see Appendix B for details of the definition. FLM constructed the natural representation $V^{\natural}$ and showed that it has the structure of a vertex operator algebra whose automorphism group is precisely the Monster $\mathbb{M}$. Their construction

¹⁷There is a generalization of the Conway-Norton conjecture, called the *generalized Monstrous Moonshine conjecture*, formulated by Queen [Que81] and Norton [Nor87] in 1987 and proved by Carnahan in a series of papers in 2010 [Car10, Car12a, Car12b]. A more thorough discussion is beyond the scope of this thesis.

¹⁸The notion of *vertex algebra* was introduced by Borchers in 1986 [Bor86a] which was not motivated by physics. Vertex operator algebras, on the other hand, introduced in [FLM88a] was motivated by string theory.

was generalized to general lattices by Dolan, Goddard and Montague [DGM90, DGM94]. Borchers then used the Monster module $V^\natural$ and introduced the notion of generalized Kac-Moody algebra to finally complete the proof of the Monstrous Moonshine conjecture [Bor86a, Bor92a]. Since then many different Moonshine phenomena have been observed but the conceptual understanding of most of them remains elusive [EOT11, CDH14, DMC14, Dun05, DMO20, DGO15].

Monstrous Moonshine raises two conceptual questions:

1. (Modularity) Why are the McKay-Thompson series T_g modular?
2. (Genus zero property) Why are the modular curves $Y(\Gamma_g)$ genus zero?

A conceptual explanation for the first question is provided by physics. VOAs and its representation theory can be understood as the algebraic formulation of *rational conformal field theories*¹⁹ (RCFTs), the precise dictionary has been summarized in Appendix B.3. In the language of CFTs, the Monster VOA $V^\natural$ is a holomorphic CFT with central charge 24. Its physical construction was given by Harvey as the orbifold of the holomorphic CFT described by strings moving on the Leech torus [Har85]²⁰. In 2d CFT, the torus partition function is modular invariant owing to conformal invariance of the theory²¹. The torus partition function of the Monster CFT is precisely $J(\tau)$. This explains the modular invariance of $J(\tau)$. Other McKay-Thompson series T_g have the interpretation of partition functions with twisted boundary conditions and hence they must be invariant under subgroups of $\mathrm{SL}(2, \mathbb{Z})$ which commutes with g . A conceptual explanation of the enhancement of the invariance groups of T_g to subgroups of $\mathrm{SL}(2, \mathbb{R})$ as well as the genus zero property²² of the Monster was attempted in [PPV16, PPV17].

¹⁹Roughly, a rational conformal field theory is defined by the finiteness of the number of *primary fields* in the theory. Mathematically, this means that there are finitely many isomorphism classes of irreducible representations of the VOA, see Appendix B for precise definitions.

²⁰The Leech lattice Λ_L is the unique 24 dimensional even selfdual lattice with no vectors of length squared 2. The Leech torus is then the space $\mathbb{R}^{24}/2\pi\Lambda_L$, see Chapter 2 for more details.

²¹More precisely, the torus partition function must only depend on the conformal class of the torus. Since the moduli space (equivalence class determined by conformal equivalence) of tori is $\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$, the partition function must be invariant under $\mathrm{SL}(2, \mathbb{Z})$.

²²A more rigorous approach to the genus zero property comes from the interpretation of the Monster CFT as the boundary dual of a 3d gravity theory [Wit07] combined with the Rademacher series expansion of [DF09]. To complete the discussion, one needs to prove the duality of 3d gravity with the boundary CFT.

Contributions of this thesis

Soon after the announcement of the work of FLM, Dixon, Ginsparg and Harvey (DGH) observed that the Monster VOA could be extended to a vertex operator superalgebra (SVOA) with a subsector containing a representation of the $\mathcal{N} = 1$ superconformal algebra [DGH88]. We will call this module the *Beauty and the Beast module* following [GJF18] and [MS23]. It was noted in [DGH88] that the extended symmetry in the Monster CFT can shed some light on the genus zero property of the Monster. As discussed in [GJF18], the analysis of [DGH88] has an important gap, it did not explicitly construct the superconformal vector. The SVOA structure of the module considered by DGH was rigorously proven by Huang in [Hua94] but the explicit construction of the superconformal vector remained open. Finally, a new method of constructing superconformal vectors appeared in the work of Harvey and Moore [HM20]. They used quantum error correcting codes to give the construction of superconformal vectors in many of the known superconformal field theories including the Conway supermoonshine SVOA of Duncan [DMC14]. One might then naturally wonder if the superconformal vector in the Beauty and the Beast module can be written using quantum error correcting codes. One of the contributions of this thesis is to give an explicit construction of a superconformal vector in the Beauty and the Beast module using the ideas from [HM20]. This construction first appeared in [MS23] and has also been used to construct superconformal vector in the entire class of holomorphic CFTs with central charge 24 [FEH25]. We give the construction using the formalism of vertex operator algebras and extensions in the sense of Dolan-Goddard-Montague [DGM94] in Chapter 2. In Chapter 2.6, we recast the proof in the more standard language of VOAs in the sense of Borcherds and Frenkel-Lepowsky-Meurmann [Bor86a, FLM88a]. We show that although there are more than one superconformal currents, the supersymmetry does not enhance beyond $\mathcal{N} = 1$. We also define the notion of *spin lift* of a CFT and show that the Beauty and the Beast module is the spin lift of the Monster module.

1.2.2 WZW models and 2d TQFTs

Another aspect of this thesis deals with a large class of RCFTs constructed from a compact semisimple Lie group G and a level $\mathbf{k} \in \mathbb{Z}_{>0}^n$, where n is the number of simple components of G . These are the so called *Wess-Zumino-Witten* (WZW) models, denoted by $\text{WZW}(G)$. It is described by the following action principle [Wit84]: for simplicity assume G to be simple and simply connected, then the action is given by

$$S_{G,\mathbf{k}}(g) = -\frac{\mathbf{k}}{8\pi} \int_{\Sigma} \text{Tr} [(g^{-1}dg) \wedge \star(g^{-1}dg)] - i\mathbf{k} \Gamma_{\text{WZ}}(\Sigma; g) , \quad (1.2.8)$$

where the *Wess-Zumino* term $\Gamma_{\text{WZ}}(\Sigma; g)$ is given by

$$\Gamma_{\text{WZ}}(\Sigma; g) := \frac{1}{12\pi} \int_B \text{Tr} [(\tilde{g}^{-1} d\tilde{g}) \wedge (\tilde{g}^{-1} d\tilde{g}) \wedge (\tilde{g}^{-1} d\tilde{g})] , \quad (1.2.9)$$

where

- Σ is a closed Riemann surface without boundary,
- $g : \Sigma \rightarrow G$ is a smooth map called the WZW field,
- B is a 3d manifold with boundary Σ ,
- $\tilde{g} : B \rightarrow G$ is an extension of g to B , i.e., $\tilde{g}|_{\Sigma} = g$.

The quantization of the theory results in the representation theory of *affine Kac-Moody algebra*. More precisely, the action has a global $(G \times G)$ -symmetry:

$$S_{G,k}(g_L g g_R^{-1}) = S_{G,k} , \quad (g_L, g_R) \in G \times G . \quad (1.2.10)$$

The generators of the affine Kac-Moody algebra are the modes of the conserved currents for this symmetry. In particular, the Hilbert space of the theory on the circle is given by the direct sum of finitely many irreducible representations of the affine Kac-Moody algebra at level k [GW86]. Algebraically, the theory is described by the representation theory of *affine vertex operator algebra* based on $\mathfrak{g} = \text{Lie}(G)$ [FZ92, LL12].

In 1984, Goddard-Kent-Olive (GKO) [GKO85, GKO86] described the algebraic construction of a new VOA starting from an affine VOA based on a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Roughly, the construction can be described as follows: given the stress tensor $T^{\mathfrak{g}}, T^{\mathfrak{h}}$ of the affine VOAs based on $\mathfrak{g}, \mathfrak{h}$ respectively, the difference

$$T^{\mathfrak{g}/\mathfrak{h}} := T^{\mathfrak{g}} - T^{\mathfrak{h}} , \quad (1.2.11)$$

is also a valid stress tensor for a 2d CFT, i.e., it satisfies the expected properties. The central charge of the resulting GKO *coset* VOA is

$$c^{\mathfrak{g}/\mathfrak{h}} := c^{\mathfrak{g}} - c^{\mathfrak{h}} , \quad (1.2.12)$$

where $c^{\mathfrak{g}}, c^{\mathfrak{h}}$ is the central charge of $T^{\mathfrak{g}}, T^{\mathfrak{h}}$ respectively. Other generators of the GKO coset VOA are those generators of the VOA based on $\mathfrak{g}$ which commutes with all generators of the VOA based on $\mathfrak{h}$. Using the representation theory of the GKO coset VOA, we can construct the minimal modular invariant RCFT, which we denote by $\text{GKO}(\mathfrak{g}/\mathfrak{h})$.

The natural question is whether this algebraic RCFT $\text{GKO}(\mathfrak{g}/\mathfrak{h})$ can be given a Lagrangian description. The obvious guess would be to consider *gauged WZW model*: let $H < G$ and gauge H diagonally embedded in the global $G \times G$ symmetry of $\text{WZW}(G)$. The gauged action can be written in the standard way by coupling the theory to H -gauge fields. The resulting theory will be denoted by $\text{WZW}(G/H)$. Strong evidence was given in [BRS88] that $\text{WZW}(G/H)$ provide a Lagrangian description for the GKO coset theory $\text{GKO}(\mathfrak{g}/\mathfrak{h})$ based on the embedding $\text{Lie}(H) =: \mathfrak{h} \subset \mathfrak{g} := \text{Lie}(G)$. Indeed, it was shown in [BRS88, ABR88] that the G/H -WZW model path integral correctly reproduces the coset central charge. Finally, in [GK88, GK89], the path integral of gauged WZW model was used to compute the torus partition function and some correlation functions of the $\text{GKO}(\mathfrak{g}/\mathfrak{h})$. In one part of this thesis, we will explore the precise relationship between gauged WZW model and GKO coset construction. Among other things, we will give a mathematically precise description of GKO coset construction using the notion of *commutant* of a VOA [FZ92]. According to our definition, the coset theory $\text{GKO}(\mathfrak{g}/\mathfrak{g})$ is trivial, i.e., the VOA is isomorphic to $\mathbb{C}$ and there are no nontrivial irreducible representations. Consequently the torus partition function is 1. On the other hand, analysis of the path integral of $\text{WZW}(G/G)$, one can show that the partition function is independent of the metric [Spi90, SY92, Wit92a, Wit93]. Consequently, $\text{WZW}(G/G)$ is a 2d *topological quantum field theory* (TQFT), not only in the sense of being metric independent but also in the sense of Atiyah [Ati89], i.e., it satisfies the axioms of a 2d TQFT. This observation should make it plain that gauged WZW model is not identical to the GKO coset theory.

Contributions of this thesis

One of the central results of Chapter 3 is that under certain assumptions, we have the following relation

$$\text{WZW}(G/H) \cong \text{GKO}(\mathfrak{g}/\mathfrak{h}) \otimes \text{End}(B) , \quad (1.2.13)$$

where $\text{End}(B)$ is the algebra of endomorphisms of an algebra object in a modular tensor category. We explicitly construct a Frobenius algebra structure on $\text{End}(B)$ turning it into a 2d TQFT. Another interpretation of the topological degeneracy in generic cases is the following relation

$$\text{WZW}(G/H) \cong \text{GKO}(\mathfrak{g}/\mathfrak{h}) \otimes_Z \text{WZW}(H/H)^Z , \quad (1.2.14)$$

where $Z = Z(G) \cap Z(H)$ is the common center of G, H , $\text{WZW}(H/H)^Z$ denotes the *equivariantization* of the 2d TQFT $\text{WZW}(H/H)$ with respect to the Z -symmetry, and the subscript $\otimes_Z$ means that $\text{GKO}(\mathfrak{g}/\mathfrak{h})$ and $\text{WZW}(H/H)$ are coupled in a precise way via the common Z -action on the two theories. We also clarify the relationship between three ways of describing the identification of modules

in GKO theory, i.e., we discuss the relationship between identification via *outer automorphism group action* [Gep89], identification via *simple currents* [SY90b] and identification via the Chern-Simons theory [MS89d]. We also clarify the *categorical coset construction* by building on the results of Fröhlich-Fuchs-Runkel-Schweigert [FFRS03b, FFRS03a]. We also include an Appendix B on the rigorous construction of a modular tensor category from a VOA.

1.3 Outline of the thesis

This thesis is based on the following papers:

[MS23] G. W. Moore and **R. K. Singh**, *Beauty and the Beast Part 2: Apprehending the Missing Supercurrent*, *Commun. Math. Phys.* **406** (2025) 303, arXiv:2309.02382 [hep-th].

[MSS24] G. W. Moore, V. Saxena, and **R. K. Singh**, *Topological Twisting of 4d $N = 2$ Supersymmetric Field Theories*, arXiv:2411.14396 [hep-th], , To appear in *Adv. Theor. Math. Phys.*

[MRS26b] G. Moore, E. Rabinovici, and **R. K. Singh**, *Comment On Topological degeneracy in gauged WZW models*, In preparation, 2026.

[MRS26a] G. Moore, E. Rabinovici, and **R. K. Singh**, *Gauged WZW models vs. the GKO construction*, In preparation, 2026.

- In Chapter 2, which is based on [MS23], we describe the explicit construction of (more than 1) superconformal current in the spin lift of the Monster module, also called the Beauty and the Beast module. We use the formalism of Hermitian vertex operator algebras and its extensions of [DGM94]. We also show, using the functorial definition of gauging finite group symmetries, that the Beauty and the Beast module is a spin lift of the Monster module. We also explain the construction of the supercurrent in the standard formalism of vertex operator algebras of Frenkel-Lepowsky-Meurman [FLM88a].
- In Chapter 3, we discuss the relationship between gauged WZW models and GKO coset construction. We give a precise definition of GKO theory and explain the construction of the category of modules of the GKO vertex operator algebra using the results of Fröhlich-Fuchs-Runkel-Schweigert [FFRS03b]. We compute the Hilbert space of the gauged WZW model on the circle and compare it with the Hilbert space of the corresponding GKO theory.
- In Chapter 4, we discuss the topological twisting of 4d $\mathcal{N} = 2$ field theories using the functorial definition of quantum field theory. We show that the correlators of the twisted theory only depend on the smooth structure on $\mathbb{X}$, 't Hooft flux and a *generalized spin-c structure*. We

apply it to Lagrangian theories and discuss the localization of the path integral to the moduli space of *non-abelian monopole equations*. Finally, we discuss the topological twisting of class $\mathcal{S}$ theories.

- Appendix [A](#) complements Chapter [2](#) and introduces some of the definitions and results required in Chapter [2](#). One of the main results of Appendix [A](#) is the proof of the locality of twist fields.
- Appendix [B](#) is a summary of the rigorous construction of a modular tensor category from a vertex operator algebra. The material in this Appendix is well known in mathematics literature but not usually discussed in the physics literature. Appendix [B.2.1](#) is a dictionary between physics and math literature, which again is well known to experts but not exactly spelled out in literature.
- Appendix [C](#) complements Chapter [3](#) and introduces notations and definitions used in Chapter [3](#). We prove the equivalence of various ways of explaining *field identifications* in GKO coset construction and also explain the degeneracy of 4 observed in [\[ABR88\]](#).
- Appendix [D](#) complements Chapter [4](#). We define and give some examples of generalized spin-c structure. We give a detailed derivation of the twisted action of Lagrangian theories and the proof of its $\mathcal{Q}$ -exactness. We also record the topological twisting of *trinion theories* with *full* and *simple punctures*.

1.4 Other works not included in this thesis

During the PhD, the author also worked on the following projects which are not included in this thesis.

[\[FS26\]](#) E. Furrer, and **R. K. Singh**, *Virtual $SU(N)$ Segre Numbers from Seiberg–Witten Theory*, In preparation, 2026.

[\[Sin26\]](#) **R.K. Singh**, *Generating Function of single-centered Black Hole Index in CHL Models*, arXiv:2606.19479 [hep-th].

[\[ABB⁺25\]](#) G. Andrews, A. Banerjee, **R. K. Singh**, and R. Tao, *Macdonald index from a refined Kontsevich-Soibelman operator*, *Phys. Rev. D* **113** (2026) 105011, arXiv:2511.07521 [hep-th].

[\[BSS25\]](#) A. Bhand, A. Sen, and **R. K. Singh**, *Generating Function of Single Centered Black Hole Index from the Igusa Cusp Form*, arXiv:2510.05219 [hep-th], To appear in *Adv. Theor. Math. Phys.*

[\[ABST25\]](#) G. Andrews, A. Banerjee, C. Bhargava, **R. K. Singh**, and R. Tao, *Argyres-Douglas Theories, Macdonald Indices and Arc Space of Zhu Algebra*, arXiv:2507.06294 [hep-th].

- [BBS25] N. Banerjee, V. Bhutra, and **R. K. Singh**, *Mock modularity of twisted index in CHL models*, [JHEP **10** \(2025\) 008](#), arXiv:2505.17182 [hep-th].
- [BS25] A. Bhand and **R. K. Singh**, *Generalisation of the Asai-Kaneko-Ninomiya identity to higher level*, [Journal of Mathematical Analysis and Applications **542** \(2025\) 128849](#).
- [SST24] **R. K. Singh**, M. Sinha, and R. Tao, *Rationality of Lorentzian lattice CFTs and the associated modular tensor category*, [Lett. Math. Phys. **115** \(2025\) 134](#), arXiv:2408.02744 [hep-th].
- [Sin24] **R. K. Singh**, *Algebraic structures in closed superstring field theory, homotopy transfer, and effective actions*, [Phys. Rev. D **110** \(2024\) 126007](#), arXiv:2405.08063 [hep-th].
- [SS23] **R. K. Singh** and M. Sinha, *Non-chiral vertex operator algebra associated to Lorentzian lattices and Narain CFTs*, [SciPost Phys. **17** \(2024\) 047](#), arXiv:2312.16296 [hep-th].
- [BSS23] A. Bhand, A. Sen, and **R. K. Singh**, *Mock modularity in CHL models*, [Res. Math. Sci. **12** \(2025\) 8](#), arXiv:2311.16252 [hep-th].
- [BRS22a] N. Banerjee, T. Rahnuma, and **R. K. Singh**, *Asymptotic symmetry algebra of $N=8$ supergravity*, [Phys. Rev. D **109** \(2024\) 046010](#), arXiv:2212.12133 [hep-th].
- [BRS22b] N. Banerjee, T. Rahnuma, and **R. K. Singh**, *Soft and collinear limits in $N = 8$ supergravity using double copy formalism*, [JHEP **04** \(2023\) 126](#), arXiv:2212.11480 [hep-th].

Chapter 2

Superconformal symmetry in the Monster module

The contents of this chapter appeared in the paper [MS23] written jointly with G.W. Moore. In this chapter, we will explain the explicit construction of superconformal current in the Monster module.

2.1 Introduction

As discussed in the Introduction, in [Har85] Harvey sketched out a clear physical interpretation of the construction of Frenkel, Lepowsky, and Meurman and a few years later, in a beautiful paper [DGH88], Dixon, Ginsparg, and Harvey elaborated on the construction and suggested that there is an underlying superconformal algebra in the FLM construction. But reference [DGH88], as was recently emphasized in [GJF18], did not provide an explicit construction of the underlying supercurrent. Although it was proved in [Hua96] that the Beauty and the Beast module does contain a vertex operator superalgebra (see Theorem 2.5.1 for the precise statement) but the problem of constructing a supercurrent remained open. The aim of this chapter is to provide explicit construction of a supercurrent. The key idea is to construct superconformal currents using the theory of (quantum and classical) error correcting codes, following ideas from [HM20]. When those ideas are applied to (a spin lift of) the Moonshine module¹ one is obliged to compute the operator products of bosonic twist fields. Such computations are technically difficult. Indeed the first such computation was a tour-de-force [DFMS87]. In our computation we make heavy use of the formalism of Dolan, Goddard, and Montague [DGM90, DGM94] (which differs slightly from the standard formalism of vertex operator algebras) and arrive at sufficiently precise formulae that we can carry out the general ideas of [HM20] and reduce the construction of a superconformal current to the existence of certain error correcting codes related to sublattices of the Leech lattice. Along the way, we clarify the notion of spin lift of a CFT and show that the CFT in [DGH88] is a spin lift of the Monster CFT.

¹There are several terms used in the literature (and in this chapter) for the Moonshine module, for example, FLM module, FLM Moonshine module, FLM VOA, Monster CFT, Monster module, Monster theory, Monster VOA, Moonshine CFT.

2.1.1 Summary for physicists

We give here an abbreviated technical summary of the chapter aimed for physicists. We start with the Leech torus theory - the holomorphic conformal field theory (CFT) described in Subsection 2.3.1 based on the Leech lattice. Let Λ_L be the Leech lattice, considered as a subset of $\mathbb{R}^{24}$, with quadratic form inherited from the standard $\mathbb{R}^{24}$ inner product $\langle \cdot, \cdot \rangle$ (see Subsection A.1.3 for the detailed construction). Consider 24 free chiral bosons $X^j(z)$, $j = 1, \dots, 24$ with target space the Leech torus $\mathbb{T}^{24} := \mathbb{R}^{24}/\Lambda_L$. The quotient $\mathbb{R}^{24}/\Lambda_L$ is defined by the equivalence relation

$$\mathbf{x} \sim \mathbf{x}' \iff \mathbf{x} - \mathbf{x}' = 2\pi\lambda \quad (2.1.1)$$

for some $\lambda \in \Lambda_L$. The field X^j is expanded as

$$X^j(z) = q^j + ip^j \ln z + i \sum_{n \neq 0} \frac{a_n^j}{n} z^{-n}. \quad (2.1.2)$$

The modes satisfy the commutation relations

$$[q^j, p^k] = i\delta^{jk}, \quad [a_n^j, a_m^k] = n\delta^{jk}\delta_{n+m,0}. \quad (2.1.3)$$

We also require that $a_n^{j\dagger} = a_{-n}^j$. For each $\lambda \in \Lambda_L$ we construct the Fock space $\mathcal{F}_\lambda$ over the highest weight vector $|\lambda\rangle$ defined by $a_n^j|\lambda\rangle = 0$ for $n > 0$ and $p^j|\lambda\rangle = \lambda^j|\lambda\rangle$, see Subsection 2.3.1 for details. The Hilbert space $\mathcal{H}(\Lambda_L)$ is then spanned by the Fock spaces. The stress tensor is given by

$$T(z) = -\frac{1}{2} : \partial_z X \cdot \partial_z X : \quad (2.1.4)$$

where the dot indicates contraction of i index using δ_{ij} . The Virasoro modes are given by

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} \cdot a_m, \quad n \neq 0 \\ L_0 &= \frac{1}{2} p^2 + \sum_{n \in \mathbb{N}} a_{-n} \cdot a_n. \end{aligned} \quad (2.1.5)$$

They satisfy the Virasoro algebra with central charge 24. The torus partition function is then given by

$$Z_{\Lambda_L}(q) = \text{Tr} q^{L_0 - c/24} = \frac{\Theta_{\Lambda_L}(q)}{\eta^{24}(q)} = \frac{1}{q} + 24 + 196884q + \dots = J(q) + 24, \quad q = e^{2\pi i\tau} \quad (2.1.6)$$

with $J(q) = j(q) - 744$ where $j(q) = 1/q + 744 + \dots$ is Klein's famous weight-zero modular function generating the field of modular functions and τ is the complex moduli of the torus. In particular $J(q)$ has no constant term.

One can give a field theory description of the theory as follows: Consider fields $\mathcal{X}^j(z, \bar{z})$ satisfying the half lattice periodicity

$$\mathcal{X}^j \sim \mathcal{X}^j + 2\pi(\lambda^j/2), \quad \lambda = (\lambda^1, \dots, \lambda^{24}) \in \Lambda_L. \quad (2.1.7)$$

Let $\{e_\mu\}_{\mu=1}^{24}$ be a generating set of Λ_L so that $\lambda = n^\mu e_\mu$ with $n^\mu \in \mathbb{Z}$ for $\lambda \in \Lambda_L$. We then start with the field theory action

$$S = \frac{1}{2\pi} \int d^2z \left(\partial \mathcal{X}^j \bar{\partial} \mathcal{X}^j + B_{jk} \partial \mathcal{X}^j \bar{\partial} \mathcal{X}^k \right) = \frac{1}{2\pi} \int d^2z \left(g_{\mu\nu} \partial \mathcal{X}^\mu \bar{\partial} \mathcal{X}^\nu + B_{\mu\nu} \partial \mathcal{X}^\mu \bar{\partial} \mathcal{X}^\nu \right), \quad (2.1.8)$$

where $g_{\mu\nu} = \langle e_\mu, e_\nu \rangle$, $\mathcal{X}^j = (e_\mu)^j \mathcal{X}^\mu$, B_{jk} is a constant antisymmetric B-field and $B_{\mu\nu} = B_{jk} (e_\mu)^j (e_\nu)^k$.

In general, the partition function for this field theory depends on both the holomorphic and antiholomorphic coordinates $q = e^{2\pi i \tau}$, $\bar{q} = e^{-2\pi i \bar{\tau}}$. If we require

$$B_{\mu\nu} \equiv \langle e_\mu, e_\nu \rangle \pmod{2}, \quad (2.1.9)$$

then the partition function takes the factorised form

$$Z(q, \bar{q}) = \left| \frac{1}{\eta(q)^{24}} \sum_{\lambda \in \Lambda_L} q^{\frac{\langle \lambda, \lambda \rangle}{2}} \right|^2, \quad (2.1.10)$$

where η is the Dedekind eta function. The Hilbert space of the theory also decomposes as a tensor product of holomorphic and antiholomorphic sectors. The restriction to the holomorphic sector gives us the Leech torus theory in the field theory description. We can relate the on-shell fields to the field $X^i(z)$ of the algebraic description above via $\mathcal{X}^i(z, \bar{z}) = \frac{1}{2}(X^i(z) + X^i(\bar{z}))$.

The Monster CFT is constructed as a $\mathbb{Z}_2$ -orbifold of the Leech torus CFT. To construct this orbifold CFT, we gauge the global $\mathbb{Z}_2$ symmetry defined by:

$$\theta X^j \theta^{-1} = -X^j, \quad \theta a_n^j \theta^{-1} = -a_n^j, \quad \theta |\lambda\rangle = |-\lambda\rangle \quad (2.1.11)$$

where θ is the generator of the $\mathbb{Z}_2$ automorphism. This defines an action on the Fock spaces and extends linearly to the Hilbert space. Now gauge it. The result is a theory of fields X^j on the orbifold $\mathbb{T}^{24}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{T}^{24}$ as $\mathbf{x} \mapsto -\mathbf{x}$. The fixed points of this action, $\frac{1}{2}\Lambda_L/\Lambda_L$, form a group isomorphic to $T_2(\Lambda_L) := \Lambda_L/2\Lambda_L$. The B-field defines a bilinear form ε on $T_2(\Lambda_L)$:

$$\varepsilon(\lambda_1, \lambda_2) = e^{\pi i \sum_{j < k} B_{jk} (\lambda_1)^j (\lambda_2)^k}. \quad (2.1.12)$$

From eq. (2.1.9) we see that

$$\frac{\varepsilon(\lambda_1, \lambda_2)}{\varepsilon(\lambda_2, \lambda_1)} = (-1)^{\langle \lambda_1, \lambda_2 \rangle}. \quad (2.1.13)$$

Following the standard rules of gauge theory, the Hilbert space of the orbifold theory is a direct sum of spaces associated with the trivial and nontrivial $\mathbb{Z}_2$ bundles over the circle. The subspace associated with the trivial bundle is the $\theta = 1$ subspace $\mathcal{H}^+(\Lambda_L)$ of $\mathcal{H}(\Lambda_L)$, called the *untwisted sector*. The subspace associated with the nontrivial bundle is known as the twisted sector.

To construct the twisted sector we start with fields $X^j(z)$ on $\mathbb{T}^{24}/\mathbb{Z}_2$ satisfying the antiperiodic boundary condition

$$X^j(e^{2\pi i} z) = -X^j(z) \pmod{\Lambda_L}. \quad (2.1.14)$$

These fields have mode expansion of the form

$$X^j(z) = \tilde{q}^j + i \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{c_k^j}{k} z^{-k} \quad (2.1.15)$$

where the modes satisfy the same algebra as eq. (2.1.3):

$$[c_k^j, c_\ell^m] = k \delta_{jm} \delta_{k+\ell, 0}. \quad (2.1.16)$$

Classically, the $\mathbb{Z}_2$ -orbifold breaks the translation symmetry on the torus to the order two points $T_2(\Lambda_L)$. The unbroken translational symmetry is realised in the quantum theory via the representation of a central extension of $T_2(\Lambda_L)$, as is often the case in quantum theory. In the present case the central extension is the Heisenberg extension and fits in an exact sequence:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \Gamma(\Lambda_L) \xrightarrow{-} T_2(\Lambda_L) \longrightarrow 0 \quad (2.1.17)$$

where $-$ is the surjective homomorphism. This central extension is characterised by a cocycle ε which is provided by the B-field (see eq. (2.1.12)). We may choose a section $T : T_2(\Lambda_L) \longrightarrow \Gamma(\Lambda_L)$ such that

$$T([\lambda_1])T([\lambda_2]) = \varepsilon(\lambda_1, \lambda_2)T([\lambda_1 + \lambda_2]) \quad (2.1.18)$$

The commutator function is given by eq. (2.1.13). There is a unique unitary irreducible real representation of the Heisenberg group $\Gamma(\Lambda_L)$ up to isomorphism. Let us denote it by $\mathcal{S}$. Since $|T_2(\Lambda_L)| = 2^{24}$, $\mathcal{S} \cong \mathbb{R}^{2^{12}}$ as a vector space and can be constructed using gamma matrices (see the discussion below eq. (2.5.87) for detailed construction). Let $\mathcal{F}_T$ be the Fock space constructed using the half-integral moded oscillators c_r^j . Then the twisted Hilbert space is

$$\mathcal{H}_T(\Lambda_L) = \text{Span}_{\mathbb{C}}\{\mathcal{F}_T \otimes_{\mathbb{R}} \mathcal{S}\}. \quad (2.1.19)$$

Note that $\mathcal{F}_T$ is complex and hence $\mathcal{H}_T(\Lambda_L)$ is a complex vector space. One can extend the action of the involution θ to $\mathcal{H}_T(\Lambda_L)$ by defining

$$\theta c_k^j \theta^{-1} = -c_k^j, \quad \theta \chi_0 = -\chi_0, \quad \chi_0 \in \mathcal{S}. \quad (2.1.20)$$

Thus $\mathcal{H}_T(\Lambda_L)$ also decomposes as a direct sum of $\theta = \pm 1$ subspaces which we denote by $\mathcal{H}_T^\pm(\Lambda_L)$. The total Hilbert space of the Monster CFT is then

$$\mathcal{H}_{\text{FLM}} := \mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}_T^+(\Lambda_L). \quad (2.1.21)$$

The vertex operators are defined precisely in Subsection 2.3.2 where the space $\mathcal{H}_{\text{FLM}}$ is also denoted $\widetilde{\mathcal{H}}(\Lambda_L)$. This theory has the famous property that the automorphism group of the operator product algebra is the Monster group. The partition function for the theory is

$$Z_{\mathcal{H}_{\text{FLM}}}(q) = J(q). \quad (2.1.22)$$

In section 2.4 below we give a careful explanation of a result of [LS19] that the theory studied in [DGH88] is a spin lift² of the Monster CFT corresponding to an involution ι of the Monster which acts as the identity on $\mathcal{H}^+(\Lambda_L)$ and as multiplication by -1 on $\mathcal{H}_T^+(\Lambda_L)$. The Hilbert space of the theory is

$$\mathcal{H}_{\text{BB}} = \mathcal{H}(\Lambda_L) \oplus \mathcal{H}_T(\Lambda_L). \quad (2.1.23)$$

The main claim of [DGH88] is that this theory is superconformal. As we will discuss, some care must be taken here since $\mathcal{H}_{\text{BB}}$ is not actually a vertex operator superalgebra (SVOA). As in [DGH88], we decompose

$$\mathcal{H}_{\text{BB}} = \mathcal{H}_{\text{BB}}^{\text{NS}} \oplus \mathcal{H}_{\text{BB}}^{\text{R}}, \quad (2.1.24)$$

where

$$\mathcal{H}_{\text{BB}}^{\text{NS}} := \mathcal{H}(\Lambda_L)^+ \oplus \mathcal{H}_T(\Lambda_L)^-, \quad \mathcal{H}_{\text{BB}}^{\text{R}} := \mathcal{H}(\Lambda_L)^- \oplus \mathcal{H}_T(\Lambda_L)^+. \quad (2.1.25)$$

We will show in Theorem 2.5.1 that the subspace

$$\mathcal{H}_{\text{BB}}^{\text{NS}} = \mathcal{H}(\Lambda_L)^+ \oplus \mathcal{H}_T(\Lambda_L)^- \quad (2.1.26)$$

is an SVOA. Then we will show that there exists a conformal dimension $\frac{3}{2}$ vertex operator that can serve as a supercurrent, thus making (2.1.26) a superconformal VOA (SCVOA). The remaining space $\mathcal{H}_{\text{BB}}^{\text{R}} = \mathcal{H}(\Lambda_L)^- \oplus \mathcal{H}_T(\Lambda_L)^+$ is a module for this SCVOA.

In some more detail, the construction of the superconformal current proceeds as follows. The spinor representation $\mathcal{S}$, which was projected out in the Monster CFT, contains states with conformal weight $24 \cdot \frac{1}{16} = \frac{3}{2}$. These are the twist fields in the theory and have the correct conformal dimension of a supercurrent. The main new observation of the in this chapter is that there does in fact exist a spinor $\Psi \in \mathcal{S}$ such that the corresponding vertex operator $V(\Psi, z)$ can be identified with $T_F(z)$ in the

²Reference [DGH88] uses the term “ $\mathbb{Z}_2$ cover,” without giving a precise definition. We make this notion precise by explaining the spin lift of a CFT, see Section 2.4.1.

OPE eq. (2.4.3). It is not difficult to see that for any $\Psi \in \mathcal{S}$ the vertex operator $V(\Psi, z)$ is a Virasoro primary of weight $3/2$. With more effort, and making use of the formalism of [DGM90, DGM94], we show that the vertex operator $V(\Psi, z)$ satisfies the OPE

$$V(\Psi, z)V(\Psi, w) \sim \frac{\langle \Psi | \Psi \rangle}{(z-w)^3} + \frac{1}{8} \frac{\langle \Psi | \Psi \rangle}{z-w} T(w) + \frac{1}{z-w} \sum_{\substack{\lambda \in \Lambda_L \\ \langle \lambda, \lambda \rangle = 4}} \kappa_\lambda(\Psi) e^{i\lambda \cdot X(w)} \quad (2.1.27)$$

where

$$\kappa_\lambda(\Psi) = \langle \Psi | T([\lambda]) | \Psi \rangle, \quad T([\lambda]) \in \Gamma(\Lambda_L). \quad (2.1.28)$$

The problem of constructing a supercurrent then reduces to finding a spinor Ψ such that $\kappa_\lambda(\Psi) = 0$ for every λ of norm-squared 4. Given such a non-zero vector we can normalise it so that $\langle \Psi | \Psi \rangle = 4$ and then $V(\Psi, z)$ is a supercurrent with central charge $\hat{c} = 16$.

Note that for any Abelian subgroup $\hat{\mathcal{L}} \subset \Gamma(\Lambda_L)$, the operator

$$P(\hat{\mathcal{L}}) := \mathcal{N} \sum_{[\lambda] \in \hat{\mathcal{L}}} T([\lambda]) \quad (2.1.29)$$

is a projection operator for a suitable normalization factor $\mathcal{N} = |\hat{\mathcal{L}}|^{-1}$. Moreover if $\hat{\mathcal{L}}$ is a maximal Abelian subgroup then $P(\hat{\mathcal{L}})$ is a rank one projection operator. We will show that there exist maximal Abelian subgroups $\hat{\mathcal{L}}$ such that, for a suitably normalized $\Psi \in \text{Im} P(\hat{\mathcal{L}})$, the vertex operator $V(\Psi, z)$ is a supercurrent. Our proof relies on the existence of sublattices $\Lambda_{\text{SC}} \subset \Lambda_L$ such that:

1. For every $\lambda_1, \lambda_2, \lambda \in \Lambda_{\text{SC}}$, $\langle \lambda_1, \lambda_2 \rangle \equiv 0 \pmod{2}$ and $\langle \lambda, \lambda \rangle \equiv 0 \pmod{4}$.
2. Λ_{SC} does not contain any vector of norm-squared 4.
3. $2\Lambda_L \subset \Lambda_{\text{SC}}$ with index 2^{12} .

Hypothesis (1) and (3) implies that $\tilde{\Gamma}(\Lambda_{\text{SC}}) := \{T([\lambda]) : [\lambda] \in \tilde{\Lambda}_{\text{SC}}\}$ is an order 2^{12} Abelian subgroup of $\Gamma(\Lambda_L)$, where $\tilde{\Lambda}_{\text{SC}} := \Lambda_{\text{SC}}/2\Lambda_L$. Hypothesis (2) is a technical requirement crucial for the construction of the supercurrent. We call such sublattices *superconformal sublattices*. We define a specific isomorphism³ $T_2(\Lambda_L) \cong \mathbb{F}_2^{24}$ using the representation of $\Gamma(\Lambda_L)$ in terms of the Dirac gamma matrices (see (2.5.95) and the discussion below it). Then the chain of maps

$$\Gamma(\Lambda_L) \longrightarrow T_2(\Lambda_L) \longrightarrow \mathbb{F}_2^{24} \quad (2.1.30)$$

determines a $[24, 12]$ binary code $\mathcal{C}_{\text{SC}}$ when restricted to $\tilde{\Gamma}(\Lambda_{\text{SC}})$. Finally we show that

$$\langle \lambda, \lambda \rangle = 4 \implies \kappa_\lambda(\Psi) = 0 \quad (2.1.31)$$

³Note that there is an obvious isomorphism between the two spaces once we choose a generating set for Λ_L .

if $\Psi \in \text{Im } P(\tilde{\Gamma}(\Lambda_{\text{SC}}))$ because of the error-correcting properties of $\mathcal{C}_{\text{SC}}$.

An example of a superconformal lattice is given by an embedding of $\sqrt{2}\Lambda_L$ into Λ_L . In fact there can be several embeddings which cannot be rotated into each other by an element of the automorphism group Co_0 of the Leech lattice. We investigate the different supercurrents that can be constructed in this way in Section 2.5.3. We give an explicit construction of two inequivalent superconformal sublattices using the theory of $\mathbb{Z}_4$ codes. One of these superconformal lattices gives rise to the binary Golay code, so that the corresponding supercurrent can be considered analogous to the Duncan spinor in Conway supermoonshine [DMC14]. In principle there might be further superconformal sublattices of Λ_L not obtained by embeddings of $\sqrt{2}\Lambda_L$, but we have nothing to say about these. Since there are inequivalent supercurrents one naturally asks if there is $\mathcal{N} \geq 2$ supersymmetry in the BB module. Unfortunately, the answer appears to be no, since there is no suitable $U(1)$ R-symmetry current. Nevertheless, there is an unusual algebraic structure in this model. A deeper understanding would most likely turn out to be fruitful.

We comment here on some related recent literature. Vertex operator superalgebras (SVOAs) of central charge $c = 24$ were classified in [HM23]. The SVOA of equation (2.1.26) is entry 968 of [HM23, Table 9]. Recently, it was also observed that most of the vertex operator superalgebras associated to the odd unimodular lattices of rank 24 have an $\mathcal{N} = 4$ superconformal structure [HM18]. Moreover, one of the results of [GJF18] was that the vertex operator superalgebra obtained by an orbifold of the VOA based on the odd Leech lattice has at least two inequivalent $\mathcal{N} = 1$ superconformal structures. Indeed, [GJF18] also shows that the orbifold of the VOA based on odd Leech lattice is isomorphic to the Beauty and the Beast SCFT, thus providing an indirect construction of a supercurrent in the Beauty and the Beast SCFT. As far as we understand, the isomorphism of their description to that used by [DGH88] is non-explicit. Our construction, which deals with the orbifold theory based on the even Leech lattice, gives a direct construction of the supercurrents in the Beauty and the Beast SCFT and hence is different from the considerations of [GJF18, HM18].

Here is a brief outline of the remainder of the chapter: In Section 2.2 we review the formalism of [DGM94] for vertex operator algebras (VOAs) and their extensions by representations. In Section 2.3 we review the twisted and untwisted construction of lattice VOAs. We discuss the Monster

VOA as the prime example. In Section 2.4 we review the notion of a superconformal vertex operator algebra (SCVOA) and the spin lift of CFTs and apply it to Monster CFT in Subsection 2.5.1 to show that the theory in [DGH88] is a spin lift of the Monster CFT. Finally in Section 2.5 we show how superconformal lattices give rise to supercurrents. In Section 2.5.3 we construct two inequivalent supercurrents by constructing two inequivalent superconformal sublattices. There are two technical appendices. In Appendix A.1 we review some aspects of binary and quaternary codes and their associated lattices that are used in this chapter. A second Appendix A.2 gives a detailed proof of a crucial locality relation satisfied by twist fields.

2.2 Vertex Operator Algebras And Their Extensions

In this section, we review vertex operator algebra in the formalism of [DGM94]. See [FLM88a] for the mathematical definition of vertex operator algebras. We refer the reader to [DGM94] for more details and the proofs of the results in this section.

2.2.1 Definition Of A Vertex Operator Algebra

We begin with the definition of a vertex operator algebra.

Definition 2.2.1. A *vertex operator algebra* (VOA) is a tuple⁴ $(\mathcal{H}, \mathcal{F}, \mathbf{V}, |0\rangle, \psi_L)$ where $\mathcal{H}$ is a Hilbert space of states, $\mathcal{F}$ a dense subspace called the *Fock space* and $\mathbf{V}$ a set of linear operators on $\mathcal{H}$ called *vertex operators* $V(\psi, z)$ which are linear in ψ and depend meromorphically on $z \in \mathbb{C}$. They in one-to-one correspondence with the states $\psi \in \mathcal{F}$. The product of vertex operators $V(\psi_1, z_1)V(\psi_2, z_2) \cdots$ is defined, *a priori*, only in the region $|z_1| > |z_2| > \dots$. There are two special states in $\mathcal{F}$, the *vacuum* $|0\rangle$ and a *conformal state* ψ_L . The first five properties listed below must be satisfied. If in addition the last property (6) is satisfied, we call it an *Hermitian vertex operator algebra*

⁴ What we call a vertex operator algebra is called a conformal field theory in [DGM94]. In the physics literature, a VOA is the *chiral* part of a CFT and the full CFT is *constructed* from the irreducible representations of the VOA, see Subsection 2.2.2 for the definition of representation of a VOA. It is in this sense that we will use the word vertex operator algebra and conformal field theories in this paper, see [MS89c, MS89a] for more details on this viewpoint. For the Monster VOA, there are no other irreducible representations other than itself. So we can (and do) use Monster CFT synonymously with Monster VOA. An attempt to formalize the full CFT in terms of the *full non-chiral algebra* and its modules was made in [HK07, Mor20, SS23]. Finally, we use the term spin CFT to mean a CFT whose state spaces on circle (a time slice of the cylinder) depends on the spin structure on the circle.

1. The *moments* of the vertex operator of ψ_L is given by

$$V(\psi_L, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \quad (2.2.1)$$

and satisfies the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \quad (2.2.2)$$

with $L_n^\dagger = L_{-n}$ and $L_n|0\rangle = 0, n \geq -1$. The vertex operator $T(z) = V(\psi_L, z)$ is called the *stress tensor*.

2. The vertex operators satisfy

$$(i) \quad V(\psi, z)|0\rangle = e^{zL_{-1}}\psi \quad (ii) \quad V(\psi, z)V(\phi, w) = V(\phi, w)V(\psi, z) \quad (2.2.3)$$

The relation (ii) is called the *bosonic locality relation* and is interpreted as follows: the left hand side is *a priori* well defined for $|z| > |w|$ and the right hand side is well defined for $|w| > |z|$. The locality relation means that the analytic continuation of the matrix elements of both sides to the whole (z, w) plane must be equal except for poles at $z = w$ and $z, w = 0, \infty$. A vertex operator is said to be *local* with respect to $\mathbf{V}$ if it satisfies the bosonic locality relation with every vertex operator in $\mathbf{V}$.

3. The operator $w^{L_0} := e^{L_0 \ln w}$ acts locally with respect to $\mathbf{V}$, that is $w^{L_0}V(\psi, z)w^{-L_0}$ is local with respect to $\mathbf{V}$.
4. L_0 is diagonalizable and the spectrum of L_0 is bounded below. The eigenvalues h_ψ of eigenstates ψ of L_0 are called *conformal weights*.
5. The vacuum is the only $\mathfrak{su}(1,1)$ invariant state in the theory, *i.e.* the only state annihilated by L_0 and $L_{\pm 1}$.
6. For all $\psi \in \mathcal{F}$, the vertex operator $V(e^{z^* L_1} z^{*-2L_0} \psi, 1/z^*)^\dagger$, where z^* is the complex conjugate of z , is local with respect to $\mathbf{V}$.

Definition 2.2.2. 1. A state $\psi \in \mathcal{H}$ is called a *quasi-primary state* if $L_1\psi = 0$. It is called a *conformal primary state* if $L_n\psi = 0$ for $n = 1, 2$ (and hence for all $n > 0$ by Virasoro algebra). The corresponding vertex operator $V(\psi, z)$ is called *quasi-primary field* and *conformal primary field* respectively.

2. The moments of a vertex operator corresponding to an eigenstate ψ of L_0 with conformal weight h_ψ are defined to be the modes in the Laurent expansion around zero:

$$V(\psi, z) = \sum_{n \in \mathbb{Z}} V_n(\psi) z^{-n-h_\psi} \quad (2.2.4)$$

where $z \neq 0, \infty$.

The following properties can be proved as a consequence of the definition of an Hermitian VOA (see [DGM94] for proofs):

Proposition 2.2.1. *Let $(\mathcal{H}, \mathcal{F}, \mathbf{V}, |0\rangle, \psi_L)$ be an Hermitian vertex operator algebra. Then the following statements hold:*

1. *We have that*

$$\frac{d}{dz}V(\psi, z) = V(L_{-1}\psi, z). \quad (2.2.5)$$

This is called the translation property.

2. *The conformal weights of an Hermitian VOA are non-negative integers.*
3. *Any conformal primary field $V(\psi, z)$ with ψ an eigenstate of L_0 with conformal weight h_ψ satisfies the commutator*

$$[L_n, V(\psi, z)] = z^n \left(z \frac{d}{dz} + (n+1)h_\psi \right) V(\psi, z). \quad (2.2.6)$$

Equivalently, the moments satisfy

$$[L_m, V_n(\psi)] = ((h_\psi - 1)m - n)V_{n+m}(\psi). \quad (2.2.7)$$

For a quasi-primary field, (2.2.6) and (2.2.7) is true only for $L_0, L_{\pm 1}$.

4. *The operator product expansion (OPE) of any two vertex operators $V(\psi, z)$ and $V(\phi, w)$ corresponding to eigenstates of L_0 is given by*

$$\begin{aligned} V(\psi, z)V(\phi, w) &= \sum_{n=0}^{\infty} (z-w)^{-n-h_\psi-h_\phi} V(\phi_n, w) \\ &= \sum_{n=0}^{\infty} (z-w)^{-h_{\phi_n}-h_\psi-h_\phi} V(\phi_n, w), \end{aligned} \quad (2.2.8)$$

where

$$\phi_n := V_{h_\phi-n}(\psi)\phi. \quad (2.2.9)$$

*Equivalently,*⁵

$$V(\psi, z)V(\phi, w) = V(V(\psi, z-w)\phi, w). \quad (2.2.10)$$

⁵The left hand side of (2.2.10) is defined in the domain $|z| > |w|$ while the right hand side of (2.2.17) is defined in the domain $|w| > |z-w|$. So the equality in (2.2.10) must be understood in the sense of analytic continuation of the matrix elements of the two sides as in the bosonic locality relation (ii) in (2.4.4).

5. For each $\psi \in \mathcal{F}$, the locality of $V(e^{z^* L_1} z^{*-2L_0} \psi, 1/z^*)^\dagger$ implies that we have an antilinear map

$$\psi \mapsto \overline{\psi}, \quad (2.2.11)$$

where

$$V(e^{z^* L_1} z^{*-2L_0} \psi, 1/z^*)^\dagger = V(\overline{\psi}, z) \quad (2.2.12)$$

so that

$$\overline{\psi} := \lim_{z \rightarrow 0} V(e^{z^* L_1} z^{*-2L_0} \psi, 1/z^*)^\dagger |0\rangle. \quad (2.2.13)$$

6. For a quasi-primary vertex operator, the moments satisfy $V_n(\psi)^\dagger = V_{-n}(\overline{\psi})$.

Remark 2.2.1. We would like to put the definition above into perspective. The notion of vertex (operator) algebras was introduced by Borchers in [Bor86b] and Frenkel-Lepowsky-Meurman in [FLM88a]. The notion of vertex operator algebra in [FLM88a] differs from ours in the sense that [FLM88a] uses formal calculus to define vertex operators and the locality axiom is replaced by Jacobi identity. In [FHL93b], it has been shown that locality (called commutativity in [FHL93b]) and OPE (2.2.10) (called associativity in [FHL93b]) of vertex operators is equivalent to the Jacobi identity. The lower boundedness of the spectrum of L_0 in item 4 corresponds to a grading-restriction condition on the VOA. Moreover, the Hermiticity axiom in item 6 above is equivalent to the statement that the VOA is isomorphic to its *contragredient* as modules for itself, see [FHL93b] for more details.

Remark 2.2.2. Usually primary fields in a VOA are defined based on their OPE with the stress tensor. A field $\phi(z)$ is called a primary field of conformal dimension h if it has the following OPE with $T(z)$:

$$T(z)\phi(w) \sim \frac{h}{(z-w)^2} \phi(w) + \frac{\partial_w \phi(w)}{z-w}. \quad (2.2.14)$$

Using the standard contour integral manipulations, one can show that this definition is equivalent to commutator (2.2.7).

Definition 2.2.3. A sub-vertex operator algebra (sub-VOA) of a VOA $\mathcal{H}$ is defined to be a subspace $\mathcal{I} \subset \mathcal{H}$ such that

1. $\mathcal{I}$ is an invariant subspace for all the vertex operators $V(\phi, z)$ for $\phi \in \mathcal{F}_{\mathcal{I}} := \mathcal{F} \cap \mathcal{I}$.
2. $\mathcal{I}$ is invariant under the $\mathfrak{su}(1,1)$ algebra spanned by $L_{\pm 1}, L_0$.
3. $\overline{\mathcal{F}_{\mathcal{I}}} := \{\overline{\phi} : \phi \in \mathcal{F}_{\mathcal{I}}\} = \mathcal{F}_{\mathcal{I}}$.

One can show that the sub-VOA contains a conformal state and that a sub-VOA of a (Hermitian) VOA is itself a (Hermitian) VOA.

Definition 2.2.4. We say that two VOAs $\mathcal{H}$ and $\mathcal{H}'$ with Fock spaces $\mathcal{F}$ and $\mathcal{F}'$ respectively and corresponding vertex operators $V(\psi, z)$ and $V'(\psi', z)$ are isomorphic if there exists a unitary map $u : \mathcal{H} \longrightarrow \mathcal{H}'$ such that

$$V'(u\psi, z) = uV(\psi, z)u^{-1}, \quad \psi \in \mathcal{F}. \quad (2.2.15)$$

An isomorphism $u : \mathcal{H} \longrightarrow \mathcal{H}$ which preserves $|0\rangle, \psi_L$ is called an automorphism. The group of all automorphisms of a VOA $\mathcal{H}$ is called the automorphism group of the VOA.

If $u : \mathcal{H} \longrightarrow \mathcal{H}'$ is an isomorphism then

$$u|0\rangle = |0'\rangle, u\psi_L = \psi'_L, \quad (2.2.16)$$

where $|0\rangle, |0'\rangle$ are the vacuum states and ψ_L, ψ'_L are the conformal states in $\mathcal{H}$ and $\mathcal{H}'$ respectively.

2.2.2 Representation Of A Vertex Operator Algebra

Definition 2.2.5. A representation $(\mathbf{U}, \mathcal{H})$ of a vertex operator algebra $(\mathcal{H}, \mathbf{V}, \mathcal{F}, |0\rangle, \psi_L)$ is a Hilbert space $\mathcal{H}$ and a set of linear operators $U(\psi, z) : \mathcal{H} \longrightarrow \mathcal{H}$ meromorphic in z and linear in ψ for $\psi \in \mathcal{F}$ and for all $\psi, \phi \in \mathcal{F}$, we have:

$$U(\psi, z)U(\phi, \zeta) = U(V(\psi, z - \zeta)\phi, \zeta) \quad (2.2.17)$$

with $U(|0\rangle, z) \equiv \mathbf{1}$. The equality is understood as in (2.2.10).

One can show that the operators $U(\psi, z)$ are local in the sense of bosonic locality relation. Moreover, the modes of $U(\psi_L, z)$ satisfies the Virasoro algebra and $U(\psi_L, z)$ also satisfies the translation property as in (2.2.5). For $\chi \in \mathcal{H}$ let us define a meromorphic family of operators $W(\chi, z) : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$W(\chi, z)\phi := e^{zL-1}U(\phi, -z)\chi. \quad (2.2.18)$$

Then we have the following proposition.

Proposition 2.2.2. [DGM94, Proposition 3.4] *The existence of a representation is equivalent to the existence of “intertwining” operators $W(\chi, z) : \mathcal{H} \longrightarrow \mathcal{H}$ for $\chi \in \mathcal{H}$ (or a dense subspace of $\mathcal{H}$) satisfying*

$$U(\psi, z)W(\chi, \zeta) = W(\chi, \zeta)V(\psi, z) \quad (2.2.19)$$

and

$$\lim_{\zeta \rightarrow 0} W(\chi, \zeta)|0\rangle = \chi. \quad (2.2.20)$$

2.2.3 Extension Of A Vertex Operator Algebra By A Real Hermitian Representation

Definition 2.2.6. A representation $\mathbf{U}$ is said to be Hermitian if

$$U(\bar{\phi}, z) = U(e^{z^* L_1} z^{*-2L_0} \phi, 1/z^*)^\dagger \quad (2.2.21)$$

for every $\phi \in \mathcal{F}$. Here L_1 is the Laurent mode of $U(\psi_L, z)$. The representation is said to be real if there is an antilinear map $\chi \mapsto \bar{\chi}$ in $\mathcal{H}$ satisfying

$$\begin{aligned} \bar{\bar{\chi}} &= \chi, \quad L_{-1}\bar{\chi} = -\overline{L_{-1}\chi}, \\ f_{\chi_1 \phi \chi_2} &= (-1)^{h_{\chi_1} + h_\phi + h_{\chi_2}} f_{\bar{\chi}_1 \bar{\phi} \bar{\chi}_2} \end{aligned} \quad (2.2.22)$$

where

$$f_{\chi_1 \phi \chi_2} := \langle \bar{\chi}_1 | U(\phi, 1) | \chi_2 \rangle. \quad (2.2.23)$$

Moreover if $L_0 \chi = h_\chi \chi$ then $L_0 \bar{\chi} = h_\chi \bar{\chi}$.

Remark 2.2.3. The above condition (2.2.21) is equivalent to

$$U(\bar{\phi}, z) = z^{-2h_\phi} U(e^{z^* L_1} \phi, 1/z^*)^\dagger \quad (2.2.24)$$

for $\phi \in \mathcal{F}$ with conformal weight h_ϕ .

Definition 2.2.7. Define the conjugate $\bar{W}$ of the operator $W(\chi, z) : \mathcal{H} \rightarrow \mathcal{H}$ by the relation:

$$\bar{W}(\bar{\chi}, z) = z^{-2h_\chi} W(e^{z^* L_1} \chi, 1/z^*)^\dagger. \quad (2.2.25)$$

Proposition 2.2.3. [DGM94, Proposition 4.3] *The operator $\bar{W} : \mathcal{H} \rightarrow \mathcal{H}$ satisfies the following intertwining relations*

$$\begin{aligned} \bar{W}(\chi, \zeta) U(\psi, z) &= V(\psi, z) \bar{W}(\chi, \zeta) \\ \bar{W}(\chi_1, \zeta) W(\chi_2, z) &= \bar{W}(\chi_2, z) W(\chi_1, \zeta). \end{aligned} \quad (2.2.26)$$

We have the following important theorem:

Theorem 2.2.1. [DGM94, Proposition 4.4] *Suppose that the operator $\bar{W}(\chi, z)$ satisfy the additional locality relation*

$$W(\chi_1, \zeta) \bar{W}(\chi_2, z) = W(\chi_2, z) \bar{W}(\chi_1, \zeta) \quad (2.2.27)$$

and the spectrum of L_0 in the representation is strictly positive, then we may extend the vertex operator algebra $\mathcal{H}$ to a Hermitian vertex operator algebra $\widetilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}$ with Fock space $\widetilde{\mathcal{F}} := \mathcal{F} \oplus \mathcal{H}$ and vertex operators defined by

$$\widetilde{V}(\psi, z) = \begin{pmatrix} V(\psi, z) & 0 \\ 0 & U(\psi, z) \end{pmatrix}, \quad \widetilde{V}(\chi, z) = \begin{pmatrix} 0 & \bar{W}(\chi, z) \\ W(\chi, z) & 0 \end{pmatrix} \quad (2.2.28)$$

where we used the notation $\psi = (\psi, 0)$ and $\chi = (0, \chi)$ for $\psi \in \mathcal{H}$ and $\mathcal{K}$ and the definition is extended to all of $\widetilde{\mathcal{F}}$ by linearity. The vacuum and conformal states are $(|0\rangle, 0)$ and $(\psi_L, 0)$ respectively.

Remark 2.2.4. The extended vertex operator algebra $\widetilde{\mathcal{H}}$ has an automorphism ι which acts as identity on $\mathcal{H}$ and multiplication by -1 on $\mathcal{K}$.

2.3 Lattice Vertex Operator Algebras And Their Orbifolds

In this section, we study VOAs associated to lattices, first studied in [Bor86b, FLM88a]. Let Λ be an even Euclidean lattice of dimension r . We can construct a VOA using this lattice. We can construct two types of VOAs, the untwisted or straight VOA and the twisted VOA. We describe this construction in this section. We follow [DGM94] for this construction.

2.3.1 Untwisted Vertex Operator Algebra

Consider the r -torus $\mathbb{T}^r := \mathbb{R}^r / \Lambda$ where the equivalence relation is

$$\mathbf{x} \sim \mathbf{x}' \iff \mathbf{x} - \mathbf{x}' = 2\pi\beta \quad (2.3.1)$$

for some $\beta \in \Lambda$. We consider r fields

$$X^j(z) = q^j + ip^j \ln z + i \sum_{n \neq 0} \frac{a_n^j}{n} z^{-n} \quad (2.3.2)$$

which we can view as in correspondence with the coordinates on the torus $\mathbb{T}^r$. We impose the commutation relations on the modes:

$$[q^j, p^k] = i\delta^{jk}, \quad [a_n^j, a_m^k] = n\delta^{jk}\delta_{n+m,0}. \quad (2.3.3)$$

We then introduce a family of orthogonal lines L_λ , one for each $\lambda \in \Lambda$. We require that a_n^j for $n > 0$ annihilates L_λ and p^j acts on L_λ as scalar multiplication by λ^j . We may choose a family of basis vectors $|\lambda\rangle$ for L_λ which are orthonormal: $\langle \lambda | \lambda' \rangle = \delta_{\lambda, \lambda'}$. Any two choices are related by a uniform multiplication by a phase. With this inner product, we have the Hermitian conjugates $a_n^{j\dagger} = a_{-n}^j$. For each $\lambda \in \Lambda$, we then construct the Fock space $\mathcal{F}_\lambda$ by creating states using a_n^j . Thus a basis is of the form:

$$\psi = \left(\prod_{a=1}^M a_{-m_a}^{j_a} \right) |\lambda\rangle, \quad (2.3.4)$$

where m_a and j_a are positive integers $1 \leq j_a \leq r$ and $M \in \mathbb{Z}_+$ with the understanding that $\psi = |\lambda\rangle$ when $M = 0$. The Hilbert space of states is then the (completion of the) span of the Fock spaces

$\mathcal{F}_\lambda$ for $\lambda \in \Lambda$. The Fock space of the VOA is defined as

$$\mathcal{F} = \bigoplus_{\lambda \in \Lambda} \mathcal{F}_\lambda. \quad (2.3.5)$$

We denote the Hilbert space by $\mathcal{H}(\Lambda)$. For $\psi \in \mathcal{F}_\lambda$ given by (2.3.4), we define the vertex operator $V(\psi, z)$ by the expression

$$V(\psi, z) = : \left(\prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} X^{j_a}}{dz^{m_a}}(z) \right) \exp \{i\lambda \cdot X(z)\} : \sigma_\lambda, \quad (2.3.6)$$

where $::$ denotes normal ordering and is defined as follows:

$$\begin{aligned} : a_n^j a_m^k : &:= a_m^k a_n^j := \begin{cases} a_m^k a_n^j & m \leq n, \\ a_n^j a_m^k & m \geq n, \end{cases} \\ : a_m^j p^k : &:= p^k a_m^j := a_m^j p^k, \\ : q^j a_n^k : &:= a_n^k q^j := a_n^k q^j, \\ : q^j p^k : &:= p^k q^j := q^j p^k. \end{aligned} \quad (2.3.7)$$

In particular

$$: \exp \{i\lambda \cdot X(z)\} : = \exp \{i\lambda \cdot X_{<}(z)\} e^{i\lambda \cdot q} z^{i\lambda \cdot p} \exp \{i\lambda \cdot X_{>}(z)\} \quad (2.3.8)$$

where

$$X_{\leq}^j(z) = i \sum_{n \leq 0} \frac{a_n^j}{n} z^{-n}. \quad (2.3.9)$$

The operator σ_λ is a cocycle operator such that

$$\widehat{\sigma}_\lambda \widehat{\sigma}_\mu = (-1)^{\lambda \cdot \mu} \widehat{\sigma}_\mu \widehat{\sigma}_\lambda \quad (2.3.10)$$

with $\widehat{\sigma}_\lambda = e^{i\lambda \cdot q} \sigma_\lambda$. Also the operator q acts on the states $|\lambda\rangle$ only via the exponential

$$e^{i\mu \cdot q} |\lambda\rangle = |\lambda + \mu\rangle, \quad \lambda, \mu \in \Lambda. \quad (2.3.11)$$

This cocycle is introduced to ensure the mutual locality and associativity of the operator product expansion. The vacuum state is $|0\rangle$ corresponding to $0 \in \Lambda$ and the conformal state ψ_L is defined as

$$\psi_L = \frac{1}{2} a_{-1} \cdot a_{-1} |0\rangle. \quad (2.3.12)$$

It can be shown that the energy momentum tensor $T(z)$ is given by

$$T(z) = V(\psi_L, z) = -\frac{1}{2} : \partial_z X \cdot \partial_z X : \quad (2.3.13)$$

and the Virasoro modes are given by

$$\begin{aligned} L_n &= \frac{1}{2} \sum_{m \in \mathbb{Z}} a_{n-m} \cdot a_m, \quad n \neq 0, \\ L_0 &= \frac{1}{2} p^2 + \sum_{n \in \mathbb{N}} a_{-n} \cdot a_n. \end{aligned} \quad (2.3.14)$$

One can also show that the Virasoro generators satisfy the Virasoro algebra with central charge $c = r$. The partition function of the theory is given by

$$Z_\Lambda(\tau) = q^{-c/24} \text{Tr}_{\mathcal{H}(\Lambda)} q^{L_0} = \frac{\Theta_\Lambda(\tau)}{\eta^r(q)}, \quad q = e^{2\pi i \tau}, \quad (2.3.15)$$

where

$$\Theta_\Lambda(\tau) = \sum_{\lambda \in \Lambda} q^{\frac{1}{2}\lambda^2} \quad (2.3.16)$$

is the theta function for the lattice Λ and $\tau \in \mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$. The partition function is said to be modular invariant if it is invariant under the modular action of $\text{SL}(2, \mathbb{Z})$:

$$Z_\Lambda\left(\frac{a\tau + b}{c\tau + d}\right) = Z_\Lambda(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \quad (2.3.17)$$

The partition function (2.3.15) is modular invariant if and only if the lattice Λ is selfdual.

Remark 2.3.1. In physics, (2.3.15) defines what is called the torus partition function and is the partition function of a conformal field theory on the torus. In general, partition function can be defined on any general Riemann surface via a path integral. Due to conformal symmetry, the path integral on the torus simplifies to the above expression involving trace over the Hilbert space.

Theorem 2.3.1. [DGM94] *If Λ is even, self-dual with dimension $r \equiv 0 \pmod{24}$, then the Hilbert space $\mathcal{H}(\Lambda)$ with the vertex operators and structures defined above forms a Hermitian vertex operator algebra with central charge $c = r$ and modular invariant partition function. Moreover the Hermitian structure is given by*

$$\overline{\phi} = e^{-i\pi L_0} \theta \phi \quad (2.3.18)$$

where ϕ is a real linear combination of states of the form (2.3.4) and can be extended to complex linear combinations using the antilinearity of the above map and θ is an involution given by

$$\theta a_n^j \theta^{-1} = -a_n^j, \quad \theta |\lambda\rangle = |-\lambda\rangle, \quad \theta X^i \theta^{-1} = -X^i. \quad (2.3.19)$$

Remark 2.3.2. It can easily be checked that the basis states (2.3.4) have conformal dimensions

$$h_\psi = \frac{1}{2}\lambda^2 + \sum_{a=1}^M m_a \quad (2.3.20)$$

which is an integer since Λ is even.

2.3.2 Twisted Vertex Operator Algebra

Given a modular invariant vertex operator algebra with a discrete subgroup G of the automorphisms group, one can construct a new vertex operator algebra by “modding out” or “twisting” or “orbifolding” by G given that G satisfies certain restrictions. This simply corresponds to gauging the

symmetry, and the restriction is simply the condition that G is not anomalous. For the lattice vertex operator algebra constructed above, the automorphism group $\mathbb{Z}_2$, consisting of the involution θ in Theorem 2.3.1 satisfies all the restrictions given that $\sqrt{2}\Lambda^*$ is even. Thus we can construct the twisted vertex operator algebra. The essential idea is to construct a representation of a sub-vertex operator algebra which satisfies the hypotheses of Theorem 2.2.1 and then consider the extended vertex operator algebra given in Theorem 2.2.1. We now describe the construction in detail.

Let $\Lambda, \mathcal{H}(\Lambda)$ and θ be as in Theorem 2.2.1. Then θ is an involution of $\mathcal{H}(\Lambda)$. We have the direct sum decomposition

$$\mathcal{H}(\Lambda) = \mathcal{H}^+(\Lambda) \oplus \mathcal{H}^-(\Lambda) \quad (2.3.21)$$

where

$$\mathcal{H}^\pm(\Lambda) := \{\psi \in \mathcal{H}(\Lambda) : \theta\psi = \pm\psi\}. \quad (2.3.22)$$

Lemma 2.3.1. [DGM94] $\mathcal{H}^+(\Lambda)$ is a sub-vertex operator algebra of $\mathcal{H}(\Lambda)$.

Note that $\mathcal{H}^+(\Lambda)$ is not a modular invariant VOA. To get a modular invariant VOA, we construct a representation of this VOA and then extend it as described in Section 2.2. To construct the representation of the VOA $\mathcal{H}^+(\Lambda)$, we first construct a Hilbert space $\mathcal{H}_T(\Lambda)$, known as the space of twisted states. To do so, we consider the bosonic oscillators c_ℓ^j indexed by $\ell \in \mathbb{Z} + \frac{1}{2}$ satisfying the same algebra as a_n^j :

$$[c_\ell^j, c_m^k] = \ell \delta^{jk} \delta_{\ell+m, 0}. \quad (2.3.23)$$

To construct the space over which these oscillators can act, we start with the $\mathbb{Z}_2$ extension of the (abelian group) lattice Λ with commutator function $(\lambda, \mu) \mapsto (-1)^{\lambda \cdot \mu}$. We choose a 2-cocycle $\varepsilon(\lambda, \mu) = \pm 1$ which describes the extension:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widehat{\Lambda} \xrightarrow{-} \Lambda \longrightarrow 0 \quad (2.3.24)$$

where $-$ denotes the projection map. Consider the subgroup⁶ K of $\widehat{\Lambda}$ given by

$$K = \{a^2(-1)^{\frac{a^2}{2}} : a \in \widehat{\Lambda}\}. \quad (2.3.25)$$

Then [FLM88a, eq. 10.3.18]

$$\bar{K} \cong 2\Lambda \quad (2.3.26)$$

and we get the central extension

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widehat{\Lambda}/K \xrightarrow{-} \Lambda/2\Lambda \longrightarrow 0 \quad (2.3.27)$$

⁶It takes a few lines of algebra to verify that K is a subgroup. It helps to choose (as one may) ε to be a bihomomorphism.

with commutator map $([\lambda], [\mu]) \mapsto (-1)^{\lambda \cdot \mu}$. Clearly the commutator map does not depend on the choice of representatives.

Lemma 2.3.2. *The extension (2.3.27) is a Heisenberg extension.*

Proof. We want to show that the commutator map of the extension is nondegenerate [Moo22, Section 15.5]. Consider the bilinear form

$$\begin{aligned} b : \Lambda/2\Lambda \times \Lambda/2\Lambda &\longrightarrow \mathbb{Z}_2 \\ ([\lambda], [\mu]) &\longmapsto \lambda \cdot \mu \bmod 2. \end{aligned} \quad (2.3.28)$$

Pick a basis $\{\lambda_i\}$ of Λ and the corresponding basis $\{[\lambda_i]\}$ of $\Lambda/2\Lambda$. Then the matrix of the bilinear form b in the basis $\{[\lambda_i]\}$ is the mod 2 reduction of the gram matrix of the basis $\{\lambda_i\}$ of Λ . Then since Λ is unimodular, the determinant of the latter matrix is ± 1 . Hence the determinant of b is 1 and hence b is nondegenerate. This implies that the commutator map is nondegenerate, that is for every $[\lambda] \neq 0$, there exists $[\mu]$ such that $(-1)^{\lambda \cdot \mu} = -1$. $\square$

We denote by $\Gamma(\Lambda)$ the Heisenberg group $\widehat{\Lambda}/K$. Let $[\lambda] \mapsto e_{[\lambda]}$ be a section of this extension. For a function $u : \Lambda/2\Lambda \longrightarrow \{\pm 1\}$, $[\lambda] \mapsto \tilde{e}_{[\lambda]} := u([\lambda])e_{[\lambda]}$ defines another section of the extension. Moreover the cocycle changes by a coboundary:

$$\tilde{e}_{[\lambda]} \tilde{e}_{[\mu]} = \tilde{\varepsilon}(\lambda, \mu) \tilde{e}_{[\lambda+\mu]} \quad (2.3.29)$$

where

$$\tilde{\varepsilon}(\lambda, \mu) = \frac{u([\lambda])u([\mu])}{u([\lambda+\mu])} \varepsilon(\lambda, \mu). \quad (2.3.30)$$

Using this “gauge freedom” we can choose⁷ $\varepsilon(0, \lambda) = 1$ for all λ and $e_{[0]} = \mathbf{1} \in \Gamma(\Lambda)$. To simplify notations, we write $e_{[\lambda]} =: e_\lambda$. We can then write

$$\begin{aligned} \Gamma(\Lambda) &= \{\pm e_\lambda : \lambda \in \Lambda\} \\ e_\lambda e_\mu &= (-1)^{\lambda \cdot \mu} e_\mu e_\lambda = \varepsilon(\lambda, \mu) e_{\lambda+\mu}, \quad e_\lambda^2 = (-1)^{\frac{1}{2}\lambda^2}, \end{aligned} \quad (2.3.31)$$

The last relation $e_\lambda^2 = (-1)^{\lambda^2/2}$ comes from the fact that K is modded out in the Heisenberg group $\Gamma(\Lambda)$. Note that these relations do not depend on the choice of representatives of $[\lambda], [\mu]$ when calculating $\lambda \cdot \mu$.

Lemma 2.3.3. *The cocycle ε satisfies*

$$\varepsilon(\lambda, \lambda) = (-1)^{\frac{\lambda^2}{2}}. \quad (2.3.32)$$

⁷Note that (2.3.26) implies that $e_{[0]} = \pm \mathbf{1}$.

Proof. To prove this, we use a result of [FLM88a]. For a central extension of an elementary abelian 2-group⁸ E by $\mathbb{Z}_2$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \widehat{E} \xrightarrow{-} E \longrightarrow 0 \quad (2.3.33)$$

with commutator map $(x, y) \mapsto (-1)^{c(x, y)}$, the *squaring map* $s : E \longrightarrow \mathbb{Z}_2$ is defined by

$$a^2 = (-1)^{s(\bar{a})}, \quad a \in \widehat{E}. \quad (2.3.34)$$

The bilinear form corresponding to s is the commutator map:

$$c(x, y) = s(x + y) - s(x) - s(y). \quad (2.3.35)$$

Then by [FLM88a, Proposition 5.3.3], for any central extension $\widehat{E}$ of an abelian 2-group E by $\mathbb{Z}_2$ with squaring map s , the cocycle satisfies

$$(-1)^{s(x)} = \varepsilon(x, x). \quad (2.3.36)$$

Moreover, by (2.3.30) cohomologous cocycles do not change this relation. For the central extension (2.3.27) at hand, it is easy to check that the squaring map is

$$[\lambda] \mapsto \frac{\lambda^2}{2} \bmod 2. \quad (2.3.37)$$

The required relation thus holds. $\square$

Let $\mathcal{S}(\Lambda)$ be the carrier space of the unique (up to isomorphism) $2^{r/2}$ -dimensional irreducible unitary representation of $\Gamma(\Lambda)$. Such a representation can be constructed using Dirac gamma matrices, see [DGM90, Appendix C] for an explicit construction. When the representation is real we choose it to be a real vector space of dimension $2^{r/2}$. The vector space $\mathcal{S}$ of the introduction is $\mathcal{S}(\Lambda_L)$ in our present notation.

The algebra of oscillators $\{c_\ell^j\}$ is represented on a Fock space $\mathcal{F}_T$ with a unique vacuum line spanned by $|0\rangle_T$, so $c_\ell^j|0\rangle_T = 0$ for $\ell > 0$. Then the twisted sector Hilbert space is the (completion of):

$$\mathcal{H}_T(\Lambda) = \mathcal{F}_T \otimes_{\mathbb{R}} \mathcal{S}(\Lambda). \quad (2.3.38)$$

A general Fock space state is a linear combination of vectors of the form:

$$\chi = \left(\prod_{a=1}^N c_{-k_a}^{j_a} \right) |0\rangle_T \otimes \chi_0, \quad (2.3.39)$$

⁸For a prime number p , an elementary abelian p -group is a direct product of cyclic groups of order p .

where $N \in \mathbb{Z}_+$ and $k_a = m_a + \frac{1}{2}$ with $m_a \in \mathbb{Z}_+$ and $1 \leq j_a \leq r$, and $\chi_0 \in \mathcal{S}(\Lambda)$. We endow $\mathcal{H}_T(\Lambda)$ with an Hermitian structure such that $(c^j)_k^\dagger = c_{-k}^j$. The Virasoro modes are given by

$$L_n^T = \frac{1}{2} \sum_{k \in \mathbb{Z} + \frac{1}{2}} : c_k \cdot c_{n-k} : + \frac{r}{16} \delta_{n,0}. \quad (2.3.40)$$

In particular, the ground states $\mathcal{S}(\Lambda)$ have conformal weight $r/16$ and the conformal weight of the state χ given by (2.3.39) is

$$h_\chi = \frac{r}{16} + \sum_{a=1}^M k_a. \quad (2.3.41)$$

To construct the vertex operators corresponding to states in $\mathcal{H}_T(\Lambda)$, we start with the an expression analogous to the untwisted vertex operator (2.3.6). We will then need to modify the naive expression in order to satisfy the translation property (2.2.5) and the conformal transformation property (2.2.6). Thus, we consider the fields

$$R^j(z) = i \sum_{k \in \mathbb{Z} + \frac{1}{2}} \frac{c_k^j}{k} z^{-k} \quad (2.3.42)$$

and we first define the naive vertex operator $V_T^0(\psi, z)$, corresponding to the state ψ in (2.3.4), to be:

$$V_T^0(\psi, z) := \left(\prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} R^{j_a}}{dz^{m_a}}(z) \right) \exp \{ i\lambda \cdot R(z) \} : e_\lambda, \quad (2.3.43)$$

where the normal ordering $::$ is defined analogous to the untwisted case. This operator fails to satisfy the L_n^T commutator (2.2.6). Including a factor of $(4z)^{-\frac{\lambda^2}{2}}$ rectifies the commutator with L_n^T [DGM90] but now the translation property (2.2.5) is not satisfied. To remedy this, we introduce the vertex operator

$$V_T(\psi, z) := V_T^0(e^{A(-z)}\psi, z), \quad (2.3.44)$$

and try to solve for $A(z)$ demanding the required properties. The constraints imposed on $A(z)$ by the L_n^T commutators and the translation property (2.2.5) is solved by the operator [DGM90, FLM88a]

$$A(z) = \frac{1}{2} \sum_{\substack{n, m \geq 0 \\ m+n > 0}} \begin{pmatrix} -\frac{1}{2} \\ m \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ n \end{pmatrix} \frac{(-z)^{-m-n}}{m+n} a_m \cdot a_n - \frac{1}{2} a_0 \cdot a_0 \ln(-4z), \quad (2.3.45)$$

where $a_0^i = p^i$. Note that the last term corresponds to the factor $(4z)^{-\frac{\lambda^2}{2}}$ which is needed for the L_n^T commutator (2.2.6).

Remark 2.3.3. The definition of the vertex operator $V_T(\psi, z)$ depends on the choice of the section in (2.3.31). Indeed if we pick a different section $\lambda \mapsto e'_\lambda$, then the new cocycle $\varepsilon'(\lambda, \mu)$ is related to the old cocycle by a coboundary $b : \Lambda/2\Lambda \mapsto \{\pm 1\}$:

$$\varepsilon'(\lambda, \mu) = \varepsilon(\lambda, \mu) \frac{b(\lambda)b(\mu)}{b(\lambda + \mu)}. \quad (2.3.46)$$

The unitary operators e'_λ and e_λ acting on $\mathcal{S}(\Lambda)$ are related by a unitary operator S [DGM94, eq. 6.10]

$$S e'_\lambda S^\dagger = b(\lambda) e_\lambda. \quad (2.3.47)$$

For the conformal weights (2.3.41) to be integers, we need both the terms to be either strictly half-integers or integers simultaneously. This immediately implies that r has to be a multiple of 8 [Ham17, Chapter 6],[Moo18]. When the second term is an integer (half-integers), the state χ is obtained by acting an even (odd) number of oscillators on a twisted ground state. To quantify this, we extend the action of the $\mathbb{C}$ -linear involution θ to $\mathcal{H}(\Lambda) \oplus \mathcal{H}_T(\Lambda)$ by

$$\theta c_k^i \theta^{-1} = -c_k^i, \quad \theta \chi_0 = (-1)^{r/8} \chi_0, \quad \theta R^j \theta^{-1} = -R^j. \quad (2.3.48)$$

It is now clear that when r is a multiple of 16, in which case the first term is an integer, h_χ can be integer only when χ is obtained by acting an even number of oscillators on a twisted ground state. Since the twisted ground states are even under θ for r a multiple of 16, the state χ is also even under θ . On the other hand if $r \equiv 8 \pmod{16}$, the twisted ground states are odd under θ and if we take χ to be an odd number of oscillators acting on a twisted ground state, χ will be even under θ . In this case the two terms in h_χ in (2.3.41) are strictly half-integers. Thus we see that the only way the conformal weights can be integers is when the following two conditions are satisfied:

1. The dimension r is a multiple of 8.
2. The state $\chi \in \mathcal{H}_T^+(\Lambda)$ where $\mathcal{H}_T^+(\Lambda)$ is $\theta = 1$ subspace with the action of θ as defined in (2.3.48).

We now need to put an Hermitian structure on the twisted sector, see Definition 2.2.6. Choose an orthonormal basis of $\mathcal{S}(\Lambda)$ and let γ_λ be the matrix of e_λ with respect to this basis. Let M be a symmetric, unitary matrix satisfying

$$M \gamma_\lambda^* = \gamma_\lambda M. \quad (2.3.49)$$

Such a matrix M exists because the unitary representation of the Clifford algebra is real in dimensions divisible by 8 and the unique (up to isomorphism) unitary representation of the Heisenberg extension (2.3.31) can be constructed using Dirac gamma matrices. Moreover one can choose M such that $M^2 = \mathbb{1}$ when $r \equiv 0 \pmod{8}$, see [Kuu19, Table 1] for more details and [DGM90, Appendix C] for an explicit construction. Define the action of M on $\mathcal{H}_T(\Lambda)$ by:

$$M : \chi = \left(\prod_{a=1}^N c_{-k_a}^{j_a} \right) \chi_0 \mapsto M(\chi) = \left(\prod_{a=1}^N c_{-k_a}^{j_a} \right) M \chi_0^*, \quad \chi_0 \in \mathcal{S}(\Lambda) \quad (2.3.50)$$

where χ_0^* is defined with respect to the chosen basis of $\mathcal{S}(\Lambda)$. We now have the following theorem.

Theorem 2.3.2. [DGM90, DGM94] *Let Λ be an even lattice and $\sqrt{2}\Lambda^*$ be even. Then the following statements are true.*

1. *The vertex operators $V_T(\psi, z)$ and the Hilbert space $\mathcal{H}_T^+(\Lambda)$ define a real, Hermitian representation of the vertex operator algebra $\mathcal{H}^+(\Lambda)$ with the conjugation map on the twisted sector $\mathcal{H}_T^+(\Lambda)$ given on states of the form (2.3.39) by*

$$\bar{\chi} = e^{-i\pi L_0^T} \theta M(\chi), \quad (2.3.51)$$

with extension to all states of the twisted sector defined by antilinearity.

2. *The extended vertex operator algebra $\widetilde{\mathcal{H}}(\Lambda) := \mathcal{H}^+(\Lambda) \oplus \mathcal{H}_T^+(\Lambda)$ is a real, Hermitian vertex operator algebra and is modular invariant if Λ is self-dual. This is called the $\mathbb{Z}_2$ -orbifold model.*

Remark 2.3.4. The extended vertex operator algebra $\widetilde{\mathcal{H}}(\Lambda)$ is independent of the choice of section e_λ used to define the operators $V_T(\psi, z)$ in (2.3.44) up to isomorphism. Indeed, if we choose a different section $\lambda \mapsto e'_\lambda$ of the Heisenberg extension and denote the operators in (2.3.44) with e_λ replaced by e'_λ by $V'_T(\psi, z)$, then the VOA $(\widetilde{\mathcal{H}}(\Lambda), \widetilde{V})$ is isomorphic to the VOA $(\widetilde{\mathcal{H}}(\Lambda), \widetilde{V}')$ where for $\psi \in \mathcal{H}^+(\Lambda)$, and $\chi \in \mathcal{H}_T^+(\Lambda)$

$$\begin{aligned} \widetilde{V}(\psi, z) &= \begin{pmatrix} V(\psi, z) & 0 \\ 0 & V_T(\psi, z) \end{pmatrix}, & \widetilde{V}(\chi, z) &= \begin{pmatrix} 0 & \overline{W}(\chi, z) \\ W(\chi, z) & 0 \end{pmatrix}, \\ \widetilde{V}'(\psi, z) &= \begin{pmatrix} V(\psi, z) & 0 \\ 0 & V'_T(\psi, z) \end{pmatrix}, & \widetilde{V}'(\chi, z) &= \begin{pmatrix} 0 & \overline{W}'(\chi, z) \\ W'(\chi, z) & 0 \end{pmatrix}, \end{aligned} \quad (2.3.52)$$

and $\overline{W}'(\chi, z)$ is defined as in (2.2.18) using V'_T . The explicit isomorphism $u : \widetilde{\mathcal{H}}(\Lambda) \rightarrow \widetilde{\mathcal{H}}(\Lambda)$ is given by

$$\begin{pmatrix} |\lambda\rangle \\ |\chi\rangle \end{pmatrix} \mapsto \begin{pmatrix} b(\lambda)|\lambda\rangle \\ S|\chi\rangle \end{pmatrix}, \quad |\chi\rangle \in \mathcal{S}(\Lambda), \quad (2.3.53)$$

where $b(\lambda), S$ is as in Remark 2.3.3 and u commutes with a_n^i, c_k^i .

Applying this theorem to the Leech lattice defines the Monster module, also known as the FLM module $\mathcal{H}_{\text{FLM}} := \widetilde{\mathcal{H}}(\Lambda_L)$, see (2.1.21) above.

Remark 2.3.5. Note that we can take $M = \mathbb{1}$ in (2.3.49) by choosing the spin representation $\mathcal{S}(\Lambda)$ of the Heisenberg group $\Gamma(\Lambda)$ to be real. In that case the representation matrices γ_λ for e_λ acting on the twisted sector Hilbert space

$$\mathcal{H}_T(\Lambda) \cong \mathcal{F}_T \otimes_{\mathbb{R}} \mathcal{S}(\Lambda) \quad (2.3.54)$$

are real, i.e., $\gamma_\lambda^* = \gamma_\lambda$. We will use this to simplify the discussion in Subsection 2.5.2.

2.4 $\mathcal{N} = 1$ Superconformal Vertex Operator Algebra And Spin Lifts

We now describe a superconformal vertex operator algebra (SCVOA). Roughly speaking, an $\mathcal{N} = 1$ SCVOA is a super-VOA with, in addition, $\mathcal{N} = 1$ conformal supersymmetry.

Definition 2.4.1. An $\mathcal{N} = 1$ superconformal vertex operator algebra (SCVOA) is a tuple $(\mathcal{H}, \mathcal{F}, \mathbf{V}, |0\rangle, \psi_L)$ where $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ is a super Hilbert space of states, $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ with $\mathcal{F}_\alpha \subset \mathcal{H}_\alpha$ a dense subspace called the *Fock space* and $\mathbf{V}$ a set of linear operators called *vertex operators* $V(\psi, z)$ which are meromorphic in z and linear in ψ . They are in one-to-one correspondence with the states $\psi \in \mathcal{F}$. We will denote the degree of $\psi \in \mathcal{H}_\alpha$ with respect to the $\mathbb{Z}_2$ -grading by $|\psi|$ and call it the *parity* of ψ . There are three special states in $\mathcal{F}$, the *vacuum* $|0\rangle \in \mathcal{F}_0$, the *conformal state* $\psi_L \in \mathcal{F}_0$ and the *superconformal state* $\psi_G \in \mathcal{F}_1$. The theory must satisfy the properties listed below, and is said to be a *Hermitian $\mathcal{N} = 1$ superconformal vertex operator algebra* if it satisfies in addition the last property (7).

1. The *moments* of the vertex operator $V(\psi_L, z)$ and $V(\psi_G, z)$ are given⁹ by

$$\begin{aligned} T_B(z) &:= V(\psi_L, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \\ T_F(z) &:= V(\psi_G, z) = \sum_r G_r z^{-r-\frac{3}{2}} \end{aligned} \tag{2.4.1}$$

and satisfy the $\mathcal{N} = 1$ superconformal algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}m(m^2-1)\delta_{m,-n} \\ [G_r, L_n] &= \left(r - \frac{n}{2}\right)G_{r+n} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \\ [L_m, \hat{c}] &= [G_r, \hat{c}] = 0 \end{aligned} \tag{2.4.2}$$

with $L_n^\dagger = L_{-n}$ and $L_n|0\rangle = 0, n \geq -1$. Equivalently, the operators T_B and T_F satisfy the OPE

$$\begin{aligned} T_B(z)T_B(w) &\sim \frac{3\hat{c}/4}{(z-w)^4} + \frac{2}{(z-w)^2}T_B(w) + \frac{1}{(z-w)}\partial_w T_B(w) \\ T_B(z)T_F(w) &\sim \frac{3/2}{(z-w)^2}T_F(w) + \frac{1}{(z-w)}\partial_w T_F(w) \\ T_F(z)T_F(w) &\sim \frac{\hat{c}/4}{(z-w)^3} + \frac{1/2}{(z-w)}T_B(w) \end{aligned} \tag{2.4.3}$$

⁹Whenever we have strictly half-integral powers of a complex number, we will choose the principal branch of square root to define it: in the principal branch, we choose $\sqrt{-1} = e^{i\pi/2}$, the square root of a complex number in the upper half plane is mapped to the first quadrant and the lower half plane is mapped to fourth quadrant.

where $\sim$ denotes that the relation is true up to a holomorphic term. When the operators G_r are integer moded, $i.e. r \in \mathbb{Z}$, the superconformal algebra is called the *Ramond* (R) algebra and for half-integer moding, $r \in \mathbb{Z} + \frac{1}{2}$, the superconformal algebra is called the *Neveu-Schwartz* (NS) algebra.

2. For homogenous ψ , the moments of vertex operators $V(\psi, z)$ are linear operators of parity $|\psi|$, where $|\psi|$ is the $\mathbb{Z}_2$ -grading of $\psi \in \mathcal{H}$.
3. The vertex operators satisfy

$$(i) \quad V(\psi, z)|0\rangle = e^{zL_0}\psi \quad (ii) \quad V(\psi, z)V(\phi, w) = (-1)^{|\psi||\phi|}V(\phi, w)V(\psi, z), \quad (2.4.4)$$

The relation (ii) is defined for vertex operators corresponding to homogeneous states and is called the *locality relation*, see Definition 2.2.1 for details. We say that the vertex operator $V(\psi, z)$ is local with degree $|\psi|$ with respect to $\mathbf{V}$.

4. The operator $w^{L_0} := e^{L_0 \ln w}$ acts locally with respect to $\mathbf{V}$, that is $w^{L_0}V(\psi, z)w^{-L_0}$ is local with parity $|\psi|$ with respect to $\mathbf{V}$.
5. L_0 is diagonalisable and its spectrum is bounded below. The eigenvalues h_ψ of eigenstates ψ of L_0 are called *conformal weights*.
6. The vacuum is the only $\mathfrak{su}(1, 1)$ invariant state in the theory, that is the only state annihilated by L_0 and $L_{\pm 1}$.
7. For all $\psi \in \mathcal{F}$, the vertex operator $V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger$ is local with parity $|\psi|$ with respect to $\mathbf{V}$.

Remark 2.4.1. If we only require the existence of a conformal vector and the associated Virasoro algebra in the above definition, it is called a *vertex operator superalgebra* (SVOA). So an SCVOA is in particular an SVOA.

Definition 2.4.2. We say that two SCVOAs $\mathcal{H}$ and $\mathcal{H}'$ with Fock spaces $\mathcal{F}$ and $\mathcal{F}'$ respectively and corresponding vertex operators $V(\psi, z)$ and $V'(\psi', z)$ are isomorphic if there exists a unitary map $u : \mathcal{H} \longrightarrow \mathcal{H}'$ such that

$$V'(u\psi, z) = uV(\psi, z)u^{-1}, \quad \psi \in \mathcal{F}, \quad u\psi_G = \psi'_G, \quad (2.4.5)$$

where ψ_G, ψ'_G are the superconformal states in $\mathcal{H}$ and $\mathcal{H}'$ respectively. An isomorphism $u : \mathcal{H} \longrightarrow \mathcal{H}$ which preserves $|0\rangle, \psi_L, \psi_G$ is called an automorphism of $\mathcal{H}$.

Note that an isomorphism $u : \mathcal{H} \longrightarrow \mathcal{H}'$ of SCVOAs (or simply VOAs) automatically satisfies [DGM94, Proposition 2.14]

$$u|0\rangle = |0'\rangle, \quad u\psi_L = \psi'_L, \quad (2.4.6)$$

where $|0\rangle, |0'\rangle$ are the vacuum states and ψ_L, ψ'_L are the conformal states of the two SCVOAs.

As will be shown in the proposition below, the conformal weights of an $\mathcal{N} = 1$ superconformal vertex operator algebra are elements of $\frac{1}{2}\mathbb{Z}$. Note that the locality relation forces the singularities in the OPEs of vertex operators to be integral powers of $(z - w)$.

In superconformal field theories in physics, one often encounters OPEs with half-integer power singularities. This clearly violates the locality relation. Indeed, the meromorphic function $(z - w)^{-\frac{1}{2}}$ defined on $|z| > |w|$ when analytically continued to $|w| > |z|$ picks up a fourth root of unity and violates the locality relation above which only allows ± 1 factor. Such fields are called non-local fields. A superconformal field theory is a spin CFT (see Footnote 4). The states in the spin CFT associated to the nonbounding circle¹⁰ are called the *Ramond* (R) sector states while the states associated to the bounding circle are called the *Neveu-Schwartz* (NS) sector states. The subspaces of R and NS states in $\mathcal{H}$ is denoted by $\mathcal{H}_R$ and $\mathcal{H}_{NS}$ respectively. The corresponding vertex operators are called the R fields and NS fields respectively. Consequently the R fields have square root singularity in their OPE with the superconformal field T_F and the NS fields have integer power singularities in their OPE with the superconformal field T_F . We will show that the “Beauty and the Beast” SCFT has a subspace which forms an SCVOA while the full space is a non-local superconformal field theory in the sense that it contains non-local fields.

The vertex operators can be Laurent-expanded as before

$$V(\psi, z) = \sum_n V_n(\psi) z^{-n-h_\psi} \quad (2.4.7)$$

where the summation index is such that $n + h_\psi \in \mathbb{Z}$ for R vertex operators and $n + h_\psi \in \mathbb{Z} + \frac{1}{2}$ for NS vertex operators. Note that if $V(\psi, z)$ is an NS vertex operator then $\psi \in \mathcal{H}_1$. The moments again satisfy

$$V_{-h_\psi}(\psi)|0\rangle = \psi, \quad V_{-n}(\psi)|0\rangle = 0, \quad n > h_\psi. \quad (2.4.8)$$

A state $\psi \in \mathcal{H}$ with conformal weight h is called a *superconformal primary* if

$$L_0\psi = h\psi, \quad L_n\psi = 0, \quad G_r\psi = 0, \quad n, r > 0. \quad (2.4.9)$$

¹⁰ Spin structures on a circle are classified by $H^1(S^1, \mathbb{Z}_2) \cong \mathbb{Z}_2$. The circle with spin structure corresponding to the trivial class is called nonbounding circle while the circle with the spin structure corresponding to the nontrivial class is called bounding circle.

Proposition 2.4.1. 1. If $U(z)$ satisfies $U(z)|0\rangle = e^{zL_{-1}}\phi$ and is local with degree $|\phi|$ with respect to $\mathbf{V}$, then $U(z) = V(\phi, z)$. In particular

$$\frac{d}{dz}V(\psi, z) = V(L_{-1}\psi, z). \quad (2.4.10)$$

2. The duality relation $V(\psi, z)V(\phi, w) = V(V(\psi, z-w)\phi, w)$ is true.

3. Skew-symmetry: $V(\psi, z)\phi = (-1)^{|\psi||\phi|}e^{zL_{-1}}V(\phi, -z)\psi$.

4. For each $\psi \in \mathcal{F}$, the locality of $V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger$ implies that we have a grading-preserving antilinear map

$$\psi \mapsto \bar{\psi},$$

where

$$V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger = V(\bar{\psi}, z) \quad (2.4.11)$$

so that

$$\bar{\psi} = \lim_{z \rightarrow 0} V(e^{z^*L_1}z^{*-2L_0}\psi, 1/z^*)^\dagger |0\rangle. \quad (2.4.12)$$

Moreover, for a quasi-primary vertex operator, the moments satisfy $V_n(\psi)^\dagger = V_{-n}(\bar{\psi})$.

5. The conformal weights are non-negative.

6. For any $\phi, \psi \in \mathcal{F}$, the following holds:

(a) if $L_0\psi = h_\psi\psi$ then $L_0\bar{\psi} = h_\psi\bar{\psi}$.

(b) $\overline{L_{-1}\psi} = -L_{-1}\bar{\psi}$.

(c) $\overline{\bar{\psi}} = \psi$.

(d) $f_{\phi_1\phi_2\phi_3} = (-1)^{|\phi_1||\phi_2|+|\phi_2||\phi_3|+|\phi_3||\phi_1|}e^{-i\pi(h_{\phi_1}+h_{\phi_2}+h_{\phi_3})}(f_{\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3})^*$ where $\phi_1, \phi_2, \phi_3 \in \mathcal{F}$
with $L_0\phi_j = h_j\phi_j$ for $1 \leq j \leq 3$ and

$$f_{\phi_1\phi_2\phi_3} = \langle \bar{\phi}_1 | V(\phi_2, 1) | \phi_3 \rangle. \quad (2.4.13)$$

(e) $\overline{L_1\psi} = -L_1\bar{\psi}$.

(f) the spectrum of L_0 is non-negative half-integers $\frac{1}{2}\mathbb{Z}$.

(g) $\langle \bar{\psi} | \phi \rangle = \langle \bar{\phi} | \psi \rangle$.

7. The operator product expansion of any two vertex operators $V(\psi, z)$ and $V(\phi, w)$ corresponding to eigenstates of L_0 is given by

$$\begin{aligned} V(\psi, z)V(\phi, w) &= \sum_n (z-w)^{n-h_\psi-h_\phi} V(\phi_n, w) \\ &= \sum_n (z-w)^{-h_{\phi_n}-h_\psi-h_\phi} V(\phi_n, w) \end{aligned} \quad (2.4.14)$$

where

$$\phi_n := V_{h_\phi - n}(\psi)\phi. \quad (2.4.15)$$

Proof. We will only prove (1), (3) and (6) since the proof of others is similar to the VOA case and can be found in [DGM94].

(1) We have

$$\begin{aligned} U(z)e^{\zeta L^{-1}}\psi &= U(z)V(\psi, \zeta)|0\rangle = (-1)^{|\phi||\psi|}V(\psi, \zeta)U(z)|0\rangle \\ &= (-1)^{|\phi||\psi|}V(\psi, \zeta)e^{zL^{-1}}\phi = (-1)^{|\phi||\psi|}V(\psi, \zeta)V(\phi, z)|0\rangle \\ &= V(\phi, z)V(\psi, \zeta)|0\rangle = V(\phi, z)e^{\zeta L^{-1}}\psi \end{aligned}$$

Thus, taking $\zeta \rightarrow 0$, we deduce $U(z) = V(\phi, z)$. The second part follows from the fact that $\frac{d}{dz}V(z, \psi)$ is local with respect to $\mathbf{V}$ and satisfies

$$\frac{d}{dz}V(z, \psi)|0\rangle = e^{zL^{-1}}L_{-1}\psi. \quad (2.4.16)$$

(3) Using duality and (3)(i) of Definition 2.4.1, we have

$$V(\psi, z)V(\phi, w)|0\rangle = V(V(\psi, z-w)\phi, w)|0\rangle = e^{wL^{-1}}V(\psi, z-w)\phi. \quad (2.4.17)$$

Using the locality (3) (ii) of Definition 2.4.1 first, we obtain

$$V(\psi, z)V(\phi, w)|0\rangle = (-1)^{|\phi||\psi|}V(\phi, z)V(\psi, w)|0\rangle = (-1)^{|\phi||\psi|}e^{zL^{-1}}V(\phi, z-w)\psi. \quad (2.4.18)$$

Taking $w \rightarrow 0$ in both expressions and comparing, we get the claimed skew-symmetry.

(6) The proof of (a), (b), (c), (e) and (g) is exactly as in [DGM94, Proposition 2.9]. We now

prove (d). We have

$$\begin{aligned}
f_{\phi_1\phi_2\phi_3} &= \langle \bar{\phi}_1 | V(\phi_2, 1) | \phi_3 \rangle \\
&= (-1)^{|\phi_2||\phi_3|} \langle \bar{\phi}_1 | e^{L_{-1}} V(\phi_3, -1) | \phi_2 \rangle \quad (\text{by skew-symmetry}) \\
&= (-1)^{|\phi_2||\phi_3|} \left\langle \phi_2 \left| V(\phi_3, -1)^\dagger e^{L_1} \right| \bar{\phi}_1 \right\rangle^* \\
&= (-1)^{|\phi_2||\phi_3| - 2h_{\phi_3}} \langle \phi_2 | V(e^{-L_1}\bar{\phi}_3, -1) e^{L_1} | \bar{\phi}_1 \rangle^* \quad (\text{by (4) and (b),(c) of (6)}) \\
&= (-1)^{|\phi_2||\phi_3| - 2h_{\phi_3} + |\phi_1||\phi_3|} \langle \phi_2 | e^{-L_{-1}} V(e^{L_1}\bar{\phi}_1, 1) e^{-L_1} | \bar{\phi}_3 \rangle^* \quad (\text{by skew-symmetry}) \\
&= (-1)^{|\phi_2||\phi_3| - 2h_{\phi_3} + |\phi_1||\phi_3|} \left\langle \phi_2 \left| e^{-L_{-1}} V(\phi_1, 1)^\dagger e^{-L_1} \right| \bar{\phi}_3 \right\rangle^* \\
&= (-1)^{|\phi_2||\phi_3| - 2h_{\phi_3} + |\phi_1||\phi_3| + |\phi_1||\phi_2|} \left\langle \phi_1 \left| V(e^{-L_1}\phi_2, -1)^\dagger \right| \bar{\phi}_3 \right\rangle^* \quad (\text{by skew-symmetry}) \\
&= (-1)^{|\phi_2||\phi_3| - 2h_{\phi_3} + |\phi_1||\phi_3| + |\phi_1||\phi_2| - 2h_{\phi_2}} \langle \phi_1 | V(\bar{\phi}_2, -1) | \bar{\phi}_3 \rangle^* \\
&= (-1)^{|\phi_2||\phi_3| - 2h_{\phi_3} + |\phi_1||\phi_3| + |\phi_1||\phi_2| - 2h_{\phi_2}} e^{i\pi(-h_{\phi_1} + h_{\phi_2} + h_{\phi_3})} \left(f_{\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3} \right)^* \\
&= (-1)^{|\phi_1||\phi_2| + |\phi_2||\phi_3| + |\phi_3||\phi_1|} e^{-i\pi(h_{\phi_1} + h_{\phi_2} + h_{\phi_3})} (f_{\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3})^*
\end{aligned}$$

where in the last step we used

$$w^{L_0} V(\psi, z) w^{-L_0} = w^{h_\psi} V(\psi, zw) \quad \text{for } w = -1. \quad (2.4.19)$$

This is proved using the axiom (4) of Definition 2.4.1 as follows: we have

$$w^{L_0} V(\psi, z) w^{-L_0} |0\rangle = w^{L_0} e^{zL_{-1}} \psi,$$

since $L_0|0\rangle = 0$ and $V(\psi, z)|0\rangle = e^{zL_{-1}}\psi$. We now use the following version of the Baker-Campbell-Hausdorff formula:

Theorem 2.4.1. *If $[X, Y] = sY$, where s is a complex number with $s \neq 2\pi i n$ for all integers n , then we have*

$$e^X e^Y = e^{\exp(s)Y} e^X. \quad (2.4.20)$$

We can write

$$w^{L_0} e^{zL_{-1}} = e^{L_0 \ln w} e^{zL_{-1}}.$$

Using the fact that $[L_0, L_{-1}] = L_{-1}$, we see that the above theorem implies

$$e^{L_0 \ln w} e^{zL_{-1}} = e^{\exp(\ln w)zL_{-1}} w^{L_0}.$$

Thus we get

$$w^{L_0} e^{zL_{-1}} = e^{zwL_{-1}} w^{L_0}.$$

Thus we get

$$w^{L_0} V(\psi, z) w^{-L_0} |0\rangle = e^{zwL_{-1}} w^{h_\psi} \psi.$$

Part (1) of this proposition then gives

$$w^{L_0} V(\psi, z) w^{-L_0} = w^{h_\psi} V(\psi, zw). \quad (2.4.21)$$

To prove (f), we follow [DGM94, Proof of Proposition 2.9 (vi)] to get

$$\langle e^{L_1} \bar{\psi} | e^{L_1} \psi \rangle = (-1)^{2h_\psi + |\psi||\bar{\psi}|} \langle e^{L_1} \psi | e^{L_1} \bar{\psi} \rangle \implies (-1)^{2h_\psi + |\psi||\bar{\psi}|} = 1.$$

If ψ is Fermionic then $|\psi| = |\bar{\psi}| = 1$ and $h_\psi \in \mathbb{Z} + \frac{1}{2}$. In other cases $h_\psi \in \mathbb{Z}$. Thus $h_\psi \in \frac{1}{2}\mathbb{Z}$. $\square$

Isomorphisms of SCVOAs and SVOAs is defined analogous to those for VOAs. A consequence of the definition is that isomorphisms also preserve the superconformal vector in addition to the conformal vector and the vacuum.

2.4.1 Spin Lifts Of A Conformal Field Theory

Given a modular invariant holomorphic conformal field theory $\mathcal{H}$ with a non-anomalous $\mathbb{Z}_2$ automorphism, there is a natural way to construct a spin CFT. The essential idea goes back to the GSO projection applied to string worldsheet superconformal field theories to obtain a consistent string theory [GSO77] and played an important role in discussions of bosonization on arbitrary Riemann surfaces [AGBM⁺87, AGMN⁺86]. More general versions of the construction of [AGBM⁺87, AGMN⁺86] have been discussed in the recent literature [GK15, JSW19, KT17, KTT19, LS19, Sha23]. This spin-conformal field theory is called a spin lift of $\mathcal{H}$. The idea is to couple the non-anomalous $\mathbb{Z}_2$ automorphism of the CFT to an invertible topological field theory sensitive to spin structure known as the Arf theory [DG18], and then gauge the diagonal global $\mathbb{Z}_2$ automorphism of the product theory. The resulting theory depends on the spin structure and in general the vertex operators do not satisfy the standard locality conditions of a super-vertex operator algebra. Nevertheless the GSO projection of the spin lift reproduces the original CFT. The spin lift also has an emergent $\mathbb{Z}_2$ symmetry which we will identify with the fermion number symmetry $(-1)^F$. Let us describe the construction in more detail.

Formal Description Of Gauging A Discrete Symmetry

Let us recall some well-known general facts. We are following [MS06, Section 7.1]. Let $\mathcal{F}$ be a d -dimensional quantum field theory, that is, a functor $\mathcal{F} : \text{Bord}_d \longrightarrow \text{Vect}$ from the bordism category, whose objects are $(d-1)$ -dimensional manifolds (possibly with extra structure) and whose morphisms are d -dimensional oriented bordisms, to the linear category of vector spaces and linear maps. We say that $\mathcal{F}$ has global symmetry G when, for every $(d-1)$ -manifold Y , the vector space $\mathcal{H}(Y) := \mathcal{F}(Y)$,

called the space of states, is a representation space of G and for any bordism $X : Y_1 \longrightarrow Y_2$, the linear map $\mathcal{F}(X) : \mathcal{H}(Y_1) \longrightarrow \mathcal{H}(Y_2)$, called the transition amplitude, is G -equivariant. Gauging the symmetry group involves two steps:

1. Construct the “equivariant theory” by enlarging the number of fields. In the present example our new fields are a principal G -bundle with connection. In our case G is a finite group so the bundle has a unique connection. ¹¹
2. “Sum” or “integrate” over the principal G -bundles with connection with a suitable measure.

To construct the equivariant theory, we define a new bordism category Bord_d^G whose objects are $(d-1)$ -manifolds Y equipped with principal G -bundles $P \rightarrow Y$ with connection and morphisms are d -dimensional bordisms $X : Y_1 \longrightarrow Y_2$ also equipped with principal G -bundles with connection $P' \rightarrow X$ such that the restriction of $P' \rightarrow X$ to the ingoing boundary is isomorphic to $P_1 \rightarrow Y_1$ and the restriction to the outgoing boundary is isomorphic to $P_2 \rightarrow Y_2$. Note that Bord_d can be regarded as a subcategory of Bord_d^G with trivial bundles associated to the objects and bordisms. Now, the equivariant theory $\mathcal{F}^{\text{eq}} : \text{Bord}_d^G \longrightarrow \text{Vect}$ assigns to every object $P \rightarrow Y$ a vector space $\mathcal{H}^{\text{eq}}(P) := \mathcal{H}(P \rightarrow Y)$ and for every bordism $(P' \rightarrow X)$ between $P_1 \rightarrow Y_1$ and $P_2 \rightarrow Y_2$ a transition amplitude

$$\mathcal{F}^{\text{eq}}(P') := \mathcal{F}^{\text{eq}}(P' \rightarrow X) : \mathcal{H}^{\text{eq}}(P_1) \longrightarrow \mathcal{H}^{\text{eq}}(P_2) \quad (2.4.22)$$

such that when $P \rightarrow Y$ is the trivial bundle $Y \times G$, $\mathcal{H}^{\text{eq}}(Y \times G) = \mathcal{H}(Y)$ and for trivial bordism $X \times G$ between trivial objects $Y_1 \times G$ and $Y_2 \times G$, $\mathcal{F}^{\text{eq}}(P')$ reduces to the original transition amplitude $\mathcal{F}(X)$. The automorphisms of the bundle $P \rightarrow Y$ must act on $\mathcal{H}^{\text{eq}}(P)$. For the trivial bundle, this corresponds to the original global symmetry G -action on $\mathcal{H}(Y)$. The amplitudes are equivariant maps.

Once we have the equivariant theories we can perform the second step of gauging, that is sum over bundles with connection. We need to take into account the fact that the isomorphic bundles have the same space of states. More precisely, if we have a bundle isomorphism $f : P \longrightarrow Q$ of bundles $P \rightarrow Y$ and $Q \rightarrow Y$ then we have an isomorphism $\tilde{f} : \mathcal{H}^{\text{eq}}(P) \longrightarrow \mathcal{H}^{\text{eq}}(Q)$ such that $\tilde{f}(\psi(P)) = \psi(Q)$ for states $\psi(P) \in \mathcal{H}^{\text{eq}}(P)$. In particular, when $f : P \longrightarrow P$ is a bundle automorphism of $P \rightarrow Y$ then $\tilde{f}(\psi(P)) = \psi(P)$. This is the mathematical equivalent of gauge invariant states. The gauged theory thus contains only gauge invariant states. The space of states of the gauged theory is then defined

¹¹We use the term “field” in the generalized sense explained in Section 3 of [FH13]. This step is the mathematical version of “coupling the theory to background gauge fields” in physics.

as a direct sum over isomorphism classes of principal bundles:

$$\mathcal{H}^{\text{gd}}(Y) := \bigoplus_{[P] \in \text{Prin}_G(Y)} \mathcal{H}^{\text{eq}}(P)^{\text{Aut}(P)} \quad (2.4.23)$$

where $\mathcal{H}^{\text{eq}}(P)^{\text{Aut}(P)}$ is the subspace of $\text{Aut}(P)$ -invariant states of $\mathcal{H}^{\text{eq}}(P)$ and $\text{Prin}_G(Y)$ denotes the set of isomorphism classes of principal G -bundles over Y . For a d -dimensional bordism without boundary, the partition function $Z(X)$ of the theory is the complex number $Z(X) := \mathcal{F}(X)$. We also have partition function $Z^{\text{eq}}(P')$ for equivariant theories corresponding to principal bundle $P' \rightarrow X$. The partition function of the gauged theory is defined as

$$Z^{\text{gd}}(X) = \sum_{[P'] \in \text{Prin}_G(X)} \frac{1}{|\text{Aut}(P')|} Z^{\text{eq}}(P'). \quad (2.4.24)$$

We now want to apply the above general procedure to $(1+1)\text{d}$ quantum field theory. We take the spatial slices to be disjoint unions of circles to determine the gauged space of states. We start by noting that any $g \in G$ we can construct a principal G -bundle $P_g \rightarrow S^1$ whose isomorphism class only depends on the conjugacy class of g , and moreover there is a 1-1 correspondence of conjugacy classes of G and isomorphism classes of principal G -bundles over the circle. More explicitly we have

$$P_g := \mathbb{R} \times_{\mathbb{Z}} G \longrightarrow S^1 \quad (2.4.25)$$

where the equivalence relation is

$$(s+1, g') \sim (s, gg'). \quad (2.4.26)$$

The projection map is $[s, g'] \mapsto e^{2\pi i s}$ and the right action of G is given by $[s, g'] \cdot g'' = [s, g'g'']$. The automorphism group of this bundle is the centraliser $Z(g)$ of g in G . Thus for each $[g] \in \text{Conj}(G)$ we have an equivariant space of states $\mathcal{H}^{\text{eq}}(P_g)$ which is a representation of the centraliser $Z(g)$ and the space of states of the gauged theory is

$$\mathcal{H}^{\text{gd}}(S^1) := \bigoplus_{[g] \in \text{Conj}(G)} \mathcal{H}^{\text{eq}}(P_g)^{Z(g)}. \quad (2.4.27)$$

Using the structure (2.4.27) of the gauged space of states, we obtain the partition function on the torus

$$Z^{\text{gd}}(\tau)(T^2) = \text{Tr}_{\mathcal{H}^{\text{gd}}(S^1)} e^{-2\pi i \text{Im}(\tau)H} e^{2\pi i \text{Re}(\tau)P} \quad (2.4.28)$$

where τ is the complex moduli of the torus and H and P are the Hamiltonian and momentum operators generating time and space translations respectively. By expanding the sum, one can also write the partition function as

$$Z^{\text{gd}}(T^2)(\tau) = \frac{1}{|G|} \sum_{[g_1, g_2]=1} Z^{\text{eq}}(P_{g_1, g_2})(\tau) \quad (2.4.29)$$

where

$$Z^{\text{eq}}(P_{g_1, g_2})(\tau) = \text{Tr}_{\mathcal{H}^{\text{eq}}(P_{g_1})} \tilde{g}_2 e^{-2\pi i \text{Im}(\tau) H} e^{2\pi i \text{Re}(\tau) P} \quad (2.4.30)$$

The g_2 action on $\mathcal{H}$ must be extended, in a consistent way, to an action by $\tilde{g}_2$ on the spaces $\mathcal{H}^{\text{eq}}(P_{g_1})$. How this is done is part of the definition of the gauged theory, but it is constrained by, e.g. modular invariance. On the left hand side of (2.4.30) P_{g_1, g_2} denotes the bundle over T^2 with monodromy g_1, g_2 around the two cycles. To be explicit, we have

$$P_{g_1, g_2} = \mathbb{R}^2 \times_{\mathbb{Z} \oplus \mathbb{Z}} G \quad (2.4.31)$$

and the equivalence relation is given by

$$(s+1, t, g) \sim (s, t, g_1 g), \quad (s, t+1, g) \sim (s, t, g_2 g). \quad (2.4.32)$$

A standard fact is that when the gauge group G is Abelian, the gauged theory has a global symmetry isomorphic to the Pontrjagin dual $\tilde{G} := \text{Hom}(G, \text{U}(1))$ of G . An element $\chi \in \tilde{G}$ acts on the equivariant Hilbert space $\mathcal{H}^{\text{eq}}(P_g)$ by multiplying $\chi(g) \in \text{U}(1)$. Note that this action is well defined since χ being a group homomorphism only depends on the conjugacy class of $g \in G$. This defines the action of χ on $\mathcal{H}^{\text{gd}}(S^1)$. We can write the action schematically as:

$$\chi \cdot \mathcal{H}^{\text{gd}}(S^1) = \bigoplus_{[g] \in \text{Conj}(G)} \chi(g) \cdot \mathcal{H}^{\text{eq}}(P_g)^{Z(g)}, \quad \chi \in \tilde{G}. \quad (2.4.33)$$

For a conformal field theory, the partition function takes the form

$$Z^{\text{gd}}(T^2) = \frac{1}{|G|} \sum_{[g_1, g_2]=1} \text{Tr}_{\mathcal{H}^{\text{eq}}(P_{g_1})} \tilde{g}_2 q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}, \quad q = e^{2\pi i \tau}, \quad \bar{q} = e^{-2\pi i \bar{\tau}} \quad (2.4.34)$$

after recognising that

$$H = L_0 + \bar{L}_0 - \frac{c}{24} - \frac{\bar{c}}{24}, \quad P = i(L_0 - \bar{L}_0). \quad (2.4.35)$$

For vertex operator algebras (*chiral algebra* of conformal field theories), we only have q dependence in the partition function. We will be mostly interested in the case when $G = \mathbb{Z}_2$. In this case, we only have two isomorphism classes of principal bundles since we only have two conjugacy classes. The space of states corresponding to the conjugacy class of -1 is called the *twisted sector*.

The Arf Theory

The Arf theory is a 2 dimensional *spin topological quantum field theory* (spin TQFT) defined on a generic Riemann surface Σ . See [MS06, Page 14] for a discussion on spin TQFT and the Arf theory as an example ([MS06, Page 17]), see also [DG18] for a nice discussion of this theory. Recall that all orientable Riemann surfaces have vanishing second Stiefel-Whitney class [Nak03] and hence are spin

manifolds. Moreover the set of spin structures on Σ is in 1-1 correspondence with $H^1(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$ [Ati71] where g is the genus of Σ . In fact, the set of spin structure on Σ is an $H^1(\Sigma, \mathbb{Z}_2)$ -torsor over $\mathbb{Z}_2$. Thanks to a beautiful paper of M. Atiyah [Ati71] we can define the Arf invariant [Arf41] of a spin structure to be the mod-two index of the Dirac operator:

$$\text{Arf}[\rho] = \text{ind}_2(D_\rho). \quad (2.4.36)$$

Thus there is a dichotomy of even and odd spin structures:

$$\text{Arf}[\rho] := \begin{cases} 1 & \rho \text{ odd} \\ 0 & \rho \text{ even.} \end{cases} \quad (2.4.37)$$

On a genus g orientable surface Σ there are $2^{g-1}(2^g - 1)$ odd and $2^{g-1}(2^g + 1)$ even spin structures.

Using the Arf invariant, one can define a 2 dimensional invertible topological field theory ¹² called the Arf theory, see [DG18] for the detailed definition. The partition function of the Arf theory on a closed oriented Riemann surface Σ with spin structure ρ is:

$$\text{Arf}(\Sigma_\rho) := Z_{\text{Arf}}[\Sigma_\rho] := e^{i\pi \text{Arf}[\rho]}. \quad (2.4.38)$$

Note that the partition function is topological in that it does not depend on the conformal structure of the Riemann surface but it is sensitive to the choice of spin structure on Σ . To describe the Hilbert space of the Arf theory, we consider the Arf theory on a cylinder $\mathbb{R} \times S^1$ where $\mathbb{R}$ is the time direction. The (complex) Hilbert space of the Arf theory on the spatial circle is one dimensional. In fact, the Hilbert space is $\mathbb{Z}_2$ -graded and the grading is defined by the action of the spin automorphism on the space. In the physical literature this operator is denoted $(-1)^F$. The Hilbert space for the NS (bounding) spin structure has $(-1)^F = 1$ and for the R (nonbounding) spin structure has $(-1)^F = -1$. Thus, as supervector spaces we have ¹³ $\text{Arf}(S_{\text{NS}}^1) \cong \mathbb{C}^{1|0}$ and $\text{Arf}(S_{\text{R}}^1) \cong \mathbb{C}^{0|1}$ and $(-1)^F$ is just the parity operator.

Coupling A CFT To The Arf Theory

Let $\mathcal{C}$ be a modular invariant CFT (i.e. a functor from Bord_2 to Vect) with statespace on a circle $\mathcal{C}(S^1) := \mathcal{H}$ and suppose $\mathcal{C}$ has a non-anomalous $\mathbb{Z}_2$ automorphism σ . We start by taking the tensor product of the σ -equivariant lift of $\mathcal{C}$ with the $(-1)^F$ -equivariant lift of the Arf theory. The product theory has as domain a bordism category $\text{Bord}_2^{\mathbb{Z}_2, \text{Spin}}$ consisting of 1-manifolds and their

¹²The notion of an invertible TQFT is discussed in [Fre14, FH16, FM04].

¹³For any field $\mathbb{F}$ we denote by $\mathbb{F}^{p|q}$ the supervector space $\mathbb{F}^{p|q} := \mathbb{F}^p \oplus \mathbb{F}^q$ where $\mathbb{F}^p$ is the even subspace ($\mathbb{Z}_2$ -parity $\bar{0}$) and $\mathbb{F}^q$ is the odd subspace ($\mathbb{Z}_2$ -parity $\bar{1}$).

bordisms equipped with a spin structure and, independently, with a principal $\mathbb{Z}_2$ -bundle. Let P_0 and P_1 be the trivial and nontrivial principal $\mathbb{Z}_2$ bundles over the circle corresponding to the trivial and nontrivial double-covers, respectively.

We need to define the “equivariant lift” of the Arf theory on the category $\text{Bord}_2^{\mathbb{Z}_2, \text{Spin}}$. We do this as follows: Isomorphism classes of principal $\mathbb{Z}_2$ bundles over any manifold X are in 1 – 1 correspondence with cohomology classes $H^1(X, \mathbb{Z}_2)$.¹⁴ For each $t \in H^1(X, \mathbb{Z}_2)$ choose a representative bundle $\pi : P_t \rightarrow X$. As noted above, the set of spin structures on X is a torsor for $H^1(X, \mathbb{Z}_2)$. If ρ denotes a spin structure and $t \in H^1(X, \mathbb{Z}_2)$ we write the action as $\rho \rightarrow \rho + t$. We define the equivariant lift of the Arf theory to be

$$\text{Arf}^{\text{eq}}[P_t, X_\rho] := \text{Arf}[X_{\rho+t}] \quad (2.4.40)$$

where X_ρ is the manifold X with spin structure ρ . In particular,

$$\begin{aligned} \text{Arf}^{\text{eq}}[P_0, S_{\text{NS}}^1] &= \mathbb{C}^{1|0} \\ \text{Arf}^{\text{eq}}[P_1, S_{\text{NS}}^1] &= \mathbb{C}^{0|1} \\ \text{Arf}^{\text{eq}}[P_0, S_{\text{R}}^1] &= \mathbb{C}^{0|1} \\ \text{Arf}^{\text{eq}}[P_1, S_{\text{R}}^1] &= \mathbb{C}^{1|0} \end{aligned} \quad (2.4.41)$$

We define the lifted global symmetry $(-1)_{\text{Arf}^{\text{eq}}}^F$ on Arf^{eq} again to be the parity operator.

The equivariant theory $\mathcal{C}^{\text{eq}}$ inherits a $\mathbb{Z}_2$ symmetry $\tilde{\sigma}$ from the action of σ on $\mathcal{C}$. The action of $\tilde{\sigma}$ on $\mathcal{H} = \mathcal{C}^{\text{eq}}(P_0)$ is just the original action of σ . The action of $\tilde{\sigma}$ on $\mathcal{H}^{\text{eq}} := \mathcal{C}^{\text{eq}}(P_1)$ is part of the definition of the gauging. In our case, as we will see presently, the action is constrained by the modular covariance of the torus partition functions when combined with the Hilbert space interpretation of torus partition functions.¹⁵ Now, the product theory $\mathcal{C}^{\text{eq}} \otimes \text{Arf}^{\text{eq}}$ has a $\mathbb{Z}_2$ automorphism given by the diagonal $\sigma_d := \tilde{\sigma} \otimes (-1)_{\text{Arf}^{\text{eq}}}^F$. The *spin lift* $\mathcal{C}^{\text{SL}}$ of $\mathcal{C}$ is the spin theory, with domain category $\text{Bord}_2^{\text{Spin}}$, obtained by gauging the group of automorphisms generated by σ_d . Thus, the Hilbert spaces on the two spin circles (S_{NS}^1 is the circle with bounding spin structure and

¹⁴Recall that for a discrete group G , isomorphism classes of principal G -bundles over a connected space X are classified by $\text{Hom}(\pi_1(X, x_0), G)/\sim$ where the equivalence relation is given by conjugation. For G Abelian we have:

$$H^1(X, G) \cong \text{Hom}(H_1(X), G) \cong \text{Hom}(\pi_1(X, x_0), G)/\sim. \quad (2.4.39)$$

Taking $G = \mathbb{Z}_2$, an alternate way to prove this result is to observe that the classifying space of $\mathbb{Z}_2$ is $\mathbb{RP}^\infty$ and the set of homotopy classes of maps from a space X to $\mathbb{RP}^\infty$ is isomorphic to $H^1(X, \mathbb{Z}_2)$.

¹⁵In topological theories this action must be the identity. See Theorem 4 of [MS06]. But in general CFT this need not be so.

S_R^1 is the circle with nonbounding spin structure, see Footnote 10) are

$$\begin{aligned}
\mathcal{H}_{\text{NS}}^{\text{SL}} &:= \mathcal{C}^{\text{SL}}(S_{\text{NS}}^1) = \left(\mathcal{H}(S^1) \otimes \mathbb{C}^{1|0} \right)^{\sigma_d=+1} \oplus \left(\mathcal{H}^{\text{eq}} \otimes \mathbb{C}^{0|1} \right)^{\sigma_d=+1} \\
&\cong \mathcal{H}^+ \oplus \mathcal{H}^{\text{eq},-} \\
\mathcal{H}_R^{\text{SL}} &:= \mathcal{C}^{\text{SL}}(S_R^1) = \left(\mathcal{H}(S^1) \otimes \mathbb{C}^{0|1} \right)^{\sigma_d=+1} \oplus \left(\mathcal{H}^{\text{eq}} \otimes \mathbb{C}^{1|0} \right)^{\sigma_d=+1} \\
&\cong \mathcal{H}^- \oplus \mathcal{H}^{\text{eq},+},
\end{aligned} \tag{2.4.42}$$

where $\mathcal{H}^{\text{eq},\pm}$ is the ± 1 eigenspace of $\mathcal{H}^{\text{eq}}$ under the action of $\tilde{\sigma}$. Now, whenever we gauge a theory by an Abelian group A the gauged theory has a global symmetry group isomorphic to the Pontryagin dual $\tilde{A}$. In our case A is generated by σ_d so the Pontryagin dual is $\tilde{A} \cong \mathbb{Z}_2$. We denote the nontrivial element by $\check{\sigma}_d$. From eq. (2.4.33) we see that the action of $\check{\sigma}_d$ on the spin lift is given by

$$\check{\sigma}_d = \begin{cases} +1 & \text{on } \mathcal{H}^+ \oplus \mathcal{H}^- \\ -1 & \text{on } \mathcal{H}^{\text{eq},+} \oplus \mathcal{H}^{\text{eq},-}. \end{cases} \tag{2.4.43}$$

Now we turn to the partition functions of the spin-lifted theory. As we remarked above, the principal $\mathbb{Z}_2$ bundles over a Riemann surface Σ are in 1-1 correspondence with $H^1(\Sigma, \mathbb{Z}_2)$. When Σ is connected the automorphism group of these bundles is isomorphic to $\mathbb{Z}_2$. When we consider the σ_d -equivariant lift of $\mathcal{C} \otimes \text{Arf}$ there are a total of $|H^1(\Sigma, \mathbb{Z}_2)| = 2^{2g}$ equivariant partition functions indexed by “background $\mathbb{Z}_2$ -connections” T . The partition function for the equivariant lift of the product theory is given by (see eq. (2.4.38) and eq. (2.4.40))

$$Z^{\text{eq}}[P_T, \Sigma_\rho] := Z_{\mathcal{C}^{\text{eq}}}[P_T] Z_{\text{Arf}^{\text{eq}}}[P_T, \Sigma_\rho] = Z_{\mathcal{C}^{\text{eq}}}[P_T] e^{i\pi \text{Arf}[\rho+T]}. \tag{2.4.44}$$

Next, we gauge the equivariant theory, so we promote the background connection T to a dynamical connection t . That is, we sum over isomorphism classes of principal $\mathbb{Z}_2$ -bundles over Σ . Applying

(2.4.24) the partition function of the gauged theory is given by ¹⁶

$$\begin{aligned} Z^{\text{SL}}[\Sigma_\rho] &= \frac{1}{2} \sum_{t \in H^1(\Sigma, \mathbb{Z}_2)} Z^{\text{eq}}[P_t, \Sigma_\rho] \\ &= \frac{1}{2} \sum_{t \in H^1(\Sigma, \mathbb{Z}_2)} Z_{\mathcal{C}^{\text{eq}}}^{\text{eq}}[P_t] \exp(i\pi \text{Arf}[\rho + t]). \end{aligned} \quad (2.4.45)$$

Now we would like to check compatibility of (2.4.45) with the Hilbert space description (2.4.42) by examining the special case where Σ is a torus. There are four spin structures on the torus and we can denote them by $T_{00}^2, T_{01}^2, T_{10}^2, T_{11}^2$ where 0 and 1 denotes the bounding and nonbounding spin structure on a cycle of the torus. The corresponding partition functions of the spin-lifted theory are denoted by $Z^{\text{SL}}[00], \dots, Z^{\text{SL}}[11]$, respectively. Interpreting the first circle factor of $T^2 = S^1 \times S^1$ as the spatial circle we can expect that the partition functions (2.4.45) can be interpreted as the NS and R partition functions with and without $\check{\sigma}_d$ insertion:

$$\begin{aligned} Z^{\text{SL}}[00] &= \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}}} q^{L_0 - \frac{c}{24}}, \\ Z^{\text{SL}}[01] &= \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}}} \check{\sigma}_d q^{L_0 - \frac{c}{24}}, \\ Z^{\text{SL}}[10] &= \text{Tr}_{\mathcal{H}_{\text{R}}^{\text{SL}}} q^{L_0 - \frac{c}{24}}, \\ Z^{\text{SL}}[11] &= \text{Tr}_{\mathcal{H}_{\text{R}}^{\text{SL}}} \check{\sigma}_d q^{L_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}. \end{aligned} \quad (2.4.46)$$

We now verify that the relations in eq. (2.4.46) is consistent with eq. (2.4.45) and eq. (2.4.42). From (2.4.45) we have

$$\begin{aligned} Z^{\text{SL}}[00] &= \frac{1}{2} [Z^{\text{eq}}(P_{0,0}) + Z^{\text{eq}}(P_{0,1}) + Z^{\text{eq}}(P_{1,0}) - Z^{\text{eq}}(P_{1,1})] \\ Z^{\text{SL}}[01] &= \frac{1}{2} [Z^{\text{eq}}(P_{0,0}) + Z^{\text{eq}}(P_{0,1}) - Z^{\text{eq}}(P_{1,0}) + Z^{\text{eq}}(P_{1,1})] \\ Z^{\text{SL}}[10] &= \frac{1}{2} [Z^{\text{eq}}(P_{0,0}) - Z^{\text{eq}}(P_{0,1}) + Z^{\text{eq}}(P_{1,0}) + Z^{\text{eq}}(P_{1,1})] \\ Z^{\text{SL}}[11] &= \frac{1}{2} [-Z^{\text{eq}}(P_{0,0}) + Z^{\text{eq}}(P_{0,1}) + Z^{\text{eq}}(P_{1,0}) + Z^{\text{eq}}(P_{1,1})] \end{aligned} \quad (2.4.47)$$

¹⁶A version of (2.4.45) appeared as equation (2.5) of [LS19], but our (2.4.45) differs from equation (2.5) of [LS19] in a few minor ways. First we have chosen $S = 0$ relative to that paper. This is inconsequential since spin structures form a torsor for $H^1(\Sigma, \mathbb{Z}_2)$. Second, there is no factor of $\exp[i\pi \text{Arf}[\rho]]$ in our version. This is merely a choice of “gravitational counterterm,” making our formula more in line with the description in terms of gauging the $\mathbb{Z}_2$ symmetry $\langle \sigma_d \rangle$. Finally we have an overall factor 1/2, again because in the description in terms of gauging the automorphism group of a principal $\mathbb{Z}_2$ bundle over a connected surface is simply $\mathbb{Z}_2$. Equation (2.5) of [LS19] has a factor of 2^{-g} , giving a factor of 2^{1-g} relative to our normalization. Again this can be interpreted as coupling to a background invertible TQFT (an Euler character theory) so the difference is not essential. The choice 2^{-g} made in [LS19] has the nice property that if the original bosonic theory couples trivially to the background $\mathbb{Z}_2$ gauge theory then the spin lift has the same partition function.

Using (2.4.30) we see that

$$\begin{aligned}
Z^{\text{SL}}[00] &= \frac{1}{2} [Z^{\text{eq}}(P_{\bar{0},\bar{0}}) + Z^{\text{eq}}(P_{\bar{0},\bar{1}}) + Z^{\text{eq}}(P_{\bar{1},\bar{0}}) - Z^{\text{eq}}(P_{\bar{1},\bar{1}})] \\
&= \text{Tr}_{\mathcal{H}} \left(\frac{1+\tilde{\sigma}}{2} \right) q^{L_0 - \frac{c}{24}} + \text{Tr}_{\mathcal{H}^{\text{eq}}} \left(\frac{1-\tilde{\sigma}}{2} \right) q^{L_0 - \frac{c}{24}} \\
&= \text{Tr}_{\mathcal{H}^+} q^{L_0 - \frac{c}{24}} + \text{Tr}_{\mathcal{H}^{\text{eq}-}} q^{L_0 - \frac{c}{24}} = \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}}} q^{L_0 - \frac{c}{24}}.
\end{aligned} \tag{2.4.48}$$

Similarly one can check all other relations in (2.4.46).

We now review how modular invariance helps us determine the equivariant Hilbert space $\mathcal{H}^{\text{eq}}(P_1)$ as a $\langle \tilde{\sigma} \rangle$ -module. To this end, note that since g_1, g_2 are involutions

$$Z^{\text{eq}}(P_{g_1, g_2})(\tau) = Z^{\text{eq}}(P_{g_2, g_1})(-1/\tau) \tag{2.4.49}$$

since exchanging the two cycles on the torus changes $\tau \rightarrow -1/\tau$ and $P_{g_1, g_2} \rightarrow P_{g_2, g_1}$. Note that this in particular implies that $Z^{\text{eq}}(P_{g, g})(\tau)$ is invariant under $\tau \rightarrow -1/\tau$. For $g = \bar{0}$, this is obvious since it is the partition function of the original theory. But for $g \neq \bar{0}$ we obtain a nontrivial identity. Taking $g_2 = \bar{0}$, we see that the trace over¹⁷ $\mathcal{H}^{\text{eq}}(P_{g_1})$ can be obtained from the trace over the original Hilbert space $\mathcal{H}$ with the insertion of g_1 and performing the modular transformation $\tau \rightarrow -1/\tau$:

$$\text{Tr}_{\mathcal{H}^{\text{eq}}(P_{g_1})} q^{L_0 - \frac{c}{24}} = Z_{\mathcal{H}}^{g_1}(-1/\tau) \tag{2.4.50}$$

where

$$Z_{\mathcal{H}}^{g_1}(\tau) := \text{Tr}_{\mathcal{H}} g_1 q^{L_0 - \frac{c}{24}}. \tag{2.4.51}$$

This relation for $g_1 = \bar{1}$ in principle can be used to determine the nontrivial equivariant Hilbert space with the understanding that

$$Z_{\mathcal{H}^{\text{eq}}}^{\bar{1}}(\tau) = \text{Tr}_{\mathcal{H}^{\text{eq}}} \tilde{\sigma} q^{L_0 - \frac{c}{24}}. \tag{2.4.52}$$

where $\tilde{\sigma}$ acts on the equivariant Hilbert space. Moreover modular invariance also implies that

$$Z^{\text{eq}}(P_{\bar{1},\bar{1}})(\tau) = Z^{\text{eq}}(P_{\bar{1},\bar{0}})(\tau + 1). \tag{2.4.53}$$

Using the equivariant Hilbert space, one can determine the $\tilde{\sigma}_d = \pm 1$ R and NS sectors as follows:

$$\begin{aligned}
Z_{\text{NS}}^{\text{SL}-}(\tau) &:= \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}-}} q^{L_0 - \frac{c}{24}} = \frac{1}{2} [Z^{\text{eq}}(P_{\bar{1},\bar{0}}) - Z^{\text{eq}}(P_{\bar{1},\bar{1}})], \\
Z_{\text{R}}^{\text{SL}}(\tau) &:= \text{Tr}_{\mathcal{H}_{\text{R}}^{\text{SL}+}} q^{L_0 - \frac{c}{24}} = \frac{1}{2} [Z^{\text{eq}}(P_{\bar{1},\bar{0}}) + Z^{\text{eq}}(P_{\bar{1},\bar{1}})]
\end{aligned} \tag{2.4.54}$$

where $\mathcal{H}_{\text{NS}}^{\text{SL}\pm}$ is the $\tilde{\sigma}_d = \pm 1$ subspace of $\mathcal{H}_{\text{NS}}^{\text{SL}}$ and so on. More explicitly

$$\mathcal{H}_{\text{NS}}^{\text{SL}} := \mathcal{H}_{\text{NS}}^{\text{SL}+} \oplus \mathcal{H}_{\text{NS}}^{\text{SL}-}, \quad \mathcal{H}_{\text{R}}^{\text{SL}} := \mathcal{H}_{\text{R}}^{\text{SL}+} \oplus \mathcal{H}_{\text{R}}^{\text{SL}-}. \tag{2.4.55}$$

¹⁷In modern terminology, $\mathcal{H}^{\text{eq}}(P_{g_1})$ is called the *defect Hilbert space* corresponding to the $\mathbb{Z}_2$ line defect inserted along the longitude of the cylinder [HLS21, PS20].

One can also check that

$$\begin{aligned} Z_{\text{NS}^+}^{\text{SL}}(\tau) &:= \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}+}} q^{L_0 - \frac{c}{24}} = \frac{1}{2} [Z^{\text{eq}}(P_{0,0}) + Z^{\text{eq}}(P_{0,1})], \\ Z_{\text{R}^-}^{\text{SL}}(\tau) &:= \text{Tr}_{\mathcal{H}_{\text{R}}^{\text{SL}-}} q^{L_0 - \frac{c}{24}} = \frac{1}{2} [Z^{\text{eq}}(P_{0,0}) - Z^{\text{eq}}(P_{0,1})] \end{aligned} \quad (2.4.56)$$

As we will show with the explicit example of the Monster CFT, these relations determine the action of $\tilde{\sigma}$ on $\mathcal{H}^{\text{eq}}$.

Finally, if we gauge the $\tilde{\sigma}_d$ symmetry of the spin conformal field theory we just obtained, we recover the original theory. Let $Z_{\mathcal{H}}$ denote the gauged torus partition function of the resulting bosonic theory. By eq. (2.4.29) we have

$$\begin{aligned} Z_{\mathcal{H}} &= \frac{1}{2} \sum_{g_1, g_2 \in \mathbb{Z}_2} Z^{\text{eq}}(P_{g_1, g_2}) = Z_{\text{NS}^+}^{\text{SL}} + Z_{\text{R}^+}^{\text{SL}} \\ &= \text{Tr}_{\mathcal{H}^+} q^{L_0 - \frac{c}{24}} + \text{Tr}_{\mathcal{H}^-} q^{L_0 - \frac{c}{24}} = \text{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \end{aligned} \quad (2.4.57)$$

where we used eq. (2.4.54) and (2.4.56) in the second equality and eq. (2.4.43) and eq. (2.4.42) for the final equality. Thus we obtain the partition function for the original theory. On the other hand using eq. (2.4.46) we see that

$$\begin{aligned} \frac{1}{2} \sum_{\rho \in H^1(T^2, \mathbb{Z}_2)} Z^{\text{SL}}[\rho] &= \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}}} \left(\frac{1 + \tilde{\sigma}_d}{2} \right) q^{L_0 - \frac{c}{24}} + \text{Tr}_{\mathcal{H}_{\text{R}}^{\text{SL}}} \left(\frac{1 + \tilde{\sigma}_d}{2} \right) q^{L_0 - \frac{c}{24}} \\ &= \text{Tr}_{\mathcal{H}_{\text{NS}}^{\text{SL}+}} q^{L_0 - \frac{c}{24}} + \text{Tr}_{\mathcal{H}_{\text{R}}^{\text{SL}+}} q^{L_0 - \frac{c}{24}} = Z_{\mathcal{H}}. \end{aligned} \quad (2.4.58)$$

This is the usual GSO projection which involves summing over spin structures with the insertion of $\frac{1 + \tilde{\sigma}_d}{2}$ in the traces over NS and R sectors [GSO77]. From this trace formula for the partition function, we see that the gauged Hilbert space is the $\tilde{\sigma}_d = 1$ space of the spin theory but without the spin dependence, i.e. we forget the supervector space structure of the RHS below:

$$\mathcal{H}_{\text{NS}}^{\text{SL}+} \oplus \mathcal{H}_{\text{R}}^{\text{SL}+} \cong \mathcal{H}^+ \oplus \mathcal{H}^- \cong \mathcal{H}. \quad (2.4.59)$$

In mathematical terms, the spin lift of a CFT is an abelian intertwining algebra, see Remark 2.5.1 below. In general the spin lift of a holomorphic CFT defined by a VOA is not an SCVOA or even an SVOA. But in some cases, one might be able to identify a subspace of the spin lift which forms an SVOA and view the spin lift as a non-local extension of the SVOA. Suppose such an SVOA subsector exists in a spin lift of a CFT, then from Proposition 2.4.1, the SVOA contains vertex

operators of half-integral conformal dimensions. It is possible that the SVOA contains a conformal weight $\frac{3}{2}$ primary which satisfies the OPE (2.4.3) and hence is in fact an SCVOA. Our goal in this paper is to show that the Monster CFT carries precisely that structure. In Subsection 2.5.1 we will show that the “Beauty and the Beast” SCFT is the spin lift of the Monster CFT with respect to a $\mathbb{Z}_2$ automorphism of the Monster CFT and has a dimension $\frac{3}{2}$ primary satisfying the OPE (2.4.3). We will also identify a subsector of the “Beauty and the Beast” SCFT which has an SCVOA structure.

2.5 Superconformal Symmetry And The Moonshine Module

In this section, we will first review the Monster VOA $\widetilde{\mathcal{H}}(\Lambda_L)$ as the $\mathbb{Z}_2$ -orbifold model of the Leech lattice Λ_L and then, following [LS19], interpret the “Beauty and the Beast” theory of [DGH88] as a spin lift of the Monster theory. We next identify a SVOA subalgebra of $\mathcal{H}_{BB}$ and then, in section 2.5.2 we proceed to the main result of the paper: The explicit construction of a dimension $\frac{3}{2}$ primary inside that SVOA which can serve as a superconformal current.

2.5.1 The Beauty And The Beast Theory As The Spin Lift Of The FLM VOA

The FLM Moonshine module is constructed, as we have reviewed, by applying Theorem 2.3.2 part (2) to the Leech lattice. The construction can be interpreted as gauging the Leech torus theory using the $\mathbb{Z}_2$ automorphism group generated by θ . The value of the gauged theory on the bundle $P_{\bar{1}}$ is the $\theta = +1$ subspace of $\mathcal{H}_T(\Lambda_L)$. The quantum automorphism group dual to $\langle \theta \rangle$ (see discussion around (2.4.33)) is the $\mathbb{Z}_2$ automorphism group of $\widetilde{\mathcal{H}}(\Lambda_L)$ generated by ι , where ι is the identity on the untwisted sector $\mathcal{H}^+(\Lambda_L)$ and acts as -1 on the twisted sector $\mathcal{H}_T^+(\Lambda_L)$.

Following [DGH88] the FLM Moonshine module can be extended by considering the Hilbert space

$$\mathcal{H}_{BB} := \mathcal{H}(\Lambda_L) \oplus \mathcal{H}_T(\Lambda_L). \quad (2.5.1)$$

The spin lift of the Monster VOA with respect to the quantum symmetry ι of the Monster theory gives $\mathcal{H}_{BB}$ [LS19] and we will now review why this is so. ¹⁸ The $\check{\sigma}_d = +1$ NS and R sector of the spin lift corresponding to ι is simply given by eq. (2.4.42):

$$\mathcal{H}_{NS}^+ = \mathcal{H}^+(\Lambda_L), \quad \mathcal{H}_R^+ = \mathcal{H}_T^+(\Lambda_L). \quad (2.5.2)$$

¹⁸Incidentally, there is another nontrivial $\mathbb{Z}_2$ automorphism σ of the Monster module called the *triality involution* in [FLM88a, DGM94]. The involution σ mixes the untwisted and twisted sectors. It is also shown in [LS19] that the spin lift of the Monster VOA with respect to σ gives the tensor product of the Baby Monster VOA [Hö7] and a Majorana-Weyl CFT.

The twisted sector corresponding to the gauging of ι can be determined using (2.4.50). The right hand side of (2.4.50) is the McKay Thompson series

$$\begin{aligned}
Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^\iota(\tau) &= \text{Tr}_{\widetilde{\mathcal{H}}(\Lambda_L)} \iota q^{L_0-1} = q^{-1} \text{Tr}_{\mathcal{H}^+(\Lambda_L)} q^{L_0} - q^{-1} \text{Tr}_{\mathcal{H}_T^+(\Lambda_L)} q^{L_0^T} \\
&= \frac{1}{2} \left[\frac{\Theta_{\Lambda_L}(\tau)}{\eta(\tau)^{24}} + q^{-1} \prod_{n=1}^{\infty} (1+q^n)^{-24} \right] - \frac{2^{12} q^{\frac{1}{2}}}{2} \left[\prod_{n=1}^{\infty} (1-q^{n-\frac{1}{2}})^{-24} - \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}})^{-24} \right] \\
&= \frac{1}{2} \frac{\Theta_{\Lambda_L}(\tau)}{\eta(\tau)^{24}} + \frac{1}{2\eta(\tau)^{24}} [(\vartheta_3\vartheta_4)^{12} - (\vartheta_2\vartheta_3)^{12} + (\vartheta_2\vartheta_4)^{12}] \\
&= \frac{1}{2}(J(\tau) + 24) - \frac{1}{2}(J(\tau) - 24) + \frac{(\vartheta_3\vartheta_4)^{12}}{\eta(\tau)^{24}} \\
&= \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24 \\
&= \frac{1}{q} + 276q - 2048q^2 + 11202q^3 + \mathcal{O}(q^4)
\end{aligned} \tag{2.5.3}$$

where we used the infinite product expansions for the Jacobi theta functions:

$$\begin{aligned}
\vartheta_2(q) &= \frac{2\eta^2(2\tau)}{\eta(\tau)} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)(1+q^n)^2 \\
\vartheta_3(q) &= \frac{\eta^5(\tau)}{\eta^2(\frac{1}{2}\tau)\eta^2(2\tau)} = \prod_{n=1}^{\infty} (1-q^n) \left(1+q^{n-\frac{1}{2}}\right)^2 \\
\vartheta_4(q) &= \frac{\eta^2(\frac{1}{2}\tau)}{\eta(\tau)} = \prod_{n=1}^{\infty} (1-q^n) \left(1-q^{n-\frac{1}{2}}\right)^2.
\end{aligned} \tag{2.5.4}$$

along with numerous identities such as $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$ and $\vartheta_3^4 = \vartheta_2^4 + \vartheta_4^4$. Now we can take the modular transform [LS19]:

$$\begin{aligned}
Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^\iota(-1/\tau) &= 2^{12} \frac{\eta(\tau)^{24}}{\eta(\tau/2)^{24}} + 24 \\
&= 24 + 4096q^{1/2} + 98304q + 1228800q^{3/2} + 10747904q^2 + \mathcal{O}(q^{5/2}) \\
&= \text{Tr}_{\mathcal{H}^{\text{eq}}(P_1)} q^{L_0-1} \\
&:= Z_{\mathcal{H}^{\text{eq}}(P_1)}(\tau),
\end{aligned} \tag{2.5.5}$$

On the other hand, we have

$$\begin{aligned}
& \text{Tr}_{\mathcal{H}^-(\Lambda_L)} q^{L_0-1} + \text{Tr}_{\mathcal{H}_T^-(\Lambda_L)} q^{L_0^T-1} \\
&= \frac{1}{2} \left[\frac{\Theta_{\Lambda_L}(\tau)}{\eta(\tau)^{24}} - q^{-1} \prod_{n=1}^{\infty} (1+q^n)^{-24} \right] + \frac{2^{12} q^{\frac{1}{2}}}{2} \left[\prod_{n=1}^{\infty} (1-q^{n-\frac{1}{2}})^{-24} + \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}})^{-24} \right] \\
&= \frac{1}{2} \frac{\Theta_{\Lambda_L}(\tau)}{\eta(\tau)^{24}} + \frac{1}{2\eta(\tau)^{24}} [(\vartheta_2\vartheta_3)^{12} + (\vartheta_2\vartheta_4)^{12} - (\vartheta_3\vartheta_4)^{12}] \\
&= \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24 - \frac{1}{2\eta(\tau)^{24}} [2(\vartheta_2\vartheta_3)^{12} - 2(\vartheta_3\vartheta_4)^{12}] \\
&= 2^{12} \frac{\eta(\tau)^{24}}{\eta(\tau/2)^{24}} + 24 \\
&= 24 + 4096q^{1/2} + 98304q + 1228800q^{3/2} + 10747904q^2 + \mathcal{O}(q^{5/2}).
\end{aligned} \tag{2.5.6}$$

Thus we identify the nontrivial equivariant Hilbert space of the Monster module as

$$\mathcal{H}^{\text{eq}}(P_1) = \mathcal{H}^-(\Lambda_L) \oplus \mathcal{H}_T^-(\Lambda_L). \tag{2.5.7}$$

Next by eq. (2.4.53) we have

$$Z_{\mathcal{H}^{\text{eq}}(P_1)}^{\iota}(\tau) := \text{Tr}_{\mathcal{H}^{\text{eq}}(P_1)} \iota q^{L_0-1} = Z_{\mathcal{H}^{\text{eq}}(P_1)}(\tau+1). \tag{2.5.8}$$

Note that we must have

$$Z_{\mathcal{H}^{\text{eq}}(P_1)}^{\iota}(-1/\tau) = Z_{\mathcal{H}^{\text{eq}}(P_1)}^{\iota}(\tau) \tag{2.5.9}$$

for consistency. It is true and can be checked using the fact that the McKay Thompson series

$Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^{\iota}(\tau)$ is a Hauptmodul for $\Gamma_0(2)$. Indeed

$$\begin{aligned}
Z_{\mathcal{H}^{\text{eq}}(P_1)}^{\iota}(-1/\tau) &= Z_{\mathcal{H}^{\text{eq}}(P_1)} \left(-\frac{1}{\tau} + 1 \right) = Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^{\iota} \left(-\frac{\tau}{\tau-1} \right) = Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^{\iota} \left(-\frac{\tau}{\tau-1} + 1 \right) \\
&= Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^{\iota} \left(-\frac{1}{\tau-1} \right) = Z_{\mathcal{H}^{\text{eq}}(P_1)}(\tau-1) = Z_{\mathcal{H}^{\text{eq}}(P_1)}(\tau+1) = Z_{\mathcal{H}^{\text{eq}}(P_1)}^{\iota}(\tau)
\end{aligned} \tag{2.5.10}$$

where we used the fact that $Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^{\iota}(\tau)$ is invariant under $\tau \rightarrow \tau+1$ and $Z_{\mathcal{H}^{\text{eq}}(P_1)}(\tau)$ is invariant under $\tau \rightarrow \tau+2$. From eq. (2.4.54) we see that

$$\begin{aligned}
\text{Tr}_{\mathcal{H}_{\text{NS}}^-} q^{L_0 - \frac{c}{24}} &= \frac{1}{2} \left[Z_{\mathcal{H}^{\text{eq}}(P_1)}(\tau) - Z_{\widetilde{\mathcal{H}}(\Lambda_L)}^{\iota}(\tau) \right] \\
&= \frac{2^{12}}{2} \left[\frac{\eta(\tau)^{24}}{\eta(\tau/2)^{24}} - \frac{\eta(\tau)^{24}}{\eta(\frac{\tau+1}{2})^{24}} \right] \\
&= \frac{2^{12} q^{\frac{1}{2}}}{2} \left[\prod_{n=1}^{\infty} (1-q^{n-\frac{1}{2}})^{-24} + \prod_{n=1}^{\infty} (1+q^{n-\frac{1}{2}})^{-24} \right] \\
&= \text{Tr}_{\mathcal{H}_T^-(\Lambda_L)} q^{L_0 - \frac{c}{24}},
\end{aligned} \tag{2.5.11}$$

where we used the infinite product expansion of the Dedekind eta function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)^{24}. \quad (2.5.12)$$

Thus we identify

$$\mathcal{H}_{\text{NS}}^- = \mathcal{H}_T^-(\Lambda_L). \quad (2.5.13)$$

This also confirms that ι acts as -1 on $\mathcal{H}_T^-(\Lambda_L)$. Similarly one can check that

$$\mathcal{H}_{\text{R}}^- = \mathcal{H}^-(\Lambda_L) \quad (2.5.14)$$

which also implies that ι acts as $+1$ on $\mathcal{H}^-(\Lambda_L)$. This completes the demonstration (at least at the level of state spaces) that the spin lift of the Monster CFT with respect to ι is the Beauty and the Beast SCFT (2.5.1).

To summarize, we have

$$\mathcal{H}_{\text{NS}} = \mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}_T^-(\Lambda_L), \quad \mathcal{H}_{\text{R}} = \mathcal{H}_T^+(\Lambda_L) \oplus \mathcal{H}^-(\Lambda_L) \quad (2.5.15)$$

in agreement with (2.5.2) and (2.5.13). The superscripts on the RHS refer to the θ eigenvalue. These formulae can be refined to:

$$\mathcal{H}_{\text{NS}}^+ \cong \mathcal{H}^+(\Lambda_L), \quad \mathcal{H}_{\text{NS}}^- \cong \mathcal{H}_T^-(\Lambda_L), \quad \mathcal{H}_{\text{R}}^+ \cong \mathcal{H}_T^+(\Lambda_L), \quad \mathcal{H}_{\text{R}}^- \cong \mathcal{H}^-(\Lambda_L), \quad (2.5.16)$$

where the superscripts on the LHS refer to the $(-1)_q^F$ eigenvalue. Here $(-1)_q^F$ is the emergent automorphism $\check{\sigma}_d$ (see discussion around (2.4.43)).

As an aside note that a corollary of the above discussion is that the ι -symmetry of the Monster theory is the quantum $\mathbb{Z}_2$ symmetry of the θ -gauged Leech torus theory, and if we gauge the ι -symmetry of the Monster theory we recover the Leech torus theory, as expected on general grounds. Similarly, if we apply the GSO projection to the BB theory we obtain the Monster theory.

Let us now begin the search for a SVOA structure within $\mathcal{H}_{BB}$. We stress that this space is not itself a SVOA.

As a preliminary, note that the vertex operators on $\mathcal{H}_{BB}$ can be described using the block form relative to the direct sum:

$$\tilde{V}(\psi, z) = \begin{pmatrix} V(\psi, z) & 0 \\ 0 & V_T(\psi, z) \end{pmatrix}, \quad \psi \in \mathcal{H}(\Lambda_L), \quad (2.5.17)$$

where $V(\psi, z)$ and $V_T(\psi, z)$ for homogeneous states are defined in eq. (2.3.6) and eq. (2.3.44) respectively and the extension to general states is defined by linearity. Vertex operators for twisted states take the form:

$$\tilde{V}(\chi, z) = \begin{pmatrix} 0 & \overline{W}(\chi, z) \\ W(\chi, z) & 0 \end{pmatrix}, \quad \chi \in \mathcal{H}_T(\Lambda_L). \quad (2.5.18)$$

Here W and $\overline{W}$ are the intertwiners associated to the representation $\mathcal{H}_T(\Lambda_L)$ of the VOA $\mathcal{H}^+(\Lambda_L)$ (see Theorem 2.3.2) defined by the relations in eq. (2.2.18) and (2.2.25). The stress tensor is defined as

$$T(z) := \tilde{V}(\psi_L, z) = \begin{pmatrix} V(\psi_L, z) & 0 \\ 0 & V_T(\psi_L, z) \end{pmatrix}. \quad (2.5.19)$$

Proposition 2.5.1. *The OPE of a vertex operator corresponding to a state in $\mathcal{H}_T^-(\Lambda_L)$ with a vertex operator corresponding to $\psi \in \mathcal{H}^-(\Lambda_L)$ or $\chi \in \mathcal{H}_T^+(\Lambda_L)$ has square root singularity and the OPE with vertex operator corresponding to $\psi \in \mathcal{H}^+(\Lambda_L)$ or $\chi \in \mathcal{H}_T^-(\Lambda_L)$ has integer power singularity.*

Proof. Consider the vertex operator $\tilde{V}(\chi, z)$, $\chi \in \mathcal{H}_T^-(\Lambda_L)$. We first analyse its OPE with an untwisted vertex operator $\tilde{V}(\psi, z)$ for $\psi \in \mathcal{H}^-(\Lambda_L)$. Note that any vector in $\mathcal{H}^-(\Lambda_L)$ is a linear combination of states of the form:

$$\Psi_\lambda = \left(\prod_{a=1}^M a^{j_a}_{-m_a} \right) \left(\frac{|\lambda\rangle \pm |-\lambda\rangle}{\sqrt{2}} \right) \quad (2.5.20)$$

where we choose $+$ sign when M is odd and $-$ sign when M is even. It is easy to see that the Laurent expansion of $V_T(\Psi_\lambda, z)$ contains $\mathbb{Z} + \frac{1}{2}$ powers of z . Indeed when M is odd, the derivatives of the fields R^{j_a} in the expression (2.3.44) gives $\mathbb{Z} + \frac{1}{2}$ powers of z while the exponential factor in (2.3.44) combines to give $\cosh(i\lambda \cdot R)$ which still contributes integer powers of z and hence the final vertex operator contains $\mathbb{Z} + \frac{1}{2}$ powers of z . Similarly when M is even the derivatives of the fields R^{j_a} contains integer powers of z while the exponentials combine to give $\sinh(i\lambda \cdot R)$ which contributes $\mathbb{Z} + \frac{1}{2}$ powers of z and hence the final vertex operator contains $\mathbb{Z} + \frac{1}{2}$ powers of z . Thus we can expand as

$$V_T(\Psi_\lambda, z) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} (V_T(\Psi_\lambda))_n z^{-n-h_{\Psi_\lambda}}. \quad (2.5.21)$$

Now the duality of this vertex operator with $\tilde{V}(\chi, z)$ implies

$$\begin{aligned} \tilde{V}(\Psi_\lambda, z) \tilde{V}(\chi, w) &= \sum_{n \in \mathbb{Z} + \frac{1}{2}} (z-w)^{n-h_{\Psi_\lambda}} \tilde{V}(\chi_n, w) \\ &= \sum_{n \in \mathbb{Z}} (z-w)^{n-h_{\Psi_\lambda}-h_\chi} \tilde{V}(\tilde{\chi}_n, w) \end{aligned} \quad (2.5.22)$$

where

$$\tilde{\chi}_n = (V_T(\Psi_\lambda))_{h_\chi - n} \chi \quad (2.5.23)$$

Noting that $h_\chi \in \mathbb{Z} + \frac{1}{2}$ and $h_{\Psi_\lambda} \in \mathbb{Z}$, we see that the OPE has square root singularity. Similarly if $\tilde{V}(\chi', z)$, $\chi' \in \mathcal{H}_T^+(\Lambda_L)$ is a twisted vertex operator then χ' has the form (2.3.39) with M even.

The duality now gives

$$\tilde{V}(\chi', z) \tilde{V}(\chi, w) = \tilde{V}(\overline{W}(\chi', z-w)\chi, w), \quad (2.5.24)$$

where we used the fact that

$$\tilde{V}(\chi', z - w)\chi = \overline{W}(\chi', z - w)\chi. \quad (2.5.25)$$

The Laurent expansion of $\overline{W}(\chi', z)$ is of the form

$$\overline{W}(\chi', z) = \sum_n \overline{W}_n(\chi') z^{-n-h_{\chi'}} \quad (2.5.26)$$

Thus we get

$$\tilde{V}(\chi', z)\tilde{V}(\chi, w) = \sum_n (z - w)^{n-h_{\chi}-h_{\chi'}} \tilde{V}(\tilde{\chi}(n), w) \quad (2.5.27)$$

where

$$\tilde{\chi}(n) = \overline{W}_{n-h_{\chi}}(\chi')\chi. \quad (2.5.28)$$

Now using (2.2.7), we have

$$L_0 \overline{W}_n(\chi) - \overline{W}_n(\chi) L_0^T = -n \overline{W}_n(\chi) \quad (2.5.29)$$

This implies that $\overline{W}_n(\chi')\chi = 0$ for $n \in \mathbb{Z}$ since the untwisted sector has integer conformal weights.

Thus we see that

$$\tilde{V}(\chi', z)\tilde{V}(\chi, w) = \sum_{n \in \mathbb{Z}} (z - w)^{n-h_{\chi}-h_{\chi'}} \tilde{V}(\tilde{\chi}(n), w) \quad (2.5.30)$$

Again since $h_{\chi'} \in \mathbb{Z}$ and $h_{\chi} \in \mathbb{Z} + \frac{1}{2}$, we see that the OPE has square-root singularity. The second part of the statement follows by similar arguments. $\square$

We will now identify a subspace of $\mathcal{H}_{BB}$ which forms an SVOA. To get an SVOA, we anticipate that we need to include the $\theta = -1$ subspace of the twisted sector. But then from Proposition 2.5.1, we see that we can neither have the $\theta = +1$ subspace of the twisted sector nor the $\theta = -1$ subspace of the untwisted sector. Thus we are led to the following theorem which was first proved in [Hua96] in the standard formalism of VOAs.

Theorem 2.5.1. *The subspace*

$$\mathcal{H}_{SC} := \mathcal{H}_{NS} = \mathcal{H}^+(\Lambda_L) \oplus \mathcal{H}_T^-(\Lambda_L) \quad (2.5.31)$$

of the “Beauty and the Beast” SCFT possesses the structure of an SVOA.

Proof. We first put a super-vector space structure on $\mathcal{H}_{SC}$ by declaring $\mathcal{H}^+(\Lambda_L)$ to be the even space and $\mathcal{H}_T^-(\Lambda_L)$ to be the odd space. We will call them respectively the untwisted (bosonic) and twisted (fermionic) sector. The Fock space of $\mathcal{H}_{SC}$ is

$$\mathcal{F}_{SC} = \mathcal{F}^+ \oplus \mathcal{F}_T^- \quad (2.5.32)$$

where $\mathcal{F}$ and $\mathcal{F}_T$ is the untwisted and twisted Fock spaces respectively and the superscript ± 1 indicates the $\theta = \pm 1$ subspaces of $\mathcal{F}, \mathcal{F}_T$. The conformal and vacuum vector is as in the BB module

$\mathcal{H}_{BB}$ and the vertex operators are also defined analogous to the BB module. We now need to prove the locality relation

$$\tilde{V}(\psi, z)\tilde{V}(\phi, w) = (-1)^{|\psi||\phi|}\tilde{V}(\phi, w)\tilde{V}(\psi, z), \quad \psi, \phi \in \mathcal{H}_{SC}. \quad (2.5.33)$$

For bosonic states ψ, ϕ , the locality relation eq. (2.5.33) is equivalent to the locality relations

$$V(\psi, z)V(\phi, w) = V(\phi, w)V(\psi, z), \quad V_T(\psi, z)V_T(\phi, w) = V_T(\phi, w)V_T(\psi, z). \quad (2.5.34)$$

The first relation is proved in [DGM90, Section 5, 6] and the second follows from [DGM94, Proposition 3.2 (1)]. This implies the duality of the untwisted vertex operators

$$\tilde{V}(\psi, z)\tilde{V}(\phi, w) = \tilde{V}(\tilde{V}(\psi, z - w)\phi, w). \quad (2.5.35)$$

The locality of an untwisted and a twisted vertex operator is

$$\tilde{V}(\psi, z)\tilde{V}(\chi, w) = \tilde{V}(\chi, w)\tilde{V}(\psi, z) \quad (2.5.36)$$

This follows from the locality relations

$$V(\psi, z)\overline{W}(\chi, w) = \overline{W}(\chi, w)V_T(\psi, z), \quad V_T(\psi, z)W(\chi, w) = W(\chi, w)V(\psi, z). \quad (2.5.37)$$

The first relation is proved in [DGM90, Section 6] while the second follows from Proposition 2.2.3 and Theorem 2.3.2 (1). This implies the duality of the twisted-untwisted vertex operators

$$\tilde{V}(\psi, z)\tilde{V}(\chi, w) = \tilde{V}(\tilde{V}(\psi, z - w)\chi, w). \quad (2.5.38)$$

The locality relation for twisted vertex operators with $\chi, \chi' \in \mathcal{H}_T^-(\Lambda_L)$ is

$$\tilde{V}(\chi, z)\tilde{V}(\chi', w) = -\tilde{V}(\chi', w)\tilde{V}(\chi, z). \quad (2.5.39)$$

For this, we need to prove the locality relations

$$\begin{aligned} \overline{W}(\chi, z)W(\chi', w) &= -\overline{W}(\chi', w)W(\chi, z), \\ W(\chi, z)\overline{W}(\chi', w) &= -W(\chi', w)\overline{W}(\chi, z), \end{aligned} \quad (2.5.40)$$

for $\chi, \chi' \in \mathcal{H}_T^-(\Lambda_L)$. These relations, especially the second, are highly nontrivial. It is proven in Appendix A.2 that the locality relations are satisfied if the conjugation operation on $\mathcal{H}_T^+(\Lambda_L)$ is extended to $\mathcal{H}_T(\Lambda_L)$ as follows:

$$\overline{\chi} = e^{-i\pi L_0^T} \theta M(\chi), \quad (2.5.41)$$

where χ is a general homogeneous state as in (2.3.39) in $\mathcal{H}_T^-(\Lambda_L)$ and the conjugation is extended to all other states by antilinearity. It is easy to check that this modification still satisfies $\overline{\overline{\chi}} = \chi$ using the fact that $\overline{z\chi} = z^*\overline{\chi}$ for a complex number z . All other axioms in Definition 2.4.1 follows from the results of [DGM90, DGM94]. $\square$

Remark 2.5.1. Let us give a broader context on Theorem 2.5.1. The first proof of Theorem 2.5.1 in the standard formalism of super vertex operator algebra appeared in [Hua96]. Alternatively, the existence of the SVOA structure on $\mathcal{H}_{SC}$ follows in general from Theorem 3.9 in [CKL19]. This requires the simple current property of the odd module, which was worked out by Dong in [Don94]. This construction of (super) vertex operator algebras can be generalized vastly using the general theory of abelian intertwining algebras. Given a nice (called strongly rational in VOA literature) holomorphic vertex operator algebra V and an automorphism g of the VOA one can consider the fixed-point vertex operator algebra V^g . Then one can show that V^g again satisfies the niceness assumptions [DM97, Miy14, CM16, McR21]. Then one can determine the representation category of V^g , which is a twisted Drinfeld double [Mö21, vEMS17] (see also [MT04, DRX17]), essentially proving a conjecture in [DPR91]. In particular, all irreducible V^g -modules are simple currents, which translates into the statement that on the direct sum of irreducible V^g -modules one gets the structure of an abelian intertwining algebra in the sense of [DL93]. What this entails is that the extensions of V^g to larger vertex operator (super)algebras (with the same conformal vector) can be fully described in terms of isotropic subgroups of discriminant forms (again, this is described in [Mö21, vEMS17] and essentially a special case of [HKL15]). Specializing to order 2 and assuming that g is anomaly-free, the structure of the representation category (and hence of the extensions) of V^g is described by the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$ with quadratic form $0, 0, 0, 1/2$. The situation is now very simple: V^g has three non-trivial extensions, all of which are holomorphic. One is the original vertex operator algebra, one is the orbifolded vertex operator algebra and the third is a vertex operator superalgebra. When we apply this general result to the Monster VOA, then the two holomorphic vertex operator algebras are the Leech lattice vertex operator algebra $\mathcal{H}(\Lambda_L)$ and the moonshine module $\widetilde{\mathcal{H}}(\Lambda_L)$ (usually denoted by $V^\natural$ in literature), and the holomorphic vertex operator superalgebra is the SVOA of Theorem 2.5.1.

In the next section, we will finally construct a superconformal vector inside $\mathcal{H}_{SC}$ making it into an $\mathcal{N} = 1$ SCVOA.

2.5.2 Constructing The Supercurrent

Our supercurrent will be a vertex operator associated to a twisted sector ground state $\mathcal{S}(\Lambda_L)$, which is the 2^{12} -dimensional representation of the Heisenberg group¹⁹ $\Gamma(\Lambda_L)$ defined analogous to (2.3.31). For any $\chi \in \mathcal{S}(\Lambda_L)$, the corresponding vertex operator (see Theorem 2.2.1 for notations) has the

¹⁹Also known in the finite group theory literature as an extraspecial group and denoted by 2_+^{1+24} .

block diagonal form relative to the direct sum decomposition $\mathcal{H}_{BB} = \mathcal{H}(\Lambda_L) \oplus \mathcal{H}_T(\Lambda_L)$:

$$\tilde{V}(\chi, z) = \begin{pmatrix} 0 & \overline{W}(\chi, z) \\ W(\chi, z) & 0 \end{pmatrix}, \quad \chi \in \mathcal{S}(\Lambda_L) \quad (2.5.42)$$

are well-defined and are present in the extended theory $\mathcal{H}_{BB}$. We Laurent expand the vertex operators $\tilde{V}(\chi, z)$:

$$\tilde{V}(\chi, z) = \sum_n \tilde{V}_n(\chi) z^{-n-\frac{3}{2}}, \quad (2.5.43)$$

where

$$\tilde{V}_n(\chi) = \begin{pmatrix} 0 & \overline{W}_n(\chi) \\ W_n(\chi) & 0 \end{pmatrix} \quad (2.5.44)$$

and

$$\overline{W}(\chi, z) = \sum_n \overline{W}_n(\chi) z^{-n-\frac{3}{2}}, \quad W(\chi, z) = \sum_n W_n(\chi) z^{-n-\frac{3}{2}} \quad (2.5.45)$$

Proposition 2.5.2. *The twisted vertex operators $\tilde{V}(\chi, z)$ are conformal primary operators of weight $3/2$.*

Proof. First note that the states χ are of conformal weight $3/2$. We now compute the OPE of $\tilde{V}(\chi, z)$ with the energy momentum tensor $\tilde{V}(\psi_L, z)$. From the locality of all vertex operators and intertwiners and using the general form (2.2.8) of the OPE of vertex operators, we get

$$\tilde{V}(\psi_L, z) \tilde{V}(\chi, w) = \sum_{\substack{n \in \mathbb{Z} + \frac{1}{2} \\ n \geq 0}} (z-w)^{n-2-3/2} \tilde{V}(\chi(n), w) \quad (2.5.46)$$

where

$$\chi(n) = L_{3/2-n}^T \chi. \quad (2.5.47)$$

We easily see that

$$\tilde{V}(\psi_L, z) \tilde{V}(\chi, w) = \frac{3/2}{(z-w)^2} \tilde{V}(\chi, w) + \frac{1}{(z-w)} \partial_w \tilde{V}(\chi, w) + \text{reg.} \quad (2.5.48)$$

□

For any $\chi_a, \chi_b \in \mathcal{S}(\Lambda_L)$, we now compute the OPE of $\tilde{V}(\chi_a, w)$ and $\tilde{V}(\chi_b, w)$. We have

$$\tilde{V}(\chi_a, z) \tilde{V}(\chi_b, w) = \sum_{n \geq 0} (z-w)^{n-3} \tilde{V}(\tilde{\chi}_{ab}(n), w) \quad (2.5.49)$$

where we can identify

$$\tilde{\chi}_{ab}(n) = \overline{W}_{3/2-n}(\chi_a) \chi_b \quad (2.5.50)$$

because the formalism identifies

$$\begin{pmatrix} \tilde{\chi}_{ab}(n) \\ 0 \end{pmatrix} = \tilde{V}_{3/2-n}(\chi_a) \begin{pmatrix} 0 \\ \chi_b \end{pmatrix} = \begin{pmatrix} 0 & \overline{W}_{3/2-n}(\chi_a) \\ W_{3/2-n}(\chi_a) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \chi_b \end{pmatrix}. \quad (2.5.51)$$

Note that using (2.2.7), we have

$$L_0 \overline{W}_n(\chi_a) - \overline{W}_n(\chi_a) L_0^T = -n \overline{W}_n(\chi_a) \quad (2.5.52)$$

and hence:

$$L_0 (\overline{W}_n(\chi_a) \chi_b) + n (\overline{W}_n(\chi_a) \chi_b) = 3/2 (\overline{W}_n(\chi_a) \chi_b) \quad (2.5.53)$$

which immediately implies that $\overline{W}_n(\chi_a) \chi_b = 0$ for $n \in \mathbb{Z}$ since the untwisted sector has integer conformal weights. In particular, the OPE of any two twist fields $\tilde{V}(\chi_a, z), \tilde{V}(\chi_b, w)$ is local. We now restrict to the case $a = b$.

Lemma 2.5.1. *The states $\overline{W}_{3/2}(\chi)\chi$ is $\mathfrak{su}(1,1)$ -invariant and the states $\overline{W}_{1/2}(\chi)\chi$ and $\overline{W}_{-1/2}(\chi)\chi$ are of conformal weights 1 and 2 respectively.*

Proposition 2.5.3. *We have that*

$$\begin{aligned} \overline{W}_{3/2}(\chi)\chi &= \alpha(\chi)|0\rangle, \\ \overline{W}_{1/2}(\chi)\chi &= 0, \\ \overline{W}_{-1/2}(\chi)\chi &= \frac{\alpha(\chi)}{8}\psi_L + \sum_{\lambda^2=4} \kappa_\lambda(\chi)|\lambda\rangle \end{aligned} \quad (2.5.54)$$

where²⁰

$$\alpha(\chi) = \langle \bar{\chi} | \chi \rangle, \quad \kappa_\lambda(\chi) = 16 \langle \chi | e_\lambda | \bar{\chi} \rangle^*. \quad (2.5.55)$$

and

Proof. By uniqueness of an $\mathfrak{su}(1,1)$ -invariant state we have

$$\overline{W}_{3/2}(\chi)\chi = \alpha(\chi)|0\rangle. \quad (2.5.56)$$

Taking the inner product with $\langle 0|$ and using the fact that $\overline{W}_{3/2}(\chi)^\dagger = W_{-3/2}(\bar{\chi})$ (see Proposition 2.2.1 (3)), we get

$$\alpha(\chi) = \langle \bar{\chi} | \chi \rangle. \quad (2.5.57)$$

Next, we prove the third relation of (2.5.54). Since $\overline{W}_{-1/2}(\chi)\chi$ has conformal dimension 2, we can expand it in a complete basis of states of conformal dimension 2 as:

$$\overline{W}_{-1/2}(\chi)\chi = \sum_i \kappa_{ii} a_{-1}^i a_{-1}^i |0\rangle + \sum_{i < j} \kappa_{ij} a_{-1}^i a_{-1}^j |0\rangle + \sum_j \kappa_j a_{-2}^j |0\rangle + \sum_{\lambda^2=4} \kappa_\lambda(\chi) |\lambda\rangle \quad (2.5.58)$$

²⁰The sign of $\kappa_\lambda(\chi)$ depends on the choice of lift of λ to the Heisenberg group. Given such a lift, $\kappa_\lambda(\chi)$ depends only on the class $[\lambda] \in \Lambda_L/2\Lambda_L$.

Under the $\mathbb{Z}_2$ -orbifold action (see [DGM90, eq. (3.11), eq. (3.27)])

$$\theta \tilde{V}(\chi, z) \tilde{V}(\chi, w) \theta^{-1} = \theta \tilde{V}(\chi, z) \theta^{-1} \theta \tilde{V}(\chi, w) \theta^{-1} = \tilde{V}(\theta \chi, z) \tilde{V}(\theta \chi, w) = \tilde{V}(\chi, z) \tilde{V}(\chi, w). \quad (2.5.59)$$

Thus the LHS of (2.5.49) is even. Thus the invariance of RHS of (2.5.49) under θ implies that $W_n(\chi)\chi$ must be invariant under θ for all n . Observing that a_n^j is anti-invariant under θ we conclude that $\kappa_i = 0$. Next using (2.2.7) for $m = 2$ and $n = -1/2$, we get

$$L_2 \overline{W}_{-1/2}(\chi) - \overline{W}_{-1/2}(\chi) L_2^T = \frac{3}{2} \overline{W}_{3/2}(\chi). \quad (2.5.60)$$

Applying it on χ , we get

$$L_2 \overline{W}_{-1/2}(\chi) \chi = \alpha(\chi) \frac{3}{2} |0\rangle \quad (2.5.61)$$

Using the above expansion of $\overline{W}_{-1/2}(\chi)\chi$, we get

$$\sum_i \kappa_{ii} L_2 a_{-1}^i a_{-1}^i |0\rangle + \sum_{i < j} \kappa_{ij} L_2 a_{-1}^i a_{-1}^j |0\rangle + \sum_{\lambda^2=4} \kappa_\lambda(\chi) L_2 |\lambda\rangle = \alpha(\chi) \frac{3}{2} |0\rangle. \quad (2.5.62)$$

Using the fact that

$$L_2 a_{-1}^i a_{-1}^j |0\rangle = \delta^{ij} |0\rangle, \quad L_2 |\lambda\rangle = 0, \quad (2.5.63)$$

we get

$$\sum_i \kappa_{ii} = \frac{3}{2} \alpha(\chi). \quad (2.5.64)$$

We now find the explicit form of κ_{ii} and κ_{ij} . Taking inner product of $\langle 0 | a_1^m a_1^n$ with $\overline{W}_{-1/2}(\chi)\chi$ and using the expression (2.5.58) for $\overline{W}_{-1/2}(\chi)\chi$, we get

$$\begin{aligned} \langle 0 | a_1^m a_1^n \overline{W}_{-1/2}(\chi) | \chi \rangle &= \kappa_{nm}; \quad n \neq m \\ &= 2\kappa_{mm}; \quad n = m. \end{aligned} \quad (2.5.65)$$

where we used the fact that

$$\langle 0 | a_1^m a_1^n a_{-1}^i a_{-1}^j | 0 \rangle = \delta^{ni} \delta^{mj} + \delta^{mi} \delta^{nj}, \quad \langle 0 | a_1^m a_1^n a_{-2}^j | 0 \rangle = 0, \quad a_1^i |\lambda\rangle = 0. \quad (2.5.66)$$

Note that

$$\langle 0 | a_1^m a_1^n \overline{W}_{-1/2}(\chi) | \chi \rangle = \langle \chi | W_{1/2}(\overline{\chi}) a_{-1}^m a_{-1}^n | 0 \rangle^*. \quad (2.5.67)$$

Using eq. (2.2.18), we get

$$W(\overline{\chi}, z) a_{-1}^m a_{-1}^n | 0 \rangle = e^{zL_{-1}^T} V_T(a_{-1}^m a_{-1}^n | 0 \rangle, -z) \overline{\chi}. \quad (2.5.68)$$

To compute $V_T(a_{-1}^m a_{-1}^n | 0 \rangle, z)$, we need to calculate $e^{A(-z)} a_{-1}^m a_{-1}^n | 0 \rangle$ (see eq. (2.3.44)), we have (see [DGM94, Page 92])

$$e^{A(-z)} a_{-1}^m a_{-1}^n | 0 \rangle = a_{-1}^m a_{-1}^n | 0 \rangle + \frac{1}{8} z^{-2} \delta^{nm} | 0 \rangle. \quad (2.5.69)$$

This gives

$$\begin{aligned}
W(\bar{\chi}, z) a_{-1}^m a_{-1}^n |0\rangle &= -e^{zL_{-1}^T} : \left(\frac{dR^m(-z)}{dz} \frac{dR^n(-z)}{dz} + \frac{1}{8} z^{-2} \delta^{nm} \right) : \bar{\chi} \\
&= e^{zL_{-1}^T} \left(\sum_{r,s \in \mathbb{Z} + \frac{1}{2}} (-1)^{-r-s} : c_r^m c_s^n : z^{-r-s-2} + \frac{1}{8} \delta^{mn} z^{-2} \right) \bar{\chi} \\
&= \left(\sum_{\substack{r,s \in \mathbb{Z} + \frac{1}{2} \\ t \in \mathbb{N}_0}} (-1)^{-r-s} \frac{(L_{-1}^T)^t}{t!} : c_r^m c_s^n : z^{t-r-s-2} + \frac{\delta^{mn}}{8} \sum_{t \in \mathbb{N}_0} \frac{(L_{-1}^T)^t}{t!} z^{t-2} \right) \bar{\chi}.
\end{aligned} \tag{2.5.70}$$

Thus we have

$$\begin{aligned}
\left(\sum_{r \in \mathbb{Z} + \frac{1}{2}} W_r(\bar{\chi}) z^{-r-3/2} \right) a_{-1}^m a_{-1}^n |0\rangle &= \left(\sum_{\substack{r,s \in \mathbb{Z} + \frac{1}{2} \\ t \in \mathbb{N}_0}} (-1)^{-r-s} \frac{(L_{-1}^T)^t}{t!} : c_r^m c_s^n : z^{t-r-s-2} \right. \\
&\quad \left. + \frac{\delta^{mn}}{8} \sum_{t \in \mathbb{N}_0} \frac{(L_{-1}^T)^t}{t!} z^{t-2} \right) \bar{\chi},
\end{aligned} \tag{2.5.71}$$

which on comparing the powers of z^{-2} gives

$$\begin{aligned}
W_{1/2}(\bar{\chi}) a_{-1}^m a_{-1}^n |0\rangle &= \left(\sum_{\substack{r \in \mathbb{Z} + \frac{1}{2} \\ t \in \mathbb{N}_0}} \frac{(-L_{-1}^T)^t}{t!} : c_r^m c_{t-r}^n : + \frac{\delta^{mn}}{8} \right) \bar{\chi}. \\
&= \frac{\delta^{mn}}{8} \bar{\chi}
\end{aligned} \tag{2.5.72}$$

This implies that

$$\kappa_{mn} = 0, \quad \kappa_{ii} = \frac{1}{16} \alpha(\chi) \quad \text{for all } i. \tag{2.5.73}$$

Note that this result is consistent with eq. (2.5.64). To compute $\kappa_\lambda(\chi)$, note that

$$\kappa_\lambda(\chi) = \langle \lambda | \bar{W}_{-\frac{1}{2}}(\chi) | \chi \rangle. \tag{2.5.74}$$

Next using (2.2.18) we get

$$W(\bar{\chi}, z) | \lambda \rangle = e^{zL_{-1}^T} V_T(|\lambda\rangle, z) \bar{\chi}. \tag{2.5.75}$$

Since

$$e^{A(-z)} | \lambda \rangle = (-4z)^{-\frac{\lambda^2}{2}} | \lambda \rangle, \tag{2.5.76}$$

we get

$$W(\bar{\chi}, z) | \lambda \rangle = e^{zL_{-1}^T} (-4z)^{-\frac{\lambda^2}{2}} : \exp(i\lambda \cdot R(-z)) : e_\lambda \bar{\chi} \tag{2.5.77}$$

Thus using the fact that $\lambda^2 = 4$ and comparing the coefficient of z^{-2} on both sides of (2.5.77) we obtain

$$W_{\frac{1}{2}}(\bar{\chi}) | \lambda \rangle = \text{coefficient of } z^{-2} \text{ in } \frac{16}{z^2} e^{zL_{-1}^T} : \exp(i\lambda \cdot R(-z)) : e_\lambda \bar{\chi}. \tag{2.5.78}$$

Because of the factor $16/z^2$, the only contribution to z^{-2} on the RHS comes from the product of terms with z^ℓ in $e^{zL_{-1}^T}$ and $z^{-\ell}$ in $:\exp(i\lambda \cdot R(-z)):$ with $\ell \geq 0$. Note that ℓ must be an integer since $e^{zL_{-1}^T}$ only has integer powers of z in its expansion. The quantity $e^{zL_{-1}^T}$ contributes $(L_{-1}^T)^\ell z^\ell$ but $:\exp(i\lambda \cdot R(-z)):$ may have nontrivial contributions. Write

$$:\exp(i\lambda \cdot R(-z)):=1+\sum_{n=1}^{\infty}\frac{1}{n!}:\left(i\sum_{k\in\mathbb{Z}+\frac{1}{2}}\frac{\lambda\cdot c_k}{k}(-z)^{-k}\right)^n:.\quad (2.5.79)$$

When $\ell = 0$, only the constant term 1 in (2.5.79) contributes. When $\ell > 0$, to get a term with $z^{-\ell}$ from the sum, we have to consider $n > 0$ terms in the sum. This involves the product of at least two oscillators c_k since ℓ is an integer and the summation index k runs over strictly half-integers. But then the normal ordering annihilates $e_\lambda \bar{\chi}$. So the only nontrivial contribution from the sum is 1 when $\ell = 0$. When $\ell > 0$, there is no nontrivial contribution. So we finally have

$$W_{\frac{1}{2}}(\bar{\chi})|\lambda\rangle = 16e_\lambda \bar{\chi}. \quad (2.5.80)$$

Thus

$$\kappa_\lambda(\chi) = \langle \lambda | \bar{W}_{-\frac{1}{2}}(\chi) | \chi \rangle = \langle \chi | W_{\frac{1}{2}}(\bar{\chi}) | \lambda \rangle^* = 16 \langle \chi | e_\lambda | \bar{\chi} \rangle^*. \quad (2.5.81)$$

Thus, we have

$$\bar{W}_{-1/2}(\chi)\chi = \frac{\alpha(\chi)}{8}\psi_L + \sum_{\lambda^2=4}\kappa_\lambda|\lambda\rangle. \quad (2.5.82)$$

It remains to prove the second equation in (2.5.54). By (2.2.7) we have

$$L_1\bar{W}_{-1/2}(\chi) - \bar{W}_{-1/2}(\chi)L_1^T = \bar{W}_{1/2}(\chi). \quad (2.5.83)$$

Applying this on χ and using the fact that $L_1^T\chi = 0$, we obtain

$$\begin{aligned} \bar{W}_{1/2}(\chi)\chi &= L_1\bar{W}_{-1/2}(\chi)\chi \\ &= \frac{\alpha(\chi)}{8}L_1\psi_L + \sum_{\lambda^2=4}\kappa_\lambda L_1|\lambda\rangle. \end{aligned} \quad (2.5.84)$$

Using the commutator $[a_m^i, L_n] = ma_{m+n}^i$, which is easy to check using the expression (2.3.14) for Virasoro generators, we see that $L_1\psi_L = 0$. Moreover, $L_1|\lambda\rangle = 0$. Thus we obtain

$$\bar{W}_{1/2}(\chi)\chi = 0. \quad (2.5.85)$$

This concludes the proof of (2.5.54) above. $\square$

We use the OPE (2.5.49) and the expression (2.5.54) to get:

$$\tilde{V}(\chi, z)\tilde{V}(\chi, w) = \frac{\alpha(\chi)}{(z-w)^3} + \frac{\alpha(\chi)/8}{(z-w)}T_B(w) + \sum_{\lambda^2=4}\frac{\kappa_\lambda(\chi)}{z-w}\begin{pmatrix} e^{i\lambda\cdot X(w)}\sigma_\lambda & 0 \\ 0 & e^{i\lambda\cdot R(w)}e_\lambda \end{pmatrix}. \quad (2.5.86)$$

Our goal is to provide a superconformal current. This will be achieved if the third term above vanishes. Therefore we now search for a twisted ground state χ such that $\alpha(\chi) = 1$ and $\kappa_\lambda(\chi) = 0$ for every $\lambda \in \Lambda_L$ with $\lambda^2 = 4$. Choose an orthonormal basis of $\mathcal{S}(\Lambda_L)$. Then we have

$$\alpha(\chi)^* = \langle \chi | \overline{\chi} \rangle = -is^\dagger M s^*, \quad (2.5.87)$$

where s is the column vector representing χ in the chosen basis and M is as in (2.3.49). We have also used (2.5.41) here. As discussed in Remark 2.3.5, we will take the representation matrices γ_λ to be real and $M = \mathbb{1}$.

We next construct an explicit matrix representation of the Heisenberg group $\Gamma(\Lambda_L)$, defined analogous to (2.3.31). Let $\{\gamma_i\}_{i=1}^{24}$ be $(2^{12} \times 2^{12})$ real gamma matrices in 24 dimensions satisfying $\gamma_i^2 = \mathbb{1}$ and $\gamma_i \gamma_j = -\gamma_j \gamma_i$ for $i \neq j$. Let Γ denote the group generated by the 24 Dirac gamma matrices. It has order $2 \cdot 2^{24}$ and is a Heisenberg extension of $\mathbb{Z}_2^{24}$. Introduce the notation:

$$\gamma_w := \gamma_1^{w^1} \cdots \gamma_{24}^{w^{24}} \quad \text{for } w = (w^1, \dots, w^{24}) \in \mathbb{F}_2^{24}. \quad (2.5.88)$$

Then clearly

$$\Gamma := \{\pm \gamma_w : w \in \mathbb{F}_2^{24}\}. \quad (2.5.89)$$

As is well-known, there is a unique irreducible representation of the Heisenberg group, up to isomorphism. Since Γ is a group of matrices it has a canonical representation, we take $\mathcal{S}(\Lambda_L)$ to be this canonical representation. See [DGM90, Appendix C] for one choice of γ_i . The group $\Gamma(\Lambda_L)$ is also a Heisenberg extension of $\mathbb{Z}_2^{24}$ and hence is isomorphic to Γ . To construct an explicit isomorphism, pick a $\mathbb{Z}_2$ -basis $\{[\alpha_i]\}_{i=1}^{24}$ for $\Lambda_L/2\Lambda_L$ using Leech lattice vectors α_i with the property that

$$\alpha_i \cdot \alpha_j \equiv (1 - \delta_{ij}) \pmod{4}. \quad (2.5.90)$$

An explicit demonstration of the existence of such a basis is given in (2.5.140) et. seq. Because of the property (2.5.90),

$$\rho(e_{\alpha_i}) = \gamma_i, \quad \rho(-e_{\alpha_i}) = -\gamma_i \quad (2.5.91)$$

defines a group isomorphism $\rho : \Gamma(\Lambda_L) \longrightarrow \Gamma$. Here $\{e_{\alpha_i}, -e_{\alpha_i}\}$ is the fiber above $[\alpha_i]$. For any $\lambda \in \Lambda_L$ with $[\lambda] = \sum_{i=1}^{24} w^i [\alpha_i]$ with $w^i \in \mathbb{Z}_2$, we get

$$e_\lambda = v([\lambda]) \prod_{i=1}^{24} e_{\alpha_i}^{w^i} \quad (2.5.92)$$

where $v(\lambda) \in \{\pm 1\}$ is determined using the value of the cocycle ε on the basis vectors. We can then easily check that

$$\rho(e_\lambda) = v([\lambda]) \prod_{i=1}^{24} \gamma_i^{w^i} = v([\lambda]) \gamma_w \quad (2.5.93)$$

where $w = (w^1 \dots w^{24}) \in \mathbb{F}_2^{24}$. As mentioned in Remark 2.2.3 we will often write

$$\gamma_\lambda := \rho(e_\lambda) . \quad (2.5.94)$$

Remark 2.5.2. A choice of basis $\{[\alpha_i]\}_{i=1}^{24}$ for $\Lambda_L/2\Lambda_L$ with the property (2.5.90) is equivalent to a choice of representation of the Heisenberg group $\Gamma(\Lambda_L)$, the representation being specified by the group isomorphism (2.5.91). The choice of representation gives the action of e_λ on $\mathcal{S}(\Lambda_L)$. A difference choice of such a basis and hence a distinct isomorphic representation of $\Gamma(\Lambda_L)$ gives another isomorphic realization of the Beauty and the Beast SCFT. The explicit isomorphism is identical to the one in Remark 2.3.4. Throughout the paper, we choose one such basis and fix it.

The choice of basis $\{[\alpha_i]\}_{i=1}^{24}$ of $\Lambda_L/2\Lambda_L$ with the property (2.5.90) gives us a map

$$\begin{aligned} f : \Lambda_L &\longrightarrow \mathbb{F}_2^{24} \\ \pm \lambda &\mapsto (w^1 \dots w^{24}) \end{aligned} \quad (2.5.95)$$

where

$$[\lambda] = \sum_{i=1}^{24} w^i [\alpha_i], \quad w^i \in \mathbb{F}_2. \quad (2.5.96)$$

This map is well defined, surjective and an $\mathbb{F}_2$ -module homomorphism. It is easy to see that $\text{Ker}(f) = 2\Lambda_L$. We denote the induced isomorphism of $\mathbb{F}_2$ -modules $\Lambda_L/2\Lambda_L$ and $\mathbb{F}_2^{24}$ by the same symbol f . We now have the following lemma.

Lemma 2.5.2. *Let $\Gamma_{12} \subset \Gamma$ be an Abelian subgroup of order 2^{12} consisting of elements of the form $\xi_w \gamma_w$, with $w \in \mathbb{F}_2^{24}$ and $\xi_w \in \{\pm 1\}$ such that $-\mathbf{1} \notin \Gamma_{12}$. Then the image $\mathcal{C}_{12}$ of Γ_{12} under the map $\Gamma \rightarrow \mathbb{F}_2^{24}$ given by $\pm \gamma_w \mapsto w$ is a $[24, 12]$ binary linear code.*

Proof. Since Γ_{12} is a subgroup, $\mathbf{1} \in \Gamma_{12}$ and hence $(00 \dots 0) \in \mathcal{C}_{12}$. Moreover since $\gamma_w \gamma_{w'} = \pm \gamma_{w+w'}$ we see that $\mathcal{C}_{12}$ is a linear subspace of $\mathbb{F}_2^{24}$. Since $-\mathbf{1} \notin \Gamma_{12}$, we cannot have both γ_w and $-\gamma_w$ in Γ_{12} . It is thus clear that $|\mathcal{C}_{12}| = 2^{12}$. The statement now follows. $\square$

Lemma 2.5.3. *Let Γ_{12} and $\mathcal{C}_{12}$ be as in Lemma 2.5.2 with the additional property that $\gamma_w^2 = \mathbf{1}$ for all $\gamma_w \in \Gamma_{12}$. Then the cocycle ε is trivialisable on $\mathcal{C}_{12}$ by a $U(1)$ coboundary $b : \mathbb{F}_2^{24} \rightarrow U(1)$. That is*

$$\varepsilon(w_1, w_2) = \frac{b(w_1 + w_2)}{b(w_1)b(w_2)}. \quad (2.5.97)$$

Proof. The isomorphism class of $U(1)$ central extensions is determined by the commutator function [FMS06, Proposition A.1]

$$\kappa(w_1, w_2) = \gamma_{w_1} \gamma_{w_2} \gamma_{w_1}^{-1} \gamma_{w_2}^{-1}. \quad (2.5.98)$$

Since elements of Γ_{12} square to identity, $\gamma_w^{-1} = \gamma_w$ for all $w \in \mathcal{C}_{12}$ and since Γ_{12} is abelian, it is clear that $\kappa(w_1, w_2) = 1$ for all $w_1, w_2 \in \mathcal{C}_{12}$. $\square$

Lemma 2.5.4. *Let $b : \mathbb{F}_2^{24} \longrightarrow U(1)$ be the coboundary from Lemma 2.5.3. Define $\tilde{\gamma}_w = b(w)\gamma_w$ for $w \in \mathbb{F}_2^{24}$. Then the operator*

$$P_{12} := \frac{1}{2^{12}} \sum_{w \in \mathcal{C}_{12}} \tilde{\gamma}_w \quad (2.5.99)$$

is a rank one projection operator acting on $\mathcal{S}(\Lambda_L)$.

Proof. Observe that

$$\tilde{\gamma}_{w_1} \tilde{\gamma}_{w_2} = b(w_1)b(w_2)\varepsilon(w_1, w_2)\gamma_{w_1+w_2} = b(w_1 + w_2)\gamma_{w_1+w_2} = \tilde{\gamma}_{w_1+w_2}. \quad (2.5.100)$$

Thus

$$P_{12}^2 = \frac{1}{2^{24}} \sum_{w \in \mathcal{C}_{12}} \sum_{w' \in \mathcal{C}_{12}} \tilde{\gamma}_{w+w'} = \frac{1}{2^{12}} \sum_{w \in \mathcal{C}_{12}} P_{12} = P_{12}. \quad (2.5.101)$$

Moreover using the identity

$$\begin{aligned} \text{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_n}) &= \delta_{\mu_1 \mu_2} \text{Tr}(\gamma_{\mu_3} \cdots \gamma_{\mu_n}) - \delta_{\mu_1 \mu_3} \text{Tr}(\gamma_{\mu_2} \gamma_{\mu_4} \cdots \gamma_{\mu_n}) + \cdots \\ &\quad + \delta_{\mu_1 \mu_n} \text{Tr}(\gamma_{\mu_2} \cdots \gamma_{\mu_{n-1}}), \end{aligned} \quad (2.5.102)$$

it is clear that

$$\text{Tr}(\gamma_w) = 0, \quad \forall \quad w \in \mathbb{F}_2^{24} \setminus \{0\}. \quad (2.5.103)$$

Thus

$$\text{Tr} P_{12} = 1. \quad (2.5.104)$$

Since the only eigenvalues of P_{12} are 0 and 1, it is clear that P_{12} is rank one. $\square$

Definition 2.5.1. A sublattice $\Lambda_{\text{SC}} \subset \Lambda_L$ is called a *superconformal sublattice* if the following properties are satisfied:

1. For every $\lambda_1, \lambda_2, \lambda \in \Lambda_{\text{SC}}$, $\langle \lambda_1, \lambda_2 \rangle \equiv 0 \pmod{2}$ and $\langle \lambda, \lambda \rangle \equiv 0 \pmod{4}$.
2. Λ_{SC} does not contain any vector of norm-squared 4.
3. $2\Lambda_L \subset \Lambda_{\text{SC}}$ with index 2^{12} .

Hypothesis (1) and (3) implies that $\tilde{\Gamma}(\Lambda_{\text{SC}}) := \{e_\lambda : [\lambda] \in \tilde{\Lambda}_{\text{SC}}\}$ is an order 2^{12} Abelian subgroup of $\Gamma(\Lambda_L)$, where $\tilde{\Lambda}_{\text{SC}} := \Lambda_{\text{SC}}/2\Lambda_L$. Hypothesis (2), as we will see below, is crucial for the construction of the supercurrent.

We can extend the map f , defined in (2.5.95) to a homomorphism $f : \Gamma(\Lambda_L) \longrightarrow \mathbb{F}_2^{24}$ as follows:

$$\begin{aligned} f : \Gamma(\Lambda_L) &\longrightarrow \mathbb{F}_2^{24} \\ \pm e_\lambda &\mapsto (w^1 \cdots w^{24}) \end{aligned} \quad (2.5.105)$$

where w^i is defined as in (2.5.96). We then claim that for any superconformal sublattice Λ_{SC}

$$\mathcal{C}_{\text{SC}} := f(\tilde{\Gamma}(\Lambda_{\text{SC}})) \quad (2.5.106)$$

is a $[24, 12]$ binary code. It is clear that the group $\rho(\tilde{\Gamma}(\Lambda_{\text{SC}}))$ satisfies all hypotheses of Lemma 2.5.2, where $\rho : \Gamma(\Lambda_L) \longrightarrow \Gamma$ is the group isomorphism described in (2.5.91). Indeed using Property (1) of Definition 2.5.1

$$e_\lambda e_\mu = (-1)^{\lambda \cdot \mu} e_\mu e_\lambda = e_\mu e_\lambda \quad \forall [\lambda], [\mu] \in \Lambda_{\text{SC}}/2\Lambda_L. \quad (2.5.107)$$

Thus $\tilde{\Gamma}(\Lambda_{\text{SC}})$ is abelian. By the same property, we have

$$e_\lambda^2 = (-1)^{\frac{\lambda^2}{2}} \mathbf{1} = \mathbf{1}, \quad \forall [\lambda] \in \Lambda_{\text{SC}}/2\Lambda_L. \quad (2.5.108)$$

Finally $-\mathbf{1} \notin \rho(\tilde{\Gamma}(\Lambda_{\text{SC}}))$ since otherwise if $e_\lambda \in \tilde{\Gamma}(\Lambda_{\text{SC}})$ then so does $-e_\lambda \in \tilde{\Gamma}(\Lambda_{\text{SC}})$ and $|\tilde{\Gamma}(\Lambda_{\text{SC}})| = |\Lambda_{\text{SC}}/2\Lambda_L| = 2^{11}$ contradicting the index of $2\Lambda_L$ in Λ_{SC} . Hence $\mathcal{C}_{\text{SC}} := f(\tilde{\Gamma}(\Lambda_{\text{SC}}))$ is a $[24, 12]$ binary code.

Theorem 2.5.2. *Let $\Lambda_{\text{SC}} \subset \Lambda_L$ be a superconformal sublattice and let $\mathcal{C}_{\text{SC}} = f(\tilde{\Gamma}(\Lambda_{\text{SC}}))$ be the $[24, 12]$ binary code with corresponding projection operator P_{SC} . Then there exists $\chi_{\text{SC}} \in \text{Im}P_{\text{SC}}$ such that $\alpha(\chi_{\text{SC}}) = 1$ and $\kappa_\lambda(\chi_{\text{SC}}) = 0$ for all $\lambda \in \Lambda_L$ with norm squared 4, where $\alpha(\chi), \kappa_\lambda(\chi)$ is as in (2.5.55).*

Proof. Choose a normalised vector χ_1 (i.e., scaled such that its norm is 1) in $\text{Im}P_{\text{SC}}$ and extend it to a basis $\{\chi_1, \chi_2, \dots, \chi_{2^{12}}\}$ of $\mathcal{S}(\Lambda_L)$. We have

$$P_{\text{SC}} = |\chi_1\rangle\langle\chi_1|, \quad (2.5.109)$$

where $|\chi\rangle\langle\chi|$ denotes the outer product of vectors. By (2.5.87) $\alpha(\chi_1)^* = -i$. So we scale χ_1 by $e^{i\pi/4}$:

$$\chi_{\text{SC}} := e^{-i\pi/4} \chi_1. \quad (2.5.110)$$

then we have

$$\alpha(\chi_{\text{SC}})^* = -i\langle e^{-i\pi/4} \chi_1 | e^{i\pi/4} \chi_1 \rangle = \langle \chi_1 | \chi_1 \rangle = 1 = \alpha(\chi_{\text{SC}}). \quad (2.5.111)$$

We now prove that $\kappa_\lambda(\chi_{\text{SC}}) = 0$. Since Λ_{SC} does not contain vectors of norm squared 4, we see that if $\gamma_\lambda = \gamma_w$ with $w \in \mathbb{F}_2^{24}$ and $\lambda' \in [\lambda]$ with $\lambda'^2 = 4$, then $w \notin \mathcal{C}_{\text{SC}}$. Thus if $\gamma_\lambda = \gamma_{w'}$ with $w' \notin \mathcal{C}_{\text{SC}}$, we have

$$\begin{aligned} \kappa_\lambda(\chi_{\text{SC}}) &= 16\langle \chi_{\text{SC}} | e_\lambda | \bar{\chi}_{\text{SC}} \rangle = -16i\langle e^{-i\pi/4} \chi_1 | e_\lambda | e^{i\pi/4} \chi_1 \rangle \\ &= 16\langle \chi_1 | e_\lambda | \chi_1 \rangle \\ &= \frac{16}{2^{12}} \text{Tr} \sum_{w \in \mathcal{C}_{\text{SC}}} b(w) \gamma_\lambda \gamma_w \\ &= \frac{16}{2^{12}} \sum_{w \in \mathcal{C}_{\text{SC}}} b(w) \text{Tr} \gamma_{w'} \gamma_w, \end{aligned} \quad (2.5.112)$$

where we used the property of outer product:

$$\langle \Psi | \Phi \rangle = \text{Tr}(|\Psi\rangle\langle\Phi|) . \quad (2.5.113)$$

Now since $w + w' \notin \mathcal{C}_{\text{SC}}$ for any $w \in \mathcal{C}_{\text{SC}}$ and $w' \in \mathbb{F}_2^{24} \setminus \mathcal{C}_{\text{SC}}$, we have that $w + w' \neq 0$ and hence $\text{Tr} \gamma_{w'} \gamma_w = 0$ since $w \in \mathcal{C}_{\text{SC}}$ and $\kappa_\lambda(\chi_{\text{SC}}) = 0$. The proof is complete. $\square$

Thus the existence of a superconformal vector reduces to the existence of a superconformal sublattice. We now prove that such a superconformal sublattice exists completing the proof of superconformal symmetry in the (spin lift of the) Moonshine module.

Theorem 2.5.3. *There exists a superconformal sublattice of Λ_L .*

Proof. It is proved in [DLMN96, Lemma 4.3] that for any Niemeier lattice L , the lattice $\sqrt{2}L$, the lattice isomorphic to L with all norms doubled, can be isometrically embedded²¹ in Λ_L . We denote this embedded sublattice by $\sqrt{2}L^\iota \subseteq \Lambda_L$ where $\iota : \sqrt{2}L \rightarrow \Lambda_L$ is the embedding map. Moreover, it is also true that $2\Lambda_L \subset \sqrt{2}L^\iota$ and $\sqrt{2}L^\iota/2\Lambda_L$ is a Lagrangian subspace of $\Lambda_L/2\Lambda_L$. In particular, if we choose the Niemeier lattice L to be the Leech lattice itself, then we have the following properties

$$\begin{aligned} (i) \quad & \lambda^2 \in 4\mathbb{N}, \quad \lambda \in \sqrt{2}\Lambda_L^\iota \setminus \{0\} \\ (ii) \quad & \lambda_1 \cdot \lambda_2 \in 2\mathbb{Z}, \quad \lambda_i \in \sqrt{2}\Lambda_L^\iota \\ (iii) \quad & \lambda^2 \neq 4 \quad \forall \lambda \in \sqrt{2}\Lambda_L^\iota. \end{aligned} \quad (2.5.114)$$

The last property follows from the fact that Leech lattice has minimal norm 4. Since $\sqrt{2}\Lambda_L^\iota/2\Lambda_L$ is a Lagrangian subspace of $\Lambda_L/2\Lambda_L$, it is clear that the subgroup of $\Gamma(\Lambda_L)$:²²

$$\tilde{\Gamma}(\sqrt{2}\Lambda_L^\iota) := \{e_\lambda : [\lambda] \in \sqrt{2}\Lambda_L^\iota/2\Lambda_L\} \quad (2.5.115)$$

has order 2^{12} . By (2.5.115) $\sqrt{2}\Lambda_L^\iota$ is a superconformal sublattice. $\square$

Theorem 2.5.4. *Let χ_{SC} be the twisted ground state of Theorem 2.5.2 for a choice of a superconformal sublattice Λ_{SC} . Then $T_F(z) := 2V(\chi_{\text{SC}}, z)$ is a superconformal current.*

Proof. From (2.5.86), the OPE is

$$\tilde{V}(\chi_{\text{SC}}, z) \tilde{V}(\chi_{\text{SC}}, w) \sim \frac{1}{(z-w)^3} + \frac{1/8}{(z-w)} T_B(w). \quad (2.5.116)$$

Any potential supercurrent for $\mathcal{H}_{\text{SC}}$ must have central charge $\hat{c} = 16$. This is obvious since the Monster VOA has central charge $c = 24$. It is now clear that T_F has the required OPE of a supercurrent with central charge $\hat{c} = 16$. $\square$

²¹For an explicit embedding, see [HK00].

²²Recall that we can choose a cocycle that is trivial on this subset

2.5.3 Classification Of $\mathcal{N} = 1$ Structures Obtained From Codes

Our construction of the supercurrent crucially depends on a choice of superconformal sublattice Λ_{SC} of the Leech lattice. If there exists one such choice of superconformal sublattice Λ_{SC} , then for any $R \in \text{Co}_0$, where Co_0 is the Conway group, $R \cdot (\Lambda_{\text{SC}})$ is again a superconformal sublattice.

Definition 2.5.2. Two supercurrents constructed from superconformal lattices are said to be *equiv-
alent* if the superconformal lattices are related by an automorphism of the Leech lattice or the codes associated to the two superconformal lattices are isomorphic.

This terminology is motivated by the fact that the $\mathcal{N} = 1$ structures obtained from Λ_{SC} and $R \cdot (\Lambda_{\text{SC}})$ (i.e., the SCVOA $\mathcal{H}_{\text{SC}}$ with superconformal states obtained from Λ_{SC} and $R \cdot (\Lambda_{\text{SC}})$) are isomorphic (see Definition 2.4.2). Similarly $\mathcal{N} = 1$ structures obtained from two isomorphic codes is isomorphic. We now explicitly demonstrate the isomorphism.

Proposition 2.5.4. *Let $\chi_{\text{SC}}, \chi'_{\text{SC}}$ be two supercurrents obtained from either two isomorphic codes or from two Conway equivalent (i.e., related by a Conway rotation) superconformal sublattices. Then there exists vertex operators $V'(\psi, z)$ such that the SCVOAs $(\mathcal{H}_{\text{SC}}, V, |0\rangle, \psi_L, \chi_{\text{SC}})$ and $(\mathcal{H}_{\text{SC}}, V', |0\rangle, \psi_L, \chi'_{\text{SC}})$ are isomorphic.*

Proof. Let us first suppose that

$$|\chi_{\text{SC}}\rangle = P|\chi_0\rangle, \quad |\chi'_{\text{SC}}\rangle = P'|\chi_0\rangle, \quad |\chi_0\rangle \in \mathcal{S}(\Lambda_L), \quad \langle \chi_0 | \chi_0 \rangle = 1, \quad (2.5.117)$$

where

$$P := \frac{1}{2^{12}} \sum_{w \in \mathcal{C}} \gamma_w, \quad P' := \frac{1}{2^{12}} \sum_{w \in \mathcal{C}'} \gamma_w, \quad (2.5.118)$$

and $\mathcal{C}, \mathcal{C}'$ are isomorphic [24, 12] codes such that P, P' are rank 1 projectors. Let $S : \mathbb{F}_2^{24} \longrightarrow \mathbb{F}_2^{24}$ be a permutation such that $\mathcal{C}' = \mathcal{C}S$. Note that

$$\Gamma' := \{\pm \gamma_{wS} : w \in \mathbb{F}_2^{24}\}, \quad (2.5.119)$$

is defines another unitary irreducible representation of the Heisenberg extension of $\mathbb{Z}_2^{24}$ acting on $\mathcal{S}(\Lambda_L)$. By uniqueness of the representation, there exists a unitary matrix $\hat{S}$ such that²³

$$\hat{S} \gamma_w \hat{S}^\dagger = \gamma_{wS}, \quad w \in \mathbb{F}_2^{24}. \quad (2.5.120)$$

Now define the map $u_S : \mathcal{H}_{BB} \longrightarrow \mathcal{H}_{BB}$ by

$$\begin{aligned} u_S a_n^j u_S^{-1} &= a_n^j, & u_S c_k^j u_S^{-1} &= c_k^j \\ u_S |\lambda\rangle &= |\lambda\rangle, & u_S |\chi\rangle &= \hat{S} |\chi\rangle. \end{aligned} \quad (2.5.121)$$

²³Note that there is no ± 1 factor because S preserves the cocycle $\varepsilon(w, w') = \varepsilon(Sw, Sw')$.

Define V' as in (2.5.17), (2.5.18) with V'_T defined as in (2.3.44) with e_λ replaced by $e_{(f^{-1} \circ S \circ f)([\lambda])}$. Then one can check that u_S satisfies $u_S V(\psi, z) u_S^{-1} = V'(u_S \psi, z)$ for all $\psi \in \mathcal{H}_{BB}$ and preserves the vacuum and conformal vector. We now show that

$$u_S |\chi_{SC}\rangle = |\chi'_{SC}\rangle . \quad (2.5.122)$$

It is clear from (2.5.120) that $\widehat{S}P = P'\widehat{S}$. Thus we have

$$u_S |\chi_{SC}\rangle = \widehat{S}P|\chi_0\rangle = P'\widehat{S}|\chi_0\rangle = |\chi'_{SC}\rangle . \quad (2.5.123)$$

Next suppose that χ_{SC} and $\chi'_{SC} =: \chi_{SC}^R$ be the superconformal vectors constructed from superconformal sublattices Λ_{SC} and $R \cdot (\Lambda_{SC})$ respectively for some $R \in \text{Co}_0$. Notice that $\{\pm \gamma_{R\lambda} : \lambda \in \Lambda_L\}$ is another unitary representation of the Heisenberg group $\Gamma(\Lambda_L)$. Thus by uniqueness of unitary representation of the Heisenberg group, there exists a unitary matrix S such that

$$S\gamma_\lambda S^\dagger = v_R(\lambda)\gamma_{R\lambda} , \quad (2.5.124)$$

for some $v_R(\lambda) = \pm 1$ which satisfies

$$\frac{v_R(\lambda)v_R(\lambda')}{v_R(\lambda + \lambda')} \varepsilon(\lambda, \lambda') = \varepsilon(R\lambda, R\lambda') . \quad (2.5.125)$$

Define the map $u_R : \mathcal{H}_{BB} \rightarrow \mathcal{H}_{BB}$ by

$$\begin{aligned} u_R a_n^\ell u_R^{-1} &= R^\ell_j a_n^j, & u_R c_k^\ell u_R^{-1} &= R^\ell_j c_k^j \\ u_R |\lambda\rangle &= v_R(\lambda) |R\lambda\rangle, & u_R |\chi\rangle &= S|\chi\rangle . \end{aligned} \quad (2.5.126)$$

One can check that u_R satisfies $u_R V(\psi, z) u_R^{-1} = V(u_R \psi, z)$ for all $\psi \in \mathcal{H}_{BB}$ and preserves the vacuum and conformal vector. We now show that

$$u_R |\chi_{SC}\rangle = |\chi_{SC}^R\rangle . \quad (2.5.127)$$

Observe that if $b(\lambda) = \pm 1$ trivializes the cocycle on $\widetilde{\Gamma}(\Lambda_{SC})$ then $b(\lambda)v_R(\lambda)$ trivializes the cocycle on $\widetilde{\Gamma}(R \cdot (\Lambda_{SC}))$. Indeed, using (2.5.125) we have

$$\frac{b(\lambda)b(\lambda')}{b(\lambda + \lambda')} \frac{v_R(\lambda)v_R(\lambda')}{v_R(\lambda + \lambda')} = \varepsilon(\lambda, \lambda') \frac{v_R(\lambda)v_R(\lambda')}{v_R(\lambda + \lambda')} = \varepsilon(R\lambda, R\lambda') . \quad (2.5.128)$$

Let P_{SC} and P_{SC}^R be the projection operators corresponding to the superconformal sublattices Λ_{SC} and $R \cdot (\Lambda_{SC})$ respectively. Then we have

$$SP_{SC}S^\dagger = \frac{1}{2^{12}} \sum_{\lambda \in \Lambda_{SC}/2\Lambda_L} b(\lambda) S\gamma_\lambda S^\dagger = \frac{1}{2^{12}} \sum_{\lambda \in \Lambda_{SC}/2\Lambda_L} b(\lambda) v_R(\lambda) \gamma_{R\lambda} = P_{SC}^R . \quad (2.5.129)$$

Since $|\chi_{SC}\rangle \in \text{Im} P_{SC}$ we have

$$S|\chi_{SC}\rangle = SP_{SC}|\chi_{SC}\rangle = P_{SC}^R S|\chi_{SC}\rangle = |\chi_{SC}^R\rangle , \quad (2.5.130)$$

where we used the fact that P_{SC}^R has rank 1. $\square$

In the previous section we used the existence of an embedding of $\sqrt{2}\Lambda_L$ into Λ_L . In fact, there can be different embeddings not related by the action of the Conway group. In our language, these give inequivalent supercurrents as long as the resulting codes are also non-isomorphic. We will now show that there can be at most 9 inequivalent supercurrents constructed using embeddings $\iota : \sqrt{2}\Lambda_L \hookrightarrow \Lambda_L$. Moreover we construct two such inequivalent supercurrents explicitly. Since there are several supercurrents in the Beauty and the Beast SCFT, a natural question to ask is whether supersymmetry can be enhanced to $\mathcal{N} > 1$. At the end of this section we show that there cannot be $\mathcal{N} > 1$ supersymmetry in the Beauty and the Beast SCFT.

Suppose that we can find another sublattice Λ' of the Leech lattice with the properties listed in (2.5.114), then we can construct another supercurrent using the code $f(\Lambda'/2\Lambda_L)$ and the associated projection operator. Our construction is based on the isometric embedding of $\sqrt{2}\Lambda_L$ into Λ_L which provides such a sublattice. Now there are many inequivalent embeddings of $\sqrt{2}\Lambda_L$ into Λ_L . We now want to classify all the supercurrents that can be constructed using these embeddings. As before, let us denote the image $\iota(\sqrt{2}\Lambda_L)$ of an embedding $\iota : \sqrt{2}\Lambda_L \rightarrow \Lambda_L$ by $\sqrt{2}\Lambda_L^\iota$. We need the following proposition.

Proposition 2.5.5. *Let $\iota : \sqrt{2}\Lambda_L \rightarrow \Lambda_L$ be an embedding and $\{[\alpha_i]\}_{i=1}^{24}$ be the basis of $\Lambda_L/2\Lambda_L$ chosen in (2.5.90). Suppose further that*

$$\sum_{i=1}^{24} [\alpha_i] \in \sqrt{2}\Lambda_L^\iota/2\Lambda_L. \quad (2.5.131)$$

Then the code

$$\mathcal{C}_{12}^\iota := \{w \in \mathbb{F}_2^{24} : \gamma_w = \gamma_\lambda = \rho(e_\lambda) \text{ for some } e_\lambda \in \tilde{\Gamma}(\sqrt{2}\Lambda_L^\iota)\} \quad (2.5.132)$$

where $\tilde{\Gamma}(\sqrt{2}\Lambda_L^\iota)$ is defined analogous to (2.5.115), is a self-dual, doubly even code.

Proof. Let $\{\alpha_i\}_{i=1}^{24}$ be a set of Leech vectors representing the basis cosets $\{[\alpha_i]\}_{i=1}^{24}$ in the hypothesis. Then we have

$$\alpha_i \cdot \alpha_j = (1 - \delta_{ij}) \bmod 4. \quad (2.5.133)$$

Then condition (2.5.131) on an embedding implies that

$$f\left(\sum_{i=1}^{24} [\alpha_i]\right) = (11 \dots 1) \in \mathbb{F}_2^{24}. \quad (2.5.134)$$

This implies that $(11 \dots 1) \in \mathcal{C}_{12}^\iota$. Next note that for any $w, w' \in \mathbb{F}_2^{24}$ [Moo22],

$$\gamma_w \gamma_{w'} = (-1)^{\text{wt}(w)\text{wt}(w') + w \cdot w'} \gamma_{w'} \gamma_w. \quad (2.5.135)$$

Now since $\tilde{\Gamma}(\sqrt{2}\Lambda_L^t)$ is abelian, using (2.5.135) with $w' = (11 \dots 1)$, we conclude that $\text{wt}(w)$ is even for every $w \in \mathcal{C}_{12}^t$. Since $\gamma_w^2 = \mathbb{1}$ for every $w \in \mathcal{C}_{12}^t$ and

$$\gamma_w^2 = (-1)^{\frac{\text{wt}(w)(\text{wt}(w)-1)}{4}} \mathbb{1}, \quad (2.5.136)$$

we must have $\text{wt}(w) = 4k$ or $4k + 1$ for k a non-negative integer. But since $\text{wt}(w)$ is even for every $w \in \mathcal{C}_{12}^t$, we see that the code $\mathcal{C}_{12}^t$ is doubly even. The fact that it is self-dual follows by the fact that $(11 \dots 1) \in \mathcal{C}_{12}^t$. $\square$

We now show that for any embedding $\iota : \sqrt{2}\Lambda_L \rightarrow \Lambda_L$, there is an equivalent embedding $R \circ \iota$, for some suitable $R \in \text{Co}_0$, such that the hypotheses in Proposition 2.5.5 are satisfied for $R \circ \iota$.

We start by constructing 24 Leech lattice vectors $\{\alpha_i\}_{i=1}^{24}$ such that $\alpha_i \cdot \alpha_j = 3\delta_{ij} + 1$. (Reference [Bas] contains some useful information for this construction.) Recall that the Mathieu group M_{24} acts on Λ_L by permuting the coordinates. In particular, M_{23} acts by permuting the first 23 coordinates. Now start with the norm-squared 4 vector [CS99]

$$\alpha_1 = \frac{1}{\sqrt{8}} \begin{pmatrix} -1, 1, 1, 1, 1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, -1, 1, -1, 1, -1, -1, -1, -1, -3 \end{pmatrix}. \quad (2.5.137)$$

Obtain 22 more vectors $\alpha_2, \dots, \alpha_{23}$ by acting on this vector by the order 23 permutation $(12 \dots 23) \in M_{23} \hookrightarrow S_{23} \hookrightarrow S_{24}$, where $S_{23} \hookrightarrow S_{24}$ is the embedding by the action on the first 23 coordinates. Notice that in these 23 vectors, the +1s and -3 is supported on dodecads which intersect at 6 places pairwise. This will be very important. Choose the last vector to be another norm squared 4 vector [CS99]

$$\alpha_{24} = \frac{1}{\sqrt{8}} \begin{pmatrix} 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, -3 \end{pmatrix}. \quad (2.5.138)$$

It is easily checked that these 24 vectors α_i have mutual inner product 1 and hence are inequivalent modulo $2\Lambda_L$. This is because if $[\alpha_i] = [\alpha_j]$ then $\alpha_i = \alpha_j + 2\lambda$ for some $\lambda \in \Lambda_L$ and $\alpha_i \cdot \alpha_j \equiv 0 \pmod{2}$ contradicting the fact that $\alpha_i \cdot \alpha_j = 1$. Thus we conclude that $\{[\alpha_i]\}$ with α_i constructed above is a basis for $\Lambda_L/2\Lambda_L$.²⁴ Alternatively, the Gram matrix for the 24 classes $[\alpha_i]$ has 1 on off diagonal and 0 on diagonal and hence is invertible implying that $[\alpha_i]$ is a basis for $\Lambda_L/2\Lambda_L$. Next, note that the norm squared of each of the 24 vectors α_i is 4 and hence any representative of $[\alpha_i]$ will also have norm-squared divisible by 4. Finally it is easily checked that

$$\sum_{i=1}^{24} [\alpha_i] = \left[\frac{1}{\sqrt{8}}(0, 0, \dots, -72) \right] = \left[\frac{1}{\sqrt{8}}(0, 0, \dots, -8) \right]. \quad (2.5.139)$$

²⁴We remark that the α_i do not form a basis for the Leech lattice. Rather, they span a sublattice of index 3^{13} .

Note that $\frac{1}{\sqrt{8}}(0, 0, \dots, -8)$ is a norm-squared 8 vector.

Now consider an arbitrary embedding $\iota : \sqrt{2}\Lambda_L \longrightarrow \Lambda_L$. Of course $\sqrt{2}\Lambda_L^\iota$ contains norm squared 8 vectors. Choose one. Since the Conway group Co_0 is transitive on norm-squared 8 vectors [CS99, Theorem 29, Chapter 10] we can postcompose the embedding map ι by an automorphism of Λ_L to obtain an equivalent embedding containing the vector $\frac{1}{\sqrt{8}}(0 \ 0 \cdots -8)$. This means that $[\frac{1}{\sqrt{8}}(0 \ 0 \cdots -8)] \in \tilde{\Gamma}(\sqrt{2}\Lambda_L^{R\circ\iota})$ and we have satisfied the key condition (2.5.131) of Proposition 2.5.5.

This proves the following proposition.

Proposition 2.5.6. *Every equivalence class of embeddings²⁵ contains a representative $\iota : \sqrt{2}\Lambda_L \longrightarrow \Lambda_L$ such that C_{12}^ι is a self-dual doubly even [24, 12] code.*

Theorem 2.5.5. *Embeddings of $\sqrt{2}\Lambda_L$ into Λ_L give rise to at most 9 inequivalent supercurrents in the “Beauty and the Beast” SCFT $\mathcal{H}_{BB}$.*

Proof. By Proposition 2.5.6, the code generating the supercurrents obtained from equivalence classes of embeddings is self-dual and doubly even. There are only 9 such codes up to equivalence [PS75]. Moreover, by Proposition 2.5.4, equivalent codes give rise to equivalent supercurrents. Hence the supercurrents constructed from such embeddings can only be one of the 9 constructed from the 9 selfdual, doubly even binary linear codes. $\square$

We now explore some explicit examples of the codes in above theorem. We pick the following representation of the Heisenberg group: for the basis $[\alpha_i]$ of $\Lambda_L/2\Lambda_L$ constructed in (2.5.137) and (2.5.138), assign

$$\rho(e_{\alpha_i}) = \gamma_{\alpha_i} = \gamma_i, \quad i = 1, \dots, 24 \quad (2.5.140)$$

where γ_i are the standard Dirac gamma matrices. This is well defined because $\alpha_i \cdot \alpha_j = 3\delta_{ij} + 1$ and hence is compatible with $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}$. For any other λ , we first write

$$[\lambda] = \sum_{i=1}^{24} x_i [\alpha_i], \quad x_i \in \{0, 1\} \quad (2.5.141)$$

and then we have

$$\gamma_\lambda = \pm \gamma_1^{x_1} \cdots \gamma_{24}^{x_{24}} \quad (2.5.142)$$

where the choice of sign is fixed by the cocycle ε of the Heisenberg extension.

Proposition 2.5.7. *With the above representation of the Heisenberg group $\Gamma(\Lambda_L)$, if $\text{wt}(w) = 4$ with $\gamma_w = \gamma_\lambda$ for $\lambda \in \sqrt{2}\Lambda_L^\iota$ for any embedding $\iota : \sqrt{2}\Lambda_L \longrightarrow \Lambda_L$ then there is a representative λ of $[\lambda]$ with norm-squared 12 and of the shape $\frac{1}{\sqrt{8}}(\pm 2^8, \pm 4^4, 0^{12})$, $\frac{1}{\sqrt{8}}(\pm 2^{12}, \pm 4^3, 0^9)$ or $\frac{1}{\sqrt{8}}(\pm 2^{16}, \pm 4^2, 0^6)$.*

²⁵Two embeddings ι_1, ι_2 are equivalent if there exists $R \in \text{Co}_0$ such that $\iota_2 = R \circ \iota_1$.

Proof. Observe that for $\text{wt}(w) = 4$, if $\gamma_\lambda = \gamma_w$ then there is a $\lambda \in [\lambda]$ such that

$$\lambda = \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} + \alpha_{i_4}, \quad (2.5.143)$$

for some $i_1, \dots, i_4 \in \{1, \dots, 24\}$. Notice that since the dodecads on which the -1 s of $\alpha_i, i = 1, \dots, 23$ are supported intersect at 6 places pairwise, for any $1 \leq i_1, i_2 \leq 23$, $\alpha_{i_1} + \alpha_{i_2}$ has the shape $8^{-1/2}(2^5, -2^6, 0^{12}, -6)$ with the nonzero coordinates supported on a dodecad. Moreover, for any $\alpha_i + \alpha_{24}$ for $i = 1, \dots, 23$ has the shape $8^{-1/2}(2^{11}, 0^{12}, -6)$. This moreover implies that λ has the shape $8^{-1/2}(\pm 2^{k_1}, \pm 4^{k_2}, 0^{23-k_1-k_2}, -12)$. Next, using the fact that $\alpha_i \cdot \alpha_j = 3\delta_{ij} + 1$, we see that $\lambda^2 = 28$. Thus the norm squared of $8^{-1/2}(\pm 2^{k_1}, \pm 4^{k_2}, 0^{12}, -12)$ being 28 implies that

$$k_1 + 4k_2 = 20. \quad (2.5.144)$$

The integral solutions to this equation are $(k_1, k_2) = (20, 0), (16, 1), (12, 2), (8, 3), (4, 4), (0, 5)$. Moreover, the only possible choices of k_1 are 8, 12, 16 in which case $k_2 = 3, 2, 1$. Other choices can be ruled out using the fact that ± 4 appears only where the dodecads in $\alpha_{i_1} + \alpha_{i_2}$ and $\alpha_{i_3} + \alpha_{i_4}$ intersect and the fact that dodecads can only intersect in 0, 4, 6, and 8 places. Adding $8^{-1/2}(0^{23}, 16)$ to λ does not change the class, so we have another representative of $[\lambda]$ of the shape $8^{-1/2}(\pm 2^{k_1}, \pm 4^{k_2+1}, 0^{23-k_1-k_2})$. Thus $[\lambda]$ can have representative of the shape $8^{-1/2}(\pm 2^8, \pm 4^4, 0^{12}), 8^{-1/2}(\pm 2^{12}, \pm 4^3, 0^9)$ or $8^{-1/2}(\pm 2^{16}, \pm 4^2, 0^6)$. $\square$

Using the construction of [HK00], we will now explicitly construct two embeddings of $\sqrt{2}\Lambda_L$ into Λ_L which will give rise to inequivalent codes and hence inequivalent supercurrents.

To construct an embedding, we start with a $\mathbb{Z}_4$ -code $\mathcal{C}$ which is generated by codewords $g \in \mathbb{Z}_4^{24}$ such that the following holds: for each generator g of $\mathcal{C}$, there is a vector $\tilde{g} \in \mathbb{Z}^{24}$ such that $\frac{\tilde{g}}{\sqrt{2}} \in \Lambda_L$ and $g \equiv \tilde{g} \pmod{4}$. In simple terms $\frac{g}{\sqrt{2}} \in \Lambda_L$ if $\frac{g}{\sqrt{2}}$ is considered as an element of $\mathbb{R}^{24}$ under the map $\mathbb{Z}_4 \rightarrow \mathbb{R}$ which sends $0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 2$ and $3 \rightarrow 3$. Pseudo Golay codes and the Harada-Kitazume codes are examples of such $\mathbb{Z}_4$ -codes, described in Appendix A.1.2. A typical element of the associated lattice $A(\mathcal{C})$ (see Theorem A.1.4) is of the form

$$\frac{g}{2} + \frac{4x}{2}, \quad x \in \mathbb{Z}^{24}. \quad (2.5.145)$$

This implies that

$$\sqrt{2} \left(\frac{g}{2} + \frac{4x}{2} \right) = \frac{g}{\sqrt{2}} + \frac{8x}{2\sqrt{2}} \in \Lambda_L \quad (2.5.146)$$

since $\frac{g}{\sqrt{2}} \in \Lambda_L$ and $\frac{8x}{2\sqrt{2}} \in \Lambda_L$ for any $x \in \mathbb{Z}^{24}$. Hence $\sqrt{2}A(\mathcal{C}) \subseteq \Lambda_L$ which gives an explicit embedding if we choose $\mathcal{C}$ such that $A(\mathcal{C}) \cong \Lambda_L$. To explicitly construct the embedding, recall that $\sqrt{2}\Lambda_L = (\Lambda_L, 2\langle \cdot, \cdot \rangle)$ where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^{24}$. This means that

$\sqrt{2}\Lambda_L \cong \sqrt{2}A(\mathcal{C})$ isometrically. The inclusion map $i : \sqrt{2}A(\mathcal{C}) \longrightarrow \Lambda_L$ then provides the embedding $\iota_{\mathcal{C}} : \sqrt{2}\Lambda_L \longrightarrow \Lambda_L$.

Duncan's Spinor Defines A Supercurrent In The BB Module

Recall that pseudo Golay codes $\mathcal{C}$ are $\mathbb{Z}_4$ codes such that $\text{Res}(\mathcal{C}) = \mathcal{G}$. From [Rai99, Theorem 9], we see that every pseudo Golay code is Type II with minimum Euclidean distance $d_E(\mathcal{C}) = 16$. Thus by Theorem A.1.4 and the uniqueness of Leech lattice, we see that $A(\mathcal{C}) \cong \Lambda_L$. We now show that the generators g of the code satisfy $\frac{g}{\sqrt{2}} \in \Lambda_L$. Indeed since $g \pmod{2} \in \mathcal{G}$, the positions of 1 and 3 in g must be an octad in the Golay code and there are even number of 2s in g . So the shape of $\frac{g}{\sqrt{2}}$ is

$$\begin{aligned} \frac{g}{\sqrt{2}} &= \frac{2g}{2\sqrt{2}} = \frac{1}{\sqrt{8}}(2^k, 6^{8-k}, 4^{2\ell}, 0^{16-2\ell}) \\ &= \frac{1}{\sqrt{8}}(\underbrace{2^k, -2^{8-k}}_{\text{octad}}, 0^{16}) + \frac{1}{\sqrt{8}}(8^{8-k}, 4^{2\ell}, 0^{16+k-2\ell}). \end{aligned} \quad (2.5.147)$$

Moreover k is even for all the 13 pseudo Golay codes [Rai99]. Clearly from the list of vectors in the Leech lattice given in [CS99, Table 4.13, Chapter 4], we see that $\frac{g}{\sqrt{2}} \in \Lambda_L$.

Proposition 2.5.8. *Let $\iota_i : \sqrt{2}A(\mathcal{C}_i) \hookrightarrow \Lambda_L$, $i = 1, \dots, 13$ be the embedding of $\sqrt{2}\Lambda_L$ into Λ_L obtained from the 13 pseudo Golay codes $\mathcal{C}_i$. Then*

$$\mathcal{C}_{12}^{\iota_i} \cong \mathcal{G}, \quad i = 1, \dots, 13 \quad (2.5.148)$$

where $\mathcal{C}_{12}^{\iota_i}$ is defined as in (2.5.149).

Proof. By Proposition 2.5.5, the code $\mathcal{C}_{12}^{\iota_i}$ is a self-dual, doubly even binary $[24, 12]$ code. It suffices to prove that for every $w \in \mathcal{C}_{12}^{\iota_i}$, $\text{wt}(w) \neq 4$. We prove this computationally using Mathematica. In view of Proposition 2.5.7, it suffices to prove that the $\binom{24}{4} = 10626$ vectors of the shape $\frac{1}{\sqrt{8}}(\pm 2^8, \pm 4^4, 0^{12})$, $\frac{1}{\sqrt{8}}(\pm 2^{12}, \pm 4^3, 0^9)$ and $\frac{1}{\sqrt{8}}(\pm 2^{16}, \pm 4^2, 0^6)$ does not belong to $\sqrt{2}A(\mathcal{C}_i)$. This is equivalent to proving that the 10626 codewords of the form ²⁶ $(\pm 1^8, \pm 2^4, 0^{12})$, $(\pm 1^{12}, \pm 2^3, 0^9)$ and $(\pm 1^{16}, \pm 2^2, 0^6)$ does not belong to $\mathcal{C}_i$. This was explicitly checked using Mathematica for each of the 13 pseudo Golay codes and we indeed found that this is true. $\square$

Remark 2.5.3. Recall that the Harada-Kitazume code $\mathcal{C}_{HK}$ is another $\mathbb{Z}_4$ code satisfying all properties required to construct an embedding $\sqrt{2}A(\mathcal{C}_{HK}) \subseteq \Lambda_L$. Using similar Mathematica calculations as in Proposition 2.5.8, we found that $\mathcal{C}_{12}^{HK}$ defined by

$$\mathcal{C}_{12}^{HK} := \{w \in \mathbb{F}_2^{24} : \gamma_w = \gamma_\lambda = \rho(e_\lambda) \text{ for some } e_\lambda \in \tilde{\Gamma}(\sqrt{2}A(\mathcal{C}_{HK}))\} \quad (2.5.149)$$

²⁶Note that $-1, -2$ are considered mod 4 in the code.

is isomorphic to the Golay code $\mathcal{G}$.

The above results imply that one of the supercurrent is generated by the projection operator constructed out of the Golay code as is the case for *Conway supermoonshine* elucidated in [HM20]. It was called the Duncan spinor in [HM20] as it was first constructed by John Duncan in [Dun05, DMC14]. Thus one of the supercurrents in the “Beauty and the Beast” SCFT is based on the Duncan spinor.

Supercurrent From Quaternary Golay Code

Using the quaternary Golay code, we now construct a supercurrent which is inequivalent to the Duncan spinor, see (A.1.12) and Table A.1 for the definition of quaternary Golay code. We first show that the quaternary Golay code $\widehat{\mathcal{Q}}_4$ gives rise to an embedding of $\sqrt{2}\Lambda_L$ into Λ_L . Depending on $n_{\pm 1} \in \{0, 8, 12, 16, 24\}$, it is clear that any element of the code $\widehat{\mathcal{Q}}_4$ is of the shape

$$g = (1^k, 3^{n_{\pm}(g)-k}, 0^{24-n_{\pm}(g)}) + (2^{2\ell}, 0^{24-2\ell}), \quad (2.5.150)$$

where k is even since otherwise the Euclidean norm of g would be odd and hence it cannot form an even lattice. This implies that

$$\begin{aligned} \frac{g}{\sqrt{2}} &= \frac{2g}{\sqrt{8}} = \frac{1}{\sqrt{8}}(2^k, 6^{n_{\pm}(g)-k}, 0^{24-n_{\pm}(g)}) + \frac{1}{\sqrt{8}}(4^{2\ell}, 0^{24-2\ell}) \\ &= \frac{1}{\sqrt{8}} \underbrace{(2^k, -2^{n_{\pm}(g)-k}, 0^{24-n_{\pm}(g)})}_{\text{Golay codeword}} + \frac{1}{\sqrt{8}}(8^{n_{\pm}(g)-k}, 0^{24+k-n_{\pm}(g)}) + \frac{1}{\sqrt{8}}(2^{2\ell}, 0^{24-2\ell}). \end{aligned} \quad (2.5.151)$$

It is now clear that $\frac{g}{\sqrt{2}} \in \Lambda_L$. Finally, by [BSC06, Theorem 4.3] we see that $A(\widehat{\mathcal{Q}}_4) = \Lambda_L$. Thus using (2.5.146) this code gives us an embedding of $\sqrt{2}\Lambda_L$ into Λ_L .

Theorem 2.5.6. *The code $\mathcal{C}_{12}^{\widehat{\mathcal{Q}}_4}$ defined as in (2.5.149) is not isomorphic to the Golay code.*

Proof. It suffices to show that the code $\mathcal{C}_{12}^{\widehat{\mathcal{Q}}_4}$ contains a weight 4 codeword and hence cannot be the Golay code. Indeed from Lemma A.1.1, we see for any dodecad $w \in \mathcal{G}$, the code $\widehat{\mathcal{Q}}_4$ contains the codeword $(\pm 1^{12}, 2^3, 0^9)$ for any choice of position of the three 2s with ± 1 supported on w . From the proof of Proposition 2.5.7, we see that there exists a vector of the shape $8^{-1/2}(\pm 2^{12}, 4^3, 0^9)$ which map to a weight 4 codeword under the homomorphism f . But then this means that $\sqrt{2}A(\widehat{\mathcal{Q}}_4)$ contains Leech vectors which under the homomorphism f of (2.5.95) map to weight 4 codewords. Thus the code $\mathcal{C}_{12}^{\widehat{\mathcal{Q}}_4}$ cannot be the Golay code. $\square$

Thus we see that the code $\mathcal{C}_{12}^{\widehat{\mathcal{Q}}_4}$ thus gives rise to a new supercurrent different from the Duncan spinor.

We have now shown that there are at least two inequivalent supercurrents in the Beauty and the Beast module. Let us now calculate the OPE of the supercurrents with dimension 1 primaries in the theory since they are required for $\mathcal{N} > 1$ supersymmetry. There are 24 linearly independent dimension 1 fields: $\tilde{V}(a_{-1}^i|0\rangle, z)$. In terms of the fields X^i, R^i it is given by

$$\tilde{V}(a_{-1}^i|0\rangle, z) = \begin{pmatrix} :\partial_z X^i(z): & 0 \\ 0 & :\partial_z R^i(z): \end{pmatrix} \quad (2.5.152)$$

Let $\tilde{V}(\chi_s, z)$ be a supercurrent. By (2.5.22) and (2.5.23) we have the OPE

$$\tilde{V}(a_{-1}^i|0\rangle, z)\tilde{V}(\chi_s, w) = \frac{1}{(z-w)^{\frac{1}{2}}}\tilde{V}(c_{-1/2}^i\chi_s, w) + \mathcal{O}((z-w)^{\frac{1}{2}}), \quad (2.5.153)$$

where we used the fact that $c_k^i\chi = 0$ for $k > 0$ and $\chi \in \mathcal{S}(\Lambda_L)$. Thus the dimension 1 primaries have a square root singularity with the supercurrents. Therefore there is no mutually local dimension one $U(1)$ R-symmetry current and therefore we cannot have extended supersymmetry in the BB module. It might be possible to define an $\mathcal{N} = 2$ supersymmetry algebra by passing to suitable covering spaces. In any case, there is a novel symmetry structure here which might be worth understanding more deeply.

We also note that similar calculations as in Section 2.5.1 give us the OPE of two vertex operators $\tilde{V}(\chi_a, z)$ and $\tilde{V}(\chi_b, z)$ corresponding to twisted ground states $\chi_a, \chi_b \in \mathcal{S}$ to be

$$\tilde{V}(\chi_a, z)\tilde{V}(\chi_b, w) \sim \frac{\alpha^{ab}}{(z-w)^3} + \frac{\alpha^{ab}/8}{(z-w)}T_B(w) + \sum_{\lambda^2=4} \frac{\kappa_\lambda^{ab}}{z-w} \begin{pmatrix} e^{i\lambda \cdot X(w)}\sigma_\lambda & 0 \\ 0 & e^{i\lambda \cdot R(w)}e_\lambda \end{pmatrix} \quad (2.5.154)$$

where

$$\alpha^{ab} := \langle \bar{\chi}_a | \chi_b \rangle, \quad \kappa_\lambda^{ab} = 16i \langle \chi_a | e_\lambda | \bar{\chi}_b \rangle. \quad (2.5.155)$$

Thus any two supercurrents are local with respect to each other. For the two supercurrents constructed above, it may be possible to calculate the coefficients α^{ab} and κ_λ^{ab} explicitly. But we will leave this calculation for future. This suggests that there is a novel chiral algebra in the BB module.

2.6 Mathematical construction of the Beauty and the Beast module

In this section, we will explain the construction in Chapter 2 in the language of vertex operator algebras (VOAs). VOAs have been reviewed in Appendix B. To be precise, we prove the following

Theorem 2.6.1. *Let V_{BB} be the Beauty and the beast module. Then V_{BB} has the structure of an $\mathcal{N} = 1$ superconformal vertex operator algebra with central charge $\hat{c} = 16$.*

See Section 2.6.1 for the precise definition of an $\mathcal{N} = 1$ superconformal vertex operator algebra.

2.6.1 Superconformal Vertex Operator Algebras

We start by defining the notion of an $\mathcal{N} = 1$ superconformal VOA, we refer the reader to Appendix B for the definition of vertex operator superalgebra and related notations.

Definition 2.6.1. An SVOA $(V, Y_V, \mathbf{1}, \omega)$ is called an $\mathcal{N} = 1$ *superconformal vertex operator algebra* with *central charge* denoted by $\hat{c}$ if there is a *superconformal vector* $\tau \in V_{(3/2)}$ with corresponding vertex operator

$$Y_V(\tau, x) = \sum_{r \in \frac{1}{2}\mathbb{Z}} \tau_r x^{-r-1} = \sum_{r \in \frac{1}{2}\mathbb{Z}} G_r x^{-r-\frac{3}{2}} \quad (2.6.1)$$

such that the operators L_n and G_r furnish a representation of the $\mathcal{N} = 1$ superconformal algebra:

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{\hat{c}}{8}m(m^2-1)\delta_{m,-n} \\ [G_r, L_n] &= \left(r - \frac{n}{2}\right)G_{r+n} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r,-s} \\ [L_m, \hat{c}] &= [G_r, \hat{c}] = 0. \end{aligned} \quad (2.6.2)$$

where $\{\cdot, \cdot\}$ is the anticommutator. We denote an $\mathcal{N} = 1$ SCVOA by the tuple $(V, Y_V, \mathbf{1}, \omega, \tau)$.

Remark 2.6.1. If the modes G_r of the vertex operator $Y_V(\tau, x)$ are zero for $r \in \mathbb{Z}$, the algebra (2.6.2) is called the $\mathcal{N} = 1$ *Neveu-Schwartz algebra*. If $G_r = 0$ for $r \in \mathbb{Z} + \frac{1}{2}$ then the algebra (2.6.2) is called the $\mathcal{N} = 1$ *Ramond algebra*.

The commutation relations for the superconformal algebra of central charge $\hat{c}$ is equivalent to the operator product expansion:

$$\begin{aligned} Y_V(\omega, x_1) Y_V(\omega, x_2) &\sim \frac{3\hat{c}/4}{(x_1 - x_2)^4} + \frac{2Y_V(\omega, x_2)}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_2)} \frac{d}{dx_2} Y_V(\omega, x_2) \\ Y_V(\omega, x_1) Y_V(\tau, x_2) &\sim \frac{3}{2} \frac{Y_V(\tau, x_2)}{(x_1 - x_2)^2} + \frac{1}{(x_1 - x_2)} \frac{d}{dx_2} Y_V(\tau, x_2) \\ Y_V(\tau, x_1) Y_V(\tau, x_2) &\sim \frac{\hat{c}/4}{(x_1 - x_2)^3} + \frac{2Y_V(\omega, x_2)}{(x_1 - x_2)}. \end{aligned} \quad (2.6.3)$$

This gives us another definition of an $\mathcal{N} = 1$ SCVOA which we record as

Proposition 2.6.1. *Suppose $(V, Y_V, \mathbf{1}, \omega)$ is an $\mathcal{N} = 1$ SVOA with central charge c . Then $\tau \in V_{(3/2)}$ is a superconformal element for the $\mathcal{N} = 1$ SCVOA V with central charge $\hat{c} = \frac{2}{3}c$ if*

$$\tau_2\tau = \frac{2c}{3}\mathbf{1}, \quad \tau_1\tau = 0, \quad \tau_0\tau = 2\omega. \quad (2.6.4)$$

2.6.2 The Monster Vertex Operator Algebra

A large class of VOAs can be constructed using even, positive definite lattice equipped with a symmetric nondegenerate bilinear form. The lattice used to construct the Monster VOA is the Leech

lattice Λ (see [FLM89, Chapter 7,8] for the construction). In this section we review the construction of the Monster VOA which will be useful in later sections. This complements the discussion in Section 2.3.

The untwisted vertex operator algebra

Let Λ be the Leech lattice. Define

$$\mathfrak{h} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \quad (2.6.5)$$

viewed as a finite dimensional $\mathbb{C}$ -vector space with a nonsingular, even, symmetric, nondegenerate form $\langle \cdot, \cdot \rangle$. Let $\{\alpha_i\}_{i=1}^{24} \subset \mathfrak{h}$ satisfying

$$\langle \alpha_i, \alpha_j \rangle = 2\delta_{ij}. \quad (2.6.6)$$

We can write thus write

$$\mathfrak{h} = \text{Span}_{\mathbb{C}}\{\alpha_1, \dots, \alpha_{24}\}. \quad (2.6.7)$$

An orthonormal basis for $\mathfrak{h}$ can then be constructed using α_i by defining

$$h_i = \frac{1}{\sqrt{2}}\alpha_i, \quad i = 1, \dots, 24. \quad (2.6.8)$$

Considering $\mathfrak{h}$ as an abelian Lie algebra, we construct the Heisenberg algebra of $\mathfrak{h}$ as:

$$\hat{\mathfrak{h}}_{\mathbb{Z}} := \bigoplus_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \mathfrak{h} \otimes t^n \oplus \mathbb{C}c \quad (2.6.9)$$

where c is a central element of the algebra and the Lie bracket is given by

$$[x \otimes t^n, y \otimes t^m] = \langle x, y \rangle m \delta_{m+n, 0} c. \quad (2.6.10)$$

Let

$$\hat{\mathfrak{h}}_{\mathbb{Z}}^- := \bigoplus_{\substack{n \in \mathbb{Z} \\ n < 0}} \mathfrak{h} \otimes t^n \quad (2.6.11)$$

and $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$ be the symmetric algebra of $\hat{\mathfrak{h}}_{\mathbb{Z}}^-$. We will consider the $\mathbb{Z}$ -grading on $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$ obtained by extending the $\mathbb{Z}$ -grading on $\hat{\mathfrak{h}}_{\mathbb{Z}}$ to the tensor product in the usual way. $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$ is an induced $\hat{\mathfrak{h}}_{\mathbb{Z}}$ -module (see [FLM89, Section 1.7]). Let $\hat{\Lambda}$ be the central extension of Λ by a cyclic group $\langle \kappa \rangle$, $\kappa^2 = 1, \kappa \neq 1$ of order 2:

$$0 \longrightarrow \langle \kappa \rangle \longrightarrow \hat{\Lambda} \xrightarrow{\pi} \Lambda \longrightarrow 0. \quad (2.6.12)$$

For any subgroup $L \subseteq \Lambda$ we define

$$\hat{L} := \{a \in \hat{\Lambda} : \bar{a} \in L\}. \quad (2.6.13)$$

Let $\mathbb{C}_\chi$ be the 1-dimensional space on which $\langle \kappa \rangle$ acts according to $\chi : \langle \kappa \rangle \longrightarrow \mathbb{C}, \chi(\kappa) = -1$. That is $\kappa \cdot 1 = -1$. Let us denote by $\mathbb{C}\{\Lambda\}$ the induced $\hat{\Lambda}$ -module of $\mathbb{C}_\chi$ by the subgroup $\langle \kappa \rangle \hookrightarrow \hat{\Lambda}$:

$$\mathbb{C}\{\Lambda\} := \mathbb{C}[\hat{\Lambda}] \otimes_{\mathbb{C}[\langle \kappa \rangle]} \mathbb{C}_\chi \quad (2.6.14)$$

where $\mathbb{C}[G]$ denotes the group algebra of G . The elements of $\mathbb{C}\{\Lambda\}$ are linear combinations of elements of the form $a \otimes 1$ where $a \in \hat{\Lambda}$. Using this, we put a $\mathbb{Z}$ -grading on $\mathbb{C}\{\Lambda\}$ by defining the *weight*²⁷

$$\text{wt}(a \otimes 1) = \frac{1}{2} \langle \bar{a}, \bar{a} \rangle, \quad a \in \hat{\Lambda} \quad (2.6.15)$$

Form the untwisted vector space

$$V_\Lambda := S(\hat{\mathfrak{h}}_\mathbb{Z}^-) \otimes \mathbb{C}\{\Lambda\}. \quad (2.6.16)$$

V_Λ inherits the $\mathbb{Z}$ -grading from that of $S(\hat{\mathfrak{h}}_\mathbb{Z}^-)$ and $\mathbb{C}\{\Lambda\}$. We consider $S(\hat{\mathfrak{h}}_\mathbb{Z}^-)$ as a trivial $\hat{\Lambda}$ -module so that V_Λ is a tensor product of $\hat{\Lambda}$ modules and hence again a $\hat{\Lambda}$ -module. Similarly we consider $\mathbb{C}\{\Lambda\}$ as a trivial $\hat{\mathfrak{h}}_\mathbb{Z}$ -module. Let us denote by $\alpha(n)$ the element $\alpha \otimes t^n \in \hat{\mathfrak{h}}_\mathbb{Z}$ for $\alpha \in \mathfrak{h}$ and for any $\alpha \in \mathfrak{h}$ define

$$\alpha(0) : \mathbb{C}\{\Lambda\} \longrightarrow \mathbb{C}\{\Lambda\}, \quad \alpha(0)(a \otimes 1) = \langle \alpha, \bar{a} \rangle (a \otimes 1). \quad (2.6.17)$$

Also for any $\alpha \in \Lambda$ and x a formal variable, define

$$x^\alpha \in (\text{End } \mathbb{C}\{\Lambda\})[x, x^{-1}], \quad x^\alpha(a \otimes 1) = x^{\langle \alpha, \bar{a} \rangle} (a \otimes 1). \quad (2.6.18)$$

For $\alpha \in \mathfrak{h}$, put

$$\alpha(x) = \sum_{n \in \mathbb{Z}} \alpha(n) x^{-n-1}. \quad (2.6.19)$$

V_Λ is generated by elements of the form

$$v = \alpha_1(-n_1) \cdot \alpha_2(-n_2) \cdots \alpha_k(-n_k) \cdot \iota(a) := \alpha_1(-n_1) \cdot \alpha_2(-n_2) \cdots \alpha_k(-n_k) \otimes \iota(a) \quad (2.6.20)$$

where $\iota(a) = 1 \otimes (a \otimes 1) \in V_\Lambda$ and $a \in \hat{\Lambda}$. For $a \in \hat{\Lambda}$ define the *untwisted vertex operator* associated to $\iota(a)$ as

$$\begin{aligned} Y_{V_\Lambda}(a, x) &:= Y_{V_\Lambda}(a, x) = \circ \exp \left(\int \bar{a}(x) \right) \circ \\ &= \circ \exp \left(- \sum_{n < 0} \frac{\bar{a}(n)}{n} x^{-n} \right) \exp \left(- \sum_{n > 0} \frac{\bar{a}(n)}{n} x^{-n} \right) a x^{\bar{a}} \circ \end{aligned} \quad (2.6.21)$$

where the normal ordering is defined as

$$\begin{aligned} \circ \alpha_1(m) \alpha_2(n) \circ &= \circ \alpha_2(n) \alpha_1(m) \circ = \begin{cases} \alpha_1(m) \alpha_2(n) & m \leq n \\ \alpha_2(n) \alpha_1(m) & m \geq n \end{cases} \\ \circ \alpha(m) a \circ &= \circ a \alpha(m) \circ = a \alpha(m), \\ \circ x^\alpha a \circ &= \circ a x^\alpha \circ = a x^\alpha \end{aligned} \quad (2.6.22)$$

²⁷The notation $\bar{a}$ denotes the image of the projection map $^- : \hat{\Lambda} \longrightarrow \Lambda$.

for $\alpha_1, \alpha_2, \alpha \in \mathfrak{h}, m, m \in \mathbb{Z}, \alpha \in \Lambda, a \in \hat{\Lambda}$. To extend this to all of V_Λ we define the vertex operators for the generating set

$$Y_{V_\Lambda}(v, x) = \prod_{i=1}^k \circ \left(\frac{1}{(n_i - 1)!} \frac{d^{n_i-1} \alpha_i(x)}{dx^{n_i-1}} \right) Y_{V_\Lambda}(a, x) \circ \quad (2.6.23)$$

where v is as in (2.6.20) and extend it to all of V_Λ by linearity. This gives us the map

$$Y_{V_\Lambda} : V_\Lambda \longrightarrow (\text{End } V_\Lambda)[[x, x^{-1}]], \quad v \mapsto Y_{V_\Lambda}(v, x). \quad (2.6.24)$$

To construct a conformal element pick an orthonormal basis $\{h_1, h_2, \dots, h_{24}\}$ of $\mathfrak{h}$ and define

$$\omega := \frac{1}{2} \sum_{i=1}^{24} h_i (-1)^2. \quad (2.6.25)$$

Then we have the following theorem

Theorem 2.6.2. [FLM89, FLM85] *The quadruple $(V_\Lambda, Y_{V_\Lambda}, \iota(1), \omega)$ is a vertex operator algebra with central charge 24.*

The twisted module

Let us now construct the V_Λ -module V_Λ^T . We consider the $\mathbb{Z} + \frac{1}{2}$ -graded twisted Heisenberg algebra

$$\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}} := \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathfrak{h} \otimes t^r \oplus \mathbb{C}c \quad (2.6.26)$$

and its subalgebra

$$\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}^- := \bigoplus_{r < 0} \mathfrak{h} \otimes t^r. \quad (2.6.27)$$

The symmetric algebra $S(\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}^-)$ is $\frac{1}{2}\mathbb{Z}$ -graded and as before an irreducible $\hat{\mathfrak{h}}_{\mathbb{Z} + \frac{1}{2}}$ -module. To construct the twisted module, we need the $\hat{\Lambda}$ -module T which we describe below. Set

$$K := \{a^2 \kappa^{\langle \bar{a}, \bar{a} \rangle / 2} : a \in \hat{\Lambda}\}. \quad (2.6.28)$$

Then K is a central subgroup of $\hat{\Lambda}$ and

$$\bar{K} = 2\Lambda. \quad (2.6.29)$$

Moreover $\hat{\Lambda}/K$ is a finite subgroup and is a central extension of $\Lambda/2\Lambda$:

$$1 \longrightarrow \langle \kappa \rangle \longrightarrow \hat{\Lambda}/K \longrightarrow \Lambda/2\Lambda \longrightarrow 1. \quad (2.6.30)$$

One can show that $\hat{\Lambda}/K$ is an extraspecial 2-group with

$$|\hat{\Lambda}/K| = 2^{25}. \quad (2.6.31)$$

The space T is then characterized as follows:

Proposition 2.6.2. [FLM89, Proposition 10.3.1] *The extraspecial group $\hat{\Lambda}/K$ has a unique (up to equivalence) irreducible module T such that*

$$\kappa K \mapsto -1 \text{ on } T. \quad (2.6.32)$$

Moreover, the corresponding representation of $\hat{\Lambda}/K$ is the unique faithful irreducible representation and

$$\dim T = 2^{12}. \quad (2.6.33)$$

To construct T , let Φ be any subgroup of Λ such that

$$\begin{aligned} 2\Lambda &\subset \Phi \subset \Lambda, \\ |\Phi/2\Lambda| &= 2^{12}, \\ \frac{1}{2}\langle \alpha, \alpha \rangle &\in 2\mathbb{Z}, \text{ for any } \alpha \in \Phi. \end{aligned} \quad (2.6.34)$$

Then $\hat{\Phi}$ is a maximal abelian subgroup of $\hat{\Lambda}$ and $\hat{\Phi}/K$ (cf. (2.6.13)) is an elementary abelian 2-group. Let

$$\psi : \hat{\Phi}/K \longrightarrow \mathbb{C}^\times \quad (2.6.35)$$

be any homomorphism such that

$$\psi(\kappa K) = -1 \quad (2.6.36)$$

and denote by $\mathbb{C}_\psi$ the one-dimensional $\hat{\Phi}$ -module with the corresponding character. Then viewed as a $\hat{\Lambda}$ -module,

$$\begin{aligned} T &= \mathbb{C}[\hat{\Lambda}] \otimes_{\mathbb{C}[\Phi]} \mathbb{C}_\psi \\ &\simeq \mathbb{C}[\Lambda/\Phi] \quad (\text{linearly}). \end{aligned} \quad (2.6.37)$$

From (2.6.37) we see that T is generated by elements of the form $t(a) := a \otimes 1$ where $a \in \hat{\Lambda}$. We then define the module V_Λ^T as

$$V_\Lambda^T := S(\hat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}}^-) \otimes T. \quad (2.6.38)$$

We give a gradation to T by defining the weight

$$\text{wt}(t) = \frac{3}{2}, \quad t \in T. \quad (2.6.39)$$

Then V_Λ^T has a $\frac{1}{2}\mathbb{Z}$ -grading induced from the tensor product gradation. Let us again denote $\alpha \otimes t^r$ by $\alpha(r)$ for $\alpha \in \mathfrak{h}$ and $r \in \mathbb{Z} + \frac{1}{2}$ and also define

$$\alpha(x) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} \alpha(r) x^{-r-1}, \quad \alpha \in \mathfrak{h}. \quad (2.6.40)$$

Let us now define the twisted vertex operators. For any $a \in \hat{\Lambda}$, define

$$\begin{aligned} Y_{V_\Lambda^T}(a, x) &:= 2^{-\langle \bar{a}, \bar{a} \rangle} \circ \exp \left(\int \bar{a}(x) \right) \circ ax^{\langle \bar{a}, \bar{a} \rangle / 2} \\ &= 2^{-\langle \bar{a}, \bar{a} \rangle} \circ \exp \left(\sum_{n \geq 0} \frac{\bar{a}(n + \frac{1}{2})}{n + \frac{1}{2}} x^{-n + \frac{1}{2}} \right) \cdot \exp \left(\sum_{n < 0} \frac{\bar{a}(n + \frac{1}{2})}{n + \frac{1}{2}} x^{-n + \frac{1}{2}} \right) \circ ax^{\langle \bar{a}, \bar{a} \rangle / 2} \end{aligned} \quad (2.6.41)$$

where the normal ordering is defined as

$$\circ \alpha_1(r) \alpha_2(s) \circ = \circ \alpha_2(s) \alpha_1(r) \circ = \begin{cases} \alpha_1(r) \alpha_2(s) & r \leq s \\ \alpha_2(s) \alpha_1(r) & r \geq s \end{cases} \quad (2.6.42)$$

for $\alpha_1, \alpha_2 \in \mathfrak{h}$, $r, s \in \mathbb{Z} + \frac{1}{2}$. For a general $v \in V_\Lambda$ of the form (2.6.20) we define

$$Y_{V_\Lambda^T}(v, x) = Y_0 \left(\exp \left(\sum_{m, n \geq 0} \sum_{i=1}^{24} c_{mn} h_i(m) h_i(n) x^{-m-n} \right) v, x \right) \quad (2.6.43)$$

where

$$Y_0(v, x) = \prod_{i=1}^k \circ \left(\frac{1}{(n_i - 1)!} \frac{d^{n_i-1} \alpha_i(x)}{dx^{n_i-1}} \right) Y_{V_\Lambda^T}(a, x) \circ, \quad (2.6.44)$$

$\{h_1, \dots, h_{24}\}$ is an orthonormal basis of $\mathfrak{h}$ and c_{mn} are complex numbers determined by the relation

$$\sum_{n, m \geq 0} c_{mn} x^m y^n = -\log \left(\frac{(1+x)^{1/2} + (1+y)^{1/2}}{2} \right). \quad (2.6.45)$$

This defines the vertex operator map

$$Y_{V_\Lambda^T}(\cdot, x) : V_\Lambda^T \longrightarrow (\text{End } V_\Lambda^T)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]. \quad (2.6.46)$$

The Monster module

To define the Monster module, we define an involution θ on V_Λ and V_Λ^T . Let $v \in V_\Lambda$ be as in (2.6.20) and $w \in V_\Lambda^T$ be of the form

$$w = \alpha_1(-r_1) \cdots \alpha_k(-r_k) \cdot t := \alpha_1(-r_1) \cdots \alpha_k(-r_k) \otimes t, \quad t \in T, \quad r_1, \dots, r_k \in \mathbb{Z} + \frac{1}{2}. \quad (2.6.47)$$

Define

$$\begin{aligned} \theta(v) &= (-1)^k \alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot \iota(a^{-1}) \\ \theta(w) &= (-1)^{k+1} \alpha_1(-r_1) \cdots \alpha_k(-r_k) \cdot t \end{aligned} \quad (2.6.48)$$

and extend it as a map $\theta : V_\Lambda \longrightarrow V_\Lambda$ and $\theta : V_\Lambda^T \longrightarrow V_\Lambda^T$ by linearity. Clearly $\theta^2 = 1$. Let $V_\Lambda^\pm$ and $(V_\Lambda^T)^\pm$ be the ± 1 eigenspaces of θ . We then define the Moonshine module $V^\natural$ as

$$V^\natural := V_\Lambda^+ \oplus (V_\Lambda^T)^+. \quad (2.6.49)$$

$V^{\natural}$ has a $\mathbb{Z}$ -grading given by the weight gradation on V_{Λ} and V_{Λ}^T :

$$V^{\natural} = \bigoplus_{\substack{n \in \mathbb{Z} \\ n \geq 0}} V_n^{\natural}. \quad (2.6.50)$$

In particular

$$V_0^{\natural} = \mathbb{C}\iota(1), \quad V_1^{\natural} = 0, \quad V_2^{\natural} = \mathfrak{t} \oplus \mathfrak{p} \quad (2.6.51)$$

where

$$\mathfrak{t} := S^2(\mathfrak{h}) \oplus \bigoplus_{a \in \hat{\Lambda}_4} \mathbb{C}x_a^+, \quad x_a^+ := \iota(a) + \iota(a^{-1}), \quad (2.6.52)$$

and

$$\mathfrak{p} := \mathfrak{h} \otimes T. \quad (2.6.53)$$

To match this with (1.2.5), we define the *degree* of a vector by

$$\deg(v) = -\text{wt}(v) + \frac{1}{24} \dim \mathfrak{h} = 1 - \text{wt}(v). \quad (2.6.54)$$

We now want to construct the vertex operator map

$$V^{\natural} \longrightarrow (\text{End} V^{\natural})[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]. \quad (2.6.55)$$

For $v \in V_{\Lambda}^+$ we define

$$Y_{V^{\natural}}(v, x) = \begin{pmatrix} Y_{V_{\Lambda}}(v, x) & 0 \\ 0 & Y_{V_{\Lambda}^T}(v, x) \end{pmatrix}. \quad (2.6.56)$$

To define vertex operators for $v \in (V_{\Lambda}^T)^+$ we first introduce a bilinear form on V_{Λ}^T . On T there is a unique positive definite hermitian bilinear form $(\cdot, \cdot)_T$ such that

$$(t(a), t(b))_T = \begin{cases} 0, & a\hat{\Phi} \neq b\hat{\Phi} \\ 1, & a = b. \end{cases} \quad (2.6.57)$$

This can be extended to a unique bilinear form $\langle \cdot, \cdot \rangle_{V_{\Lambda}^T}$ on V_{Λ}^T as follows:

$$\begin{aligned} \langle 1 \otimes t(a), 1 \otimes t(b) \rangle_{V_{\Lambda}^T} &= (t(a), t(b))_T \\ \langle \alpha(n) \cdot u, v \rangle_{V_{\Lambda}^T} &= \langle u, \alpha(-n) \cdot v \rangle_{V_{\Lambda}^T} \end{aligned} \quad (2.6.58)$$

where $\alpha(n) \cdot u$ is the module multiplication in the $\hat{\mathfrak{h}}_{\mathbb{Z}+\frac{1}{2}}$ -module V_{Λ}^T . Now for $v \in (V_{\Lambda}^T)^+$ we define

$$Y_{V^{\natural}}(v, x) = \begin{pmatrix} 0 & \bar{Z}(v, x) \\ Z(v, x) & 0 \end{pmatrix} \quad (2.6.59)$$

where $Z(v, x) : V_\Lambda^+ \longrightarrow (V_\Lambda^T)^+[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]$ is defined as

$$Z(v, x)u = e^{xL(-1)}Y_{V_\Lambda^T}(u, -x)v, \quad u \in V_\Lambda^+ \quad (2.6.60)$$

and $\bar{Z}(v, x) : (V_\Lambda^T)^+ \longrightarrow (V_\Lambda^+)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]$ is defined by

$$\langle w, \bar{Z}(v, x)u \rangle_{V_\Lambda} = \left\langle Y_{V_\Lambda^T}(w, -x^{-1})e^{xL(1)}(-x^2)^{-L(0)}v, e^{x^{-1}L(1)}u \right\rangle_{V_\Lambda^T}, \quad \text{for all } w \in V_\Lambda. \quad (2.6.61)$$

Theorem 2.6.3. [FLM89] *The quadruple $(V^\natural, Y_{V^\natural}, \iota(1), \omega)$ is a vertex operator algebra of central charge 24 with automorphism group the Monster $\mathbb{M}$.*

2.6.3 $\mathcal{N} = 1$ Neveu-Schwartz algebra in the Beauty and the Beast module

In this section, we explicitly construct a superconformal vector in the meromorphic extension of the Monster module called the Beauty and the Beast module in [MS23]. We will show that the superconformal vector is contained in the image of a rank 1 projection operator acting on the module T .

The Beauty and the Beast SVOA

In this section, we recall the SVOA structure of the Beauty and the Beast module. We consider the extended space

$$V_{BB} := V_\Lambda \oplus V_\Lambda^T. \quad (2.6.62)$$

Clearly V_{BB} is $\frac{1}{2}\mathbb{Z}$ -graded. Let us write

$$V_{BB} = \bigoplus_{\substack{n \in \frac{1}{2}\mathbb{Z} \\ n \geq 0}} (V_{BB})_n = \left(\bigoplus_{\substack{n \in \mathbb{Z} \\ n \geq 0}} (V_{BB})_n \right) \oplus \left(\bigoplus_{\substack{n \in \mathbb{Z} + \frac{1}{2} \\ n > 0}} (V_{BB})_n \right) \quad (2.6.63)$$

and define

$$(V_{BB})_{\bar{0}} := \left(\bigoplus_{\substack{n \in \mathbb{Z} \\ n \geq 0}} (V_{BB})_n \right), \quad (V_{BB})_{\bar{1}} := \left(\bigoplus_{\substack{n \in \mathbb{Z} + \frac{1}{2} \\ n > 0}} (V_{BB})_n \right). \quad (2.6.64)$$

This defines a $\mathbb{Z}_2$ -grading so that V_{BB} has a super vector space structure. Let us now define the vertex operator map

$$Y_{V_{BB}} : V_{BB} \otimes V_{BB} \longrightarrow V_{BB}[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]].$$

For $v \in V_\Lambda$ we define

$$Y_{V_{BB}}(v, x) = \begin{pmatrix} Y_{V_\Lambda}(v, x) & 0 \\ 0 & Y_{V_\Lambda^T}(v, x) \end{pmatrix}. \quad (2.6.65)$$

For $v \in V_\Lambda^T$ we define

$$Y_{V_{BB}}(v, x) = \begin{pmatrix} 0 & \bar{Z}(v, x) \\ Z(v, x) & 0 \end{pmatrix} \quad (2.6.66)$$

where $Z(v, x) : V_\Lambda \longrightarrow (V_\Lambda^T)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]$ is defined as

$$Z(v, x)u = e^{xL(-1)}Y_{V_\Lambda^T}(u, e^{\pi i}x)v, \quad u \in V_\Lambda \quad (2.6.67)$$

and $\bar{Z}(v, x) : V_\Lambda^T \longrightarrow (V_\Lambda)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]$ is defined by

$$\langle w, \bar{Z}(v, x)u \rangle_{V_\Lambda} = \left\langle Y_{V_\Lambda^T}(w, e^{\pi i}x^{-1})e^{xL(1)}(e^{\pi i}x^2)^{-L(0)}v, e^{x^{-1}L(1)}u \right\rangle_{V_\Lambda^T}, \quad \text{for all } w \in V_\Lambda \quad (2.6.68)$$

and then extend to all of V_{BB} by linearity. We thus have the following theorem.

Theorem 2.6.4. *The quadruple $(V_{BB}, Y_{V_{BB}}, \iota(1), \omega)$ is a super vertex operator algebra of central charge 24.*

Sketch of Proof: The statement follows from general theory of logarithmic tensor category [HLZ10a, HLZ10b, HLZ10c, HLZ10d, HLZ10e, HLZ10f, HLZ11a, HLZ11b]. To be specific, one can use [CKL19, Theorem 3.9] with the odd module $(V_{BB})_{\bar{1}}$ as a simple current for $(V_{BB})_{\bar{0}}$. Note that the fact that $(V_{BB})_{\bar{1}}$ is a simple current for $(V_{BB})_{\bar{0}}$ follows from the results in [Don94]. See also Remark 2.5.1. Alternatively, one can also prove it in the formalism of [DGM90, DGM94] as done in [MS23] by taking the formal variable to be a non zero complex number and using analytic continuation. The only nontrivial part of the proof is the Jacobi identity for the vertex operators $Y_{V_{BB}}(u, x_1)$ and $Y_{V_{BB}}(v, x_2)$ with $u, v \in (V_{BB})_{\bar{1}}$. For other cases, the Jacobi identity follows from similar calculations as in the proof of the same for the Monster VOA [FLM89, DGM90, DGM94]. This has been reviewed in Appendix A.2. $\square$

2.6.4 Superconformal vector in the Beauty and the Beast module

The superconformal vector τ is a weight $\frac{3}{2}$ vector, this means that $\tau \in T \hookrightarrow V_{BB}$. To construct τ we must show that $Y_{V_{BB}}(\tau, x)$ satisfies the OPE in (2.6.3) or equivalently show that τ satisfies Proposition 2.6.1 with $\hat{c} = 16$.

Lemma 2.6.1. *Suppose we have*

$$Y_{V_{BB}}(v, x) = \sum_{n \in \frac{1}{2}\mathbb{Z}} v_n x^{-n-1}, \quad v_n \in \text{End}(V_{BB}). \quad (2.6.69)$$

Then for $v \in V_\Lambda^+$ we have $v_n = 0$ when $n \in \mathbb{Z} + \frac{1}{2}$ and

$$v_n = \begin{pmatrix} \tilde{v}_n & 0 \\ 0 & \check{v}_n \end{pmatrix}, \quad n \in \mathbb{Z}, \tilde{v}_n \in \text{End}(V_\Lambda), \check{v}_n \in \text{End}(V_\Lambda^T). \quad (2.6.70)$$

For $v \in V_\Lambda^-$

$$v_n = \begin{pmatrix} \tilde{v}_n & 0 \\ 0 & 0 \end{pmatrix}, n \in \mathbb{Z}, \quad v_n = \begin{pmatrix} 0 & 0 \\ 0 & \check{v}_n \end{pmatrix}, n \in \mathbb{Z} + \frac{1}{2}. \quad (2.6.71)$$

Proof. We expand the vertex operator as

$$Y_{V_{BB}}(v, x) = \sum_{n \in \frac{1}{2}\mathbb{Z}} v_n x^{-n-1} = \begin{pmatrix} \sum_{n \in \mathbb{Z}} \tilde{v}_n x^{-n-1} & 0 \\ 0 & \sum_{n \in \frac{1}{2}\mathbb{Z}} \check{v}_n x^{-n-1} \end{pmatrix}, \quad v \in V_\Lambda, \tilde{v}_n \in \text{End}(V_\Lambda), \check{v}_n \in \text{End}(V_\Lambda^T). \quad (2.6.72)$$

We want to see when the sum in the formal series for $Y_{V_\Lambda^T}(v, x)$ runs over $\mathbb{Z}$ and $\mathbb{Z} + \frac{1}{2}$. Using the action of θ on V_Λ we see that $V_\Lambda^\pm$ is generated by elements of the form

$$\alpha_1(-n_1) \cdots \alpha_k(-n_k) \cdot (\iota(a) \pm \iota(a^{-1})) \quad (2.6.73)$$

with $(k, \pm) \equiv (\text{odd}, -), (\text{even}, +)$ for V_Λ^+ and $(k, \pm) \equiv (\text{odd}, +), (\text{even}, -)$ for V_Λ^- . Using the definition of the vertex operator $Y_{V_\Lambda^T}(v, x)$ in (2.6.40)–(2.6.45) the result follows. $\square$

Lemma 2.6.2. *Let $t \in T$ and write the corresponding vertex operator as*

$$Y_{BB}(t, x) = \sum_{n \in \frac{1}{2}\mathbb{Z}} t_n x^{-n-1}, \quad t_n \in \text{End}(V_{BB}). \quad (2.6.74)$$

Then we have

$$\text{wt}(t_2 t) = 0, \quad \text{wt}(t_1 t) = 1, \quad \text{wt}(t_0 t) = 2. \quad (2.6.75)$$

Proof. Using (B.1.30) and (B.1.29) for $v = t$ we get

$$[L_0, t_n] = \left(\frac{3}{2} - n - 1 \right) t_n \quad (2.6.76)$$

and the result follows. $\square$

Proposition 2.6.3. *For any $t \in T$, we have that*

$$t_2 t = \langle 1 \otimes t, 1 \otimes t \rangle \iota(t), \quad t_1 t = 0 \quad (2.6.77)$$

and

$$t_0 t = \frac{1}{8} \langle 1 \otimes t, 1 \otimes t \rangle \omega + 16 \sum_{a \in \hat{\Lambda}_4} \langle 1 \otimes t, 1 \otimes [a] \cdot t \rangle (\iota(a) + \iota(a^{-1})) \quad (2.6.78)$$

where $[a] \cdot t$ denotes the action of the image $[a]$ of a under the map $\hat{\Lambda} \rightarrow \hat{\Lambda}/K$ (see Proposition 2.6.2) on T .

Proof. By Lemma 2.6.2 and (2.6.51) we can write

$$t_2 t = \kappa(t) \iota(1), \quad t_1 t = \sum_{i=1}^{24} \kappa_i(t) \alpha_i(-1) \cdot \iota(1), \quad \kappa(i), \kappa_i(t) \in \mathbb{C}. \quad (2.6.79)$$

Next since $t \in T \subset V_\Lambda^T$, we have²⁸

$$Y_{V_{BB}}(t, x) t = \bar{Z}(t, x) t \in ((V_\Lambda)_2)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]]. \quad (2.6.80)$$

Using (2.6.51) we can write

$$\begin{aligned} t_0 t &= \sum_{i=1}^{24} \delta_i(t) \alpha_i(-1) \alpha_i(-1) \cdot \iota(1) + \sum_{\substack{i,j=1 \\ i \neq j}}^{24} \kappa_{ij}(t) \alpha_i(-1) \alpha_j(-1) \cdot \iota(1) \\ &+ \sum_{i=1}^{24} \kappa'_i(t) \alpha_i(-2) \cdot \iota(1) + \sum_{a \in \hat{\Lambda}_4} \kappa_a(t) (\iota(a) + \iota(a^{-1})), \quad \delta_i(t), \kappa_{ij}(t), \kappa'_i(t), \kappa_a(t) \in \mathbb{C}. \end{aligned} \quad (2.6.81)$$

Using the relation

$$\langle w, \bar{Z}(t, x) t \rangle_{V_\Lambda} = \left\langle Y_{V_\Lambda^T}(w, e^{\pi i} x^{-1}) e^{xL(1)} (e^{\pi i} x^2)^{-L(0)} t, e^{x^{-1}L(1)} t \right\rangle_{V_\Lambda^T}, \quad w \in V_\Lambda. \quad (2.6.82)$$

from (2.6.68) and performing similar calculations as in Chapter 2 we get the required result. $\square$

We now want to show that there exists a $t \in T$ such that for every $a \in \hat{\Lambda}_4$

$$\langle 1 \otimes t, 1 \otimes [a] \cdot t \rangle = 0. \quad (2.6.83)$$

We will explicitly construct such an element. Let $\{t^i\}_{i=1}^{2^{12}}$ be an orthonormal basis for T , that is

$$\langle 1 \otimes t^i, 1 \otimes t^j \rangle = \delta^{ij}, \quad i, j = 1, \dots, 2^{12}. \quad (2.6.84)$$

Let γ_a be the matrix representing $[a] \in \hat{\Lambda}/K$. Then it easy to see that

$$\gamma_a^2 = (-1)^{\frac{\langle a, a \rangle}{2}}. \quad (2.6.85)$$

Let us choose a section $e : \Lambda \longrightarrow \hat{\Lambda}$. Then since the commutator map for the central extension $\hat{\Lambda}$ is

$$c_0(\alpha, \beta) = \langle \alpha, \beta \rangle + 2\mathbb{Z}, \quad \alpha, \beta \in \Lambda \quad (2.6.86)$$

we see that (see [FLM89, Chapter 5] for details)

$$\gamma_a \gamma_b = (-1)^{\langle \bar{a}, \bar{b} \rangle} \gamma_b \gamma_a = \varepsilon_0(\bar{a}, \bar{b}) \gamma_{e(\bar{a} + \bar{b})}. \quad (2.6.87)$$

where ε_0 is the cocycle defining the central extension. It is known that the matrix representation of this algebra is unique upto unitary equivalence. It is constructed recursively using the standard Dirac gamma matrices (see [DGM90, Appendix C]). This means that

$$\text{Tr} \gamma_a = 0 \quad \text{for } a \neq 1. \quad (2.6.88)$$

²⁸We have abused notation a little to write a the vector $(0, t) \in V_{BB}$ simply as t .

Let us define a *superconformal sublattice* $\Lambda_{SC} \subset \Lambda$ to be a sublattice of the Leech lattice with the following properties:

$$\begin{aligned} (i) \quad & \langle \alpha, \alpha \rangle \in 4\mathbb{N}, \quad \alpha \in \Lambda_{SC} \setminus \{0\} \\ (ii) \quad & \langle \alpha, \beta \rangle \in 2\mathbb{Z}, \quad \alpha, \beta \in \Lambda_{SC} \\ (iii) \quad & \langle \alpha, \alpha \rangle \neq 4 \quad \forall \alpha \in \Lambda_{SC}. \end{aligned} \tag{2.6.89}$$

The proof of the following lemma is immediate from [FMS06] and the assumptions in (2.6.89).

Lemma 2.6.3. *The cocycle ε_0 is trivialisable over Λ_{SC} by a coboundary $b : \Lambda_{SC} \rightarrow U(1)$. That is*

$$\varepsilon_0(\alpha, \beta) = \frac{b(\alpha + \beta)}{b(\alpha)b(\beta)}. \tag{2.6.90}$$

Define the operator

$$P_{12} := \frac{1}{2^{12}} \sum_{a \in \hat{\Lambda}_{SC}} \tilde{\gamma}_a, \tag{2.6.91}$$

where $\tilde{\gamma}_a = b(\bar{a})\gamma_a$. It is easy to check that P_{12} is a rank 1 projection operator, that is

$$P_{12}^2 = P_{12}, \quad \text{Tr} P_{12} = 1. \tag{2.6.92}$$

This means that $\text{Im} P_{12}$ is one dimensional.

Theorem 2.6.5. *Let $\tau' \in \text{Im} P_{12}$ be the unique vector such that $\langle 1 \otimes \tau', 1 \otimes \tau' \rangle = 1$. Then*

$$1 \otimes \tau := 2(1 \otimes \tau') \tag{2.6.93}$$

is a superconformal vector with central charge $\hat{c} = 16$.

Proof. It is clear from Proposition 2.6.3 that

$$\tau_2 \tau = \langle 1 \otimes \tau, 1 \otimes \tau \rangle = 4. \tag{2.6.94}$$

It remains to show that

$$\langle 1 \otimes \tau, 1 \otimes [a] \cdot \tau \rangle = 0, \quad \forall a \in \hat{\Lambda}_4. \tag{2.6.95}$$

Let us write

$$\tau' = \sum_{i=1}^{2^{12}} s_i t^i, \quad s_i \in \mathbb{C} \tag{2.6.96}$$

where

$$\sum_{i=1}^{2^{12}} |s_i|^2 = 1. \tag{2.6.97}$$

The matrix for the projection operator can be written as $P_{12} = ss^\dagger$ where $s = (s_1, \dots, s_{2^{12}})^{tr}$ and $\dagger$ denotes conjugate-transpose. Then we see that

$$\begin{aligned}
\langle 1 \otimes \tau, 1 \otimes [a] \cdot \tau \rangle &= \sum_{i=1}^{2^{12}} s_i^* (\gamma_a)_{ij} s_j \\
&= \text{Tr} \gamma_a P_{12} \\
&= \frac{1}{2^{12}} \text{Tr} \sum_{a' \in \hat{\Lambda}_{SC}} \gamma_a \tilde{\gamma}_{a'} \\
&= \frac{1}{2^{12}} \text{Tr} \sum_{b \in \hat{\Lambda}_{SC}} \varepsilon_0(\bar{a} + \bar{a}') b(\bar{a}') \gamma_{e(\bar{a} + \bar{a})} \\
&= 0
\end{aligned} \tag{2.6.98}$$

where we used the fact that since $a \notin \hat{\Lambda}_{SC}$, $\bar{a} + \bar{a}' \neq 0$ for any $a' \in \hat{\Lambda}_{SC}$ and hence each of the 2^{12} terms in the above sum is 0 because of the fact that $\text{Tr} \gamma_a = 0$ for each $a \in \hat{\Lambda}/K$. $\square$

Thus the problem of constructing a superconformal vector is equivalent to constructing superconformal sublattices of the Leech lattice. For this we use the important theorem of Dong-Li-Mason-Norton:

Theorem 2.6.6. [DLMN96, Theorem 4.1] *Let L be any Niemeier lattice and let $\sqrt{2}L$ be the same lattice with the bilinear form scaled by a 2. Then there is at least one (and in general many) isometric embedding $\sqrt{2}L \hookrightarrow \Lambda$.*

If we take $L = \Lambda$ in this theorem, then we get an isometric embedding of $\sqrt{2}\Lambda$ into Λ and this embedding satisfies all the properties of a superconformal sublattice. This finishes our proof of the existence of a superconformal vector in V_{BB} .

From the above construction, it is clear that there can be more than one superconformal vectors in the Beauty and the Beast module. One just needs to construct a superconformal sublattice, but all such sublattices do not give rise to inequivalent superconformal vectors. It is proved in [MS23] that there can be at most 9 inequivalent such superconformal vectors using the classification of binary linear codes. Explicit construction of Λ_{SC} using certain $\mathbb{Z}_4$ -codes corresponding to inequivalent superconformal vectors is shown in Section 2.5.3. It turns out that one of the superconformal vector is same as that in the Conway supermoonshine module of Duncan [DMC14].

Chapter 3

Topological degeneracy in gauged WZW models

The contents of this chapter is based on an upcoming paper written jointly with E. Rabinovici and G.W. Moore. In this chapter, we discuss the relation between gauged Wess-Zumino-Witten (WZW) models and Goddard-Kent-Olive (GKO) algebraic coset construction. We give a precise categorical construction of GKO coset theory. We present evidence that the spectrum of gauged WZW model differs from that of GKO coset theory via coupling with a nontrivial 2d topological quantum field theory (TQFT).

3.1 Introduction

WZW models are an important class of Lagrangian conformal field theory from which several rational conformal field theories can be derived [Wit84, GW86, KZ84]. Algebraically, given any semisimple Lie algebra, one can construct a vertex operator algebra associated to it, called the affine VOA. Then the modules of this affine VOA at integer level can be organized into a modular invariant CFT [DFMS97, FZ92, LL12]. WZW models give a Lagrangian description for the diagonal modular invariant algebraic CFT¹.

Goddard-Kent-Olive described a procedure to construct a new VOA from any affine VOA based on a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h} \subset \mathfrak{g}$. This new VOA was called the algebraic GKO coset VOA. One can study the modules of this VOA and organize them into a Hilbert space on S^1 with a modular invariant torus partition function from this the coset VOA. We will only consider the diagonal modular invariant theory. Let us denote the algebraic GKO coset theory by $\text{GKO}(\mathfrak{g}/\mathfrak{h})$. On the other hand, given a WZW model based on a semisimple Lie group G , it has a global $(G \times G)$ -symmetry arising from the left-action of $G \times G$ on G where (g_1, g_2) maps $g \mapsto g_1 g g_2^{-1}$. For any subgroup $H < G$, we can choose the diagonally embedded $H < (G \times G)$. Gauging this subgroup H , we obtain a gauged

¹The WZW field $g(z, \bar{z})$ is in the representation $V \otimes V^\vee$ if we try to separate into left- and right-moving degrees of freedom. Here V is a “fundamental” representation of G (considered as a matrix Lie group). If ρ_λ is a representation of G we can make fields $\rho_\lambda(g(z, \bar{z}))$ and some of these will be primary fields. But again these fields are in the representation $V_\lambda \otimes V_\lambda^\vee$. So for the primaries we always have the “same” representation on the left and right leading to diagonal modular invariant.

WZW model often called the G/H -WZW model.² We will denote it by $\text{WZW}(G/H)$. As discussed in Chapter 1, [BRS88] gave some evidence that gauged G/H -WZW models provide a Lagrangian description for the GKO coset CFT based on the embedding $\text{Lie}(H) =: \mathfrak{h} \subset \mathfrak{g} := \text{Lie}(G)$. Indeed, it was shown in [BRS88, ABR88] that the G/H -WZW model path integral correctly reproduces the coset central charge and the various modular invariants of the torus partition function of the $c < 1$ unitary minimal models. Finally, in [GK88, GK89], the path integral of gauged WZW model was used to compute the torus partition function and some correlation functions of the coset CFT.³

It is known from [Spi92, Spi90, SY92, Wit92a, Wit93] that the G/G -WZW model is a 2d topological quantum field theory (TQFT)⁴. The Frobenius algebra⁵ describing this 2d TQFT is the Verlinde algebra⁶ $V_k(G)$ of the G_k -WZW model. Schematically, we write

$$\text{WZW}(G_k/G_k) \cong V_k(G) . \quad (3.1.1)$$

On the other hand, the algebraic GKO coset construction for this case gives a trivial $c = 0$ theory. Thus, the gauged WZW and GKO models have long been known to be different. In this chapter, we address the more general question of the relation of the gauged G/H -WZW model to the related GKO coset construction. Under certain assumptions on the GKO theory, we argue that they are related via a 2d topological quantum field theory described by the algebra of endomorphisms of an algebra object in a modular tensor category:

$$G/H\text{-WZW} \cong \text{GKO}(\mathfrak{g}/\mathfrak{h}) \otimes \text{End}(B) , \quad (3.1.2)$$

²The condition for a subgroup of $(G \times G)$ to be nonanomalous is presented in (3.4.30). In particular, the diagonally embedded subgroup H is non-anomalous and hence can be gauged. Anomalous 2d gauge theories have been studied in [HRSC86], and an example of anomalous gauged WZW model has been studied in [RS11]. Interestingly, anomalous gauged WZW models have been used in [Wit92a, Wit93] to prove holomorphic factorization of WZW models and to relate the conformal blocks of gauged WZW models to nonabelian theta functions.

³In [GK88, GK89], various constant factors were ignored or absorbed in the path integral measure. Thus the path integral of gauged WZW model on torus agrees with the torus partition function of the coset CFT only up to a multiplicative constant factor.

⁴The first evidence of G/G -WZW model being a TQFT was given in [Spi90, Spi92, SY92] where it was shown that the correlation functions of G/G -WZW models are the fusion rules of G -WZW model. It was then explicitly proved from path integrals in [Wit92a].

⁵The folklore that specifying a 2d TQFT is equivalent to specifying a Frobenius algebra goes back to ideas of Daniel Friedan, Graeme Segal, and Robbert Dijkgraaf. It was stated by Dijkgraaf in his PhD thesis [Dij89] and made mathematically precise in [Dub94, Qui91, Abr96], see also the book [Koc04] for a nice exposition of this result. A full mathematical proof of the assertion was first given by Moore and Segal using Morse theory and appeared in [MS06] and [Koc04].

⁶The Verlinde algebra of an RCFT is the formal $\mathbb{C}$ -span of the irreducible modules of the chiral algebra with the chiral algebra being the identity element and the multiplication being the fusion product [Ver88]. It is a commutative associative algebra.

see (3.3.63) for the precise definition of the algebra object B . Our argument is based on the explicit computation of the Hilbert space of G/H -WZW model on the circle. In many cases, there is an isomorphism

$$\text{End}(B) \cong \mathbb{C}[Z] , \quad (3.1.3)$$

of Frobenius algebras, where $Z := Z(G) \cap Z(H)$ is the common center and $\mathbb{C}[Z]$ is the group algebra of Z which can be endowed with a natural Frobenius algebra structure. When the isomorphism (3.1.3) holds, we have another interpretation of the topological degeneracy relating the gauged WZW models and GKO coset theory. In these cases, we also argue that

$$\text{WZW}(G/H) \cong \text{GKO}(\mathfrak{g}/\mathfrak{h}) \otimes_Z \text{WZW}(H/H)^Z , \quad (3.1.4)$$

where $\text{WZW}(H/H)^Z$ is the Z -equivariant H/H -WZW model and the subscript on $\otimes_Z$ indicates that $\text{WZW}(H/H)^Z$ and $\text{GKO}(\mathfrak{g}/\mathfrak{h})$ are intertwined via the Z -action.

One of the important aspects of this chapter is to give a precise definition of the GKO coset theory. To do this, we employ the notion of commutant of a VOA, studied mostly in the VOA literature. In Section 3.2, we define the notation of commutant of a VOA and prove a general categorical theorem regarding the category of modules of the commutant of a VOA under reasonable assumptions. In Section 3.3.1, we then define the GKO vertex operator algebra as the commutant of the $\mathfrak{h}$ -subalgebra in the $\mathfrak{g}$ -affine VOA. We then define the GKO theory to be the diagonal modular invariant theory obtained from the irreducible modules of the GKO vertex operator algebra. Applying then the general theorem of Section 3.2, we prove a general result about the category of modules of the GKO vertex operator algebra, essentially complementing the results of [FFRS03a]. We also recover some of the known results from [CGS23, CGS24].

3.2 Commutant Of A Vertex Operator Subalgebra

In this section, we collect some basic facts about the commutant of a sub-VOA of a VOA. We also discuss the category of modules of the commutant VOA. We will assume some basic definitions and results about VOAs, we refer the reader to [FHL93a, LL12] for details.

3.2.1 Commutant VOA And Its Modules

Let us begin by recalling the general notion of a coset VOA. We will expand the vertex operators as

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} . \quad (3.2.1)$$

Let $(V, Y, \omega^V, \mathbf{1})$ be a VOA.

Let $(U \subseteq V, Y, \omega^U)$ be a sub-VOA with possibly a *different* conformal vector

$$\omega^U \in U \cap V_{(2)} . \quad (3.2.2)$$

Set

$$Y(\omega^U, z) = \sum_{n \in \mathbb{Z}} L^U(n) z^{-n-2} , \quad (3.2.3)$$

acting on all of V . Define the commutant of U in V by [FZ92, LL12]

$$\begin{aligned} C_V(U) &:= \{v \in V : [Y(v, z_1), Y(u, z_2)] = 0 \text{ for all } u \in U\} \\ &= \{v \in V : v_n \cdot u = 0 \text{ for all } u \in U, n \geq 0\} \\ &= \{v \in V : u_n \cdot v = 0 \text{ for all } u \in U, n \geq 0\} . \end{aligned} \quad (3.2.4)$$

If we set

$$\omega^{V/U} := \omega^V - \omega^U . \quad (3.2.5)$$

Then we have the following result [FZ92, LL12].

Proposition 3.2.1. *Let $V \equiv (V, Y, \mathbf{1}, \omega^V)$ be a VOA and $U \equiv (U, Y, \mathbf{1}, \omega^U)$ be a sub-VOA such that $L^V(1)\omega^U = 0$. Then the following is true:*

1. *We have*

$$L^V(n) = L^U(n), \quad n \geq -1 \text{ on } U . \quad (3.2.6)$$

In particular, the gradings of U and V are compatible in the sense that $L^V(0) = L^U(0)$ acting on U .

2. *We have*

$$C_V(U) = \text{Ker}_V L^U(-1) . \quad (3.2.7)$$

In particular, $\mathbf{1} \in C_V(U)$.

3. *On any V -module, the action of $U, C_V(U)$ commutes, i.e.,*

$$Y(u, z_1)Y(v, z_2) = Y(v, z_2)Y(u, z_1) , \quad u \in U, v \in C_V(U) . \quad (3.2.8)$$

4. *$C_V(U) \equiv (C_V(U), Y, \mathbf{1}, \omega^{V/U})$ is a vertex operator subalgebra of V with central charge*

$$c^{V/U} := c^V - c^U , \quad (3.2.9)$$

and

$$\omega^{V/U} \in C_V(U) \cap V_{(2)} , \quad L^V(1)\omega^{V/U} = 0 . \quad (3.2.10)$$

5. We have

$$U \subseteq C_V(C_V(U)) , \quad C_V(C_V(C_V(U))) = C_V(U) . \quad (3.2.11)$$

Moreover

$$C_V(C_V(U)) = U , \quad (3.2.12)$$

if and only if $C_V(U)$ is maximal in the sense that if (U', Y, ω^U) is another sub-VOA of V , then $U' \subseteq U$.

6. $U \otimes C_V(U)$ is a vertex operator subalgebra of V with the same conformal vector as the VOA V .

It is expected that if U, V are both rational⁷ then so is $C_V(U)$. Let M be an irreducible V -module which is completely reducible considered as a U -module. Then we have

$$M \cong \bigoplus_i \text{Hom}_U(W^i, M) \otimes W^i , \quad (3.2.13)$$

where W_i are irreducible U -modules occurring in M and $\text{Hom}_U(W^i, M)$ is the space of U -module homomorphisms. Then we have the following result [DRX24]:

Lemma 3.2.1. *Let M, W^i be as above. Then the following is true:*

1. For any homogeneous vector $0 \neq x \in W^i$, the map $\iota_i^x : \text{Hom}_U(W^i, M) \longrightarrow M$ given by $f \mapsto f(x)$ is an injective homomorphism.
2. $\text{Hom}_U(W^i, M)$ is a $C_V(U)$ -module such that

$$(v_n \cdot f)(w) = v_n \cdot f(w), \quad v \in C_V(U), n \in \mathbb{Z}, f \in \text{Hom}_U(W^i, M), w \in W^i . \quad (3.2.14)$$

In particular, (3.2.13) is the decomposition of M as a $(C_V(U) \otimes U)$ -module.

3. The image M^i of $\text{Hom}_U(W^i, M)$ under ι_i^x is a $C_V(U)$ -submodule of M and ι_i^x is a $C_V(U)$ -module isomorphism.

Assuming the V is completely reducible U -module and considering V as a $(C_V(U) \otimes U)$ -module, we can decompose

$$V \cong \bigoplus_i C_V(U)^i \otimes U^i = C_V(U) \otimes U \oplus \bigoplus_{i \neq 0} C_V(U)^i \otimes U^i , \quad (3.2.15)$$

where U^i are the U -modules occurring in V and $C_V(U)^i$ is the multiplicity space of U^i in V , and we have used the notation $C_V(U)^0 \otimes U^0 = C_V(U) \otimes U$. In favorable cases, all simple modules of $C_V(U)$ can be obtained from the $(C_V(U) \otimes U)$ -module decomposition of some simple V -module as the following theorem asserts.

⁷See (3.3.44) for some examples.

Theorem 3.2.1. [KM15, Theorem 2] *Let $V \equiv (V, Y, \mathbf{1}, \omega^V)$ be a VOA and $U \equiv (U, Y, \mathbf{1}, \omega^U)$ be a sub-VOA. Assume that $C_V(C_V(U)) = U$ and that $U, V, C_V(U)$ are simple, regular⁸, CFT-type, that is,*

$$V_{(0)} = \mathbb{C}\mathbf{1}, \quad V_{(n)} = 0 \quad n < 0, \quad (3.2.16)$$

and self-dual. Then all simple U -modules appear in some simple V -module.

Remark 3.2.1. In the setting of Theorem 3.2.1, replacing U by $C_V(U)$ and using the fact that $C_V(C_V(C_V(U))) = C_V(U)$, we conclude that all simple $C_V(U)$ -modules appear in some simple V -module.

3.2.2 Commutant Of Conformally Embedded Sub-VOA

A particularly interesting example is the commutant of a $U \subset V$ with the property that $c^V = c^U$.

Definition 3.2.1. Let (V, ω^V) be a VOA. An embedding of a VOA $(U, \omega^U) \subseteq (V, \omega^V)$ is called a *conformal embedding* if $c^U = c^V$ or equivalently $c^{V/U} = 0$. We say that (U, ω^U) is *conformally embedded* in V .

The following proposition shows that under suitable conditions, the commutant of a conformally embedded sub-VOA is trivial.

Proposition 3.2.2. *Let (V, ω^V) be a unitary selfdual⁹ VOA and (U, ω^U) be a conformally embedded unitary selfdual sub-VOA. Then*

$$\omega^{V/U} = 0. \quad (3.2.17)$$

Moreover, $C_V(U)$ is a commutative associative algebra independent of U in the sense that $C_V(U) \cong C_V(U')$ for any U, U' satisfying the above hypotheses. If, in addition, V is a simple vertex operator algebra then

$$C_V(U) = \mathbb{C}\mathbf{1}. \quad (3.2.18)$$

In particular, $C_V(V) = \mathbb{C}\mathbf{1}$.

⁸A VOA V is called *rational* if every admissible V -module is isomorphic to the direct sum of (finitely many) admissible irreducible modules, and is *regular* if every weak V -module is isomorphic to the direct sum of (finitely many) admissible irreducible modules. See [DLM95] for details.

⁹A VOA is said to be selfdual if it is isomorphic to its contragredient. See [FHL93a] for the definition of contragredient of a module.

Proof. We begin by noting that

$$\left\langle \mathbf{1}, L^{V/U}(2)L^{V/U}(-2)\mathbf{1} \right\rangle = \frac{c^{V/U}}{2} = 0 . \quad (3.2.19)$$

Unitarity of the VOA U, V implies sesquilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate and positive-definite and selfduality implies that

$$L^V(n)^\dagger = L^V(-n) , \quad L^U(n)^\dagger = L^U(-n) , \quad n \in \mathbb{Z} . \quad (3.2.20)$$

In particular,

$$L^{V/U}(n)^\dagger = L^{V/U}(-n), \quad n \in \mathbb{Z} . \quad (3.2.21)$$

Then non-degeneracy implies that

$$L^{V/U}(-2)\mathbf{1} = \omega^{V/U} = 0 . \quad (3.2.22)$$

This implies that

$$\omega^V = \omega^U . \quad (3.2.23)$$

[LL12, Proposition 3.11.2] implies that

$$C_V(U) = \text{Ker}_V L^V(-1) , \quad (3.2.24)$$

which also indicates the independence of $C_V(U)$ from U . Then for any $v \in C_V(U)$, the $L(-1)$ -derivative property

$$\frac{d}{dz} Y(v, z) = Y(L^V(-1)v, z) = 0 , \quad (3.2.25)$$

implies that all vertex operators of $C_V(U)$ are z -independent. Then we define the multiplication on $C_V(U)$ by

$$v \cdot u := Y(v, z)u = v_{-1}u . \quad (3.2.26)$$

This product is easily seen to be commutative:

$$v \cdot u = v_{-1}u = v_{-1}u_{-1}\mathbf{1} = u_{-1}v_{-1}\mathbf{1} = u_{-1}v = u \cdot v , \quad (3.2.27)$$

where we used the fact that the vertex operators have regular OPE (since they are independent of z). Associativity follows from Jacobi identity for vertex operators. Finally, when V is a simple VOA, we apply [LL12, Proposition 3.11.4] to conclude the result. $\square$

An Explicit Example: The Lattice VOA

We now present an explicit example of the commutant of a sub-VOA using lattice VOAs. We will follow the exposition in [LL12, FLM88b]. Let L be an even lattice with respect to some nondegenerate positive definite bilinear form $\langle \cdot, \cdot \rangle$. Let $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{R}$ and extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ be the Heisenberg algebra associated to $\mathfrak{h}$ equipped with the bracket

$$\begin{aligned} [\alpha \otimes t^m, \beta \otimes t^n] &= m \langle \alpha, \beta \rangle \delta_{m+n,0} \mathbf{k} , \quad \alpha, \beta \in \mathfrak{h} , \quad m, n \in \mathbb{Z} , \\ [\alpha \otimes t^m, \mathbf{k}] &= 0 . \end{aligned} \tag{3.2.28}$$

Consider the central extension

$$1 \rightarrow \mathbb{Z}_2 \rightarrow \hat{L} \rightarrow L \rightarrow 1 , \tag{3.2.29}$$

with commutator map

$$c(\alpha, \beta) = \langle \alpha, \beta \rangle + 2\mathbb{Z} , \tag{3.2.30}$$

and let e_α be a section of the extension. Let $\varepsilon : L \times L \rightarrow \mathbb{Z}$ be a cocycle for this central extension. Then $e_\alpha e_\beta = (-1)^{\varepsilon(\alpha, \beta)} e_{\alpha+\beta}$. Let $\mathbb{C}[L]$ be the group algebra of L and we will write $\alpha \mapsto e^\alpha$ for the embedding of $L \hookrightarrow \mathbb{C}[L]$. Define the action

$$e_\alpha \cdot e^\beta = (-1)^{\varepsilon(\alpha, \beta)} e^{\alpha+\beta} , \quad \theta \cdot e^\beta = \theta e^\beta , \quad \alpha, \beta \in L, \quad \theta \in \mathbb{Z}_2 . \tag{3.2.31}$$

This makes $\mathbb{C}[L]$ into an $\hat{L}$ -module. Let us decompose

$$\hat{\mathfrak{h}} = \hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \hat{\mathfrak{h}}^0 , \tag{3.2.32}$$

where

$$\hat{\mathfrak{h}}^+ := \mathfrak{h} \otimes t\mathbb{C}[t] , \quad \hat{\mathfrak{h}}^- := \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}] , \quad \hat{\mathfrak{h}}^0 := \mathfrak{h} \otimes t^0 \oplus \mathbb{C}\mathbf{k} . \tag{3.2.33}$$

Then we define the vector space

$$V_L := S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[L] , \tag{3.2.34}$$

where $S(\hat{\mathfrak{h}}^-)$ is the symmetric algebra. Let us introduce the notation

$$\alpha(n) := \alpha \otimes t^n , \quad \alpha \in \mathfrak{h}, \quad n \in \mathbb{Z} . \tag{3.2.35}$$

Then V_L is spanned by elements of the form

$$\alpha_1(-n_1) \cdots \alpha_k(-n_k) e^\beta := \alpha_1(-n_1) \cdots \alpha_k(-n_k) \otimes e^\beta , \quad n_i \in \mathbb{N}, \alpha_i \in \mathfrak{h}, \beta \in L . \tag{3.2.36}$$

Let $\hat{\mathfrak{h}}^+$ act as 0 on $\mathbb{C}[L]$ and $\hat{\mathfrak{h}}^0$ act on $\mathbb{C}[L]$ as

$$(\alpha \otimes t^0) \cdot e^\beta = \langle \alpha, \beta \rangle e^\beta, \quad \mathbf{k} \cdot e^\beta = e^\beta, \quad \alpha \in \mathfrak{h}, \beta \in L. \quad (3.2.37)$$

V_L is an $\hat{\mathfrak{h}}$ -module. The action of $\alpha(n)$ for $n < 0$ is defined by

$$\alpha(n) \cdot (\alpha_1(-n_1) \cdots \alpha_k(-n_k) e^\beta) = \alpha(n) \alpha(n) \alpha_1(-n_1) \cdots \alpha_k(-n_k) e^\beta, \quad (3.2.38)$$

and the action of $\alpha(n)$ for $n \leq 0$ is defined by commuting it past $\alpha_i(-n_i)$ using the bracket (3.2.28) and then acting it on $\mathbb{C}[L]$. We also define vacuum and conformal vector by

$$\mathbf{1} := e^0, \quad \omega^L := \frac{1}{2} \sum_{i=1}^{\dim(\mathfrak{h})} \circ u^i(-1) u^i(-1) \circ, \quad (3.2.39)$$

where $\{u^i\}$ is an orthonormal basis of $\mathfrak{h}$ and the normal ordering is defined by putting operators $\alpha(m)$ to the right of operators $\beta(n)$ whenever $m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\leq 0}$. Define the operator z^α by

$$z^\alpha \cdot (u \otimes e^\beta) = z^{\langle \alpha, \beta \rangle} (u \otimes e^\beta), \quad \alpha, \beta \in L, u \in S(\hat{\mathfrak{h}}^-). \quad (3.2.40)$$

Let z be a formal variable. Define vertex operators

$$\begin{aligned} Y(e^\alpha, z) &= \exp\left(-\sum_{n < 0} \frac{\alpha(n)}{n} z^{-n}\right) \exp\left(-\sum_{n > 0} \frac{\alpha(n)}{n} z^{-n}\right) e_\alpha z^\alpha, \\ Y(\alpha_1(-n_1) \cdots \alpha_k(-n_k) e^\alpha, z) &= \circ \left(\prod_{i=1}^k \frac{1}{(n_i - 1)!} \frac{d^{n_i-1}}{dz^{n_i-1}} \alpha_i(z) \right) Y(e^\alpha, z) \circ, \end{aligned} \quad (3.2.41)$$

where

$$\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n) z^{-n-1}, \quad \alpha \in \mathfrak{h}. \quad (3.2.42)$$

In particular, note that

$$Y(\alpha(-1)\mathbf{1}, z) = \alpha(z). \quad (3.2.43)$$

Recall that for formal variables, we define

$$(z_1 - z_2)^s = \sum_{n=0}^{\infty} \binom{s}{n} z_1^{s-n} z_2^n, \quad (3.2.44)$$

where

$$\binom{s}{n} = \frac{s(s-1) \cdots (s-n+1)}{n!}. \quad (3.2.45)$$

In particular, the series has positive integral powers of z_2 . With this notation, one can prove that

$$\begin{aligned}
[\alpha(z_1), \beta(z_2)] &= \langle \alpha, \beta \rangle ((z_1 - z_2)^{-2} - (-z_2 + z_1)^{-2}) , \\
[\alpha(z_1), Y(e^\beta, z_2)] &= \langle \alpha, \beta \rangle ((z_1 - z_2)^{-1} - (-z_2 + z_1)^{-1}) Y(e^\alpha, z_2) , \\
[Y(e^\alpha, z_1), Y(e^\beta, z_2)] &= \left((z_1 - z_2)^{\langle \alpha, \beta \rangle} - (-z_2 + z_1)^{\langle \alpha, \beta \rangle} \right) \\
&\quad \times \exp \left(- \sum_{n < 0} \frac{\alpha(n)}{n} z_1^{-n} \right) \exp \left(- \sum_{n < 0} \frac{\beta(n)}{n} z_1^{-n} \right) \\
&\quad \times \exp \left(- \sum_{n > 0} \frac{\alpha(n)}{n} z_1^{-n} \right) \exp \left(- \sum_{n > 0} \frac{\beta(n)}{n} z_1^{-n} \right) e_\alpha e_\beta z_1^\alpha z_2^\beta .
\end{aligned} \tag{3.2.46}$$

Using these calculations, one can prove that $(V_L, Y, \mathbf{1}, \omega^L)$ is a VOA with central charge $c^L := \dim(L)$.

Consider now a sublattice $M \subseteq L$ of L . Then, we have a sub-VOA (V_M, Y, ω^M, c^M) of V_L . We would like to compute the commutant $C_{V_L}(V_M)$. The commutators (3.2.46) imply that $\alpha(-1)\mathbf{1} \in C_{V_L}(V_M)$ if

$$\begin{aligned}
[\alpha(z_1), \beta(z_2)] &= [\alpha(z_1), Y(e^\gamma, z_2)] = 0 \\
\iff \langle \alpha, \beta \rangle &= \langle \alpha, \gamma \rangle = 0 \text{ for all } \beta \in M \otimes_{\mathbb{Z}} \mathbb{R} =: \mathfrak{h}_M, \gamma \in M .
\end{aligned} \tag{3.2.47}$$

Similarly, $e^\alpha \in C_{V_L}(V_M)$ if

$$[Y(e^\alpha, z_1), Y(e^\beta, z_2)] = 0 \iff \langle \alpha, \beta \rangle = 0 \text{ for all } \beta \in L. \tag{3.2.48}$$

In fact using the general form of the vertex operator (3.2.41), we see that these conditions are enough to ensure commutativity of $\alpha(z)$ and $Y(e^\alpha, z)$ for $\alpha(-1)\mathbf{1}, e^\alpha \in C_{V_L}(V_M)$ with all vertex operators in V_M . Using the commutators (3.2.46) repeatedly, we conclude that

$$C_{V_L}(V_M) = S((\hat{\mathfrak{h}}_M^\perp)^-) \otimes \mathbb{C}[M^\perp] , \tag{3.2.49}$$

where

$$\begin{aligned}
\mathfrak{h}_M^\perp &:= \{ \alpha \in \mathfrak{h} : \langle \alpha, \beta \rangle = 0 \ \forall \ \beta \in \mathfrak{h}_M \} , \\
M^\perp &:= \{ \alpha \in L : \langle \alpha, \beta \rangle = 0 \ \forall \ \beta \in \mathfrak{h}_M \} .
\end{aligned} \tag{3.2.50}$$

If we now take M to be a full rank sublattice, then we have $M \otimes_{\mathbb{Z}} \mathbb{R} = \mathfrak{h}$ and hence

$$\omega^M = \omega^L . \tag{3.2.51}$$

In particular, (V_M, Y, ω^M, c^M) is conformally embedded in V_L . We claim that $C_{V_L}(V_M) = \mathbb{C}\mathbf{1}$ in this case. Indeed, since the pairing is nondegenerate, we conclude that $\mathfrak{h}_M^\perp = M^\perp = \{0\}$. Thus the commutant is trivial.

3.2.3 The Category Of Modules Of The Commutant VOA

In this section, we discuss the categorical commutant, see Appendix C.4 for notations relevant definitions.

The Category Of Modules Of A VOA

Let V be a VOA and $\mathcal{C}_V$ be the category of V -modules with objects being V -modules and morphisms being the set of VOA homomorphisms. Since the space of V -module homomorphisms is a complex vector space, the V -module category is naturally an abelian category. In fact, $\mathcal{C}_V$ can be given the structure of a modular tensor category for sufficiently “nice” VOA. This was first discussed in [MS89c, MS89a] and later formalized in [HL95, Hua95a, HL93, Hua05]. The resolution is to define a z -dependent tensor product, what is called a $P(z)$ -tensor product. We will call a VOA a “nice” VOA if it satisfies the following properties:

1. V is CFT type i.e., $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V is selfdual $V \cong V'$.
2. V is regular, i.e., every $\mathbb{N}_0$ -gradable weak V -module is completely reducible, i.e., isomorphic to direct sum of finitely many irreducible V -modules.
3. V is C_2 -cofinite, i.e.,

$$\dim V/C_2(V) < \infty , \quad (3.2.52)$$

where

$$C_2(V) = \text{Span}_{\mathbb{C}}\{u_{-2} \cdot v : u, v \in V\} . \quad (3.2.53)$$

Then it was shown in [Hua05] that the module category $\mathcal{C}_V$ of a “nice” VOA V has the structure of a modular tensor category. We give a detailed summary of the construction in Appendix B. We will denote the tensor product of two modules W_1, W_2 by $W_1 \boxtimes W_2$. The unit object in the category is the VOA V considered as a module over itself. The left dual ${}^\vee W$ and right dual $W^\vee$ of a V -module W is the contragredient module

$${}^\vee W \cong W^\vee \cong W' . \quad (3.2.54)$$

This requires the VOA to be selfdual. The construction of creation and annihilation morphisms again requires the VOA to satisfy “niceness” properties. This makes $\mathcal{C}_V$ into a rigid braided tensor category. The fusion structure of $\mathcal{C}_V$ requires the full “niceness” properties: the simple objects of $\mathcal{C}_V$ are the (finitely many) isomorphism classes of irreducible V -modules. We will denote the set of

isomorphism classes of irreducible V -modules by $\{W^a\}_{a \in \mathcal{I}_V}$ with $W^0 = V$. Complete reducibility of V -modules implies semisimplicity of the category. An important feature of the tensor product is the following decomposition

$$W^a \boxtimes W^b \cong \bigotimes_{c \in \mathcal{I}_V} \mathcal{N}_{ab}^c W^c, \quad (3.2.55)$$

where $\mathcal{N}_{ab}^c$ are the fusion rules given by the dimension of the space of intertwining operators of type $\left(\begin{smallmatrix} W^c \\ W^a W^b \end{smallmatrix} \right)$. The twist map on the simple objects is given by

$$\theta_a := \theta_{W^a} = e^{2\pi i L^a(0)}, \quad (3.2.56)$$

where $L^a(0)$ is the Virasoro zero mode acting on W^a . If W^a admits a grading of the form $W^a = \bigoplus_{n \in \mathbb{Z} + h} W_{(n)}^a$ for some h , then the twist map is multiplication by phase $\theta_a = e^{2\pi i h}$. The S -matrix of the MTC can be written in terms of the S -matrix coming from the $\tau \rightarrow -1/\tau$ transformation of the characters.

Module Category Of The Commutant VOA

We would now like to understand the construction of the module category of the commutant of a VOA in terms category theory. Let $U \subset V$ be a sub-VOA. Assuming that U and $C_V(U)$ are “nice” VOAs, we have the modular tensor categories $\mathcal{C}_{C_V(U)}$ and $\mathcal{C}_U$. The first question we ask is: under what conditions do we have an algebra object A in $\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U$ such that

$$\mathcal{C}_V \cong (\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U)_A^0. \quad (3.2.57)$$

Here $\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U$ denotes the *Deligne product* of categories, and $\mathcal{C}_A^0$ denotes the category of local A -modules in $\mathcal{C}$, see Appendix C.4 for notations and definitions. Our starting point is a theorem from [KJO02, HKL15] adopted to our setting.

Theorem 3.2.2. [KJO02, Theorem 5.2], [HKL15, Theorem 3.6] *Let $\mathcal{V}$ be a “nice”¹⁰ VOA with the property that the conformal dimension of all irreducible V -modules are ≥ 0 with equality only for $\mathcal{V}$ itself. Let $\mathcal{C}_{\mathcal{V}}$ be its module category, Then the following are equivalent:*

1. An extension¹¹ $\mathcal{V} \hookrightarrow \mathcal{V}_e$ where $\mathcal{V}_e$ is also a “nice” VOA.
2. An algebra object A with $\theta_A = \mathbb{1}_A$.

Under this correspondence, the category $\mathcal{C}_{\mathcal{V}_e}$ is equivalent to $(\mathcal{C}_{\mathcal{V}})_A^0$.

¹⁰The “niceness” conditions on the VOA can be relaxed for this theorem to hold, see [KJO02] and [HKL15].

¹¹A VOA $\mathcal{V}_e$ is called an extension of $\mathcal{V}$ if $\mathcal{V} \hookrightarrow \mathcal{V}_e$ is embedded as a sub-VOA.

The algebra object A in Theorem 3.2.2 is given by $\mathcal{V}_e$ itself. More precisely, one first proves that $\mathcal{V}_e$ is in fact a $\mathcal{V}$ -module [KJO02] and hence an object in the category $\mathcal{C}_{\mathcal{V}}$. The vertex operators of $\mathcal{V}_e$ can then be used to give $\mathcal{V}_e$ the structure of an algebra. The fact that $\theta_A = \mathbb{1}_A$ follows from the fact that the conformal dimensions of vectors in a VOA are integers.

The physical interpretation of equivalence of MTCs in Theorem 3.2.2 is the following: given an MTC, one can construct a 3d TFT [Tur94]. The simple objects of the MTC map to anyons in the 3d TFT. The algebra object A corresponds to *gaugable anyons* and the fact that the twist map is trivial means that we consider $\mathcal{V}$ -modules of integer conformal dimensions, i.e., *bosonic anyons*. The category $(\mathcal{C}_{\mathcal{V}})_A^0$ is then obtained by *gauging*¹² the bosonic anyons corresponding to A . Consequently, the notation

$$\mathcal{C}_{\mathcal{V}_e} \cong \mathcal{C}_{\mathcal{V}}/A , \quad (3.2.58)$$

is common in physics literature.

The following theorem gives us a description of the module category of the commutant VOA¹³.

Theorem 3.2.3. *Let $U, V, C_V(U)$ be as above. Assume that U, V and $C_V(U)$ are “nice” VOAs. Then V is an algebra object in $\mathcal{C}_{C_V(U)}$ and*

$$\mathcal{C}_V \cong \frac{\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U}{V} := (\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U)_V^0 . \quad (3.2.59)$$

If in addition, the only simple subobjects of V of the form $W \boxtimes_{\mathbb{D}} U$ is $C_V(U) \boxtimes_{\mathbb{D}} U$, then there exists a special symmetric commutative Frobenius algebra object B in $\mathcal{C}_V \boxtimes_{\mathbb{D}} \mathcal{C}_U^{\text{opp}}$ such that

$$\mathcal{C}_{C_V(U)} \cong \frac{\mathcal{C}_V \boxtimes_{\mathbb{D}} \mathcal{C}_U^{\text{opp}}}{B} := (\mathcal{C}_V \boxtimes_{\mathbb{D}} \mathcal{C}_U^{\text{opp}})_B^0 , \quad (3.2.60)$$

where $\mathcal{C}_U^{\text{opp}}$ is the opposite category¹⁴ of $\mathcal{C}_U$.

Proof. By Proposition 3.2.1 (5), $C_V(U) \otimes U$ can be embedded in V as a sub-VOA. This means that V is an extension of $C_V(U) \otimes U$. Since both $C_V(U)$ and U are “nice” VOAs, the tensor product VOA $C_V(U) \otimes U$ is also a “nice” VOA. Thus, by Theorem 3.2.2 there exists an algebra object A in $\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U$ with trivial twist such that

$$\mathcal{C}_V \cong (\mathcal{C}_{C_V(U)} \boxtimes_{\mathbb{D}} \mathcal{C}_U)_A^0 . \quad (3.2.61)$$

¹²This procedure of gauging of bosonic anyons has been called *non-abelian anyon condensation* in [BS09]. Reference [BS09] provides a heuristic procedure for such a gauging which was formalized in [Kon14] in terms of the local A -modules.

¹³A rough version of this result was presented in [FFRS03a].

¹⁴The opposite category has been called the dual category in [FFRS03b, Section 6.2].

By the discussion below Theorem 3.2.2, we conclude that the algebra object is simply $A = V$. For the next part, we can use a version of the main theorem of [FFRS03b] (reviewed in Appendix C.4). We first show that V is a special symmetric commutative Frobenius algebra object. The algebra structure on V is given as follows: considering V as a $(C_V(U) \otimes U)$ -module, the vertex operator map $Y_V : V \otimes V \rightarrow V[[z, z^{-1}]]$ is an intertwining operator of type $\left(\begin{smallmatrix} V \\ V \end{smallmatrix} V\right)$. By the universal property of the $P(z)$ -tensor product [HL95], we get a $(C_V(U) \otimes U)$ -module homomorphism $\mu_V : V \boxtimes V \rightarrow V$. Since $C_V(U) \otimes U$ is embedded as a sub-VOA in V , the inclusion map $\eta_V := \iota : C_V(U) \otimes U \rightarrow V$ is a $(C_V(U) \otimes U)$ -module homomorphism. Then [KJO02, HKL15] shows that (V, μ_V, η_V) is a commutative associative algebra object in $\mathcal{C}_{C_V(U) \otimes U} \cong \mathcal{C}_{C_V(U)} \boxtimes_{\mathcal{D}} \mathcal{C}_U$. The vacuum vector in V is the unit of the algebra. The coalgebra structure on V can be obtained using the construction in [Kon06]. Since V is selfdual, we have a V -module isomorphism $\varphi_V : V \xrightarrow{\cong} V'$. Then we can define the comultiplication as a $(C_V(U) \otimes U)$ -module homomorphisms $\Delta_V : V \rightarrow V \otimes V$ in the category $\mathcal{C}_{C_V(U)} \boxtimes_{\mathcal{D}} \mathcal{C}_U$ and $\varepsilon_V : V \rightarrow C_V(U) \otimes U$ by the following composition of maps

$$\Delta_A : V \xrightarrow{r_V^{-1}} V \boxtimes (C_V(U) \boxtimes_{\mathcal{D}} U) \xrightarrow{1_V \boxtimes i_V} V \boxtimes (V \boxtimes V') \xrightarrow{\alpha_{V,V,V'}} (V \boxtimes V) \boxtimes V' \xrightarrow{\mu_V \boxtimes \varphi_V^{-1}} V \boxtimes V, \quad (3.2.62)$$

where $r_V, i_V, \alpha_{V,V,V'}$ are right unit isomorphism, creation morphism, associativity isomorphism respectively in the tensor category $\mathcal{C}_{C_V(U)} \boxtimes_{\mathcal{D}} \mathcal{C}_U$. The counit morphism $\varepsilon_V : V \rightarrow C_V(U) \otimes U$ is given as follows:

$$\varepsilon_V : V \xrightarrow{\varphi_V} V' \xrightarrow{\iota^V} C_V(U)' \boxtimes_{\mathcal{D}} U' \xrightarrow{\varphi_{C_V(U)}^{-1} \boxtimes \varphi_U^{-1}} C_V(U) \boxtimes_{\mathcal{D}} U, \quad (3.2.63)$$

where ι^V is the contragredient of the inclusion map $\iota : C_V(U) \otimes U \rightarrow V$ and $\varphi_{C_V(U)} : C_V(U) \rightarrow C_V(U)', \varphi_U : U \rightarrow U'$ are isomorphisms. Then, the computations in [Kon06, Proof of Theorem 4.15] shows that $(V, \mu_V, \eta_V, \Delta_V, \varepsilon_V)$ is a commutative Frobenius algebra object. Moreover, since $\theta_V = 1_V$, by [FFRS03b, Proposition 2.25], V is a symmetric Frobenius algebra object. Finally, we show that V is special in the sense that

$$\varepsilon_V \circ \eta_V = \beta_1(1_{C_V(U)} \boxtimes_{\mathcal{D}} 1_U), \quad \mu_V \circ \Delta_V = \beta_V 1_V, \quad (3.2.64)$$

for nonzero complex numbers β_1, β_V . We will use [FRS02, Lemma 3.11]: we will show that there exists a nonzero constant β such that

$$\varepsilon_V = \beta \varepsilon_{\mathfrak{h}}, \quad (3.2.65)$$

where $\varepsilon_{\mathfrak{h}}$ is defined by the following chain of compositions

$$\begin{aligned} \varepsilon_{\mathfrak{h}} : V &\xrightarrow{\ell_V^{-1}} (C_V(U) \boxtimes_{\mathcal{D}} U) \boxtimes V \xrightarrow{i_{V'}} (V' \boxtimes V) \boxtimes V \xrightarrow{\alpha_{V',V,V}^{-1}} V' \boxtimes (V \boxtimes V) \xrightarrow{1_{V'} \boxtimes \mu_V} V' \boxtimes V \\ &\xrightarrow{e_V} C_V(U) \boxtimes_{\mathcal{D}} U, \end{aligned} \quad (3.2.66)$$

where ℓ_V is the left unit isomorphism and e_V is the annihilation isomorphism. Now since the only subobjects of V of the form $W \boxtimes_D U$ is $C_V(U) \boxtimes_D U$, we conclude that $\text{Hom}(V, C_V(U) \boxtimes_D U)$ is one-dimensional. This means that there exists $\beta \in \mathbb{C}$ such that $\varepsilon_V = \beta \varepsilon_{\mathfrak{g}}$. From construction, we see that both $\varepsilon_V, \varepsilon_{\mathfrak{g}} \neq 0$ and hence $\beta \neq 0$. Thus by [FRS02, Lemma 3.11], we conclude that V is a special symmetric commutative Frobenius algebra object. We now apply Theorem C.4.2 to conclude the existence of B such that (3.2.60) holds. We can use [FFRS03b, Lemma 7.8] to give an explicit description of the algebra object B in (3.2.60): Assume that $U, V, C_V(U)$ are “nice” VOAs and let $\{U^a\}_{a \in \mathcal{I}_U}, \{W^a\}_{a \in \mathcal{I}_V}$ be the set of (finitely many) isomorphism classes of irreducible representations. Then we have

$$B = \bigoplus_{a \in \mathcal{I}_V, b \in \mathcal{I}_U} \dim \left[\text{Hom}^{C_{C_V(U) \boxtimes_D U}}(W^a, C_V(U) \boxtimes_D U^b) \right] (W^a \boxtimes_D (U^{\text{opp}})^b) , \quad (3.2.67)$$

where $(U^{\text{opp}})^b$ denotes the object corresponding to U^b in the opposite category. $\square$

We will give some examples where we can apply Theorem 3.2.3 in the next section.

3.3 Goddard-Kent-Olive (GKO) Coset Construction

In this section, we discuss commutants of affine VOA which give rise to the Goddard-Kent-Olive (GKO) coset construction. We also discuss the Lagrangian representations of affine VOA in terms of Wess-Zumino-Witten (WZW) models and the putative Lagrangian description of cosets of affine VOA in terms of gauged WZW models.

3.3.1 Affine VOA

Let us begin by briefly recalling the construction of the affine vertex operator algebra and refer the reader to [LL12] for more details.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra equipped with a nondegenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$. Consider the affine Lie algebra

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k , \quad (3.3.1)$$

equipped with the Lie bracket

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m \langle a, b \rangle \delta_{m+n, 0} k , \quad (3.3.2)$$

$$[a \otimes t^m, k] = 0 , \quad a, b \in \mathfrak{g}, \quad m, n \in \mathbb{Z} . \quad (3.3.3)$$

The vector space $\hat{\mathfrak{g}}$ can be decomposed as

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+ , \quad (3.3.4)$$

where

$$\hat{\mathfrak{g}}_{\pm} := \mathfrak{g} \otimes t^{\pm 1} \mathbb{C}[t^{\pm}] , \quad \hat{\mathfrak{g}}_0 := \mathfrak{g} \otimes t^0 \oplus \mathbb{C}k . \quad (3.3.5)$$

We will identify the subspace $\mathfrak{g} \otimes t^0$ of $\hat{\mathfrak{g}}$ with $\mathfrak{g}$. Let $\mathbb{C}_k$ be the one-dimensional vector space on which $\hat{\mathfrak{g}}_+, \mathfrak{g}$ acts trivially and k acts by scalar multiplication by k . This makes $\mathbb{C}_k$ into a $(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)$ -module. Then by the induced module construction, we get the $\hat{\mathfrak{g}}$ -module

$$V_{\mathfrak{g}}(k, 0) := \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)} \mathbb{C}_k , \quad (3.3.6)$$

where $\mathcal{U}(\hat{\mathfrak{g}})$ denotes the universal enveloping algebra of $\hat{\mathfrak{g}}$. By the Poincaré-Birkhoff-Witt (PBW) theorem, we have

$$V_{\mathfrak{g}}(k, 0) \cong \mathcal{U}(\hat{\mathfrak{g}}_-) \otimes \mathbb{C}_k . \quad (3.3.7)$$

Let us denote by $a(n)$ the action of $a \otimes t^n$ on $V_{\mathfrak{g}}(k, 0)$. Thus, $V_{\mathfrak{g}}(k, 0)$ is spanned by elements of the form

$$a_1(-n_1) \dots a_{\ell}(-n_{\ell}) \mathbf{1} , \quad (3.3.8)$$

where $\mathbf{1}$ is the element $1 \in \mathbb{C}$ considered as an element of $V_{\mathfrak{g}}(k, 0)$. We endow $V_{\mathfrak{g}}(k, 0)$ a grading by declaring that the weight of the above vector is

$$\text{wt}(a_1(-n_1) \dots a_{\ell}(-n_{\ell}) \mathbf{1}) = \sum_{i=1}^{\ell} n_i . \quad (3.3.9)$$

We now define the vertex operator map

$$\begin{aligned} Y(\cdot, z) : V_{\mathfrak{g}}(k, 0) \otimes V_{\mathfrak{g}}(k, 0) &\longrightarrow V_{\mathfrak{g}}(k, 0)[[z, z^{-1}]] , \\ (u, v) &\longmapsto Y(u, z)v = \sum_{n \in \mathbb{Z}} (u_n \cdot v) z^{n-1} , \end{aligned} \quad (3.3.10)$$

where

$$u_n = a(n) , \quad \text{for } u = a(-1) \mathbf{1} . \quad (3.3.11)$$

We will often write

$$Y(a(-1) \mathbf{1}, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} =: a(z) . \quad (3.3.12)$$

For $u = a_1(-n_1) \dots a_{\ell}(-n_{\ell}) \mathbf{1}$, we have

$$Y(u, z) = \circ \prod_{i=1}^{\ell} \frac{1}{(n_i - 1)!} \frac{d^{n_i-1} a_i(z)}{dz^{n_i-1}} \circ , \quad (3.3.13)$$

where, for any $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, the normal ordering is defined as

$$\circ a(m) b(n) \circ = \begin{cases} a(m) b(n) & \text{if } m < 0 , \\ b(n) a(m) & \text{if } m \geq 0 . \end{cases} \quad (3.3.14)$$

The conformal vector is defined using the Sugawara construction:

$$\omega_{\mathfrak{g}} := \frac{1}{2(k + h_{\mathfrak{g}}^{\vee})} \sum_{a=1}^{\dim(\mathfrak{g})} u^a(-1)u^a(-1)\mathbf{1} , \quad (3.3.15)$$

where $h_{\mathfrak{g}}^{\vee}$ is the dual Coxeter number (see Appendix C.1 for definition) of $\mathfrak{g}$ and $\{u^i\}_{i=1}^{\dim(\mathfrak{g})}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the nondegenerate symmetric bilinear form. One can show that $L^{\mathfrak{g}}(n) \in \text{End}(V_{\mathfrak{g}}(k, 0))$ defined by

$$T^{\mathfrak{g}}(z) := Y(\omega_{\mathfrak{g}}, z) =: \sum_{n \in \mathbb{Z}} L^{\mathfrak{g}}(n) z^{-n-2} , \quad (3.3.16)$$

satisfies the Virasoro algebra with central charge

$$c_{\mathfrak{g}} = \frac{k \dim(\mathfrak{g})}{k + h_{\mathfrak{g}}^{\vee}} . \quad (3.3.17)$$

An explicit formula for $L^{\mathfrak{g}}(n)$ is given by

$$L^{\mathfrak{g}}(n) = \frac{1}{2(k + h_{\mathfrak{g}}^{\vee})} \sum_{i=1}^{\dim(\mathfrak{g})} \sum_{m \in \mathbb{Z}} {}^{\circ} u^i(m) u^i(n - m) {}^{\circ} . \quad (3.3.18)$$

Theorem 3.3.1. [LL12, Theorem 6.2.18] *The tuple $(V_{\mathfrak{g}}(k, 0), Y, \mathbf{1}, \omega_{\mathfrak{g}})$ is a VOA of central charge $c_{\mathfrak{g}}$.*

In general, $V_{\mathfrak{g}}(k, 1)$ as a module for itself is not simple. We thus mod out by the largest proper ideal $I_{\mathfrak{g}}(k, 0) \subset V_{\mathfrak{g}}(k, 0)$ to obtain a simple vertex operator algebra

$$L_{\mathfrak{g}}(k, 0) := V_{\mathfrak{g}}(k, 0) / I_{\mathfrak{g}}(k, 0) . \quad (3.3.19)$$

One can show that $\mathbf{1}, \omega_{\mathfrak{g}} \notin I_{\mathfrak{g}}(k, 0)$. $L_{\mathfrak{g}}(k, 0)$ along with the vertex operator map, vacuum vector and conformal vector is a simple vertex operator algebra which we call the *affine vertex operator algebra* associated to $\mathfrak{g}$.

Let

$$P_k^+(\mathfrak{g}) := \{ \lambda \in \Lambda_{\mathfrak{g}}^{\mathbf{w}} : \langle \lambda, \alpha^i \rangle \geq 0, \ i = 1, \dots, \text{rank}(\mathfrak{g}), \ \langle \lambda, \theta \rangle \leq k \} , \quad (3.3.20)$$

where $\Lambda_{\mathfrak{g}}^{\mathbf{w}}$ is the weight lattice of $\mathfrak{g}$, α^i are the simple roots of $\mathfrak{g}$ and θ is the highest root of $\mathfrak{g}$, denote the set of dominant integrable weights of $\mathfrak{g}$. Our Lie algebra conventions have been reviewed in Appendix C.1. Corresponding to every dominant integral weights of $\mathfrak{g}$ is a simple module of $L_{\mathfrak{g}}(k, 0)$. The construction goes as follows. Let us denote the irreducible module of $\mathfrak{g}$ corresponding to $\lambda \in \Lambda_{\mathfrak{g}}^{\mathbf{w}}$ by $L_{\mathfrak{g}}(\lambda)$. We let $\hat{\mathfrak{g}}_+$ act on $L_{\mathfrak{g}}(\lambda)$ trivially and k act as multiplication by k . Then $L_{\mathfrak{g}}(\lambda)$ is a $(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)$ -module. By induced module construction, we get a $V_{\mathfrak{g}}(k, 0)$ -module

$$V_{\mathfrak{g}}(k, \hat{\lambda}) := \mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_0)} L_{\mathfrak{g}}(\lambda) , \quad (3.3.21)$$

where now $\hat{\lambda}$ is the corresponding weight in the affine Lie algebra $\hat{\mathfrak{g}}$, see Appendix C.1 for conventions. It is spanned by elements of the form

$$a_1(-n_1) \dots a_\ell(-n_\ell) \otimes m_\lambda, \quad a_i \in \mathfrak{g}, n_i > 0, \ell > 0, m_\lambda \in L_{\mathfrak{g}}(\lambda). \quad (3.3.22)$$

The grading on $V_{\mathfrak{g}}(k, \hat{\lambda})$ is given as follows: let

$$\Omega^{\mathfrak{g}} := \sum_{a=1}^{\dim(\mathfrak{g})} u^a u^a \in \mathcal{U}(\mathfrak{g}), \quad (3.3.23)$$

be the Casimir element associated to $\mathfrak{g}$ and $\langle \cdot, \cdot \rangle$. It commutes with every element of $\mathcal{U}(\mathfrak{g})$ and hence by Schur's lemma it acts as a scalar on $L_{\mathfrak{g}}(\lambda)$. Let us write

$$\frac{1}{2(k + h_{\mathfrak{g}}^{\vee})} \Omega^{\mathfrak{g}}(\lambda) =: h_{\lambda}^{\mathfrak{g}}, \quad \text{on } L_{\mathfrak{g}}(\lambda), \quad (3.3.24)$$

where $\Omega^{\mathfrak{g}}(\lambda) = \Omega^{\mathfrak{g}}$ on $L_{\mathfrak{g}}(\lambda)$. The grading on the space $V_{\mathfrak{g}}(k, \hat{\lambda})$ is defined by declaring the weight as

$$\text{wt}(a_1(-n_1) \dots a_\ell(-n_\ell) \otimes m_\lambda) = h_\lambda + \sum_{i=1}^{\ell} n_i. \quad (3.3.25)$$

Extend the action of Y defined in (3.3.13) to $V_{\mathfrak{g}}(k, \hat{\lambda})$. To get a $L_{\mathfrak{g}}(k, 0)$ -module, we need to mod out by the direct sum of all proper $\hat{\mathfrak{g}}$ -submodules of $V_{\mathfrak{g}}(k, \hat{\lambda})$ [LL12]. Let $I_{\mathfrak{g}}(k, \hat{\lambda}) \subset V_{\mathfrak{g}}(k, \hat{\lambda})$ be the direct sum of all proper $\hat{\mathfrak{g}}$ -submodules of $V_{\mathfrak{g}}(k, \hat{\lambda})$. Then one can show that

$$L_{\mathfrak{g}}(k, \hat{\lambda}) := V_{\mathfrak{g}}(k, \hat{\lambda}) / I_{\mathfrak{g}}(k, \hat{\lambda}), \quad (3.3.26)$$

is an irreducible $L_{\mathfrak{g}}(k, 0)$ -module [LL12]. In fact, we have the following classification theorem, originally due to Frenkel-Zhu [FZ92].

Theorem 3.3.2. [LL12, Section 6.6] *Let $L_{\mathfrak{g}}(k, \hat{\lambda})$ be as above. Then $(L_{\mathfrak{g}}(k, \hat{\lambda}), Y)$ is an irreducible module for the simple vertex operator algebra $L_{\mathfrak{g}}(k, 0)$. Moreover, every irreducible $L_{\mathfrak{g}}(k, 0)$ -module is isomorphic to $L_{\mathfrak{g}}(k, \hat{\lambda})$ for some $\lambda \in P_k^+(\mathfrak{g})$.*

This theorem tells us that the affine VOA is *rational* in the physics language¹⁵. In fact $L_{\mathfrak{g}}(k, 0)$ is a rational vertex operator algebra even in mathematical sense [FZ92]. Actually, a stronger statement holds: $L_{\mathfrak{g}}(k, 0)$ is a *regular* VOA [DLM95].

The construction above is readily extended to a semisimple Lie algebra. Let

$$\mathfrak{g} = \bigoplus_{i=1}^n \mathfrak{g}_i, \quad (3.3.27)$$

¹⁵See [DLM95] for the mathematical definition of rational VOA. The physics definition is a consequence of the mathematical definition.

with $\mathfrak{g}_i$ simple. Then the affine VOA $L_{\mathfrak{g}}(\mathbf{k}, 0)$ is isomorphic to the tensor product of the affine VOAs of the individual factors:

$$L_{\mathfrak{g}}(\mathbf{k}, 0) \cong \bigotimes_{i=1}^n L_{\mathfrak{g}_i}(k_i, 0) , \quad (3.3.28)$$

where

$$\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 1}^n . \quad (3.3.29)$$

The conformal vector is given by

$$\begin{aligned} \omega_{\mathfrak{g}} &:= \sum_{i=1}^n \omega_{\mathfrak{g}_i} \\ &= \sum_{i=1}^n \frac{1}{2(k_i + h_{\mathfrak{g}_i}^{\vee})} \sum_{a_i=1}^{\dim(\mathfrak{g}_i)} u^{a_i}(-1) u^{a_i}(-1) \mathbf{1} , \end{aligned} \quad (3.3.30)$$

where we have embedded

$$L_{\mathfrak{g}_i}(k_i, 0) \ni \omega_{\mathfrak{g}_i} \mapsto \mathbf{1} \otimes \dots \otimes \omega_{\mathfrak{g}_i} \otimes \dots \otimes \mathbf{1} \in L_{\mathfrak{g}}(\mathbf{k}, 0) . \quad (3.3.31)$$

The central charge is given by

$$c_{\mathfrak{g}} = \frac{\mathbf{k} \dim(\mathfrak{g})}{\mathbf{k} + h_{\mathfrak{g}}^{\vee}} := \sum_{i=1}^n \frac{k_i \dim(\mathfrak{g}_i)}{k_i + h_{\mathfrak{g}_i}^{\vee}} . \quad (3.3.32)$$

The irreducible modules of the affine VOA are in one-to-one correspondence with the dominant integral weights of $\mathfrak{g}$ given by

$$P_{\mathbf{k}}^+(\mathfrak{g}) := \bigoplus_{i=1}^n P_{k_i}^+(\mathfrak{g}_i) , \quad (3.3.33)$$

where $P_{k_i}^+(\mathfrak{g}_i)$ is defined as in (3.3.20). We will denote the module corresponding to the highest dominant integral weight $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ with ¹⁶ $\lambda_i \in P_{k_i}^+(\mathfrak{g}_i)$ by

$$L_{\mathfrak{g}}(\mathbf{k}, \hat{\boldsymbol{\lambda}}) \cong \bigotimes_{i=1}^n L_{\mathfrak{g}_i}(k_i, \hat{\lambda}_i) . \quad (3.3.34)$$

By [DLM95, Proposition 3.3], $L_{\mathfrak{g}}(\mathbf{k}, 0)$ is also a regular VOA.

Using these irreducible modules, we can construct modular invariant CFTs. We will denote the diagonal modular invariant CFT by $\text{affine}(\mathfrak{g})$.

3.3.2 Goddard-Kent-Olive (GKO) Coset Construction

We now want to apply the consider the commutants of affine VOA.

Let $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$ be a subalgebra. If $\mathbf{k}$ is the level of $\hat{\mathfrak{g}}$ then the level $\tilde{\mathbf{k}}$ of $\hat{\mathfrak{h}}$ is given by

$$\tilde{\mathbf{k}} := \mathbf{x}_e \mathbf{k} := (x_{e,1} k_1, \dots, x_{e,n} k_n) , \quad (3.3.35)$$

¹⁶Throughout this chapter, we will follow the convention that quantities for semisimple Lie algebra will be written in bold.

where $\mathbf{x}_e = (x_{e,1}, \dots, x_{e,n}) \in (\mathbb{Z}_{\geq 1})^n$ is determined by the embedding of $\hat{\mathfrak{h}} \hookrightarrow \hat{\mathfrak{g}}$, see (C.1.51) for the definition. Let us choose an orthonormal basis $\{u^{a'}\}_{a'=1}^{\dim(\mathfrak{h})}$ of $\mathfrak{h}$ and extend it to an orthonormal basis $\{u^{a'}\}_{a'=1}^{\dim(\mathfrak{h})} \cup \{u^a\}_{a=\dim(\mathfrak{h})+1}^{\dim(\mathfrak{g})}$ of $\mathfrak{g}$. We then consider the affine VOA corresponding to $\mathfrak{h}$ and $\mathfrak{g}$ with their corresponding conformal vectors $\omega_{\mathfrak{h}}$ and $\omega_{\mathfrak{g}}$ as described above.

It is clear that $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ is a sub-VOA of $L_{\mathfrak{g}}(\mathbf{k}, 0)$ with possibly a different conformal vector. We can then consider the commutant VOA

$$V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) := C_{L_{\mathfrak{g}}(\mathbf{k}, 0)}(L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)) . \quad (3.3.36)$$

Then $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ is a VOA with conformal vector

$$\omega_{\mathfrak{g}/\mathfrak{h}} := \omega_{\mathfrak{g}} - \omega_{\mathfrak{h}} , \quad (3.3.37)$$

and central charge

$$c_{\mathfrak{g}/\mathfrak{h}} = c_{\mathfrak{g}} - c_{\mathfrak{h}} = \frac{\mathbf{k} \dim(\mathfrak{g})}{\mathbf{k} + \mathbf{h}_{\mathfrak{g}}^{\vee}} - \frac{\mathbf{x}_e \mathbf{k} \dim(\mathfrak{h})}{\mathbf{x}_e \mathbf{k} + \mathbf{h}_{\mathfrak{h}}^{\vee}} := \sum_{i=1}^n \left(\frac{k_i \dim(\mathfrak{g}_i)}{k_i + \mathbf{h}_{\mathfrak{g}_i}^{\vee}} - \frac{x_{e,i} k_i \dim(\mathfrak{h}_i)}{x_{e,i} k_i + \mathbf{h}_{\mathfrak{h}_i}^{\vee}} \right) . \quad (3.3.38)$$

We will call $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ the GKO coset VOA or simply coset VOA. The GKO coset VOA corresponding to $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h} = \mathfrak{su}(2)$ realize the unitary Virasoro VOAs [GKO85, GKO86]. We now define the GKO coset CFT.

Definition 3.3.1. We will call the *minimal unitary* modular invariant CFT obtained from the irreducible modules of $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ to be the GKO conformal field theory. We will denote the *diagonal* GKO CFT by $\text{GKO}(\mathfrak{g}/\mathfrak{h})$.

We will also see that our definition differs from what is usually understood in literature. But this is suitable for the relation between GKO CFT and gauged WZW model we want to study.

We now want to construct the modules of the VOA $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$.

Theorem 3.3.3. *Let $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}), L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}), V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ be as above. Then we have the decomposition*

$$L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \cong \bigoplus_{\mu \in P_{\tilde{\mathbf{k}}}^+(\mathfrak{h})} L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) , \quad (3.3.39)$$

where

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) := \text{Hom}_{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)}(L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}), L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})) , \quad (3.3.40)$$

is a module of the coset VOA $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$. In particular, $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ is a simple VOA.

Proof. Since $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ is a regular VOA, the $L_{\mathfrak{g}}(\mathbf{k}, 0)$ -module $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})$ considered as a $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ -module is completely reducible. This gives us the decomposition (3.3.39). The fact that $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is a

$V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -module follows from Lemma 3.2.1. The simplicity of $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ follows from the identification

$$V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \cong L_{\mathfrak{g}/\mathfrak{h}}(0, 0) , \quad (3.3.41)$$

and [DRX24, Proposition 2.5]. $\square$

The question that we want to ask is:

Question 2. *Is the $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -module $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ irreducible?*

In general, the answer to this question is not known but computational evidence suggests that the irreducibility of $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is related to the issue of *fixed point resolution* which we briefly describe below. Irrespective of that, in view of Remark 3.2.1, Theorem 3.2.1 tells us that in favorable cases, all irreducible modules of the coset VOA $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ occurs in some irreducible $L_{\mathfrak{g}}(\mathbf{k}, 0)$ -module.

Note that not all $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is nontrivial. More precisely, $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is nontrivial if and only if the irreducible module $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \tilde{\mu})$ of $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ appears in the decomposition of the irreducible module $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})$ of $L_{\mathfrak{g}}(\mathbf{k}, 0)$. The nontrivial modules $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ are determined by *selection rules* which is briefly reviewed in Appendix C.2. Moreover, it may turn out that

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}', \hat{\mu}') , \quad (3.3.42)$$

for $(\lambda, \mu) \neq (\lambda', \mu')$. This leads to *field identifications*, also reviewed in Appendix C.2. These isomorphisms are understood in terms of spectral flow isomorphism corresponding to the common center of $\tilde{G} \times \tilde{H}$ [MS89d], where $\tilde{G}, \tilde{H}$ is the unique simply connected Lie group with Lie algebra $\mathfrak{g}, \mathfrak{h}$ respectively. Algebraically, these identifications result from the action of outer automorphisms on the dominant integral weights. Since the center of $\tilde{G}$ is isomorphic to the outer automorphism group $\mathcal{O}(\hat{\mathfrak{g}})$ of $\mathfrak{g}$, the common center of $\tilde{G} \times \tilde{H}$ corresponds to outer automorphisms $(A, \tilde{A}) \in \mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathfrak{h}})$ such that A branches to $\tilde{A}$, see (C.2.6) for the precise definition. Then, one can show that [SY90b, DFMS97]

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong L_{\mathfrak{g}/\mathfrak{h}}(A\hat{\lambda}, \tilde{A}\hat{\mu}) . \quad (3.3.43)$$

It is widely believed (and computationally supported) that the modules $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ are irreducible if $(\hat{\lambda}, \hat{\mu})$ is not fixed by any branching outer automorphism. If $(\hat{\lambda}, \hat{\mu})$ is fixed by some branching outer automorphism, then $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is not irreducible and decomposes into other irreducible $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -modules. A construction of these irreducible $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -modules was proposed in [FSS95b], and has been reviewed in Appendix C.2.4.

An important class of examples of coset VOAs are the *parafermionic cosets*. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. Let $L_{\mathfrak{g}}(\mathbf{k}, 0), L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$

be the affine VOA associated to $\mathfrak{g}$ and $\mathfrak{h}$ respectively. For the Cartan subalgebra, one can show that the induced level $\tilde{\mathbf{k}} = \mathbf{k}$. The parafermionic coset VOA is then defined to be the commutant of the Heisenberg VOA $L_{\mathfrak{h}}(\mathbf{k}, 0)$ in $L_{\mathfrak{g}}(\mathbf{k}, 0)$:

$$K(\mathfrak{g}, \mathbf{k}) := C_{L_{\mathfrak{g}}(\mathbf{k}, 0)}(L_{\mathfrak{h}}(\mathbf{k}, 0)) . \quad (3.3.44)$$

It has been proved that parafermionic VOA $K(\mathfrak{g}, \mathbf{k})$ are C_2 -cofinite [ALY14, DW11]. The rationality of parafermionic VOA obtained from $\mathfrak{sl}_2$ and $\mathfrak{sl}_n$ Lie algebras has been proved in [ALY19, JL13, JL14].

3.3.3 Characters And Partition Functions Of Affine VOA And Its Cosets

The characters $\chi_{\hat{\lambda}}^{\mathfrak{g}}(\tau)$ of the irreducible modules $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})$ is defined as

$$\chi_{\hat{\lambda}}^{\mathfrak{g}}(\tau) := \text{Tr}_{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})} q^{L^{\mathfrak{g}}(0) - \frac{c_{\mathfrak{g}}}{24}} , \quad q := e^{2\pi i \tau} . \quad (3.3.45)$$

Under modular transformations, these characters transform as a vector-valued modular form:

$$\chi_{\hat{\lambda}}^{\mathfrak{g}}(\tau + 1) = \sum_{\hat{\lambda}' \in P_{\mathbf{k}}^+(\mathfrak{g})} T_{\hat{\lambda}\hat{\lambda}'}^{\mathfrak{g}} \chi_{\hat{\lambda}'}^{\mathfrak{g}}(\tau) , \quad (3.3.46)$$

$$\chi_{\hat{\lambda}}^{\mathfrak{g}}(-1/\tau) = \sum_{\hat{\lambda}' \in P_{\mathbf{k}}^+(\mathfrak{g})} S_{\hat{\lambda}\hat{\lambda}'}^{\mathfrak{g}} \chi_{\hat{\lambda}'}^{\mathfrak{g}}(\tau) , \quad (3.3.47)$$

where $T^{\mathfrak{g}}, S^{\mathfrak{g}}$ are the modular T, S -matrices. The fusion rules are defined as

$$\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\hat{\lambda}_3} := \dim I_{L_{\mathfrak{h}}(\mathbf{k}, 0)} \left(\begin{matrix} L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_3) \\ L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_1) \quad L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_2) \end{matrix} \right) . \quad (3.3.48)$$

Then the S -matrix satisfies the Verlinde formula:

$$\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\hat{\lambda}_3} = \sum_{\hat{\lambda} \in P_{\mathbf{k}}^+(\mathfrak{g})} \frac{S_{\hat{\lambda}_1 \hat{\lambda}} S_{\hat{\lambda}_2 \hat{\lambda}} S_{\hat{\lambda}_3 \hat{\lambda}}^*}{S_{\hat{\lambda} 0}} . \quad (3.3.49)$$

The Hilbert space of the full theory is given by

$$\mathcal{H}^{\mathfrak{g}} := \bigoplus_{\hat{\lambda}_L, \hat{\lambda}_R \in P_{\mathbf{k}}^+(\mathfrak{g})} \mathcal{M}_{\hat{\lambda}_L, \hat{\lambda}_R}^{\mathfrak{g}} L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_L) \otimes \widetilde{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_R)} , \quad (3.3.50)$$

where $\widetilde{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_R)}$ is a copy of $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_R)$. Then the partition function of the affine VOA $L_{\mathfrak{g}}(\mathbf{k}, 0)$ is given by

$$Z_{\mathfrak{g}}(\tau, \bar{\tau}) := \sum_{\hat{\lambda}_L, \hat{\lambda}_R \in P_{\mathbf{k}}^+(\mathfrak{g})} \mathcal{M}_{\hat{\lambda}_L, \hat{\lambda}_R}^{\mathfrak{g}} \chi_{\hat{\lambda}_L}^{\mathfrak{g}}(\tau) \overline{\chi_{\hat{\lambda}_R}^{\mathfrak{g}}(\tau)} , \quad \bar{q} = e^{-2\pi i \bar{\tau}}, \quad q = e^{2\pi i \tau} \quad (3.3.51)$$

where $\mathcal{M}_{\hat{\lambda}_L, \hat{\lambda}_R}^{\mathfrak{g}} \in \mathbb{Z}_{\geq 0}$ is a matrix satisfying certain properties imposed by modular invariance. There are several choices of the matrix $\mathcal{M}^{\mathfrak{g}}$ which give modular invariant partition function. We will mostly stick to the *diagonal modular invariant*:

$$\mathcal{M}_{\hat{\lambda}_L, \hat{\lambda}_R}^{\mathfrak{g}} = \delta_{\hat{\lambda}_L, \hat{\lambda}_R} . \quad (3.3.52)$$

Consider now the embedding of $\hat{\mathfrak{h}} \subset \hat{\mathfrak{g}}$. Then one can also decompose the characters $\chi_{\hat{\lambda}}^{\mathfrak{g}}(\tau)$ in terms of the characters $\chi_{\hat{\mu}}^{\mathfrak{h}}(\tau)$:

$$\chi_{\hat{\lambda}}^{\mathfrak{g}}(\tau) = \sum_{\hat{\mu} \in P_k^+(\mathfrak{h})} \chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau) \chi_{\hat{\mu}}^{\mathfrak{h}}(\tau) . \quad (3.3.53)$$

The coefficient functions $\chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau)$ are called *branching functions* and are identified with the character of the module $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ of the coset VOA $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$. Let $\mathcal{E}$ be the set

$$\mathcal{E} = \left\{ (\hat{\lambda}, \hat{\mu}) \in P_k^+(\mathfrak{g}) \times P_k^+(\mathfrak{h}) : L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \neq 0 \right\} . \quad (3.3.54)$$

Let $\sim$ be the equivalence relation

$$(\hat{\lambda}, \hat{\mu}) \sim (\hat{\lambda}', \hat{\mu}') \iff L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}', \hat{\mu}') . \quad (3.3.55)$$

Then according to our definition, the torus partition function of the GKO coset CFT is given by

$$Z_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}}(\tau, \bar{\tau}) = \sum_{(\hat{\lambda}, \hat{\mu}) \in \mathcal{E}/\sim} \chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau) \overline{\chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau)} . \quad (3.3.56)$$

One can construct non-diagonal modular invariants for the coset CFT using the modular invariants $\mathcal{M}^{\mathfrak{g}}, \mathcal{M}^{\mathfrak{h}}$ for the affine VOAs $L_{\mathfrak{g}}(\mathbf{k}, 0)$ and $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ respectively, but we will restrict to the diagonal models. An interesting class of examples of GKO cosets are the *conformal embeddings*. Recall that an embedding $\mathfrak{h}_{\tilde{\mathbf{k}}} \subset \mathfrak{g}_{\mathbf{k}}$ is said to be a *conformally embedding* if

$$c^{\mathfrak{g}} = c^{\mathfrak{h}} . \quad (3.3.57)$$

In this case, the GKO coset VOA is a central charge 0 VOA. Since the affine VOAs $L_{\mathfrak{g}}(\mathbf{k}, 0)$ and $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ satisfy the hypotheses of Proposition 3.2.2, we conclude that the GKO coset VOA is trivial:

$$V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \cong \mathbb{C} . \quad (3.3.58)$$

In particular, according to our definition of GKO coset CFT in Definition 3.3.1, we have

$$Z_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}}(\tau, \bar{\tau}) = 1 , \quad (3.3.59)$$

for conformal embeddings. This is in contrast with the usual partition function of the GKO coset theory corresponding to conformal embedding, for example in [ABI90, KORS20, CGS23, CGS24].

3.3.4 Categorical Coset Construction

As we discussed in Section 3.2.3, under some assumptions the category of modules of the commutant VOAs can be obtained from the category of modules of U and V using the notion of nonabelian gauging. In this section, we give some concrete examples of Theorem 3.2.3 using affine VOAs.

As already noted, affine VOAs $L_{\mathfrak{g}}(\mathbf{k}, 0)$ are “nice” in the sense that they satisfy the properties [item 1–item 3](#). Thus the category of modules of $L_{\mathfrak{g}}(\mathbf{k}, 0)$, denoted by $\mathcal{C}(\mathbf{k}, \mathfrak{g})$, is a modular tensor category.

Corollary 3.3.4. *Assume that $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ is a “nice” VOA and denote the category of $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -modules by $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$. Then we have*

$$\mathcal{C}(\mathbf{k}, \mathfrak{g}) \cong \left(\mathcal{C}(\mathfrak{g}, \mathfrak{h}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h}) \right)_{L_{\mathfrak{g}}(\mathbf{k}, 0)}^0. \quad (3.3.60)$$

Let $\{(0, 0)\}$ be the set:

$$\{(0, 0)\} = \left\{ (\hat{\lambda}, \hat{\mu}) \in \mathcal{E} : n_{(\hat{\lambda}, \hat{\mu})} := \dim \operatorname{Hom} \left(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}), V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \right) \neq 0 \right\}. \quad (3.3.61)$$

Then

$$\mathcal{C}(\mathfrak{g}, \mathfrak{h}) \cong \frac{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}^{\operatorname{opp}}(\tilde{\mathbf{k}}, \mathfrak{h})}{B} := \left(\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}^{\operatorname{opp}}(\tilde{\mathbf{k}}, \mathfrak{h}) \right)_B^0, \quad (3.3.62)$$

where

$$B = \bigoplus_{(\hat{\lambda}, \hat{\mu}) \in \{(0, 0)\}} n_{(\hat{\lambda}, \hat{\mu})} \left(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \otimes L_{\mathfrak{h}}^{\operatorname{opp}}(\tilde{\mathbf{k}}, \hat{\mu}) \right), \quad (3.3.63)$$

where $L_{\mathfrak{h}}^{\operatorname{opp}}(\tilde{\mathbf{k}}, \hat{\mu})$ denotes the object corresponding to $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})$ in the opposite category $\mathcal{C}^{\operatorname{opp}}(\tilde{\mathbf{k}}, \mathfrak{h})$.

Proof. By assumption, $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ is a “nice” VOA, the category $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$ is a modular tensor category. The Deligne product $\mathcal{C}(\mathfrak{g}, \mathfrak{h}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})$ corresponding to the tensor product VOA $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$, with the objects being vector space tensor product of modules is then also a modular tensor category. Since $L_{\mathfrak{g}}(\mathbf{k}, 0)$ is also a $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ -module, $L_{\mathfrak{g}}(\mathbf{k}, 0)$ is an object in the category $\mathcal{C}(\mathfrak{g}, \mathfrak{h}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})$. The decomposition

$$L_{\mathfrak{g}}(\mathbf{k}, 0) \cong \bigoplus_{\mu \in P_{\mathbf{k}}^+(\mathfrak{h})} L_{\mathfrak{g}/\mathfrak{h}}(0, \hat{\mu}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}), \quad (3.3.64)$$

makes this fact evident. Thus, by Theorem [3.2.3](#), we conclude that

$$\mathcal{C}(\mathbf{k}, \mathfrak{g}) \cong \left(\mathcal{C}(\mathfrak{g}, \mathfrak{h}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h}) \right)_{L_{\mathfrak{g}}(\mathbf{k}, 0)}^0. \quad (3.3.65)$$

From the decomposition [\(3.3.60\)](#), we see that the subobjects of $L_{\mathfrak{g}}(\mathbf{k}, 0)$ are $L_{\mathfrak{g}/\mathfrak{h}}(0, \hat{\mu}) \otimes L_{\mathfrak{g}}(\tilde{\mathbf{k}}, \hat{\mu})$. Thus, the only subobject of $L_{\mathfrak{g}}(\mathbf{k}, 0)$ of the form $W \otimes L_{\mathfrak{g}}(\tilde{\mathbf{k}}, 0)$ is $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \otimes L_{\mathfrak{g}}(\tilde{\mathbf{k}}, 0)$. Thus there exists a special symmetric commutative Frobenius algebra object B in $\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})$ such that [\(3.3.62\)](#) is true. Using [\(3.2.67\)](#), we can give an explicit description of B . We look at the $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -modules of the form $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$. Suppose first $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is irreducible then $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})$

is a simple object in the category $\mathcal{C}(\mathfrak{g}, \mathfrak{h}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathfrak{k}}, \mathfrak{h})$ and

$$\dim \text{Hom} \left(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}), V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \right) = \begin{cases} 1, & \text{if } L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) , \\ 0, & \text{otherwise .} \end{cases} \quad (3.3.66)$$

If $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is not irreducible, then assuming that we can decompose it as direct sum of irreducible $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -modules, we write

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong n_{(\hat{\lambda}, \hat{\mu})} V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \oplus \dots . \quad (3.3.67)$$

Note that we can have $n_{(\hat{\lambda}, \hat{\mu})} = 0$. Then it is clear that

$$\dim \text{Hom} \left(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}), V(\mathfrak{g}, \mathfrak{h}; \mathbf{k}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \right) =: n_{(\hat{\lambda}, \hat{\mu})} . \quad (3.3.68)$$

This description of B allows for its easy computation as we will demonstrate in examples below. $\square$

Remark 3.3.1. 1. In physical terms, (3.3.60) means that the chiral $\hat{\mathfrak{g}}$ theory can be obtained gauging the nonabelian anyon corresponding to $L_{\mathfrak{g}}(\mathbf{k}, 0)$ in the MTC of the tensor product of the chiral coset theory and the chiral $\hat{\mathfrak{h}}$ theory. If we can now show that $L_{\mathfrak{g}}(\mathbf{k}, 0)$ is a special symmetric commutative Frobenius algebra object, then we can use Theorem 3.2.3 to get the MTC $\mathcal{C}(\mathfrak{g}, \mathfrak{h})$ in terms of a nonabelian gauging of $\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})$.

2. In principal, the algebra object B in (3.3.63) should have all the information about field identifications, selection rules and fixed point resolutions.

We will now see this in some examples.

Explicit Examples

Unitary Minimal Models.

It is well known that the unitary minimal models are realized by the GKO coset CFT [GKO85, GKO86]¹⁷

$$\mathfrak{g} = \mathfrak{su}(2)_k \oplus \mathfrak{su}(2)_1 , \quad \mathfrak{h} = \mathfrak{su}(2)_{k+1} , \quad k \geq 1 . \quad (3.3.69)$$

Explicit evaluation of branching rules and analysis of outer automorphism group action implies that there are nontrivial selection rules and field identifications [DFMS97]. Let us denote the *minimal Virasoro* VOA with central charge

$$c_{p,q} := 1 - \frac{6(p-q)^2}{pq} , \quad (3.3.70)$$

¹⁷There are other coset realizations of unitary minimal models. In Appendix C.3, we review the coset realization of [BRS88, ABR88].

by $\text{Vir}(c_{p,q}, 0)$. There are finitely many isomorphism classed of irreducible $\text{Vir}(c_{p,q}, 0)$ -modules specified by the conformal dimension $h_{r,s}$ of the highest weight vector. These are denoted by $\text{Vir}(c_{p,q}, h_{r,s})$ and the conformal dimension take the values

$$h_{r,s} := \frac{(pr - qs)^2 - (p - q)^2}{4pq}, \quad 1 \leq r \leq q - 1, \quad 1 \leq s \leq p - 1, \quad (3.3.71)$$

with the isomorphism

$$\text{Vir}(c_{p,q}, h_{r,s}) \cong \text{Vir}(c_{p,q}, h_{q-r, p-s}). \quad (3.3.72)$$

Let us also denote the modular tensor category of $\text{Vir}(c_{p,q}, 0)$ by $\mathcal{C}_{\mathcal{M}(p,q)}$. The unitary minimal models correspond to $p = k + 3, q = k + 2, k \geq 1$. Here $k = 1$ corresponds to the Ising model.

Let us recall that $\widehat{\mathfrak{su}}(2)$ at level k has $k + 1$ dominant integrable representations which we will label by $i = 0, \dots, k$. By abuse of notation, we will write

$$P_k^+(\mathfrak{su}(2)) = \{0, 1, \dots, k\}. \quad (3.3.73)$$

Then dominant integrable weights of $\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_1$ are labeled as

$$P_k^+(\mathfrak{su}(2)) \times P_1^+(\mathfrak{su}(2)) = \{(i, j) : 0 \leq i \leq k, 0 \leq j \leq 1\}. \quad (3.3.74)$$

We have the decomposition [GKO86, GKO85]

$$L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (i, j)) \cong \bigoplus_{\substack{\ell=0 \\ i-\ell \equiv j \pmod{2}}}^{k+1} \text{Vir}(c_{k+3, k+2}, h_{i+1, \ell+1}) \otimes L_{\mathfrak{su}(2)}(k+1, \ell). \quad (3.3.75)$$

In particular, we have

$$L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (0, 0)) \cong \bigoplus_{\substack{\ell=0 \\ \ell \text{ even}}}^{k+1} \text{Vir}(c_{k+3, k+2}, h_{1, \ell+1}) \otimes L_{\mathfrak{su}(2)}(k+1, \ell). \quad (3.3.76)$$

By Theorem 3.2.3, we have

$$\mathcal{C}((k, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \cong \frac{\mathcal{C}_{\mathcal{M}(k+3, k+2)} \boxtimes_{\mathbb{D}} \mathcal{C}(k+1, \mathfrak{su}(2))}{A_k}, \quad (3.3.77)$$

where

$$A_k = \bigoplus_{\substack{\ell=0 \\ \ell \text{ even}}}^{k+1} \text{Vir}(c_{k+3, k+2}, h_{1, \ell+1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(k+1, \ell) \cong L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (0, 0)). \quad (3.3.78)$$

We can use (3.3.62) and the isomorphism (3.3.72) to write

$$\mathcal{C}_{\mathcal{M}(k+3, k+2)} \cong \frac{\mathcal{C}((k, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(k+1, \mathfrak{su}(2))}{B_k}, \quad (3.3.79)$$

where

$$B_k = L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (0, 0)) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(k+1, 0) \oplus L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (k, 1)) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(k+1, k+1) . \quad (3.3.80)$$

Using the $\widehat{\mathfrak{su}}(2)_k$ fusion rules [DFMS97], we have the following $P(z)$ -tensor products:

$$\begin{aligned} L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (k, 1)) \boxtimes L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (k, 1)) &\cong L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((k, 1), (0, 0)) , \\ L_{\mathfrak{su}(2)}^{\text{opp}}(k+1, k+1) \boxtimes L_{\mathfrak{su}(2)}^{\text{opp}}(k+1, k+1) &\cong L_{\mathfrak{su}(2)}^{\text{opp}}(k+1, 0) . \end{aligned} \quad (3.3.81)$$

In the language of 3d TFT, A generates a $\mathbb{Z}_2$ abelian anyon. We can thus schematically write

$$\mathcal{C}_{\mathcal{M}(k+3, k+2)} \cong \frac{\mathcal{C}((k, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(k+1, \mathfrak{su}(2))}{\mathbb{Z}_2} , \quad (3.3.82)$$

which is in agreement with the proposal of [MS89d] for obtaining the GKO coset theory $\text{GKO}(\mathfrak{g}/\mathfrak{h})$ from Chern-Simons theory for gauge group $(G_k \times H_{-\bar{k}})/Z$, where G, H are the unique simply connected Lie groups with $\text{Lie}(G) = \mathfrak{g}, \text{Lie}(H) = \mathfrak{h}$ and Z is the common center of G, H . See Appendix C.2.2 for a quick review of the proposal.

For $k = 1$,

$$A_1 = \text{Vir}\left(\frac{1}{2}, 0\right) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(2, 0) \oplus \text{Vir}\left(\frac{1}{2}, \frac{1}{2}\right) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(2, 1) . \quad (3.3.83)$$

Using the fusion rules [DFMS97], we have the following $P(z)$ -tensor products:

$$\begin{aligned} \text{Vir}\left(\frac{1}{2}, \frac{1}{2}\right) \boxtimes \text{Vir}\left(\frac{1}{2}, \frac{1}{2}\right) &\cong \text{Vir}\left(\frac{1}{2}, 0\right) , \\ L_{\mathfrak{su}(2)}(2, 1) \boxtimes L_{\mathfrak{su}(2)}(2, 1) &\cong L_{\mathfrak{su}(2)}(2, 0) . \end{aligned} \quad (3.3.84)$$

We can thus schematically write

$$\mathcal{C}((1, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \cong \frac{\mathcal{C}_{\mathcal{M}(4,3)} \boxtimes_{\mathbb{D}} \mathcal{C}(k+1, \mathfrak{su}(2))}{\mathbb{Z}_2} . \quad (3.3.85)$$

This is in agreement with [CGS23]¹⁸. From (3.3.62), we have

$$\mathcal{C}_{\mathcal{M}(4,3)} \cong \frac{\mathcal{C}((1, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(k+1, \mathfrak{su}(2))}{B_1} , \quad (3.3.86)$$

where

$$B_1 = L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((1, 1), (0, 0)) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(2, 0) \oplus L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((1, 1), (1, 1)) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(2, 2) . \quad (3.3.87)$$

The $k = 2$ case is interesting because B is a direct sum of nonabelian anyons in this case. We have

$$\mathcal{C}((2, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \cong \frac{\mathcal{C}_{\mathcal{M}(5,4)} \boxtimes_{\mathbb{D}} \mathcal{C}(3, \mathfrak{su}(2))}{A_2} , \quad (3.3.88)$$

¹⁸In the notation of [CGS23] $\mathcal{A} = (0, 0, 0) + (1, 1, 2)$.

with

$$A_2 = \text{Vir} \left(\frac{3}{5}, 0 \right) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(3, 0) \oplus \text{Vir} \left(\frac{3}{5}, \frac{3}{5} \right) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(3, 2) . \quad (3.3.89)$$

This isomorphism has been demonstrated in [CGS23] using the heuristic rules for anyon condensation of [BS09]¹⁹. We can also write

$$\mathcal{C}_{\mathcal{M}(5,4)} \cong \frac{\mathcal{C}((2, 1), \mathfrak{su}(2) \oplus \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(3, \mathfrak{su}(2))}{B_2} , \quad (3.3.90)$$

where

$$B_2 = L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((2, 1), (0, 0)) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(3, 0) \oplus L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((2, 1), (2, 1)) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(3, 3) . \quad (3.3.91)$$

$$\hat{\mathfrak{g}} = (\hat{\mathfrak{g}}_2)_1, \hat{\mathfrak{h}} = \hat{\mathfrak{su}}(2)_3, \hat{\mathfrak{su}}(2)_1.$$

The affine VOA for the exceptional Lie algebra²⁰ $\mathfrak{g}_2$ at level 1 has two irreducible modules which we denote by $L_{\mathfrak{g}_2}(1, \mathbf{1}), L_{\mathfrak{g}_2}(1, \mathbf{7})$ where $\mathbf{1}, \mathbf{7}$ denotes the trivial and 7-dimensional representations of $\mathfrak{g}_2$. There is an embedding of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into $\mathfrak{g}_2$ under which the level of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is $(1, 3)$. One can check that this is a conformal embedding. We have the decomposition [CGS23]

$$\begin{aligned} L_{\mathfrak{g}_2}(1, \mathbf{1}) &\cong L_{\mathfrak{su}(2)}(1, 0) \otimes L_{\mathfrak{su}(2)}(3, 0) \oplus L_{\mathfrak{su}(2)}(1, 1) \otimes L_{\mathfrak{su}(2)}(3, 3), \\ L_{\mathfrak{g}_2}(1, \mathbf{7}) &\cong L_{\mathfrak{su}(2)}(1, 0) \otimes L_{\mathfrak{su}(2)}(3, 2) \oplus L_{\mathfrak{su}(2)}(1, 1) \otimes L_{\mathfrak{su}(2)}(3, 1) . \end{aligned} \quad (3.3.92)$$

Let us first understand the coset²¹ $L_{\mathfrak{g}_2/\mathfrak{su}(2)_1}(1, 0)$. From the decomposition (3.3.92), it is clear that $L_{\mathfrak{g}_2/\widehat{\mathfrak{su}}(2)_1}(1, 0) \cong L_{\mathfrak{su}(2)}(3, 0)$. By (3.3.60), we have

$$\mathcal{C}(1, \mathfrak{g}_2) \cong \frac{\mathcal{C}(3, \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}(1, \mathfrak{su}(2))}{A} , \quad (3.3.93)$$

where

$$A = L_{\mathfrak{su}(2)}(1, 0) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(3, 0) \oplus L_{\mathfrak{su}(2)}(1, 1) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(3, 3) . \quad (3.3.94)$$

Moreover

$$\mathcal{C}(3, \mathfrak{su}(2)) \cong \mathcal{C}(1, \mathfrak{g}_2) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(1, \mathfrak{su}(2)) , \quad (3.3.95)$$

since $B = L_{\mathfrak{su}(2)}(3, 0)$ which is the unit object. This agrees with [CGS23]. On the other hand

$$\mathcal{G}(1, \mathfrak{su}(2)) \cong \frac{\mathcal{C}(1, \mathfrak{g}_2) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(3, \mathfrak{su}(2))}{B} , \quad (3.3.96)$$

¹⁹In the notation of [CGS23] $\mathcal{A}_2 = (0, 0) + (\epsilon', 2)$.

²⁰The notation is unfortunate since the subscript in $\mathfrak{g}_2$ is not the level!

²¹Here, we have differed from our usual notation $L_{\mathfrak{g}/\mathfrak{h}}(0, 0)$ of coset VOA because we need to indicate with respect to which level $\widehat{\mathfrak{su}}(2)$ we are considering the commutant. So $L_{\mathfrak{g}_2/\mathfrak{su}(2)_1}(1, 0)$ denoted the commutant of the affine VOA $L_{\mathfrak{su}(2)}(1, 0)$ in the affine VOA $L_{\mathfrak{g}_2}(1, \mathbf{1})$.

where

$$B = L_{\mathfrak{g}_2}(1, \mathbf{1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(3, 0) \oplus L_{\mathfrak{g}_2}(1, \mathbf{7}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(3, 2) . \quad (3.3.97)$$

This was again demonstrated in [CGS23] using heuristic rules of anyon condensation from [BS09]²².

Conformal embeddings.

As we have already proved in Proposition 3.2.2, if a unitary VOA U is conformally embedded in a unitary VOA V , i.e, $c^U = c^V$, then the commutant $C_V(U) \cong \mathbb{C}$. Consequently, the category of $C_V(U)$ -modules is trivial. Theorem 3.2.3 in this case means that

$$\mathcal{C}_V \cong \mathcal{C}_U/V := (\mathcal{C}_U)_V^0 . \quad (3.3.98)$$

Or equivalently, by Theorem 3.2.2 V provides an extension of U . This is the well-known result that conformal embeddings lead to chiral algebra extension [DFMS97]. By (3.3.62), we see that

$$\frac{\mathcal{C}_V \boxtimes_{\mathbb{D}} \mathcal{C}_U^{\text{opp}}}{B} , \quad (3.3.99)$$

where

$$B = \bigoplus_{\substack{a \in \mathcal{I}_V \\ b \in \mathcal{I}_U}} (\dim \text{Hom}(W^a, U^b)) (W^a \boxtimes_{\mathbb{D}} (U^{\text{opp}})^b) , \quad (3.3.100)$$

is the trivial MTC. Let us now discuss some concrete examples.

$$\widehat{\mathfrak{su}}(N)_k \oplus \widehat{\mathfrak{su}}(k)_N \hookrightarrow \widehat{\mathfrak{su}}(Nk)_1 , \quad N, k \geq 2$$

This is an infinite family of conformal embeddings. By (3.3.60), we have

$$\mathcal{C}(1, \mathfrak{su}(Nk)) \cong \frac{\mathcal{C}(N, \mathfrak{su}(k)) \boxtimes_{\mathbb{D}} \mathcal{C}(k, \mathfrak{su}(N))}{A_{k,N}} , \quad (3.3.101)$$

where

$$A_{k,N} = \bigoplus_{(\lambda_1, \lambda_2) \in K} L_{\mathfrak{su}(k)}(N, \hat{\lambda}_1) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(N)}(k, \hat{\lambda}_2) \cong L_{\mathfrak{su}(Nk)}(1, 0) , \quad (3.3.102)$$

and the set $K \subset P_N^+(\mathfrak{su}(k)) \times P_k^+(\mathfrak{su}(N))$ can be read off from [ABI90, Theorem 1]. This proves [CGS23, Eq. (5.3)]. We can easily find $A_{k,N}$ for small values of k, N . For example, for $N = k = 2$, we find that

$$L_{\mathfrak{su}(4)}(1, 0) \cong L_{\mathfrak{su}(2)}(2, 0) \otimes L_{\mathfrak{su}(2)}(2, 0) \oplus L_{\mathfrak{su}(2)}(2, 2) \otimes L_{\mathfrak{su}(2)}(2, 2) . \quad (3.3.103)$$

From the fusion rules for $\mathcal{C}(2, \mathfrak{su}(2))$, one can show that the simple objects $L_{\mathfrak{su}}(2, 0), L_{\mathfrak{su}}(2, 2)$ generate a $\mathbb{Z}_2$ fusion group. Thus we can schematically write

$$\mathcal{C}(1, \mathfrak{su}(4)) \cong \frac{\mathcal{C}(2, \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}(2, \mathfrak{su}(2))}{\mathbb{Z}_2} , \quad (3.3.104)$$

²²In the notation of [CGS23], $B = (1, 0) + (\phi, 2)$ which generates the Drinfeld center $\mathcal{Z}(\text{Fib})$ of the Fibonacci MTC.

which agrees with [CGS23, Eq. (5.2)]. For $N = 3, k = 2$, we can use the isomorphism [ABI90, Theorem 1]

$$L_{\mathfrak{su}(6)}(1, \mathbf{1}) \cong L_{\mathfrak{su}(3)}(2, \mathbf{1}) \otimes L_{\mathfrak{su}(2)}(3, 0) \oplus L_{\mathfrak{su}(3)}(2, \mathbf{8}) \otimes L_{\mathfrak{su}(2)}(3, 2) , \quad (3.3.105)$$

where the bold numbers $\mathbf{n}$ indicate the n -dimensional representation of the algebra, to obtain the isomorphism

$$\mathcal{C}(1, \mathfrak{su}(6)) \cong \frac{\mathcal{C}(2, \mathfrak{su}(3)) \boxtimes_{\mathbb{D}} \mathcal{C}(3, \mathfrak{su}(2))}{A_{3,2}} , \quad (3.3.106)$$

with

$$A_{3,2} = L_{\mathfrak{su}(3)}(2, \mathbf{1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(3, 0) \oplus L_{\mathfrak{su}(3)}(2, \mathbf{8}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(3, 2) . \quad (3.3.107)$$

This again agrees with the computations in [CGS23, Section 5.1.1].

If we restrict the embedding to $\widehat{\mathfrak{su}}(N)_k \hookrightarrow \widehat{\mathfrak{su}}(Nk)_1$, then we expect that the coset VOA will be the affine $\mathfrak{su}(N)$ VOA:

$$V_{\mathfrak{su}(Nk)/\mathfrak{su}(N)} \cong L_{\mathfrak{su}(k)}(N, \mathbf{1}) . \quad (3.3.108)$$

This can be proved using the character decomposition worked out in [ABI90, Theorem 1]. Then by (3.3.62), we have

$$\mathcal{C}(N, \mathfrak{su}(k)) \cong \frac{\mathcal{C}(1, \mathfrak{su}(Nk)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(k, \mathfrak{su}(N))}{B} , \quad (3.3.109)$$

where the algebra object B can be obtained using [ABI90, Theorem 1]. We find that the weights $\lambda \in P_1^+(\mathfrak{su}(Nk)), \mu \in P_k^+(\mathfrak{su}(N))$ such that

$$L_{\mathfrak{su}(Nk)/\mathfrak{su}(N)}(\hat{\lambda}, \hat{\mu}) \cong L_{\mathfrak{su}(k)}(N, 0) , \quad (3.3.110)$$

are in the orbit of a $\mathbb{Z}_N$ -action on $P_k^+(\mathfrak{su}(N))$. This suggests that the fusion algebra of simple subobjects of B form the $\mathbb{Z}_N$ group. This provides evidence for the level rank duality of [HS16]. A detailed mathematical proof of this equivalence can be found in [OS14].

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{su}}(3)_1, \widehat{\mathfrak{h}} = \widehat{\mathfrak{su}}(2)_4$$

There are three dominant integral weights of $\widehat{\mathfrak{su}}(3)_1$. We denote them by $\mathbf{1}, \mathbf{3}, \overline{\mathbf{3}}$ as in [CGS23]. Then the decomposition of modules for this conformal embedding is given by

$$\begin{aligned} L_{\mathfrak{su}(3)}(1, \mathbf{1}) &\cong L_{\mathfrak{su}(2)}(4, 0) \oplus L_{\mathfrak{su}(2)}(4, 4) , \\ L_{\mathfrak{su}(3)}(1, \mathbf{3}) &\cong L_{\mathfrak{su}(2)}(4, 2) , \\ L_{\mathfrak{su}(3)}(1, \overline{\mathbf{3}}) &\cong L_{\mathfrak{su}(2)}(4, 2) . \end{aligned} \quad (3.3.111)$$

(3.3.60) then implies

$$\frac{\mathcal{C}(1, \mathfrak{su}(3))}{A} \cong \mathcal{C}(4, \mathfrak{su}(2)) , \quad (3.3.112)$$

where

$$A = L_{\mathfrak{su}(2)}(4, 0) \oplus L_{\mathfrak{su}(2)}(4, 4) . \quad (3.3.113)$$

Moreover, since the coset VOA is trivial, the MTC

$$\frac{\mathcal{C}(1, \mathfrak{su}(3)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(4, \mathfrak{su}(2))}{B} , \quad (3.3.114)$$

with

$$\begin{aligned} B = & L_{\mathfrak{su}(3)}(1, \mathbf{1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(4, 0) \oplus L_{\mathfrak{su}(3)}(1, \mathbf{1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(4, 0) \\ & \oplus L_{\mathfrak{su}(3)}(1, \mathbf{3}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(4, 2) \oplus L_{\mathfrak{su}(3)}(1, \bar{\mathbf{3}}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(4, 2) , \end{aligned} \quad (3.3.115)$$

is the trivial MTC. This was argued in [CGS24, Eq. (4.4)].

Maverick Cosets.

These are a set of coset VOA for which the selection rules for $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ to be nontrivial obtained from algebraic considerations is not enough [DJ93b, PSW99]. Moreover, field identification method also fails for such cosets [DJ93b, PSW99]. A list of all known Maverick cosets can be found in [CGS23]. We now give an example of a Maverick coset for which our category theory methods still works.

$$\hat{\mathfrak{g}} = \widehat{\mathfrak{su}(3)}_2, \hat{\mathfrak{h}} = \widehat{\mathfrak{su}(2)}_8$$

From the explicit analysis of branching functions [DJ93a], one can show that the coset VOA in this case is the chiral algebra of the critical three-state-Potts model [DJ93b, DFMS97]²³. We want to identify the algebra object such that

$$\mathcal{C}(2, \mathfrak{su}(3)) \cong \frac{\mathcal{C}(\mathfrak{su}(3), \mathfrak{su}(2)) \boxtimes_{\mathbb{D}} \mathcal{C}(8, \mathfrak{su}(2))}{A} . \quad (3.3.116)$$

There are six dominant integral weights of $\widehat{\mathfrak{su}(3)}$ at level 2 which we will denote by the dimension (in bold) of the corresponding $\mathfrak{su}(3)$ representation. From [DJ93b, Table 1], we find that

$$\begin{aligned} A = & L_{\mathfrak{su}(3)/\mathfrak{su}(2)}(\mathbf{1}, 0) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(8, 0) \oplus L_{\mathfrak{su}(3)/\mathfrak{su}(2)}(\mathbf{1}, 8) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(8, 8) \\ & \oplus L_{\mathfrak{su}(3)/\mathfrak{su}(2)}(\mathbf{1}, 4) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}(8, 4) . \end{aligned} \quad (3.3.117)$$

Inverting the coset using (3.3.62), we find

$$\mathcal{C}(\mathfrak{su}(3), \mathfrak{su}(2)) \cong \frac{\mathcal{C}(2, \mathfrak{su}(3)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(8, \mathfrak{su}(2))}{B} , \quad (3.3.118)$$

²³It is an extension of $\text{Vir}(\frac{4}{5}, 0)$ by dimension 3 holomorphic field.

where, again using [DJ93b, Table 1], we find

$$B = L_{\mathfrak{su}(3)}(2, \mathbf{1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(8, 0) \oplus L_{\mathfrak{su}(3)}(2, \mathbf{1}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(8, 8) \oplus L_{\mathfrak{su}(3)}(2, \mathbf{8}) \boxtimes_{\mathbb{D}} L_{\mathfrak{su}(2)}^{\text{opp}}(8, 4) . \quad (3.3.119)$$

This is in agreement with the heuristic calculation of [BS09].

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{su}}(4)_1, \widehat{\mathfrak{h}} = \widehat{\mathfrak{su}}(2)_{10}$$

From the explicit computation of branching functions, it was shown in [PSW99] that the coset VOA for this Maverick coset is the Ising VOA $\text{Vir}(\frac{1}{2}, 0)$. Thus, by Corollary 3.3.4, we have

$$\mathcal{C}(1, \mathfrak{su}(4)) \cong \frac{\mathcal{C}_{\mathcal{M}(4,3)} \boxtimes_{\mathbb{D}} \mathcal{C}(10, \mathfrak{su}(2))}{A} , \quad (3.3.120)$$

where [PSW99]

$$\begin{aligned} A = & \text{Vir}\left(\frac{1}{2}, 0\right) \boxtimes_{\mathbb{D}} (L_{\mathfrak{su}(2)}(10, 0) \oplus L_{\mathfrak{su}(2)}(10, 6)) \\ & \oplus \text{Vir}\left(\frac{1}{2}, \frac{1}{2}\right) \boxtimes_{\mathbb{D}} (L_{\mathfrak{su}(2)}(10, 4) \oplus L_{\mathfrak{su}(2)}(10, 10)) \cong L_{\mathfrak{su}(4)}(1, \mathbf{1}) . \end{aligned} \quad (3.3.121)$$

By (3.3.62), we have

$$\mathcal{C}_{\mathcal{M}(4,3)} \cong \frac{\mathcal{C}(1, \mathfrak{su}(4)) \boxtimes_{\mathbb{D}} \mathcal{C}^{\text{opp}}(10, \mathfrak{su}(2))}{B} , \quad (3.3.122)$$

where [PSW99]

$$\begin{aligned} B = & L_{\mathfrak{su}(4)}(1, \mathbf{1}) \boxtimes_{\mathbb{D}} (L_{\mathfrak{su}(2)}^{\text{opp}}(10, 0) \oplus L_{\mathfrak{su}(2)}^{\text{opp}}(10, 6)) \\ & \oplus L_{\mathfrak{su}(4)}(1, \mathbf{6}) \boxtimes_{\mathbb{D}} (L_{\mathfrak{su}(2)}^{\text{opp}}(10, 4) \oplus L_{\mathfrak{su}(2)}^{\text{opp}}(10, 10)) . \end{aligned} \quad (3.3.123)$$

This is in agreement with the analysis of [CGS23, Appendix C.3].

3.4 Review Of WZW Models And Their Gauging

In this section, we will review WZW models and their gauging from path integral as well as in the algebraic formalism. We follow the conventions of [Gaw91, Ebe19]. Let G be a compact semisimple Lie group of the form

$$G = \left(\prod_{i=1}^n G_i \right) / Z , \quad (3.4.1)$$

where each G_i is simple, simply connected and Z is a subgroup of the center of $\prod_i G_i$. Fix a vector of positive integers $\mathbf{k} = (k_1, \dots, k_n) \in (\mathbb{Z}_{>0})^n$. Let $\langle \cdot, \cdot \rangle_{\mathbf{k}}$ be the Killing form on $\mathfrak{g} := \text{Lie}(G)$ extended to the complexification $\mathfrak{g}^{\mathbb{C}} := \oplus_i \mathfrak{g}_i^{\mathbb{C}}$ normalized such that half the length-squared of long roots of $\mathfrak{g}_i$ is equal to k_i^{-1} . On any Riemann surface Σ and a smooth map $g : \Sigma \rightarrow G$, define the functional

$$S_{G, \mathbf{k}}(g) := \frac{i}{4\pi} \int_{\Sigma} d^2z \langle g^{-1} \partial_z g, g^{-1} \partial_{\bar{z}} g \rangle_{\mathbf{k}} - i\Gamma_{\text{WZ}}(\Sigma; g) , \quad (3.4.2)$$

where

$$\Gamma_{\text{WZ}}(\Sigma; g) := \frac{1}{24\pi} \int_B \langle \tilde{g}^{-1} d\tilde{g}, [\tilde{g}^{-1} d\tilde{g}, \tilde{g}^{-1} d\tilde{g}] \rangle_{\mathbf{k}} , \quad (3.4.3)$$

is the Wess-Zumino (WZ) term, B is 3-manifold such that $\partial B = \Sigma$ and $\tilde{g} : B \rightarrow G$ is an extension²⁴ of the map g , i.e., $\tilde{g}|_{\partial B} = g$. The action is manifestly conformally invariant. We will denote the corresponding CFT by $\text{WZW}(G)$. The partition function of WZW model is given by

$$Z_{G,\mathbf{k}} = \int [Dg] \exp(-S_{G,\mathbf{k}}(g)) . \quad (3.4.4)$$

The action has a $(G_L \times G_R)$ -symmetry which acts on g as

$$(g_L, g_R) \cdot g = g_L g g_R^{-1} . \quad (3.4.5)$$

Up to a constant multiple, the currents for these symmetries in complex coordinates are

$$J = g^{-1} \partial_z g, \quad \tilde{J} = -g^{-1} \partial_{\bar{z}} g . \quad (3.4.6)$$

Using the equations of motion, one can show that

$$\partial_z \tilde{J} = \partial_{\bar{z}} J = 0 . \quad (3.4.7)$$

The conservation of these currents implies that the global $(G_L \times G_R)$ -symmetry of the action is enhanced to a local symmetry: the action is invariant under the transformation²⁵

$$g(z, \bar{z}) \mapsto g_L(z) g(z, \bar{z}) g_R(\bar{z})^\dagger , \quad (3.4.8)$$

by an arbitrary holomorphic map²⁶ $g_L : \Sigma \rightarrow G^\mathbb{C}$ and an antiholomorphic map $g_R : \Sigma \rightarrow G^\mathbb{C}$. On the cylinder, parametrized by (σ, τ) with $\sigma \sim \sigma + 2\pi$ being the spatial coordinate and τ the Euclidean time and related to the complex coordinate on the plane by $z = e^{-i\sigma + \tau}$, the local $(G_L \times G_R)$ -symmetry at a fixed value of the Euclidean time $\tau = \tau_0$ is the loop group $(LG_L \times LG_R)$:

$$g(z, \bar{z}) \mapsto \gamma_L(\sigma) g(z, \bar{z}) \gamma_R(\sigma)^{-1} , \quad (3.4.9)$$

where $\gamma_{L,R} : S^1 \rightarrow G$ are arbitrary smooth maps from the spatial circle to G . Again, this symmetry enhances to the loop group $LG_L^\mathbb{C} \times LG_R^\mathbb{C}$ of the complexification $G^\mathbb{C}$.

Upon quantization, the loop group symmetry gets centrally extended to $(\widetilde{LG}_L \times \widetilde{LG}_R)$ -symmetry. To see this, let us expand $J(z)$ and $\tilde{J}(\bar{z})$ in an orthonormal basis (with respect to the Killing form)

²⁴When G is simply connected, then the existence of such an extension follows from $\pi_1(G) = \pi_2(G) = 0$. For general G , see [Gaw87, FGK88, ADPW91].

²⁵Here $\dagger$ denotes the complex conjugation operation in $\mathfrak{g}^\mathbb{C} = \text{Lie}(G^\mathbb{C})$ which leaves $G \subset G^\mathbb{C}$ invariant. Note that our convention is $G = e^{\mathfrak{g}}$.

²⁶Note that the map must be valued in the complexification $G^\mathbb{C}$ to be (anti)holomorphic since there are no nonconstant holomorphic maps from a compact Riemann surface Σ to a compact group G .

of $\mathfrak{g}$ and denote the components by $J^a(z)$ and $\tilde{J}^a(\bar{z})$ respectively. Here $a = 1, \dots, \dim(\mathfrak{g})$. Then one can prove the OPE

$$\begin{aligned} J^{a_i}(z)J^{b_i}(w) &\sim \frac{k_i\delta^{a_ib_i}}{(z-w)^2} + \frac{if^{a_ib_i}_{c_i}J^{c_i}(w)}{(z-w)}, \\ \tilde{J}^{a_i}(\bar{z})\tilde{J}^{b_i}(\bar{w}) &\sim \frac{k_i\delta^{a_ib_i}}{(\bar{z}-\bar{w})^2} + \frac{if^{a_ib_i}_{c_i}\tilde{J}^{c_i}(\bar{w})}{(\bar{z}-\bar{w})}, \\ J^{a_i}(z)\tilde{J}^{b_i}(\bar{w}) &\sim \text{regular} \quad , \end{aligned} \quad (3.4.10)$$

where a_i, b_i, c_i are indices labelling a basis of $\mathfrak{g}_i$, $f^{a_ib_i}_{c_i}$ are structure constants of $\mathfrak{g}_i$. Expanding the currents as

$$\begin{aligned} J^a(z) &= \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1} \quad , \\ \tilde{J}^a(\bar{z}) &= \sum_{n \in \mathbb{Z}} \tilde{J}_n^a \bar{z}^{-n-1} \quad , \end{aligned} \quad (3.4.11)$$

and using the OPE (3.4.10), we obtain the *affine Kac-Moody algebra*

$$\begin{aligned} [J_m^{a_i}, J_n^{b_i}] &= k_i m \delta^{a_ib_i} \delta_{m+n,0} + if^{a_ib_i}_{c_i} J_{m+n}^{c_i} \quad , \\ [\tilde{J}_m^{a_i}, \tilde{J}_n^{b_i}] &= k_i m \delta^{a_ib_i} \delta_{m+n,0} + if^{a_ib_i}_{c_i} \tilde{J}_{m+n}^{c_i} \quad , \\ [J_m^{a_i}, \tilde{J}_n^{b_i}] &= 0 \quad . \end{aligned} \quad (3.4.12)$$

The affine Kac-Moody algebra is the Lie algebra of the centrally extended loop group $\widetilde{LG}$ [PS86].

The Lagrangian is conformally invariant classically. The stress tensor is given by

$$T^{\mathfrak{g}}(z) = \frac{1}{2(\mathbf{k} + \mathbf{h}^\vee)} : J^a(z)J^a(z) : , \quad \bar{T}^{\mathfrak{g}}(\bar{z}) = \frac{1}{2(\mathbf{k} + \mathbf{h}^\vee)} : \tilde{J}^a(\bar{z})\tilde{J}^a(\bar{z}) : , \quad (3.4.13)$$

where

$$\frac{1}{2(\mathbf{k} + \mathbf{h}^\vee)} := \sum_{i=1}^n \frac{1}{2(k_i + h_i^\vee)} \quad , \quad (3.4.14)$$

and $h_i^\vee$ is the dual Coxeter number of $\mathfrak{g}_i$. They satisfy the expected OPE with central charge

$$c_{\mathfrak{g}} := \frac{\mathbf{k} \dim(\mathfrak{g})}{\mathbf{k} + \mathbf{h}^\vee} := \sum_{i=1}^n \frac{k_i \dim(\mathfrak{g}_i)}{k_i + h_i^\vee} \quad . \quad (3.4.15)$$

Consequently, if we expand

$$T^{\mathfrak{g}}(z) = \sum_{n \in \mathbb{Z}} L_n^{\mathfrak{g}} z^{-n-2} \quad , \quad \bar{T}^{\mathfrak{g}}(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{L}_n^{\mathfrak{g}} \bar{z}^{-n-2} \quad , \quad (3.4.16)$$

then the modes satisfy the Virasoro algebra. This is the *Sugawara construction*. The currents $J^a, \tilde{J}^a$ are primary fields of weights $(1,0), (0,1)$ respectively. The algebra above is thus two copies of the affine Kac-Moody algebra $\hat{\mathfrak{g}}_{\mathbf{k}}$ and the Sugawara construction shows that $\mathcal{U}(\hat{\mathfrak{g}}_{\mathbf{k}})$ contains the Virasoro algebra. Thus the WZW model $WZW(G)$ model at level $\mathbf{k}$ gives a realization of the affine vertex operator algebra $\text{affine}(\mathfrak{g})$ discussed in Section 3.3.1.

3.4.1 Gauged WZW Model

The GKO coset VOA described in Section 3.3.2 is an algebraic construction. It is not apriori clear whether it can be given a Lagrangian description. It was proposed that one can gauge some subgroup of $(G_L \times G_R)$ -symmetry of the G -WZW model to obtain the GKO coset CFT [ABR88, BRS88, GK88, GK89]. The main goal of this chapter is to show that gauged WZW model indeed realizes GKO coset VOA $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ but only up to a 2d TQFT. In this section, we will set up the gauging procedure.

Recall that the action of G -WZW model has a $(G_L \times G_R)$ -symmetry. We would like to gauge a subgroup $F \subset G_L \times G_R$. To gauge the subgroup F , we introduce a principal F -bundle $P \rightarrow \Sigma$ along with a connection A on P . Recall that for any F -space S , we have an associated F -bundle P_S defined by

$$P_S := P \times_F S := (P \times S) / \sim, \quad (3.4.17)$$

where $\sim$ is the equivalence relation

$$(p, s) \sim (p \cdot f^{-1}, f \cdot s), \quad f \in F. \quad (3.4.18)$$

Moreover, any connection A on P induces a connection on P_S . In our case, we take $S = G$ to be the F -space. Let

$$\pi_L : G_L \times G_R \longrightarrow G, \quad \pi_R : G_L \times G_R \longrightarrow G_R, \quad (3.4.19)$$

be the projection to the left and right components of $G_L \times G_R$. For $h \in H$, we write

$$f_L := \pi_L(f), \quad f_R := \pi_R(f), \quad f \in F \subset G_L \times G_R. \quad (3.4.20)$$

Consider the associated bundle P_G with F acting on G as

$$g \longmapsto f_L g f_R^{-1}, \quad f \in F \subset G_L \times G_R. \quad (3.4.21)$$

More explicitly,

$$P_G = P \times_F G := (P \times G) / \sim \quad (3.4.22)$$

where the equivalence relation is

$$(p, g) \sim (p \cdot f^{-1}, f_L g f_R^{-1}), \quad f \in F \subset G_L \times G_R. \quad (3.4.23)$$

The WZW field $g : \Sigma \longrightarrow G$ is now replaced by a section of the bundle P_G . Since A is Lie algebra valued, we have the embedding

$$A \mapsto A_L \oplus A_R, \quad (3.4.24)$$

given by

$$A^a t^a \mapsto A_L \oplus A_R := A^a (t_L^a \oplus t_R^a) , \quad (3.4.25)$$

where $t^a \in \mathfrak{h}$ is embedded in $\mathfrak{g} \oplus \mathfrak{g}$ as $t^a \mapsto (t_L^a \oplus t_R^a)$. We can also decompose these connections into the holomorphic and antiholomorphic parts

$$A = A^{(1,0)} + A^{(0,1)} , \quad A^{(1,0)} = -(A^{(0,1)})^* , \quad (3.4.26)$$

where $A^{(1,0)}, A^{(0,1)}$ is the dz and $d\bar{z}$ parts and we have also imposed a reality condition on the connection²⁷. To write the gauged action, we promote the exterior derivative in the first term of $S_{G,\mathbf{k}}(g)$ in (3.4.2) to gauge covariant derivative:

$$\partial \rightarrow \partial + A^{(1,0)} , \quad \bar{\partial} \rightarrow \bar{\partial} + A^{(0,1)} , \quad (3.4.27)$$

where $\partial, \bar{\partial}$ are the $(1,0)$ and $(0,1)$ part of the exterior derivative respectively. Moreover, writing

$$A^{(1,0)} = A_z dz , \quad A^{(0,1)} = A_{\bar{z}} d\bar{z} , \quad (3.4.28)$$

we can write the action of A on the WZW field as

$$A_z \cdot g = (A_z)_L g - g(A_z)_R , \quad A_{\bar{z}} \cdot g = (A_{\bar{z}})_L g - g(A_{\bar{z}})_R . \quad (3.4.29)$$

The Wess-Zumino term does not admit a straightforward gauge invariant extension. It turns out that Γ_{WZ} can be extended to a gauge invariant term if the subgroup H is *anomaly-free* in the following sense [Wit92a, Wit93]: We embed $t^a \in \mathfrak{h}$ into $\mathfrak{g} \oplus \mathfrak{g}$ by $t^a \mapsto (t_L^a, t_R^a)$ and then demand

$$\langle t_L^a, t_L^b \rangle = \langle t_R^a, t_R^b \rangle . \quad (3.4.30)$$

Using the extension of the WZ term given in [Wit92a, Wit93], the H -gauged WZW action is given by

$$\begin{aligned} S_{G,\mathbf{k}}(g, A) := S_{G,\mathbf{k}}(g) &+ \frac{i}{2\pi} \int_{\Sigma} d^2z \left(\langle (A_z)_R, g^{-1} \partial_{\bar{z}} g \rangle_{\mathbf{k}} + \langle g \partial_z g^{-1}, (A_{\bar{z}})_L \rangle_{\mathbf{k}} \right. \\ &\left. + \langle g(A_z)_R g^{-1}, (A_{\bar{z}})_L \rangle_{\mathbf{k}} - \langle (A_z)_R, (A_{\bar{z}})_L \rangle_{\tilde{\mathbf{k}}} \right) . \end{aligned} \quad (3.4.31)$$

Let $\text{Prin}_H(\Sigma)$ denote the isomorphism classes of principal H -bundles over Σ , $\mathcal{A}_P$ denote the space of unitary connections on P and $\mathcal{G}_P$ be the group gauge transformations, and $G^{\Sigma} := \text{Map}(\Sigma, G)$. Then the partition function of the theory on Σ is given by

$$Z_{G/H}^{\text{WZW}}(\Sigma) = \sum_{[P] \in \text{Prin}_H(\Sigma)} \int_{G^{\Sigma}} [Dg] \int_{\mathcal{A}_P / \mathcal{G}_P} [DA] \exp(-S_{G,\mathbf{k}}(g, A)) . \quad (3.4.32)$$

²⁷ Here $*$ is complex conjugation in $\mathfrak{h}^{\mathbb{C}}$ which leaves $\mathfrak{h}$ invariant. We also use the convention that $H = e^{\mathfrak{h}}$.

The action $S_{G,\mathbf{k}}(g, A)$ transforms according to the *Polyakov-Wiegmann formula* [PW84] under the gauge transformation: consider $h : \Sigma \longrightarrow H^{\mathbb{C}} \subset G_L^{\mathbb{C}} \times G_R^{\mathbb{C}}$ and introduce the notation

$${}^h g := h_L g h_R^{\dagger}, \quad ({}^h A)_z := h^{\dagger} A_z (h^{\dagger})^{-1} - h^{\dagger} \partial_z (h^{\dagger})^{-1}, \quad ({}^h A)_{\bar{z}} := h A_{\bar{z}} h^{-1} + h \partial_{\bar{z}} h^{-1}, \quad (3.4.33)$$

then we have [Wit92a, Wit93, GK89]

$$S_{G,\mathbf{k}}(g, A) = S_{G,\mathbf{k}}({}^h g, {}^h A) - S_{H,\tilde{\mathbf{k}}}({}^h h, {}^h A), \quad (3.4.34)$$

where the second term on the RHS is the gauged H -WZW action with the diagonal embedding of H in $H_L \times H_R$. When we take h to be valued in H , then $S_{H,\tilde{\mathbf{k}}}({}^h h, {}^h A)$ vanishes and the gauged WZW action is gauge invariant.

In the remainder of the chapter, we will be working in the following simplified setting: given any subgroup $H < G$, the diagonal embedding of H in $G_L \times G_R$ given by

$$\begin{aligned} \Delta : H &\longrightarrow G_L \times G_R \\ h &\longmapsto (h, h), \end{aligned} \quad (3.4.35)$$

is anomaly-free. Indeed, for diagonal embedding, we have $t_L^a = t_R^a = t^a \in \mathfrak{h}$ and (3.4.30) is trivially satisfied. From now on, by G/H -WZW model, we will mean this particular H -gauged G -WZW model. We will denote the resulting CFT by $\text{WZW}(G/H)$.

3.5 H/H -WZW Model As A Topological Quantum Field Theory

In this section, we discuss some features of the H/H -WZW model and the relevance of a theorem of Freed-Hopkins-Teleman [FHT07] about the twisted equivariant K-theory.

The action of the H/H -WZW model is given by (3.4.31) with g replaced by the H -WZW field $h \in \text{Map}(\Sigma, H)$. For H simple and simply connected, this theory was shown to be topological in [Wit92a, Wit93]. More precisely, [Wit84] shows that under an infinitesimal change of the metric and changing the path integral variable of the measure DA appropriately, the path integral for the partition function remains invariant. [Wit84] also shows that the partition function $Z_{H/H,\tilde{\mathbf{k}}}(\Sigma)$ defined in (3.4.32) is equal to the dimension of the space of nonabelian theta functions on Σ . In particular, the torus partition function $Z_{H/H,\tilde{\mathbf{k}}}(\mathbb{T}^2)$ is equal to the dimension of Verlinde algebra $V_{\tilde{\mathbf{k}}}(H)$.

As we have mentioned, it is a theorem that to give a 2d TQFT is to give a commutative Frobenius algebra. Recall that a Frobenius algebra is a $\mathbb{C}$ -vector space A equipped with a nondegenerate bilinear pairing $\eta : A \otimes A \longrightarrow \mathbb{C}$, called the Frobenius pairing. In case the 2d TQFT is described by a semisimple Frobenius algebra, the partition function of the 2d TQFT on any compact Riemann

surface Σ_g of genus g takes a simple form. To describe the partition function, we note that for a semisimple Frobenius algebra, there exists a basis $\{\varepsilon_x\}_{x \in \mathcal{X}}$ of idempotents of A such that

$$A = \text{Span}_{\mathbb{C}}\{\varepsilon_x\} \ , \quad \varepsilon_x \varepsilon_y = \delta_{xy} \mathbb{1} \ . \quad (3.5.1)$$

Then there exists $\theta_x \in \mathbb{C}$ such that

$$\eta(\varepsilon_x \otimes \mathbb{1}) = \theta_x \ . \quad (3.5.2)$$

The partition function of the associated TQFT is given by

$$Z(\Sigma_g) = \sum_{x \in \mathcal{X}} \theta_x^{1-g} \ . \quad (3.5.3)$$

We now describe the Frobenius algebra structure emerging from H/H -WZW model. Recall that the Verlinde algebra $V_{\tilde{\mathbf{k}}}(H)$ is isomorphic to the $\mathbb{C}$ -span of $P_{\tilde{\mathbf{k}}}^+(\mathfrak{h})$:

$$V_{\tilde{\mathbf{k}}}(H) = \text{Span}_{\mathbb{C}} \left\{ \mathcal{O}_{\mu} : \mu \in P_{\tilde{\mathbf{k}}}^+(\mathfrak{h}) \right\} \ . \quad (3.5.4)$$

The multiplication on $V_{\tilde{\mathbf{k}}}(\mathfrak{h})$ is given by

$$\mathcal{O}_{\mu_1} \cdot \mathcal{O}_{\mu_2} = \sum_{\mu_3 \in P_{\tilde{\mathbf{k}}}^+(\mathfrak{h})} \mathcal{N}_{\mu_1 \mu_2}^{\mathfrak{h}, \mu_3} \mathcal{O}_{\mu_3} \ , \quad (3.5.5)$$

where the $\mathcal{N}_{\mu_1 \mu_2}^{\mathfrak{h}, \mu_3} \in \mathbb{Z}_{\geq 0}$ are the fusion rules. Note that the product is commutative and $\mathcal{O}_0$ is the identity of the multiplication. Then $V_{\tilde{\mathbf{k}}}(H)$ can be endowed with a Frobenius algebra structure by defining the Frobenius pairing to be

$$\langle \mathcal{O}_{\mu_1}, \mathcal{O}_{\mu_2} \rangle = \delta_{\mu_2, \mu_1^*} \ . \quad (3.5.6)$$

From the point of view of H/H -WZW, the Frobenius algebra structure can be obtained from H/H -path integral as follows: the multiplication is defined as

$$\mathcal{N}_{\mu_1 \mu_2 \mu_3}^{\mathfrak{h}} := \left\langle \prod_{i=1}^3 \text{Tr } h_{\mu_i}(z_i) \right\rangle_{\mathbb{CP}^1}^{H/H} \ , \quad (3.5.7)$$

along with the identification

$$\mathcal{N}_{\mu_1 \mu_2 \mu_3} = \mathcal{N}_{\mu_1 \mu_2}^{\mathfrak{h}, \mu_3^*} \ . \quad (3.5.8)$$

The Frobenius pairing is defined by the two-point function:

$$\langle \mathcal{O}_{\mu_1}, \mathcal{O}_{\mu_2} \rangle = \langle \mathcal{O}_{\mu_1}(z_1) \mathcal{O}_{\mu_2}(z_2) \rangle_{\mathbb{CP}^1}^{H/H} \ , \quad (3.5.9)$$

where $\mathcal{O}_{\mu_1}(z_1), \mathcal{O}_{\mu_2}(z_2)$ are the primary fields corresponding to the irreducible modules $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \mu_i)$, $i = 1, 2$. To write down the partition function of the theory, we need to find a basis of idempotents.

From (3.5.5), we see that it is equivalent to diagonalizing the matrix $[\mathcal{N}_{\mu_1}^h]^{\mu_3}_{\mu_2}$. This is precisely done by Verlinde formula. More precisely, Verlinde's formula is equivalent to the fact that the S -matrix diagonalizes fusion rules in the sense of the following equation:

$$\sum_{\mu_1, \mu_3 \in P_k^+(\mathfrak{h})} [(S^h)^{-1}]_{\mu_1 \nu} \mathcal{N}_{\mu_1 \mu_2}^{h, \mu_3} [S^h]_{\mu \mu_3} = \delta_{\nu}^{\mu} \frac{[S^h]_{\nu \mu_2}}{[S^h]_{\mu_2 0}}. \quad (3.5.10)$$

Thus, the basis of idempotents is

$$\varepsilon_{\mu} := \sum_{\nu \in P_k^+(\mathfrak{h})} [S^h]_{\mu 0} [(S^h)^{-1}]_{\nu \mu} \mathcal{O}_{\nu}. \quad (3.5.11)$$

From (3.5.6), it follows that the partition function is given by

$$Z_{H/H}^{\text{WZW}}(\Sigma_g) = \sum_{\mu \in P_k^+(\mathfrak{h})} ([S^h]_{\mu 0} [(S^h)^{-1}]_{0 \mu})^{1-g}. \quad (3.5.12)$$

Using the fact that $(S^h)^{\dagger} = (S^h)^{-1}$ and $S_{0\mu}^h$ is real, we get

$$Z_{H/H}^{\text{WZW}}(\Sigma_g) = \sum_{\mu \in P_k^+(\mathfrak{h})} ([S^h]_{\mu 0})^{2-2g}. \quad (3.5.13)$$

A theorem of Freed-Hopkins-Teleman [FHT07] identifies the Verlinde algebra $V_{\tilde{k}}(H)$ with the twisted equivariant K-theory ${}^{\tau}K_H(H)$ where the twisting τ is determined by the level $\tilde{k}$. A physical interpretation for this result was provided by [Moo03] using symmetry preserving D-branes. More precisely, using the description of D-branes as boundary conditions in target space H , it was argued in [Moo03] that the D-brane charges in H -WZW models are valued²⁸ in the K-theory ${}^{\tau}K_H(H)$. On the other hand, symmetry preserving branes in the worldsheet description in H -WZW models are integer linear combinations of the Cardy boundary states [Car89]. Since Cardy states are classified by elements of $V_{\tilde{k}}(H)$, we see that the target space and worldsheet description of D-branes give a physical perspective to the FHT theorem:

$$V_{\tilde{k}}(H) \cong {}^{\tau}K_H(H). \quad (3.5.14)$$

This is summarized schematically in Figure 3.1.

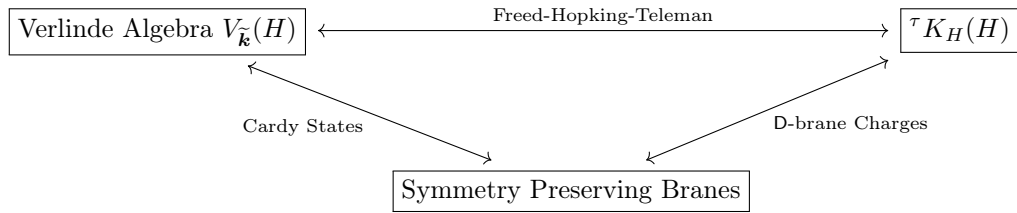


Figure 3.1: Schematic diagram illustrating the physical perspective to the FHT theorem.

²⁸See also [MM97, Wit98, MW99] for earlier work on relation of D-branes charges with K-theory.

This isomorphism induces a commutative Frobenius algebra structure²⁹ on ${}^\tau K_H(H)$.

3.5.1 The $U(1)/U(1)$ -WZW Model

In this section, we will discuss the $U(1)/U(1)$ -WZW model. Naively, since $U(1)$ acts trivially on itself by conjugation, the currents are trivial and hence the gauged WZW action is simply the non-linear sigma model action with $U(1)$ target. Moreover, the WZ term vanishes. This is problematic since we expect $U(1)/U(1)$ -WZW model to be a TQFT. So, to write the $U(1)/U(1)$ -WZW action, we will take the general H/H -WZW model action (3.4.31) and then specialize to $H = U(1)$ case. Let Σ be a compact Riemann surface with genus g . We will denote real coordinates on Σ by (x, y) and complex coordinates by $z, \bar{z}$ with the relation $z = x + iy$. We want to compute the path integral

$$Z_{U(1)/U(1)}^{\text{WZW}}(\Sigma) = \sum_{c_1(P) \in H^2(\Sigma, \mathbb{Z})} \int_{\text{Map}(\Sigma, U(1))} [Dg] \int_{\mathcal{A}_P / \mathcal{G}_P} [DA] \exp(-S_{U(1),k}(g, A)) , \quad (3.5.15)$$

where the sum is over isomorphism class of $U(1)$ -bundles $P \rightarrow \Sigma$ classified by the first Chern class $c_1(P)$, $\mathcal{A}_P$ is the space of connections on P , $\mathcal{G}_P := \text{Map}(\Sigma, U(1))$ is the group of local gauge transformations and $S_{U(1),k}(g, A)$ is the gauged WZW action (3.4.31) specialized to $G = H = U(1)$.

We will use the convention

$$\text{Lie}(U(1)) = i\mathbb{R} , \quad (3.5.16)$$

and $U(1) = e^{i\mathbb{R}}$. We will parametrize the WZW field $g(z, \bar{z}) : \Sigma \rightarrow U(1)$ as

$$g(z, \bar{z}) = e^{i\phi(z, \bar{z})} , \quad (3.5.17)$$

where $\phi(z, \bar{z}) : \Sigma \rightarrow \mathbb{R}$ is a periodic field:

$$\phi \sim \phi + 2\pi . \quad (3.5.18)$$

Then

$$\begin{aligned} g^{-1} \partial_z g &= e^{-i\phi} (i \partial_z \phi) e^{i\phi} = i \partial_z \phi , \\ \partial_{\bar{z}} g g^{-1} &= i \partial_{\bar{z}} \phi . \end{aligned} \quad (3.5.19)$$

Thus the gauged WZW action (3.4.31) is given by

$$S_{U(1),k}(g, A) = -\frac{ik}{4\pi} \int_{\Sigma} d^2 z \partial_z \phi \partial_{\bar{z}} \phi - \frac{k}{2\pi} \int_{\Sigma} d^2 z (\partial_{\bar{z}} \phi A_z - \partial_z \phi A_{\bar{z}}) . \quad (3.5.20)$$

The $U(1)/U(1)$ -action can then be written as

$$S_{U(1),k}(g, A) = \frac{k}{8\pi} \int_{\Sigma} d\phi \wedge \star d\phi + \frac{k}{2\pi} \int_{\Sigma} d\phi \wedge A , \quad (3.5.21)$$

²⁹[FHT07] had their own way of deriving a commutative Frobenius algebra structure on ${}^\tau K_H(H)$.

where we used the fact that

$$\begin{aligned}
d\phi \wedge \star d\phi &= (\partial_z \phi dz + \partial_{\bar{z}} \phi d\bar{z}) \wedge (\partial_z \phi (-idz) + \partial_{\bar{z}} \phi (id\bar{z})) \\
&= i\partial_z \phi \partial_{\bar{z}} \phi (dz \wedge d\bar{z}) - i\partial_{\bar{z}} \phi \partial_z \phi (d\bar{z} \wedge dz) \\
&= 2i\partial_z \phi \partial_{\bar{z}} \phi (dz \wedge d\bar{z}) ,
\end{aligned} \tag{3.5.22}$$

and

$$\begin{aligned}
d\phi \wedge A &= \partial\phi \wedge A^{(0,1)} + \bar{\partial}\phi \wedge A^{(1,0)} \\
&= (\partial_z \phi A_{\bar{z}} - \partial_{\bar{z}} \phi A_z) dz \wedge d\bar{z} .
\end{aligned} \tag{3.5.23}$$

Before proceeding, let us notice that the correct way to write the $d\phi \wedge A$ term is by using differential cohomology [MS25]. The reason is that A is not a globally defined 1-form. One may use Stokes theorem and write it as

$$\int_{\Sigma} d\phi \wedge A = \int_{\Sigma} F(A) \wedge \phi , \tag{3.5.24}$$

where $F(A)$ is the curvature of the connection which is globally defined. But now ϕ is ill-defined since it is defined only up to 2π period. The resolution is to treat the class $c_1(P) = \left[\frac{F}{2\pi}\right]$ as an element of $\check{H}^2(\Sigma)$, and the periodic scalar $e^{i\phi}$ as an element of $\check{H}^1(\Sigma)$ and use the pairing

$$\begin{aligned}
\check{H}^2(\Sigma) \times \check{H}^1(\Sigma) &\longrightarrow \check{H}^1(\text{pt}) = \mathbb{R}/\mathbb{Z} \\
\left(\left[\frac{F}{2\pi}\right], e^{i\phi}\right) &\longmapsto \int^{\check{H}} \left[\frac{F}{2\pi}\right] \odot e^{i\phi} ,
\end{aligned} \tag{3.5.25}$$

see [MS25] for details of this pairing. When $c_1(P) = 0$, then

$$\int^{\check{H}} \left[\frac{F}{2\pi}\right] \odot e^{i\phi} = \exp\left(\int_{\Sigma} d\phi \wedge A\right) . \tag{3.5.26}$$

When ϕ has no winding modes, then

$$\int^{\check{H}} \left[\frac{F}{2\pi}\right] \odot e^{i\phi} = \exp\left(\int_{\Sigma} F \wedge \phi\right) . \tag{3.5.27}$$

Let us first integrate over the zero modes of ϕ . To this end, let $\{\varphi_n\}_{n \geq 0} \subset C^\infty(\Sigma, \mathbb{R})$ be a basis of eigenfunctions of the Laplacian with eigenvalues λ_n :

$$\Delta \varphi_n = \lambda_n \varphi_n , \quad 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \tag{3.5.28}$$

Write

$$\phi = \phi_0 + \phi' , \tag{3.5.29}$$

where

$$\phi_0 = a_0 \varphi_0 , \quad \phi' = \sum_{n=1}^{\infty} a_n \varphi_n , \quad a_0, a_n \in \mathbb{R} . \tag{3.5.30}$$

Then the action is given by

$$S_{U(1),k}(g, A) = \frac{k}{8\pi} \int_{\Sigma} d\phi' \wedge \star d\phi' + \frac{k}{2\pi} \int_{\Sigma} d\phi' \wedge A + \frac{k}{2\pi} \int_{\Sigma} F(A) \wedge \phi_0 , \quad (3.5.31)$$

where we used Stokes theorem. Here $F(A)$ is the curvature of the connection and is nonzero for a nontrivial bundle. More precisely, the cohomology class $\left[\frac{F(A)}{2\pi}\right] = c_1(P) \in H^2(\Sigma, \mathbb{Z})$ is the first Chern class of the $U(1)$ -bundle $P \rightarrow \Sigma$. For a Riemann surface Σ , $H^2(\Sigma, \mathbb{Z}) \cong \mathbb{Z}$ and the integer corresponding to the first Chern class $c_1(P)$ is given by the Chern number defined by the integral of $F(A)$ on Σ :

$$n := \int_{\Sigma} c_1(P) = \frac{1}{2\pi} \int_{\Sigma} F(A) . \quad (3.5.32)$$

The Chern number classifies the bundle. The integral over ϕ_0 is given by

$$I(n) = \int_{\mathbb{R}/\mathbb{Z}} d\phi_0 \exp\left(-\frac{k}{2\pi} \int_{\Sigma} F(A) \wedge \phi_0\right) . \quad (3.5.33)$$

Notice that if we make a change of variable $\phi_0 \rightarrow \phi_0 + \gamma$ for some constant γ , since the measure is invariant, we get

$$\begin{aligned} I(n) &= \int_{\mathbb{R}/\mathbb{Z}} d(\phi_0 + \gamma) \exp\left(-\frac{k}{2\pi} \int_{\Sigma} F(A) \wedge (\phi_0 + \gamma)\right) \\ &= e^{-k\gamma n} I(n) . \end{aligned} \quad (3.5.34)$$

Thus, we have

$$I(n) = 2\pi \delta_{n,0} , \quad (3.5.35)$$

where the factor of 2π comes from the integral over ϕ_0 because of the periodicity $\phi_0 \sim \phi_0 + 2\pi$. This collapses the sum over isomorphism classes of bundles in the path integral to the trivial bundle. We now expand the field ϕ' in winding modes plus oscillators and also parametrize the connection appropriately. Let us choose a symplectic integral basis $\{\alpha^i, \beta_j\}_{i,j=1}^g$ of $H^1(\Sigma, \mathbb{Z})$ and an integral basis $\{A_i, B^j\}_{i,j=1}^g$ of $H_1(\Sigma, \mathbb{Z})$:

$$\begin{aligned} \int_{A_i} \alpha^j &= \delta_i^j, & \int_{A_i} \beta_j &= 0, & \int_{B^i} \alpha^j &= 0, & \int_{B^i} \beta_j &= \delta_j^i, \\ \int_{\Sigma} \alpha^i \wedge \alpha^j &= 0, & \int_{\Sigma} \alpha^i \wedge \beta_j &= \delta_j^i . \end{aligned} \quad (3.5.36)$$

Then we can write

$$d\phi' = n_i \alpha^i + m^j \beta_j + d\phi'', \quad n_i, m^i \in \mathbb{Z} . \quad (3.5.37)$$

where ϕ'' is a globally well-defined scalar orthogonal to the zero mode. The affine space of connections on a principal $U(1)$ -bundle $P \rightarrow \Sigma$ is given by

$$\mathcal{A}_P = A_0 + i\Omega^1(\Sigma, \mathbb{R}) , \quad (3.5.38)$$

where A_0 is a reference connection whose curvature satisfies $F(A_0) = -2\pi i n \text{vol}(g)$ (this follows from Donaldson's Theorem [Don83a], which is reformulation of results of [NS65, AB83]), and $\Omega^1(\Sigma, \mathbb{R})$ is the space of 1-forms on Σ . Since we want to restrict to trivial bundles, we can choose $A_0 = 0$. The gauge group $\mathcal{G}_P = \text{Map}(\Sigma, \text{U}(1))$ acts on $\mathcal{A}_P$ by

$$A \longmapsto {}^h A := A + i d\sigma, \quad h = e^{i\sigma} \in \text{Map}(\Sigma, \text{U}(1)) , \quad (3.5.39)$$

We will fix the gauge to the Landau gauge

$$d \star A = 0 . \quad (3.5.40)$$

Then, the partition takes the form

$$Z_{\text{U}(1)/\text{U}(1)}^{\text{WZW}}(\Sigma) = 2\pi \int [D\phi'] \int_{\mathcal{A}_P} [DA] \delta(d \star A) \exp(-S_{\text{U}(1),k}(\phi', A)) , \quad (3.5.41)$$

where $S_{\text{U}(1),k}(\phi', A)$ is given by

$$S_{\text{U}(1),k}(\phi', A) := \frac{k}{8\pi} \int_{\Sigma} d\phi' \wedge \star d\phi' + \frac{k}{2\pi} \int_{\Sigma} d\phi' \wedge A . \quad (3.5.42)$$

We now need a convenient parametrization of $\mathcal{A}_P$. The Hodge decomposition for $\Omega^1(\Sigma, \mathbb{R})$ states that

$$\Omega^1(\Sigma, \mathbb{R}) = \text{Im } d \oplus \text{Im } d^\dagger \oplus \mathcal{H}^1(\Sigma) , \quad (3.5.43)$$

where $\mathcal{H}^1(\Sigma)$ is the 2g-dimensional space of harmonic 1-forms on Σ . So we can write a generic local connection in $\mathcal{A}_P$ as

$$A = i d\omega_1 + i d^\dagger \omega_2 + i \omega^{\text{harmonic}} , \quad (3.5.44)$$

where $\omega^{\text{harmonic}} \in \mathcal{H}^1(\Sigma)$ and $\omega_1 \in \Omega^0(\Sigma, \mathbb{R})$, $\omega_2 \in \Omega^2(\Sigma, \mathbb{R})$. By Hodge's theorem, $\mathcal{H}^1(\Sigma) \cong H^1(\Sigma, \mathbb{R})$ and hence we can write

$$\omega^{\text{harmonic}} = \phi_i \alpha^i + \theta^j \beta_j, \quad \phi_i \sim \phi_i + 2\pi, \quad \theta^j \sim \theta^j + 2\pi . \quad (3.5.45)$$

Since $\Omega^0(\Sigma, \mathbb{R}) = C^\infty(\Sigma, \mathbb{R})$, we take $\omega_1 = \sigma \in C^\infty(\Sigma, \mathbb{R})$. Moreover, using $d^\dagger = -\star d\star$, and the fact that $\Omega^2(\Sigma, \mathbb{R}) \cong \Omega^0(\Sigma, \mathbb{R}) = C^\infty(\Sigma, \mathbb{R})$ we can write $\omega_2 = \alpha(dx \wedge dy)$ and

$$\begin{aligned} d^\dagger \omega_2 &= -\star d(\alpha \star (dx \wedge dy)) \\ &= -\star d\alpha \\ &= -\partial_z \alpha(\star dz) - \partial_{\bar{z}} \alpha(\star d\bar{z}) \\ &= -\partial_z \alpha(-i dz) - \partial_{\bar{z}} \alpha(i d\bar{z}) \\ &= i \partial_z \alpha dz - i \partial_{\bar{z}} \alpha d\bar{z} . \end{aligned} \quad (3.5.46)$$

If we write $h = e^{i\sigma+\alpha} : \Sigma \longrightarrow \mathbb{C}^*$ with $\sigma, \alpha \in C^\infty(\Sigma, \mathbb{R})$ then we can write A as

$$A = i\phi_i \alpha^i + i\theta^j \beta_j - (h^\dagger)^{-1} \partial h^\dagger + h^{-1} \bar{\partial} h . \quad (3.5.47)$$

The path integral measure then factorizes as

$$[DA] = [Dh] \frac{\partial(A)}{\partial(h)} \prod_{i,j=1}^g [D\phi_i][D\theta^j] , \quad (3.5.48)$$

where the Jacobian of the change of variables is given by [GK89]:

$$\frac{\partial(A)}{\partial(h)} = \frac{\det'(-\Delta)}{2\pi} \exp(S_{\text{U}(1),2}(hh^\dagger)) . \quad (3.5.49)$$

The action (3.5.42) in the variables (3.5.37) and (3.5.47) is given by

$$\begin{aligned} S_{\text{U}(1),k}(\phi', A) &= \frac{k}{8\pi} \int_\Sigma d\phi' \wedge \star d\phi' + \frac{ik}{2\pi} \int_\Sigma (n_i \alpha^i + m^j \beta_j) \wedge (\phi_\ell \alpha^\ell + \theta^m \beta_m) \\ &+ \frac{k}{2\pi} \int_\Sigma (n_i \alpha^i + m^j \beta_j) \wedge \left(- (h^\dagger)^{-1} \partial h^\dagger + h^{-1} \bar{\partial} h \right) . \end{aligned} \quad (3.5.50)$$

Let us now perform the integral over ϕ_i, θ^j . The relevant part of the action is

$$\frac{ik}{2\pi} \int_\Sigma (n_i \alpha^i + m^j \beta_j) \wedge (\phi_\ell \alpha^\ell + \theta^m \beta_m) = \frac{ik}{2\pi} (n_i \theta^i - m^j \phi_j) . \quad (3.5.51)$$

The integral is then

$$\int \prod_{i,j=1}^g [D\phi_i][D\theta^j] \exp \left[-\frac{ik}{2\pi} (n_i \theta^i - m^j \phi_j) \right] , \quad (3.5.52)$$

which again enforces

$$n_i = m^j = 0 , \quad (3.5.53)$$

and evaluates to

$$\int \prod_{i,j=1}^g [D\phi_i][D\theta^j] \exp \left[-\frac{ik}{2\pi} (n_i \theta^i - m^j \phi_j) \right] = (2\pi)^{2g} \prod_{i,j=1}^g \delta_{n_i,0} \delta_{m^j,0} . \quad (3.5.54)$$

Using $h = e^{i\sigma+\alpha}$, we compute

$$d \star A = d(i d\alpha + \star i d\sigma) = d(\partial_z \sigma dz - \partial_{\bar{z}} \sigma d\bar{z}) = -(\partial_z \partial_{\bar{z}} \sigma) dz \wedge d\bar{z} . \quad (3.5.55)$$

Thus, the Landau gauge (3.5.40) is equivalent to σ being harmonic:

$$d \star A = 0 \iff \Delta \sigma = 0 . \quad (3.5.56)$$

We will take σ to be orthogonal to constants. The ghost action for this gauge fixing is easily seen to be

$$S_{\text{gh}}(b, c) = \int_\Sigma d^2 z b \Delta c , \quad (3.5.57)$$

where b, c are the anticommuting ghosts and also orthogonal to constants, see [MS25, Section 47] and references therein for discussion on ghost zeromodes. Thus, the partition function takes the form

$$Z_{\mathrm{U}(1)/\mathrm{U}(1)}^{\mathrm{WZW}}(\Sigma) = (2\pi)^{2g+1} \frac{\det'(-\Delta)}{2\pi} \int [D\phi'] \int [D\alpha][D\sigma] \int [DbDc] \delta(\Delta\sigma) \exp \left[-\frac{k}{8\pi} \int_{\Sigma} d\phi' \wedge \star d\phi' + S_{\mathrm{U}(1),2}(hh^\dagger) - \int_{\Sigma} d^2z b\Delta c \right]. \quad (3.5.58)$$

Remembering that $h = e^{i\sigma+\alpha}$, and using the WZW action (3.4.2) with g replaced by $hh^\dagger$, we get

$$\begin{aligned} S_{\mathrm{U}(1),2}(hh^\dagger) &= \frac{2i}{4\pi} \int_{\Sigma} d^2z (hh^\dagger)^{-1} \partial_z (hh^\dagger) (hh^\dagger)^{-1} \partial_{\bar{z}} (hh^\dagger) \\ &= \frac{1}{2\pi} \int_{\Sigma} d^2z (2\partial_z \alpha) (2\partial_{\bar{z}} \alpha) \\ &= \frac{1}{\pi} \int_{\Sigma} d\alpha \wedge \star d\alpha \end{aligned} \quad (3.5.59)$$

Thus we have

$$\begin{aligned} Z_{\mathrm{U}(1)/\mathrm{U}(1)}^{\mathrm{WZW}}(\Sigma) &= (2\pi)^{2g} \det'(-\Delta) \int [D\phi'] [D\alpha] \exp \left[-\frac{k}{8\pi} \int_{\Sigma} d\phi' \wedge \star d\phi' + \frac{1}{\pi} \int_{\Sigma} d\alpha \wedge \star d\alpha \right] \\ &\quad \times \int [D\sigma] \int [DbDc] \delta(\Delta\sigma) \exp \left[- \int_{\Sigma} d^2z b\Delta c \right]. \end{aligned} \quad (3.5.60)$$

To compute the path integral over σ, b, c , we expand them in a basis of eigenfunctions (3.5.28) of Δ :

$$\sigma = \sum_{n=1}^{\infty} \sigma_n \varphi_n, \quad b = \sum_{n=1}^{\infty} b_n \varphi_n, \quad c = \sum_{n=1}^{\infty} c_n \varphi_n. \quad (3.5.61)$$

Note that since σ and b, c are orthogonal to constants, the expansion does not contain the zeromodes.

Then the path integral becomes

$$\begin{aligned} \int [D\sigma] \int [DbDc] \delta(\Delta\sigma) \exp \left[- \int_{\Sigma} d^2z b\Delta c \right] &= \prod_{n=1}^{\infty} d\sigma_n \delta(\lambda_n \sigma_n) \int \prod_{m=1}^{\infty} db_m dc_m \exp(-b_m \lambda_m c_m) \\ &= \frac{\det(\Delta)}{\prod_{i=1}^{\infty} \lambda_i} \int \prod_{n=1}^{\infty} d\sigma_n \delta(\sigma_n) \\ &= 1. \end{aligned} \quad (3.5.62)$$

Thus we get

$$\begin{aligned} Z_{\mathrm{U}(1)/\mathrm{U}(1)}^{\mathrm{WZW}}(\Sigma) &= (2\pi)^{2g} \det'(-\Delta) \int [D\phi'] [D\alpha] \exp \left[-\frac{k}{8\pi} \int_{\Sigma} d\phi' \wedge \star d\phi' + \frac{1}{\pi} \int_{\Sigma} d\alpha \wedge \star d\alpha \right] \\ &= (2\pi)^{2g} \det'(-\Delta) \det' \left(-\frac{k}{8\pi} \Delta \right)^{-1/2} \det' \left(\frac{1}{\pi} \Delta \right)^{-1/2} \\ &= \text{constant}. \end{aligned} \quad (3.5.63)$$

This proves that the $\mathrm{U}(1)/\mathrm{U}(1)$ -WZW model is a TQFT.

3.6 State Space Of Gauged WZW Model

In this section, we will describe the finite-energy gauge-invariant states in the G/H -WZW model. We will see that the Ishibashi states of H -WZW model naturally appear when we impose gauge invariance. We will add the Maxwell term to the G/H -WZW action and then take the gauge coupling to infinity to flow to the G/H -WZW model. We will then identify *finite-energy gauge invariant states* of the theory. For simplicity, we assume that G, H are compact, connected, semisimple and simply connected. To this end, consider the action

$$S(A, g) := \frac{1}{e^2} \int_{\Sigma} \text{Tr}(F(A) \wedge \star F(A)) + S_{G, \mathbf{k}}(g, A) , \quad (3.6.1)$$

where $F(A)$ is the curvature of the connection A and $S_{G, \mathbf{k}}(g, A)$ is the G/H -WZW action (3.4.31). The gauge coupling e^2 has dimensions of mass² and triggers an RG flow. To find the IR fixed point states, we want to identify the lowest energy gauge invariant states of the theory governed by (3.6.1) in the $e^2 \rightarrow \infty$ limit. The Hilbert space of the theory is given by [Wit91, Wit92b, CMR94]

$$\mathcal{H}_G^{\text{WZW} \otimes \text{YM}} := (\mathcal{H}_G^{\text{WZW}} \otimes L^2(\mathcal{A}))^{LH} , \quad (3.6.2)$$

where $\mathcal{H}_G^{\text{WZW}}$ is the Hilbert space of G -WZW model and $L^2(\mathcal{A})$ is the space of square integrable functions on the space of H -connections. The Hamiltonian of the theory is given by³⁰

$$H = (L_0^{\mathfrak{g}} + \bar{L}_0^{\mathfrak{g}}) + e^2 \int dx \frac{\delta}{\delta A^a(x)} \frac{\delta}{\delta A^a(x)} . \quad (3.6.3)$$

In the $e^2 \rightarrow \infty$ limit, we look for functions $f \in L^2(\mathcal{A})$ such that

$$\lim_{e^2 \rightarrow \infty} e^2 \int dx \frac{\delta}{\delta A^a(x)} \frac{\delta}{\delta A^a(x)} f(A) < \infty . \quad (3.6.4)$$

This requires

$$\int dx \frac{\delta}{\delta A^a(x)} \frac{\delta}{\delta A^a(x)} f(A) = 0 . \quad (3.6.5)$$

This forces $f(A)$ to be a constant as a function of A . We thus conclude that

$$\mathcal{H}_G^{\text{WZW} \otimes \text{YM}} \cong (\mathcal{H}_G^{\text{WZW}})^{LH} \otimes \mathbb{C} \cong (\mathcal{H}_G^{\text{WZW}})^{LH} . \quad (3.6.6)$$

We will now determine the gauge invariant states. To do so, we will impose the Gauss law constraints on the cylinder. We parameterize the cylinder by (σ, τ) where $\sigma \sim \sigma + 2\pi$ parametrizes the spatial direction and τ is the Euclidean time. The complex coordinate on the plane is given by $z = e^{-i\sigma + \tau}$. Note that

$$\partial_{\sigma} = i(\bar{z}\partial_{\bar{z}} - z\partial_z) . \quad (3.6.7)$$

³⁰We are considering the Hamiltonian on the cylinder with the circumference of the cylinder normalized to 1.

We are gauging the symmetry $H \hookrightarrow G \times G$ embedded diagonally and acts on the WZW field $g \in \text{Map}(\Sigma, G)$ as

$$g \mapsto hgh^{-1} . \quad (3.6.8)$$

Let us write the gauged WZW action (3.4.31) for this diagonal gauging of H in terms of differential forms. We have

$$\begin{aligned} S_{G,\mathbf{k}}(g, A) = S_{G,\mathbf{k}}(g) + \frac{i}{2\pi} \sum_{a'=1}^{\dim(\mathfrak{h})} \int_{\Sigma} A^{a'} \wedge \langle t^{a'}, g^{-1} dg \rangle_{\mathbf{k}} + \frac{i}{4\pi} \sum_{a',b'=1}^{\dim(\mathfrak{h})} \int_{\Sigma} (A^{a'} \wedge A^{b'}) \langle gt^{a'} g^{-1}, t^{b'} \rangle_{\mathbf{k}} \\ + \frac{i}{4\pi} \sum_{a',b'=1}^{\dim(\mathfrak{h})} \int_{\Sigma} (A^{a'} \wedge A^{b'}) \langle t^{a'}, t^{b'} \rangle_{\tilde{\mathbf{k}}} , \end{aligned} \quad (3.6.9)$$

where $\{t^{a'}\}_{a'=1}^{\dim(\mathfrak{h})}$ is an orthonormal basis of $\mathfrak{h}$. We make the gauge choice $A_{\tau} = 0$. The Gauss law constraint is imposed on physical states as an operator equation of the field constraint

$$\frac{\delta S_{G,\mathbf{k}}(g, A)}{\delta A_{\tau}} = 0 . \quad (3.6.10)$$

Note that this constraint is to be imposed on the states in the Hilbert space at fixed Euclidean time $\tau = \tau_0$. We will take $\tau_0 = 0$. It is clear from (3.6.9) that the Gauss law constraint on physical states is

$$\widehat{\mathcal{G}}|\text{phys}\rangle = 0 , \quad (3.6.11)$$

where

$$\widehat{\mathcal{G}} := \sum_{a'=1}^{\dim(\mathfrak{h})} \langle t^{a'}, g^{-1} \partial_{\sigma} g \rangle_{\mathbf{k}} + \frac{1}{2} \sum_{a',b'=1}^{\dim(\mathfrak{h})} (A_{\sigma}^{b'} - A_{\sigma}^{a'}) \left(\langle gt^{a'} g^{-1}, t^{b'} \rangle_{\mathbf{k}} + \langle t^{a'}, t^{b'} \rangle_{\mathbf{k}} \right) . \quad (3.6.12)$$

Note that the A -dependent terms annihilate constant functions on $L^2(\mathcal{A})$ and hence the Gauss law constraint becomes

$$\sum_{a'=1}^{\dim(\mathfrak{h})} \langle t^{a'}, g^{-1} \partial_{\sigma} g \rangle_{\mathbf{k}} |\text{phys}\rangle = 0 . \quad (3.6.13)$$

From (3.4.6) and (3.6.7), the Gauss law constraint takes the form

$$\sum_{a'=1}^{\dim(\mathfrak{h})} \left(z \langle t^{a'}, J(z) \rangle_{\mathbf{k}} + \bar{z} \langle t^{a'}, \tilde{J}(\bar{z}) \rangle_{\mathbf{k}} \right) |\text{phys}\rangle = 0 . \quad (3.6.14)$$

We identify

$$J_{\mathfrak{h}}(z) := \sum_{a'=1}^{\dim(\mathfrak{h})} \langle t^{a'}, J(z) \rangle_{\mathbf{k}} , \quad \tilde{J}_{\mathfrak{h}}(\bar{z}) := \sum_{a'=1}^{\dim(\mathfrak{h})} \langle t^{a'}, \tilde{J}(\bar{z}) \rangle_{\mathbf{k}} , \quad (3.6.15)$$

with the Kac-Moody current for $\mathfrak{h}$. Expanding the $\mathfrak{h}$ Kac-Moody currents into modes, the Gauss law constraints at $\tau = 0$ takes the form

$$\sum_{n \in \mathbb{Z}} e^{-in\sigma} (J_n^{a'} + \tilde{J}_{-n}^{a'}) |\text{phys}\rangle = 0, \quad a' = 1, \dots, \dim(\mathfrak{h}). \quad (3.6.16)$$

Equivalently, the Gauss law constraints can be written as

$$(J_n^{a'} + \tilde{J}_{-n}^{a'}) |\text{phys}\rangle = 0, \quad n \in \mathbb{Z}, \quad a' = 1, \dots, \dim(\mathfrak{h}). \quad (3.6.17)$$

Note that (3.6.17) imposes gauge invariance only for the infinitesimal gauge transformations. But it is known that the image of the exponential map in LH is dense [PS86]. Thus (3.6.17) is also sufficient for gauge invariance. All in all, we want to compute the following space:

$$\mathcal{H}_{G/H}^{\text{WZW}} := \left\{ |\psi\rangle \in \mathcal{H}_G^{\text{WZW}} : (J_n^{a'} + \tilde{J}_{-n}^{a'}) |\psi\rangle = 0 \right\}, \quad (3.6.18)$$

is the minimum energy configuration in the gauge sector.

3.6.1 Solution To (3.6.17): Ishibashi States And Equivariant H/H -WZW Model

We begin by noting that (3.6.17) are the well-known *Ishibashi constraints* for H -WZW model which one solves to determine conformal boundary conditions of a CFT when studying the theory on the upper half plane. The solutions to Ishibashi constraints in the diagonal H -WZW model are well known – these are the Ishibashi states which lives in the bulk Hilbert space³¹. But we want to solve for $|\psi\rangle \in \mathcal{H}_G^{\text{WZW}}$. To do so, we consider the diagonal G -WZW model. We have the decomposition:

$$\begin{aligned} \mathcal{H}_G^{\text{WZW}} &= \bigoplus_{\lambda \in P_k^+(\mathfrak{g})} L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \otimes \widetilde{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})} \\ &\cong \bigoplus_{\substack{(\lambda, \mu) \in \mathcal{E} \\ (\tilde{\lambda}, \tilde{\mu}) \in \mathcal{E}}} \left(L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \right) \otimes \left(\widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \otimes \widetilde{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})} \right) \\ &\cong \bigoplus_{\substack{(\lambda, \mu) \in \mathcal{E} \\ (\tilde{\lambda}, \tilde{\mu}) \in \mathcal{E}}} \left(L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \right) \otimes \left(L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})} \right). \end{aligned} \quad (3.6.19)$$

Now note that the action $J_m^{a'} + \tilde{J}_n^{b'}$ on the last line of the RHS of (3.6.19) is given by

$$J_m^{a'} + \tilde{J}_n^{b'} \mapsto (\mathbb{1} \otimes \tilde{\mathbb{1}}) \otimes (J_m^{a'} \otimes \mathbb{1}) + (\mathbb{1} \otimes \tilde{\mathbb{1}}) \otimes (\mathbb{1} \otimes \tilde{J}_n^{b'}). \quad (3.6.20)$$

So, the solutions to

$$(J_n^{a'} + \tilde{J}_{-n}^{a'}) |\psi\rangle = 0, \quad n \in \mathbb{Z}, \quad a' = 1, \dots, \dim(\mathfrak{h}), \quad |\psi\rangle \in \mathcal{H}_G^{\text{WZW}}, \quad (3.6.21)$$

³¹To be precise, Ishibashi states are elements of a bigger space, called *rigged* Hilbert space formulated in [GS64, GV64].

can be written as

$$|\psi\rangle = \sum_{\substack{(\lambda, \mu) \in \mathcal{E} \\ (\lambda, \tilde{\mu}) \in \mathcal{E}}} \left| \psi_{(\lambda, \mu), (\lambda, \tilde{\mu})}^{\mathfrak{g}/\mathfrak{h}} \right\rangle \otimes |\mu, \tilde{\mu}\rangle, \quad (3.6.22)$$

where

$$\left| \psi_{(\lambda, \mu), (\lambda, \tilde{\mu})}^{\mathfrak{g}/\mathfrak{h}} \right\rangle \in L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})}, \quad |\mu, \tilde{\mu}\rangle \in L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})}, \quad (3.6.23)$$

such that $|\mu, \tilde{\mu}\rangle$ satisfies the Ishibashi constraint

$$\left[(J_n^{a'} \otimes \mathbb{1}) + (\mathbb{1} \otimes \tilde{J}_{-n}^{a'}) \right] |\mu, \tilde{\mu}\rangle = 0. \quad (3.6.24)$$

We now show that a solution to (3.6.24) exists if and only if

$$\mu = \tilde{\mu}. \quad (3.6.25)$$

This follows from the observation that any element $v \otimes f \in \mathcal{H}_1 \otimes \mathcal{H}_2^\vee$ of a tensor product of the Hilbert space $\mathcal{H}_1$ and the dual $\mathcal{H}_2^\vee$ of the Hilbert space $\mathcal{H}_2$ corresponds to a linear map

$$\varphi_{v \otimes f} : \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \quad (3.6.26)$$

$$u \longmapsto \varphi_{v \otimes f}(u) := f(u)v,$$

More generally, any finite (and possibly infinite subject to formal convergence) linear combination of elements of $\mathcal{H}_1 \otimes \mathcal{H}_2^\vee$ can be treated as a linear map in $\text{Hom}(\mathcal{H}_2, \mathcal{H}_1)$. In particular, if we identify $\widetilde{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})}$ with the dual $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})^\vee$, then the state $|\mu, \tilde{\mu}\rangle$ can be treated as an element $\varphi_{\mu, \tilde{\mu}} \in \text{Hom}(L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}), L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}))$. The condition (3.6.24) is equivalent to

$$J_n^{a'} \circ \varphi_{\mu, \tilde{\mu}} = \varphi_{\mu, \tilde{\mu}} \circ \text{Ad}_{\mathcal{U}}(J_n^{a'}) , \quad (3.6.27)$$

where $\mathcal{U}$ is an operator satisfying

$$\mathcal{U} \tilde{J}_n^{a'} = -\tilde{J}_n^{a'} \mathcal{U}, \quad a' = 1, 2, \dots, \dim(\mathfrak{h}). \quad (3.6.28)$$

Thus $\varphi_{\mu, \tilde{\mu}}$ is intertwiner of $\hat{\mathfrak{h}}$ -modules $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}), L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})$. By Schur's lemma, we conclude that

$$\varphi_{\mu, \tilde{\mu}} = C \mathbb{1} \delta_{\mu, \tilde{\mu}}, \quad (3.6.29)$$

for some nonzero constant C . This proves the desired result. The explicit solution

$$|\mu\rangle := |\mu, \mu\rangle, \quad (3.6.30)$$

satisfying (3.6.24) are the Ishibashi states which we now describe.

Let $\{|\mu, i\rangle\}_{i=0}^{\infty}$ denote an orthonormal basis of homogeneous vectors $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})$, then

$$|\mu\rangle\rangle = \sum_{i=0}^{\infty} |\mu, i\rangle \otimes \widetilde{\mathcal{U}|\mu, i\rangle}, \quad (3.6.31)$$

is the Ishibashi state corresponding to the module $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \mu)$ and solves³² (3.6.17). We conclude that any state of the form

$$(v \otimes \tilde{v}) \otimes |\mu\rangle\rangle, \quad v \in L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}), \quad \tilde{v} \in \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})}, \quad \mu \in P_{\tilde{\mathbf{k}}}^+(\mathfrak{h}), \quad (3.6.33)$$

is a solution to (3.6.17). The space of gauge-invariant states takes the form³³

$$\mathcal{H}_{G/H}^{\text{WZW}} := \bigoplus_{(\lambda, \mu) \in \mathcal{E}} \left(L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \right) \otimes |\mu\rangle\rangle. \quad (3.6.34)$$

Notice that for $G = H$,

$$\mathcal{H}_{H/H}^{\text{WZW}} \cong V_{\tilde{\mathbf{k}}}(H). \quad (3.6.35)$$

Examples

We now work out some simple examples. Let us look at the coset models of the Ising CFT.

$$G = \text{SU}(2) \times \text{SU}(2), H = \text{SU}(2).$$

H is embedded diagonally in G .

$\mathbf{k} = (1, 1), \tilde{\mathbf{k}} = 2$. Let us label the chiral primaries of G -WZW theory by (i, j) with $i, j = 0, 1$ and the primaries of H by $\ell = 0, 1, 2$. Then from the decomposition (3.3.75), we have

$$\mathcal{E} = \{((0, 0), 0), ((0, 0), 2), ((1, 0), 1), ((0, 1), 1), ((1, 1), 0), ((1, 1), 2)\}. \quad (3.6.36)$$

³²A succinct description of Ishibashi states is as follows: identify the right moving sector $\widetilde{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})}$ with the dual $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})^{\vee}$. Then the Ishibashi states $|\mu\rangle\rangle$ is identified with the identity endomorphism

$$\mathbb{1}_{\mu} \in \text{End}(L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})) \cong L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \otimes L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})^{\vee}. \quad (3.6.32)$$

³³Field identifications can be explained using gauge-invariance under a larger group of *discontinuous loops* $L_Z H$ defined as $L_Z H = \{\zeta \in C^{\infty}(\mathbb{R}, H) : \zeta(x + 2\pi)\zeta(x)^{-1} \in Z\}$ with $Z = Z(G) \cap Z(H)$. More precisely, one can show that a discontinuous loop $\zeta \in L_Z H / L H \cong Z$ acts on $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})$ and $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})$ by fusion with simple current corresponding to $\zeta \in Z$ [TL99]. Consequently, we must identify $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ and $L_{\mathfrak{g}/\mathfrak{h}}(\zeta \cdot \hat{\lambda}, \zeta \cdot \hat{\mu})$. As shown in Appendix C.2.2 and Appendix C.2.3, identification by this action of Z is precisely field identifications.

Using the decomposition (3.3.75), we find that

$$\begin{aligned} \mathcal{H}_{G/H}^{\text{WZW}} = & \left(\text{Vir} \left(\frac{1}{2}, 0 \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, 0 \right) \right) \otimes |0\rangle \oplus \left(\text{Vir} \left(\frac{1}{2}, 0 \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, 0 \right) \right) \otimes |2\rangle \\ & \oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{2} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{2} \right) \right) \otimes |2\rangle \oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{2} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{2} \right) \right) \otimes |0\rangle \\ & \oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{16} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{16} \right) \right) \otimes |1\rangle \oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{16} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{16} \right) \right) \otimes |1\rangle, \end{aligned} \quad (3.6.37)$$

The partition function of the gauged WZW model is

$$Z_{G/H}^{\text{WZW}}(\tau) = 2 \left(|\chi_0(\tau)|^2 + |\chi_{1/2}(\tau)|^2 + |\chi_{1/16}(\tau)|^2 \right) = 2Z_{\text{Ising}}(\tau), \quad (3.6.38)$$

where $Z_{\text{Ising}}(\tau)$ is the partition function of the Ising CFT with unique vacuum. Note that the factor of 2 is the order of the common center $Z = Z(G) \cap Z(H)$ which in this case is $\mathbb{Z}_2$.

$\mathbf{k} = (2, 1), \tilde{\mathbf{k}} = 3$. We have in this case

$$\begin{aligned} \mathcal{E} = & \{((0, 0), 0), ((0, 0), 2), ((1, 0), 1), ((1, 0), 3), ((0, 1), 1), ((0, 1), 3), ((1, 1), 0), ((1, 1), 2), \\ & ((2, 0), 0), ((2, 0), 2), ((0, 2), 0), ((0, 2), 2), ((2, 1), 1), ((2, 1), 3)\}. \end{aligned} \quad (3.6.39)$$

Using the decomposition (3.3.75), we get

$$\begin{aligned} \mathcal{H}_{G/H}^{\text{WZW}} = & \left(\text{Vir} \left(\frac{7}{10}, 0 \right) \otimes \widetilde{\text{Vir}} \left(\frac{7}{10}, 0 \right) \right) \otimes (\mathbb{C}|0\rangle \oplus \mathbb{C}|3\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{7}{10}, \frac{3}{5} \right) \otimes \widetilde{\text{Vir}} \left(\frac{7}{10}, \frac{3}{5} \right) \right) \otimes (\mathbb{C}|2\rangle \oplus \mathbb{C}|1\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{7}{10}, \frac{1}{10} \right) \otimes \widetilde{\text{Vir}} \left(\frac{7}{10}, \frac{1}{10} \right) \right) \otimes (\mathbb{C}|1\rangle \oplus \mathbb{C}|2\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{7}{10}, \frac{3}{2} \right) \otimes \widetilde{\text{Vir}} \left(\frac{7}{10}, \frac{3}{2} \right) \right) \otimes (\mathbb{C}|3\rangle \oplus \mathbb{C}|0\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{7}{10}, \frac{3}{80} \right) \otimes \widetilde{\text{Vir}} \left(\frac{7}{10}, \frac{3}{80} \right) \right) \otimes (\mathbb{C}|1\rangle \oplus \mathbb{C}|2\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{7}{10}, \frac{7}{16} \right) \otimes \widetilde{\text{Vir}} \left(\frac{7}{10}, \frac{7}{16} \right) \right) \otimes (\mathbb{C}|0\rangle \oplus \mathbb{C}|3\rangle). \end{aligned} \quad (3.6.40)$$

$$G = \text{Spin}(8), H = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2).$$

The embedding of H in G and the corresponding branching functions is discussed in Appendix C.3. The level of G is 1 and the level of H is $(1, 1, 2)$. There are four dominant integrable representations of $\text{Spin}(8)$ at level 1 which we will label by $\mathbf{o}, \mathbf{v}, \mathbf{s}, \mathbf{t}$. Then from [BRS88, Table 1] we have

$$\begin{aligned} \mathcal{E} = & \{(\mathbf{o}, (0, 0, 0)), (\mathbf{o}, (1, 1, 2)), (\mathbf{o}, (1, 1, 0)), (\mathbf{o}, (0, 0, 2)) \\ & (\mathbf{v}, (0, 1, 1)), (\mathbf{v}, (1, 0, 1)), (\mathbf{s}, (0, 0, 0)), (\mathbf{s}, (0, 0, 2)) \\ & (\mathbf{s}, (1, 1, 0)), (\mathbf{s}, (1, 1, 2)), (\mathbf{t}, (1, 0, 1)), (\mathbf{t}, (0, 1, 1))\}. \end{aligned} \quad (3.6.41)$$

Looking at the Virasoro modules corresponding to these pair of G, H -weights from [BRS88, Table 1], we find that

$$\begin{aligned} \mathcal{H}_{G/H}^{\text{WZW}} = & \left(\text{Vir} \left(\frac{1}{2}, 0 \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, 0 \right) \right) \otimes (\mathbb{C}|0, 0, 0\rangle \oplus \mathbb{C}|1, 1, 2\rangle \oplus \mathbb{C}|0, 0, 2\rangle \oplus \mathbb{C}|1, 1, 0\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{2} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{2} \right) \right) \otimes (\mathbb{C}|1, 1, 0\rangle \oplus \mathbb{C}|0, 0, 2\rangle \oplus \mathbb{C}|0, 0, 0\rangle \oplus \mathbb{C}|1, 1, 2\rangle) \\ & \oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{16} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{16} \right) \right) \otimes (\mathbb{C}|0, 1, 1\rangle \oplus \mathbb{C}|1, 0, 1\rangle \oplus \mathbb{C}|1, 0, 1\rangle \oplus \mathbb{C}|0, 1, 1\rangle) . \end{aligned} \quad (3.6.42)$$

The partition function of the G/H -WZW model is thus given by

$$Z_{G/H}^{\text{WZW}}(\tau) = 4 (|\chi_0(\tau)|^2 + |\chi_{1/2}(\tau)|^2 + |\chi_{1/16}(\tau)|^2) = 4Z_{\text{Ising}}(\tau) . \quad (3.6.43)$$

Again the factor of 4 is the order of the common center $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Parafermion: $G = \text{SU}(2), H = \text{U}(1)$.

We have seen that the affine VOA $L_{\text{su}(2)}(k, 0)$ has $k + 1$ irreducible modules which we labeled by $i = 0, \dots, k$ and denote by $L_{\text{su}(2)}(k, i)$. The affine VOA $L_{\text{u}(1)}(k, 0)$ has³⁴ $2k$ irreducible modules which we label by $n = -k + 1, \dots, 0, \dots, k - 1, k$ and denote by $L_{\text{u}(1)}(k, n)$. Under the embedding $e^{i\theta} \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ of $\text{U}(1)$ in $\text{SU}(2)$, the induced level of $\text{u}(1)$ is same as the level of $\text{SU}(2)$. We have the following decomposition of irreducible $L_{\text{su}(2)}(k, 0)$ -modules in terms of irreducible $L_{\text{u}(1)}(k, 0)$ modules [DFMS97]:

$$L_{\text{su}(2)}(k, i) \cong \bigoplus_{n=-k+1}^k L_{\text{su}(2)/\text{u}(1)}(i, n) \otimes L_{\text{u}(1)}(k, n) , \quad (3.6.44)$$

where the multiplicity spaces $L_{\text{su}(2)/\text{u}(1)}(i, n)$ are nontrivial as long as $2i + n \equiv 0 \pmod{2}$. Thus, we have

$$\mathcal{E} = \{(i, n) : -k + 1 \leq n \leq k, 0 \leq i \leq k, 2i + n \equiv 0 \pmod{2}\} . \quad (3.6.45)$$

Field identification by outer automorphism yields [DFMS97]

$$L_{\text{su}(2)/\text{u}(1)}(i, n) \cong L_{\text{su}(2)/\text{u}(1)}(k - i, k - n) \cong L_{\text{su}(2)/\text{u}(1)}(k - i, n - k) . \quad (3.6.46)$$

Note that when $k - n > k$ then the field identification identifies $(i, n) \sim (k - i, n - k)$. This field identification can be understood from simple currents as follows: the conformal dimension of the

³⁴This is the extended chiral algebra of the rational Gaussian model, i.e., the compact boson at rational radius $R = \sqrt{p/q}$. See Section 3.6.4 for more details.

respective irreducible modules are given by

$$\begin{aligned} h_i &= \frac{i(i+2)}{4(k+2)} \quad \text{for } L_{\mathfrak{su}(2)}(k, i) , \\ h_n &= \frac{n^2}{4k} \quad \text{for } L_{\mathfrak{u}(1)}(k, n) . \end{aligned} \quad (3.6.47)$$

From the S -matrix

$$S_{nn'}^{\mathfrak{u}(1)} = e^{-\frac{\pi i n n'}{k}} , \quad (3.6.48)$$

we see that all the irreducible $L_{\mathfrak{u}(1)}(k, 0)$ -modules $L_{\mathfrak{u}(1)}(k, n)$ are simple currents while only $L_{\mathfrak{su}(2)}(k, 0)$ and $L_{\mathfrak{su}(2)}(k, k)$ are simple currents. From the conformal dimensions (3.6.47), we see that

$$G_{\text{id}} = \{(0, 0), (k, k)\} . \quad (3.6.49)$$

Using the fusion

$$L_{\mathfrak{u}(1)}(k, n_1) \boxtimes L_{\mathfrak{u}(1)}(k, n_2) \cong L_{\mathfrak{u}(1)}(k, n_1 + n_2 \bmod 2k) , \quad (3.6.50)$$

we see that G_{id} is isomorphic to the common $\mathbb{Z}_2$ center of $\text{SU}(2)$ and $\text{U}(1)$. Using this fusion, the field identification (3.6.46) is easily seen to be equivalent to (3.6.58). Let $\mathcal{E}/\sim$ denote the equivalence classes under the action of G_{id} . We then have

$$\mathcal{H}_{G/H}^{\text{WZW}} = \bigoplus_{[(i,n)] \in \mathcal{E}/\sim} \left(L_{\mathfrak{su}(2)/\mathfrak{u}(1)}(i, n) \otimes \widetilde{L_{\mathfrak{su}(2)/\mathfrak{u}(1)}(i, n)} \right) \otimes (\mathbb{C}|n\rangle \oplus \mathbb{C}|\dot{\pm}(k-n)\rangle) , \quad (3.6.51)$$

where $\dot{\pm}$ means that we choose the sign such that $-k+1 \leq \pm(k-n) \leq k$. The partition function of the G/H -WZW model is given by

$$Z_{G/H}^{\text{WZW}}(\tau) = 2 \sum_{[(i,n)] \in \mathcal{E}/\sim} \left| \chi_{(i,n)}^{\mathfrak{su}(2)/\mathfrak{u}(1)}(\tau) \right|^2 = 2Z_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}}(\tau) , \quad (3.6.52)$$

where $\chi_{(i,n)}^{\mathfrak{su}(2)/\mathfrak{u}(1)}(\tau)$ is the character of the module $L_{\mathfrak{su}(2)/\mathfrak{u}(1)}(i, n)$. The factor of 2 is again coming from the common center $|\mathbb{Z}_2| = 2$ of $\text{SU}(2)$ and $\text{U}(1)$.

A Maverick example.

We now consider a Maverick example: $G = \text{SU}(3)_2, H = \text{SU}(2)_8$. There are six dominant integral weights of $\mathfrak{su}(3)$ at level 2, which will label by the dimension of their corresponding finite-dimensional representation:

$$\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}, \mathbf{6}, \bar{\mathbf{6}}, \mathbf{8} . \quad (3.6.53)$$

The dominant integral weights of $\mathfrak{su}(2)$ at level 8 will be labeled as usual by $i = 0, \dots, 8$. The corresponding GKO VOA is the W_3 algebra, which is an extension of $\text{Vir}(\frac{4}{5}, 0)$ by a dimension 3

operator. The GKO theory is the three-state-Potts model (TPSM) and the irreducible modules of the W_3 algebra along with the corresponding pair of $V(\mathfrak{su}(3), \mathfrak{su}(2); 2)$ -module is shown in Table 3.1. The table is obtained by combining the computations of [DJ93a, BS09].

Module	Conformal dimension	$V(\mathfrak{su}(3), \mathfrak{su}(2); 2)$ -module
$V_{\mathbb{I}}$	0	$(\mathbf{1}, 0) \cong (\mathbf{1}, 8) \cong (\mathbf{8}, 4)$
V_{ϵ}	$\frac{2}{5}$	$(\mathbf{1}, 4) \cong (\mathbf{8}, 6) \cong (\mathbf{8}, 2)$
V_{ψ_1}	$\frac{2}{3}$	$(\mathbf{6}, 0) \cong (\mathbf{6}, 8) \cong (\bar{\mathbf{3}}, 4)$
V_{ψ_2}	$\frac{2}{3}$	$(\bar{\mathbf{6}}, 0) \cong (\bar{\mathbf{6}}, 8) \cong (\mathbf{3}, 4)$
V_{σ_1}	$\frac{1}{15}$	$(\mathbf{6}, 4) \cong (\bar{\mathbf{3}}, 6) \cong (\bar{\mathbf{3}}, 2)$
V_{σ_2}	$\frac{1}{15}$	$(\bar{\mathbf{6}}, 4) \cong (\mathbf{3}, 6) \cong (\mathbf{3}, 2)$

Table 3.1: Primaries and their conformal dimension of the three-state-Potts model

We thus find that

$$\begin{aligned}
\mathcal{H}_{\mathrm{SU}(3)_2/\mathrm{SU}(2)_8}^{\mathrm{WZW}} \cong & (V_{\mathbb{I}} \otimes \tilde{V}_{\mathbb{I}}) \otimes (\mathbb{C}|0\rangle \oplus \mathbb{C}|8\rangle \oplus \mathbb{C}|4\rangle) \\
& \oplus (V_{\epsilon} \otimes \tilde{V}_{\epsilon}) \otimes (\mathbb{C}|4\rangle \oplus \mathbb{C}|6\rangle \oplus \mathbb{C}|2\rangle) \\
& \oplus (V_{\psi_1} \otimes \tilde{V}_{\psi_1}) \otimes (\mathbb{C}|0\rangle \oplus \mathbb{C}|8\rangle \oplus \mathbb{C}|4\rangle) \\
& \oplus (V_{\psi_2} \otimes \tilde{V}_{\psi_2}) \otimes (\mathbb{C}|0\rangle \oplus \mathbb{C}|8\rangle \oplus \mathbb{C}|4\rangle) \\
& \oplus (V_{\sigma_1} \otimes \tilde{V}_{\sigma_1}) \otimes (\mathbb{C}|4\rangle \oplus \mathbb{C}|6\rangle \oplus \mathbb{C}|2\rangle) \\
& \oplus (V_{\sigma_2} \otimes \tilde{V}_{\sigma_2}) \otimes (\mathbb{C}|4\rangle \oplus \mathbb{C}|6\rangle \oplus \mathbb{C}|2\rangle) .
\end{aligned} \tag{3.6.54}$$

The partition function of the G/H -WZW model is thus given by

$$Z_{G/H}^{\mathrm{WZW}}(\tau) = 3 (|\chi_{\mathbb{I}}(\tau)|^2 + |\chi_{\epsilon}(\tau)|^2 + 2|\chi_{\psi_1}(\tau)|^2 + 2|\chi_{\sigma_1}(\tau)|^2) = 3Z_{\mathrm{TPSM}}(\tau) . \tag{3.6.55}$$

A conformal embedding example.

We finally consider the conformal embedding $G = (\mathrm{G}_2)_1, H = \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_3$. The GKO theory is trivial and from the decomposition (3.3.92), we get

$$\mathcal{H}_{(\mathrm{G}_2)_1/\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_3}^{\mathrm{WZW}} \cong \mathbb{C}|0, 0\rangle \oplus \mathbb{C}|1, 3\rangle \oplus \mathbb{C}|0, 2\rangle \oplus \mathbb{C}|1, 1\rangle . \tag{3.6.56}$$

Thus, the topological degeneracy is 4 which is again not the order of the center of $G_2, \text{SU}(2) \times \text{SU}(2)$ since the latter is empty.

3.6.2 2d TQFT relating $\text{WZW}(G/H)$ and $\text{GKO}(\mathfrak{g}/\mathfrak{h})$

In this section, we will show that under certain assumptions, the Hilbert space on the circle of $\text{WZW}(G/H)$ and $\text{GKO}(\mathfrak{g}/\mathfrak{h})$ are related by the endomorphism algebra $\text{End}(B)$ of the algebra object B in (3.3.63):

$$\mathcal{H}_{G/H}^{\text{WZW}} \cong \mathcal{H}_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}} \otimes \text{End}(B) . \quad (3.6.57)$$

Recall that the identification group G_{id} is defined as the pair of simple currents (J, J') of $L_{\mathfrak{g}}(\mathbf{k}, 0)$ and $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ such that $h_J - h_{J'} \in \mathbb{Z}$ and the branching function $\chi_{(J, J')}^{\mathfrak{g}/\mathfrak{h}}(\tau) \neq 0$, see Appendix C.2.3 for details. One can show that

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong L_{\mathfrak{g}/\mathfrak{h}}(J\hat{\lambda}, J'\hat{\mu}) , \quad (J, J') \in G_{\text{id}} , \quad (3.6.58)$$

where $L_{\mathfrak{g}}(\mathbf{k}, J\hat{\lambda})$ is isomorphic to the irreducible $L_{\mathfrak{g}}(\mathbf{k}, 0)$ -module $J \boxtimes L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})$ and $J'\hat{\mu}$ is defined similarly. This also defines an action of G_{id} on $\mathcal{E}$. We can thus write

$$\mathcal{H}_{G/H}^{\text{WZW}} \cong \bigoplus_{(\lambda, \mu) \in \mathcal{E}/\sim} \left[\left(L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \right) \otimes \bigoplus_{(J, J') \in G_{\text{id}}} \mathbb{C} |J'\mu\rangle \right] . \quad (3.6.59)$$

Assumption 3.6.1. The set of isomorphism classes of irreducible $V(\mathfrak{g}, \mathfrak{h}, \mathbf{k})$ -modules are in 1-1 correspondence with G_{id} -orbits of $\mathcal{E}$.

Remark 3.6.1. This is a reasonable assumption since a large class of GKO models satisfy this assumption. For example, the unitary minimal model series $V_{\frac{\widehat{\mathfrak{su}}(2)_k \oplus \widehat{\mathfrak{su}}(2)_1}{\widehat{\mathfrak{su}}(2)_{k+1}}}$ and $V_{\frac{\widehat{\mathfrak{so}}(4n)_1}{\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1 \oplus \widehat{\mathfrak{su}}(2)_n}}$ satisfy the assumption. There also nontrivial examples where this assumption fails, for example, the Maverick coset $V_{\widehat{\mathfrak{su}}(3)_2 / \widehat{\mathfrak{su}}(2)_8}$ or when there are fixed points of the outer automorphism action. In principal, we know field identifications and selection rules from the categorical results discussed in Section 3.2.3. To generalize the result discussed below to such examples, we need to precisely understand what global symmetry the algebra object B in Corollary 3.3.4 corresponds to in the $(G_k \times H_{-\tilde{\mathbf{k}}})$ -Chern-Simons theory. We leave this for future investigation.

Under this assumption, it is easy to see from (3.6.59) and the fact that $G_{\text{id}} \cong Z$ (see Appendix C.2.2) that the partition function of G/H -WZW and GKO theory is related by $|Z|$:

$$Z_{G/H}^{\text{WZW}}(\tau) = |Z| Z_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}}(\tau) . \quad (3.6.60)$$

Let $\mathcal{A}_{\mathfrak{g}/\mathfrak{h}}$ be the set of labels of isomorphism classes of irreducible $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -modules. Then, because of our assumption, for each $a \in \mathcal{A}_{\mathfrak{g}/\mathfrak{h}}$ there are projection maps

$$\pi_1 : \mathcal{O}_a \longrightarrow P_{\mathbf{k}}^+(\mathfrak{g}) , \quad \pi_2 : \mathcal{O}_a \longrightarrow P_{\mathbf{k}}^+(\mathfrak{h}) , \quad (3.6.61)$$

where $\mathcal{O}_a$ is the corresponding G_{id} -orbit. Then we have

$$\mathcal{H}_{G/H}^{\text{WZW}} \cong \bigoplus_{a \in \mathcal{A}_{\mathfrak{g}/\mathfrak{h}}} \left[\left(W^a \otimes \widetilde{W}^a \right) \otimes \bigoplus_{\mu \in \pi_2(\mathcal{O}_a)} \mathbb{C}|\mu\rangle\right] . \quad (3.6.62)$$

We can rewrite this as

$$\mathcal{H}_{G/H}^{\text{WZW}} \cong \bigoplus_{a \in \mathcal{A}_{\mathfrak{g}/\mathfrak{h}}} \left[\left(W^a \otimes \widetilde{W}^a \right) \otimes \bigoplus_{\mu \in \pi_2(\mathcal{O}_a)} \mathbb{C}[\mathcal{S}_{\mu}] \right] , \quad (3.6.63)$$

where $\mathcal{S}_{\mu} < G_{\text{id}}$ is the stabilizer subgroup of μ and $\mathbb{C}[\mathcal{S}_{\mu}]$ is the group algebra of $\mathcal{S}_{\mu}$, since

$$\mathbb{C}[\mathcal{S}_{\mu}] \cong \underbrace{\mathbb{C}|\mu\rangle\langle\mu| \oplus \cdots \oplus \mathbb{C}|\mu\rangle\langle\mu|}_{|\mathcal{S}_{\mu}| \text{ times}} . \quad (3.6.64)$$

Under our assumption, we see that

$$\bigoplus_{\mu \in \pi_2(\mathcal{O}_a)} \mathbb{C}[\mathcal{S}_{\mu}] \cong \mathbb{C}[G_{\text{id}}] . \quad (3.6.65)$$

Consider now the algebra object B in (3.3.63). Under our assumption, we have

$$B = \bigoplus_{(J, J') \in G_{\text{id}}} (J \boxtimes_{\text{D}} (J')^{\text{opp}}) . \quad (3.6.66)$$

Moreover, since $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\text{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu})$ is a simple object in $\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\text{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})$, we have

$$\dim \text{Hom}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\text{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})} \left(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\text{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu}), L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}') \boxtimes_{\text{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu}') \right) = \delta_{\hat{\lambda}, \hat{\lambda}'} \delta_{\hat{\mu}, \hat{\mu}'} . \quad (3.6.67)$$

This gives us a basis for $\text{End}(B)$

$$\text{End}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\text{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})}(B) = \text{End}(B) = \bigoplus_{(J, J') \in G_{\text{id}}} \mathbb{C}f_{(J, J')} , \quad (3.6.68)$$

where we defined

$$f_{(J, J')} := \iota_{(J, J')} \circ r_{(J, J')} , \quad (3.6.69)$$

where

$$\iota_{(J, J')} : J \boxtimes_{\text{D}} (J')^{\text{opp}} \longrightarrow B , \quad (3.6.70)$$

$$r_{(J, J')} : B \longrightarrow J \boxtimes_{\text{D}} (J')^{\text{opp}} ,$$

are the inclusion and projection maps respectively. This implies that

$$\mathbb{C}[G_{\text{id}}] \cong \text{End}(B) \cong \bigoplus_{(J, J') \in G_{\text{id}}} \mathbb{C}f_{(J, J')} . \quad (3.6.71)$$

Combining (3.6.65), (3.6.71) with (3.6.63) we have

$$\mathcal{H}_{G/H}^{\text{WZW}} \cong \mathcal{H}_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}} \otimes \text{End}(B) . \quad (3.6.72)$$

The Case of Conformal Embedding

The derivation of the space of gauge invariant states in gauged WZW model is modified when H is embedded conformally in G . In this case, the decomposition the multiplicity spaces $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ are isomorphic to $\mathbb{C}^{n(\hat{\lambda}, \hat{\mu})}$ for some $n(\hat{\lambda}, \hat{\mu}) \in \mathbb{Z}_{\geq 0}$ defined in (3.3.68). The state space of $\text{WZW}(G/H)$ from the above discussion is given by

$$\mathcal{H}_{G/H}^{\text{WZW}} := \bigoplus_{(\lambda, \mu) \in \mathcal{E}} \left(\mathbb{C}^{n(\hat{\lambda}, \hat{\mu})} \otimes \tilde{\mathbb{C}}^{n(\hat{\lambda}, \hat{\mu})} \right) \otimes |\mu\rangle\rangle . \quad (3.6.73)$$

We can give a Frobenius algebra structure on $\mathcal{H}_{G/H}^{\text{WZW}}$ and hence turn it into a 2d TQFT. We claim that with

$$\mathcal{H}_{G/H}^{\text{WZW}} \cong \text{End}(B) , \quad (3.6.74)$$

where B is the special symmetric commutative Frobenius algebra object defined in (3.3.63) and $\text{End}(B)$ denotes the space of endomorphisms which are also algebra morphisms. To see this, note that since $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu})$ is a simple object in $\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})$, we have

$$\dim \text{Hom}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})} \left(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu}), L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}') \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu}') \right) = \delta_{\hat{\lambda}, \hat{\lambda}'} \delta_{\hat{\mu}, \hat{\mu}'} . \quad (3.6.75)$$

Clearly then for conformal embedding

$$\text{End}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})}(B) = \text{End}(B) , \quad (3.6.76)$$

and we have

$$\text{End}(B) \cong \bigoplus_{(\lambda, \mu) \in \mathcal{E}} \text{End}(\mathbb{C}^{n(\hat{\lambda}, \hat{\mu})}) \cong \mathcal{H}_{G/H}^{\text{WZW}} . \quad (3.6.77)$$

The Maverick coset $G = \text{SU}(3)_2, H = \text{SU}(2)_8$

It was shown in [DJ93a, DJ93b] that the Maverick coset $\widehat{\mathfrak{su}}(3)_2 / \widehat{\mathfrak{su}}(2)_8$ does not satisfy the Assumption 3.6.1. Indeed, from (3.6.54), we find that the topological degeneracy is 3, which is *not* the order of the common center, since $Z(\text{SU}(3)) \cap Z(\text{SU}(2)) = \emptyset$. The vacuum degeneracy

$$V_{\mathbb{I}} \otimes (\mathbb{C}|0\rangle\rangle \oplus \mathbb{C}|8\rangle\rangle \oplus \mathbb{C}|4\rangle\rangle) , \quad (3.6.78)$$

is still governed by $\text{End}(B)$ since

$$\begin{aligned} \text{End}(B) &\cong \mathbb{C}f_{(1,0)} \oplus \mathbb{C}f_{(1,8)} \oplus f_{(8,4)} \\ &\cong \mathbb{C}|0\rangle\rangle \oplus \mathbb{C}|8\rangle\rangle \oplus \mathbb{C}|4\rangle\rangle , \end{aligned} \quad (3.6.79)$$

where we introduced the notation

$$f_{(\hat{\lambda}, \hat{\mu})} := \iota_{(\hat{\lambda}, \hat{\mu})} \circ r_{(\hat{\lambda}, \hat{\mu})} , \quad (3.6.80)$$

where

$$\begin{aligned} \iota_{(\hat{\lambda}, \hat{\mu})} : L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu}) &\longrightarrow B , \\ r_{(\hat{\lambda}, \hat{\mu})} : B &\longrightarrow L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu}) , \end{aligned} \quad (3.6.81)$$

are the inclusion and projection maps. They satisfy

$$r_{(\hat{\lambda}, \hat{\mu})} \circ \iota_{(\hat{\lambda}', \hat{\mu}')} = \delta_{\hat{\lambda}, \hat{\lambda}'} \delta_{\hat{\mu}, \hat{\mu}'} \mathbb{1}_{(\hat{\lambda}, \hat{\mu})} := \delta_{\hat{\lambda}, \hat{\lambda}'} \delta_{\hat{\mu}, \hat{\mu}'} \mathbb{1}_{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathbb{D}} L_{\mathfrak{h}}^{\text{opp}}(\tilde{\mathbf{k}}, \hat{\mu})} . \quad (3.6.82)$$

As a vector space, we can still write

$$\mathcal{H}_{\text{SU}(3)/\text{SU}(2)}^{\text{WZW}} \cong \mathcal{H}_{\mathfrak{su}(3)_2/\mathfrak{su}(2)_8}^{\text{GKO}} \otimes \text{End}(B) . \quad (3.6.83)$$

Frobenius algebra structure on $\text{End}(B)$

We now describe the Frobenius algebra structure on ³⁵ $\text{End}(B)$. The multiplication map is given by

$$f \cdot g := \mu_B \circ (f \boxtimes g) \circ \Delta_B . \quad (3.6.85)$$

The unit is given by

$$\mathbf{1}_{\text{End}(B)} := \eta_B \circ \varepsilon_B . \quad (3.6.86)$$

Frobenius pairing is defined as follows:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \text{End}(B) \boxtimes \text{End}(B) &\longrightarrow \mathbb{C} \\ f \boxtimes g &\longmapsto \langle f, g \rangle := \text{Tr}_B(f \cdot g) , \end{aligned} \quad (3.6.87)$$

where the trace

$$\text{Tr}_B(f \cdot g) \in \text{End}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})} \left(\mathbf{1}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathbb{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})} \right) \cong \mathbb{C} , \quad (3.6.88)$$

is defined by the following composition of maps

$$\text{Tr}_B(f \cdot g) := \varepsilon_B \circ (f \cdot g) \circ \Delta_B \circ \eta_B = \varepsilon_B \circ \mu_B \circ (f \boxtimes g) \circ \Delta_B \circ \eta_B . \quad (3.6.89)$$

³⁵ There is an obvious algebra structure: the composition of morphisms and the unit element is the identity morphism.

But this Frobenius algebra structure is not consistent with the G/G case. Indeed, in this case it is easy to see that

$$\mathcal{H}_{G/G}^{\text{WZW}} \cong \bigoplus_{\lambda \in P_{\mathbf{k}}^+(\mathfrak{g})} \mathbb{C}(\mathbf{1}_{L_{\mathfrak{g}}(\mathbf{k}, \lambda)} \boxtimes_{\mathbb{D}} \mathbf{1}_{L_{\mathfrak{g}}^{\text{opp}}(\mathbf{k}, \lambda)}) , \quad (3.6.84)$$

and the multiplication map can never involve the fusion rules of $\text{WZW}(G)$ which arise when we consider the $\text{WZW}(G/G)$.

Proposition 3.6.1. $\text{End}(B)$ equipped with multiplication (3.6.85) and the pairing $\langle \cdot, \cdot \rangle$ is a commutative Frobenius algebra.

Proof. First we prove commutativity. Since B is commutative and cocommutative, we have

$$\begin{aligned}
 f \cdot g &= \mu_B \circ (f \boxtimes g) \circ \Delta_B = \mu_B \circ c_{B,B} \circ (f \boxtimes g) \circ \Delta_B \\
 &= \mu_B \circ (g \boxtimes f) \circ c_{B,B} \circ \Delta_B \\
 &= \mu_B \circ (g \boxtimes f) \circ \Delta_B \\
 &= g \cdot f .
 \end{aligned} \tag{3.6.90}$$

Next, we show that $\mathbf{1}_{\text{End}(B)}$ is the unit. We have

$$\begin{aligned}
 \mathbf{1}_{\text{End}(B)} \cdot f &= (\eta_B \circ \varepsilon_B) \cdot f \\
 &= \mu_B \circ ((\eta_B \circ \varepsilon_B) \boxtimes f) \circ \Delta_B \\
 &= \mu_B \circ ((\eta_B \boxtimes \mathbf{1}_B) \circ (\mathbf{1}_B \boxtimes f) \circ (\varepsilon_B \boxtimes \mathbf{1}_B)) \circ \Delta_B \\
 &= \ell_B \circ (\mathbf{1}_B \boxtimes f) \circ \ell_B^{-1} \\
 &= f ,
 \end{aligned} \tag{3.6.91}$$

where in the second last step we used (C.4.6), (C.4.11) and (C.4.8) and in the last step we used the fact that the left unitor map in $\mathcal{C}_V$ for a “nice” VOA V satisfies $\ell_W(\mathbf{1} \boxtimes w) = w$ for any V -module W and the vacuum vector $\mathbf{1}$. Next, we prove that the pairing $\langle \cdot, \cdot \rangle$ is a symmetric Frobenius pairing:

$$\langle f \cdot h, g \rangle = \langle f, h \cdot g \rangle , \quad f, g, h \in \text{End}(B) . \tag{3.6.92}$$

The fact $\langle \cdot, \cdot \rangle$ is symmetric follows from the definition and the fact that $f \cdot g = g \cdot f$. For the Frobenius condition, we calculate

$$\begin{aligned}
 \langle f \cdot h, g \rangle &= \varepsilon_B \circ \mu_B \circ ((\mu_B \circ (f \boxtimes h) \circ \Delta_B) \boxtimes g) \circ \Delta_B \circ \eta_B \\
 &= \varepsilon_B \circ \mu_B \circ ((\mu_B \boxtimes \mathbf{1}_B) \circ ((f \boxtimes h) \circ \Delta_B) \boxtimes g) \circ \Delta_B \circ \eta_B \\
 &= \varepsilon_B \circ \mu_B \circ ((\mu_B \boxtimes \mathbf{1}_B) \circ \alpha_{B,B,B} \circ \alpha_{B,B,B}^{-1} \circ ((f \boxtimes h) \boxtimes g) \circ (\Delta_B \boxtimes \mathbf{1}_B) \circ \Delta_B \circ \eta_B \\
 &= \varepsilon_B \circ \mu_B \circ (\mathbf{1}_B \boxtimes \mu_B) \circ (f \boxtimes (h \boxtimes g)) \circ (\mathbf{1}_B \boxtimes \Delta_B) \circ \Delta_B \circ \eta_B \\
 &= \varepsilon_B \circ \mu_B \circ (f \boxtimes (\mu_B \circ (h \boxtimes g) \circ \Delta_B)) \circ \Delta_B \circ \eta_B \\
 &= \langle f, h \cdot g \rangle ,
 \end{aligned} \tag{3.6.93}$$

where we used the associativity isomorphism, (C.4.8) and (C.4.11). Finally, we prove the nondegeneracy. We want to show that

$$\text{Tr}_B(f \cdot g) = 0 \quad \text{for all } g \in \text{End}(B) \implies f = 0 . \tag{3.6.94}$$

We first show that $\varepsilon_B \circ \mu_B : B \boxtimes B \rightarrow \mathbf{1}_{\mathcal{C}(\mathbf{k}, \mathfrak{g}) \boxtimes_{\mathcal{D}} \mathcal{C}(\tilde{\mathbf{k}}, \mathfrak{h})}$ is invertible. To prove this, we use the fact that the morphism Φ_1 defined in (C.4.15) is invertible [FFRS03b, Remark 2.23 (iv)]. We have

$$(\varepsilon_B \circ \mu_B) \boxtimes \mathbf{1}_{B^\vee} = \Phi_1 \circ r_B \circ (\mathbf{1}_B \boxtimes i_B^{-1}) \circ \alpha_{B, B, B^\vee}^{-1} . \quad (3.6.95)$$

This implies that $\varepsilon_B \circ \mu_B$ is invertible. We can then write

$$\begin{aligned} \mathrm{Tr}_B(f \cdot g) &= (\varepsilon_B \circ \mu_B) \circ (f \boxtimes g) \circ (\varepsilon_B \circ \mu_B)^{-1} \circ (\varepsilon_B \circ \mu_B) \circ \Delta_B \circ \eta_B \\ &= \beta_1 \beta_B (\varepsilon_B \circ \mu_B) \circ (f \boxtimes g) \circ (\varepsilon_B \circ \mu_B)^{-1} , \end{aligned} \quad (3.6.96)$$

where β_1, β_B are nonzero complex numbers for special Frobenius algebras, see Definition C.4.4. The nondegeneracy is now clear. $\square$

The space $\mathrm{End}(B)$ is thus a 2d TQFT.

The G/G Frobenius algebra from $\mathrm{End}(B)$

It is instructive to work out the Frobenius algebra structure on $\mathrm{End}(B)$ for $\mathfrak{h} = \mathfrak{g}$. We expect that

$$\mathrm{End}(B) \cong V_{\mathbf{k}}(G) , \quad (3.6.97)$$

in this case. Let us begin by noting that in this case

$$B = \bigoplus_{\lambda \in P_{\mathbf{k}}^+(\mathfrak{g})} L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathcal{D}} L_{\mathfrak{g}}^{\mathrm{opp}}(\mathbf{k}, \hat{\lambda}) , \quad (3.6.98)$$

is a Frobenius algebra object in the category $\mathcal{C}(\mathbf{k}; \mathfrak{g}) \boxtimes_{\mathcal{D}} \mathcal{C}(\mathbf{k}; \mathfrak{g})^{\mathrm{opp}}$. In fact, B is the *trivializing algebra* discussed in [FFRS03b, Lemma 6.19]. The multiplication on B is defined as follows. From the universal property of the tensor product $L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_1) \boxtimes L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_2)$ in the category $\mathcal{C}(\mathbf{k}; \mathfrak{g})$, a basis $\left\{ \mathcal{Y}_{\hat{\lambda}_1 \hat{\lambda}_2; i}^{\hat{\lambda}_3} \right\}_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\hat{\lambda}_3}}$ for the space of intertwining operator of the type $\left(\begin{smallmatrix} L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_3) \\ L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_1), L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_2) \end{smallmatrix} \right)$ implies the existence of a basis $\left\{ e_{\hat{\lambda}_1 \hat{\lambda}_2; i}^{\hat{\lambda}_3} \right\}_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\hat{\lambda}_3}}$ for the space $\mathrm{Hom}_{\mathcal{C}(\mathbf{k}; \mathfrak{g})}(L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_1), L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_2), L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}_3))$. The multiplication morphism is given by

$$\mu_B := \sum_{\lambda_1, \lambda_2, \lambda_3 \in P_{\mathbf{k}}^+(\mathfrak{g})} \sum_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\hat{\lambda}_3}} \iota_{\hat{\lambda}_3} \circ \left(e_{\hat{\lambda}_1 \hat{\lambda}_2; i}^{\hat{\lambda}_3} \boxtimes_{\mathcal{D}} \left(e_{\hat{\lambda}_1 \hat{\lambda}_2; i}^{\hat{\lambda}_3} \right)^{\mathrm{opp}} \right) \circ (r_{\hat{\lambda}_1} \boxtimes r_{\hat{\lambda}_2}) , \quad (3.6.99)$$

where

$$\begin{aligned} \iota_{\hat{\lambda}} &: L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathcal{D}} L_{\mathfrak{g}}^{\mathrm{opp}}(\mathbf{k}, \hat{\lambda}) \longrightarrow B , \\ r_{\hat{\lambda}} &: B \longrightarrow L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathcal{D}} L_{\mathfrak{g}}^{\mathrm{opp}}(\mathbf{k}, \hat{\lambda}) , \end{aligned} \quad (3.6.100)$$

are the inclusion and projection maps respectively. Clearly

$$r_{\hat{\lambda}} \circ \iota_{\hat{\lambda}'} = \delta_{\hat{\lambda}, \hat{\lambda}'} \mathbf{1}_{\hat{\lambda}} := \delta_{\hat{\lambda}, \hat{\lambda}'} \mathbf{1}_{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \boxtimes_{\mathcal{D}} L_{\mathfrak{g}}^{\mathrm{opp}}(\mathbf{k}, \hat{\lambda})} . \quad (3.6.101)$$

The unit map η_B is simply the inclusion map

$$\eta_B := \iota_0 . \quad (3.6.102)$$

Let $\left\{ f_{\hat{\lambda}_3}^{\hat{\lambda}_1 \hat{\lambda}_2; i} \right\}_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\mathfrak{g}, \hat{\lambda}_3}}$ be the dual basis of $\left\{ e_{\hat{\lambda}_1 \hat{\lambda}_2; i}^{\hat{\lambda}_3} \right\}_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\mathfrak{g}, \hat{\lambda}_3}}$ satisfying

$$f_{\hat{\lambda}_3}^{\hat{\lambda}_1 \hat{\lambda}_2; i} \circ e_{\hat{\lambda}_1 \hat{\lambda}_2; j}^{\hat{\lambda}_3} = \delta_{ij} \mathbb{1}_{L_{\mathfrak{g}}(\mathfrak{k}, \hat{\lambda}_3)} . \quad (3.6.103)$$

The coproduct is given by

$$\Delta_B := \sum_{\lambda_1, \lambda_2, \lambda_3 \in P_{\mathfrak{k}}^+(\mathfrak{g})} \sum_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\mathfrak{g}, \hat{\lambda}_3}} (\iota_{\hat{\lambda}_1} \boxtimes \iota_{\hat{\lambda}_2}) \circ \left(f_{\hat{\lambda}_3}^{\hat{\lambda}_1 \hat{\lambda}_2; i} \boxtimes_{\mathbb{D}} \left(f_{\hat{\lambda}_3}^{\hat{\lambda}_1 \hat{\lambda}_2; i} \right)^{\text{opp}} \right) \circ r_{\hat{\lambda}_3} , \quad (3.6.104)$$

A basis for $\text{End}(B)$ is $\{f_{\hat{\lambda}}\}_{\lambda \in P_{\mathfrak{k}}^+(\mathfrak{g})}$ given by

$$f_{\hat{\lambda}} := \iota_{\hat{\lambda}} \circ r_{\hat{\lambda}} . \quad (3.6.105)$$

Using (3.6.101) and (3.6.103), we find

$$\begin{aligned} f_{\hat{\lambda}_1} \cdot f_{\hat{\lambda}_2} &= \sum_{\substack{\lambda'_1, \lambda'_2, \lambda'_3 \in P_{\mathfrak{k}}^+(\mathfrak{g}) \\ \lambda''_1, \lambda''_2, \lambda''_3}} \sum_{i=1}^{\mathcal{N}_{\hat{\lambda}'_1 \hat{\lambda}'_2}^{\mathfrak{g}, \hat{\lambda}'_3}} \sum_{j=1}^{\mathcal{N}_{\hat{\lambda}''_1 \hat{\lambda}''_2}^{\mathfrak{g}, \hat{\lambda}''_3}} \iota_{\hat{\lambda}'_3} \circ \left(e_{\hat{\lambda}'_1 \hat{\lambda}'_2; i}^{\hat{\lambda}'_3} \boxtimes_{\mathbb{D}} \left(e_{\hat{\lambda}'_1 \hat{\lambda}'_2; i}^{\hat{\lambda}'_3} \right)^{\text{opp}} \right) \\ &\quad \circ (r_{\hat{\lambda}'_1} \boxtimes r_{\hat{\lambda}'_2}) \circ (f_{\hat{\lambda}_1} \boxtimes f_{\hat{\lambda}_2}) \circ (\iota_{\hat{\lambda}''_1} \boxtimes \iota_{\hat{\lambda}''_2}) \circ \left(f_{\hat{\lambda}''_3}^{\hat{\lambda}''_1 \hat{\lambda}''_2; j} \boxtimes_{\mathbb{D}} \left(f_{\hat{\lambda}''_3}^{\hat{\lambda}''_1 \hat{\lambda}''_2; j} \right)^{\text{opp}} \right) \circ r_{\hat{\lambda}''_3} \\ &= \sum_{\substack{\lambda'_1, \lambda'_2, \lambda'_3 \in P_{\mathfrak{k}}^+(\mathfrak{g}) \\ \lambda''_1, \lambda''_2, \lambda''_3}} \sum_{i=1}^{\mathcal{N}_{\hat{\lambda}'_1 \hat{\lambda}'_2}^{\mathfrak{g}, \hat{\lambda}'_3}} \sum_{j=1}^{\mathcal{N}_{\hat{\lambda}''_1 \hat{\lambda}''_2}^{\mathfrak{g}, \hat{\lambda}''_3}} \delta_{\hat{\lambda}_1, \hat{\lambda}'_1} \delta_{\hat{\lambda}_2, \hat{\lambda}'_2} \delta_{\hat{\lambda}_1, \hat{\lambda}''_1} \delta_{\hat{\lambda}_2, \hat{\lambda}''_2} \delta_{ij} (\iota_{\hat{\lambda}'_3} \circ r_{\hat{\lambda}''_3}) \\ &= \sum_{\hat{\lambda}_3 \in P_{\mathfrak{k}}^+(\mathfrak{g})} \mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\mathfrak{g}, \hat{\lambda}_3} f_{\hat{\lambda}_3} . \end{aligned} \quad (3.6.106)$$

Finally, we compute the trace of $f_{\hat{\lambda}}$. We have

$$\begin{aligned} \text{Tr}_B(f_{\hat{\lambda}}) &= \varepsilon_B \circ (\mathbf{1}_{\text{End}(B)} \cdot f_{\hat{\lambda}_1}) \circ \eta_B \\ &= \varepsilon_B \circ f_{\hat{\lambda}_1} \circ \eta_B \\ &= \delta_{\hat{\lambda}, 0} \varepsilon_B \circ \iota_{\hat{\lambda}} \\ &= \delta_{\hat{\lambda}, 0} \varepsilon_B \circ \eta_B . \end{aligned} \quad (3.6.107)$$

One can now rescale the counit³⁶ ε_B appropriately so that

$$\text{Tr}_B(f_{\hat{\lambda}}) = \delta_{\hat{\lambda}, 0} . \quad (3.6.108)$$

This establishes (3.6.97).

³⁶We will need to rescale the unit $\mathbf{1}_{\text{End}(B)}$ appropriately so that (3.6.91) holds.

The Frobenius algebra isomorphism $\mathbb{C}[G_{\text{id}}] \cong \text{End}(B)$

For any finite group K , the group algebra $\mathbb{C}[K]$ has a natural Frobenius algebra structure. Let us write an element $x \in \mathbb{C}[K]$ as

$$x = \sum_{k \in K} x_k k . \quad (3.6.109)$$

The multiplication is given by

$$x \cdot y = \left(\sum_{k \in K} x_k k \right) \cdot \left(\sum_{k' \in K} y_{k'} k' \right) = \sum_{k, k'} x_k y_{k'} k k' , \quad (3.6.110)$$

and the Frobenius trace is given by

$$\theta(x) = x_e , \quad (3.6.111)$$

where e is the identity of K . The Frobenius pairing is determined in terms of the trace as

$$\langle x, y \rangle = \theta(x \cdot y) = \sum_{k \in K} x_k y_{k^{-1}} . \quad (3.6.112)$$

We now show that the isomorphism (3.6.71) is an isomorphism of Frobenius algebras with the Frobenius algebra on $\text{End}(B)$ as described in Section 3.6.2. Under the Assumption 3.6.1, we have

$$B = \bigoplus_{(J, J') \in G_{\text{id}}} (J \boxtimes_{\text{D}} (J')^{\text{opp}}) . \quad (3.6.113)$$

We first describe the multiplication on $\text{End}(B)$. Let $e_{J_1, J_2}^{J'_1, J'_2}, e_{J'_1, J'_2}^{J_1, J_2}$ be a basis for the space $\text{Hom}_{\mathcal{C}(\mathbf{k}; \mathbf{g})}(J_1 \boxtimes J_2, J_1 \cdot J_2)$ and $\text{Hom}_{\mathcal{C}(\tilde{\mathbf{k}}; \mathbf{h})}(J'_1 \boxtimes J'_2, J'_1 \cdot J'_2)$ respectively. Let $f_{J_1 \cdot J_2}^{J'_1, J'_2}, f_{J'_1 \cdot J'_2}^{J_1, J_2}$ be their dual basis. Let us also define

$$\iota_B := \bigoplus_{(J, J') \in G_{\text{id}}} \iota_{(J, J')} , \quad r_B := \bigoplus_{(J, J') \in G_{\text{id}}} r_{(J, J')} . \quad (3.6.114)$$

The multiplication and comultiplication morphisms on B are given by

$$\begin{aligned} \mu_B &:= \sum_{(J_1, J'_1), (J_2, J'_2) \in G_{\text{id}}} \iota_B \circ \left(e_{J_1, J_2}^{J'_1, J'_2} \boxtimes_{\text{D}} \left(e_{J'_1, J'_2}^{J_1, J_2} \right)^{\text{opp}} \right) \circ \left(r_{(J_1, J'_1)} \boxtimes r_{(J_2, J'_2)} \right) , \\ \Delta_B &:= \sum_{(J_1, J'_1), (J_2, J'_2) \in G_{\text{id}}} \left(\iota_{(J_1, J'_1)} \boxtimes \iota_{(J_2, J'_2)} \right) \circ \left(f_{J_1 \cdot J_2}^{J'_1, J'_2} \boxtimes_{\text{D}} \left(f_{J'_1 \cdot J'_2}^{J_1, J_2} \right)^{\text{opp}} \right) \circ r_B . \end{aligned} \quad (3.6.115)$$

The unit is given by

$$\eta_B := \iota_{(\mathbf{1}, \mathbf{1}')} . \quad (3.6.116)$$

It is now easy to check that

$$f_{(J_1, J'_1)} \cdot f_{(J_2, J'_2)} = f_{(J_1 \cdot J_2, J'_1 \cdot J'_2)} . \quad (3.6.117)$$

Moreover, we have

$$\mathrm{Tr}_B(f_{(J,J')}) = \delta_{J,1} \delta_{J',1'} \varepsilon_B \circ \eta_B . \quad (3.6.118)$$

Again, we can rescale the counit such that

$$\mathrm{Tr}_B(f_{(J,J')}) = \delta_{J,1} \delta_{J',1'} . \quad (3.6.119)$$

$(G_2)_1/SU(2)_1 \times SU(3)_3$ **TQFT**

We discuss one more example: $G = (G_2)_1, H = SU(2)_1 \times SU(3)_3$. This is a conformal embedding and the topological degeneracy is 4. For this case, we have

$$\begin{aligned} B = & (L_{\mathfrak{g}_2}(1, \mathbf{1}) \boxtimes_{\mathrm{D}} L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((1, 3), (0, 0))) \oplus (L_{\mathfrak{g}_2}(1, \mathbf{1}) \boxtimes_{\mathrm{D}} L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((1, 3), (1, 3))) \\ & \oplus (L_{\mathfrak{g}_2}(1, \mathbf{7}) \boxtimes_{\mathrm{D}} L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((1, 3), (0, 2))) \oplus (L_{\mathfrak{g}_2}(1, \mathbf{7}) \boxtimes_{\mathrm{D}} L_{\mathfrak{su}(2) \oplus \mathfrak{su}(2)}((1, 3), (1, 1))) . \end{aligned} \quad (3.6.120)$$

Define

$$\mathbb{I} := f_{(\mathbf{1}, (0, 0))} , \quad \phi := f_{(\mathbf{1}, (1, 3))} , \quad \psi := f_{(\mathbf{7}, (0, 2))} , \quad \chi := f_{(\mathbf{7}, (1, 1))} . \quad (3.6.121)$$

Then we have

$$\mathrm{End}(B) \cong \mathrm{Span}_{\mathbb{C}}\{\mathbb{I}, \phi, \psi, \chi\} . \quad (3.6.122)$$

Define

$$\iota_B := \iota_{(\mathbf{1}, (0, 0))} \oplus \iota_{(\mathbf{1}, (1, 3))} \oplus \iota_{(\mathbf{7}, (0, 2))} \oplus \iota_{(\mathbf{7}, (1, 1))} , \quad (3.6.123)$$

$$r_B := r_{(\mathbf{1}, (0, 0))} \oplus r_{(\mathbf{1}, (1, 3))} \oplus r_{(\mathbf{7}, (0, 2))} \oplus r_{(\mathbf{7}, (1, 1))} .$$

Then the multiplication and comultiplication on B is given by

$$\begin{aligned} \mu_B := & \sum_{\substack{\lambda_1, \lambda_2, \lambda_3 \in P_{\mathfrak{k}}^+(\mathfrak{g}) \\ \mu_1, \mu_2, \mu_3 \in P_{\mathfrak{k}}^+(\mathfrak{h})}} \sum_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\mathfrak{g}, \hat{\lambda}_3}} \sum_{j=1}^{\mathcal{N}_{\hat{\mu}_1 \hat{\mu}_2}^{\mathfrak{h}, \hat{\mu}_3}} \iota_B \circ \left(e_{\hat{\lambda}_1 \hat{\lambda}_2; i}^{\hat{\lambda}_3} \boxtimes_{\mathrm{D}} \left(e_{\hat{\mu}_1 \hat{\mu}_2; j}^{\hat{\mu}_3} \right)^{\mathrm{opp}} \right) \circ (r_{(\hat{\lambda}_1, \hat{\mu}_1)} \boxtimes r_{(\hat{\lambda}_2, \hat{\mu}_2)}) , \\ \Delta_B := & \sum_{\substack{\lambda_1, \lambda_2, \lambda_3 \in P_{\mathfrak{k}}^+(\mathfrak{g}) \\ \mu_1, \mu_2, \mu_3 \in P_{\mathfrak{k}}^+(\mathfrak{h})}} \sum_{i=1}^{\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\mathfrak{g}, \hat{\lambda}_3}} \sum_{j=1}^{\mathcal{N}_{\hat{\mu}_1 \hat{\mu}_2}^{\mathfrak{h}, \hat{\mu}_3}} (\iota_{(\hat{\lambda}_1, \hat{\mu}_1)} \boxtimes \iota_{(\hat{\lambda}_2, \hat{\mu}_2)}) \circ \left(f_{\hat{\lambda}_3}^{\hat{\lambda}_1 \hat{\lambda}_2; i} \boxtimes_{\mathrm{D}} \left(f_{\hat{\mu}_3}^{\hat{\mu}_1 \hat{\mu}_2; j} \right)^{\mathrm{opp}} \right) \circ r_B , \end{aligned} \quad (3.6.124)$$

where the symbols have their usual meaning. Now, a calculation similar to (3.6.106) gives

$$\begin{aligned} \mathbb{I} \cdot \phi &= \phi, & \mathbb{I} \cdot \psi &= \psi, & \mathbb{I} \cdot \chi &= \chi, \\ \phi \cdot \phi &= \mathbb{I}, & \phi \cdot \psi &= \chi, & \phi \cdot \chi &= \psi, \\ \psi \cdot \psi &= \mathbb{I} + \psi, & \psi \cdot \chi &= \phi + \chi, \\ \chi \cdot \chi &= \mathbb{I} + \psi. \end{aligned} \quad (3.6.125)$$

We thus find that

$$\text{End}(B) \cong V_1(\mathbf{G}_2) \otimes V_1(\text{SU}(2)) , \quad (3.6.126)$$

with the map

$$\begin{aligned} \mathbb{I} &\mapsto \mathcal{O}_1 \otimes \mathcal{O}_0 , & \phi &\mapsto \mathcal{O}_1 \otimes \mathcal{O}_1 , \\ \psi &\mapsto \mathcal{O}_7 \otimes \mathcal{O}_0 , & \chi &\mapsto \mathcal{O}_7 \otimes \mathcal{O}_1 . \end{aligned} \quad (3.6.127)$$

3.6.3 Topological degeneracy in terms of Z -equivariant H/H theory

In this section, under Assumption 3.6.1, we show that the gauged WZW model Hilbert space (3.6.59) can be understood as a coupling between the GKO Hilbert space and the Z -equivariant H/H -TQFT, where Z as before is the common center of G, H .

As discussed in the introduction, we will now recast the key result (3.6.63) in terms of a coupling of the GKO Hilbert space and the G_{id} -equivariant H/H -TQFT. First, we show that G_{id} is a global symmetry group of the G/H -WZW model. Choose a $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -module isomorphism³⁷

$$f_{J,J'} : L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \longrightarrow L_{\mathfrak{g}/\mathfrak{h}}(J\hat{\lambda}, J'\hat{\mu}) , \quad (3.6.128)$$

and define the operator $\mathcal{O}_{J,J'}$ as follows:

$$\begin{aligned} \mathcal{O}_{J,J'} : \mathcal{H}_{G/H}^{\text{WZW}} &\longrightarrow \mathcal{H}_{G/H}^{\text{WZW}} \\ L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \otimes |\mu'\rangle &\ni (v \otimes \tilde{v}) \otimes |\mu'\rangle \longmapsto f_{J,J'}(v) \otimes \tilde{f}_{J,J'}(\tilde{v}) \otimes |J'\mu'\rangle , \end{aligned} \quad (3.6.129)$$

where $\tilde{f}_{J,J'} : \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \rightarrow \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(J\hat{\lambda}, J'\hat{\mu})}$ is another copy of $f_{J,J'}$. Since $f_{J,J'}$ is a $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$ -module map, it commutes with $L_m^{\mathfrak{g}/\mathfrak{h}}$ and similarly, $\tilde{f}_{J,J'}$ commutes with $\bar{L}_m^{\mathfrak{g}/\mathfrak{h}}$. Thus, we have

$$[\mathcal{O}_{J,J'}, L_m^{\mathfrak{g}/\mathfrak{h}}] = [\mathcal{O}_{J,J'}, \bar{L}_m^{\mathfrak{g}/\mathfrak{h}}] = 0 . \quad (3.6.130)$$

Thus, $\mathcal{O}_{J,J'}$ is a global symmetry of the G/H -WZW model. In fact, $\mathcal{O}_{J,J'}$ commutes with all generators of $V(\mathfrak{g}, \mathfrak{h}; \mathbf{k})$. It is also easy to check that the set of all such operators $\mathcal{O}_{J,J'}$ form a group isomorphic to G_{id} .

Z -equivariant H/H theory

Recall that a group K is a global symmetry of a semisimple Frobenius algebra A (with idempotents $\{\varepsilon_x\}_{x \in \mathcal{X}}$) if $\mathcal{X}$ is a K -set such that

$$\theta_x = \theta_{g \cdot x} , \quad g \in K, x \in \mathcal{X} . \quad (3.6.131)$$

³⁷Since $\mathcal{N}_{(0,0),(\lambda,\mu)}^{(J\lambda,J'\mu)} = 1$, this isomorphism is unique up to scalar multiples.

Then the K -equivariant 2d TQFT is described by a subspace of the *algebra of little loops* as described in [MS06, Section 7.3.2]. K -equivariant TQFT is described by the *Turaev algebra* [MS06]

$$A^K := \bigoplus_{\substack{x \in \mathcal{X}, g \in K \\ g \cdot x = x}} \mathbb{C} \varepsilon_{x,g} \cong \bigoplus_{x \in \mathcal{X}} \mathbb{C}[\mathcal{S}_x] , \quad (3.6.132)$$

where $\mathcal{S}_x < K$ is the stabilizer group of x . This space can be equipped with a Frobenius trace, see [MS06] for details. Now G_{id} acts on weights μ as

$$(J, J') \cdot |\mu\rangle\rangle = |J'\mu\rangle\rangle . \quad (3.6.133)$$

Moreover, it can be proved that (see (C.2.66))

$$S_{J'\mu,0}^{\mathfrak{h}} = S_{\mu,0}^{\mathfrak{h}} , \quad (3.6.134)$$

and hence G_{id} is a global symmetry of the H/H -TQFT. The G_{id} -equivariant H/H -TQFT is then described by the Turaev algebra

$$V_{\tilde{k}}(H)^{G_{\text{id}}} := \bigoplus_{\mu \in P_{\tilde{k}}^+(\mathfrak{h})} \mathbb{C}[\mathcal{S}_{\mu}] . \quad (3.6.135)$$

It is now clear that (3.6.63) is a coupling of GKO Hilbert space to the G_{id} -equivariant H/H -TQFT described by the Turaev algebra $V_{\tilde{k}}(H)^{G_{\text{id}}}$.

We now rewrite the two coset descriptions of Ising CFT discussed above in this language. Consider first the $\text{SU}(2)$ -coset description of the unitary minimal model. In this case,

$$G_{\text{id}} \cong \mathbb{Z}_2 \cong \{((0,0),0), ((k,1),k+1)\} , \quad (3.6.136)$$

where $((i,j),\ell)$ represents the simple current pair $(L_{\text{su}(2) \oplus \text{su}(2)}(k, (i,j)), L_{\text{su}(2)}(k+1, \ell))$. We find that the nontrivial element of G_{id} acts on $V_{k+1}(\text{SU}(2))$ as

$$((k,1),k+1) \cdot \ell = k+1-\ell , \quad 0 \leq \ell \leq k+1 . \quad (3.6.137)$$

Writing $\mathbb{Z}_2 = \{\pm 1\}$, the $\mathbb{Z}_2$ -equivariant $\text{SU}(2)/\text{SU}(2)$ theory is given by

$$V_{k+1}(\text{SU}(2))^{\mathbb{Z}_2} = \left\{ \begin{array}{ll} \bigoplus_{0 \leq \ell \leq k+1} \mathbb{C} \varepsilon_{\ell,1} & k \text{ even} \\ \bigoplus_{0 \leq \ell \leq k+1} \mathbb{C} \varepsilon_{\ell,1} \oplus \mathbb{C} \varepsilon_{\frac{k+1}{2},-1} & k \text{ odd} \end{array} \right\} \cong \left\{ \begin{array}{ll} \mathbb{C}^{k+2} & k \text{ even} \\ \mathbb{C}^{k+1} \oplus \mathbb{C}[\mathbb{Z}_2] & k \text{ odd} \end{array} \right\} . \quad (3.6.138)$$

Thus we see that the dimension of $V_{k+1}(\text{SU}(2))^{\mathbb{Z}_2}$ is $k+2$ when k is even and $k+3$ when k is odd.

In particular, we have

$$V_2(\text{SU}(2))^{\mathbb{Z}_2} = \mathbb{C} \varepsilon_{0,1} \oplus \mathbb{C} \varepsilon_{1,1} \oplus \mathbb{C} \varepsilon_{2,1} \oplus \mathbb{C} \varepsilon_{1,-1} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}[\mathbb{Z}_2] . \quad (3.6.139)$$

The Hilbert space (3.6.37) can be written as

$$\begin{aligned}\mathcal{H}_{G/H}^{\text{WZW}} &= \left(\text{Vir} \left(\frac{1}{2}, 0 \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, 0 \right) \right) \otimes (\mathbb{C}\varepsilon_{0,1} \oplus \mathbb{C}\varepsilon_{2,1}) \\ &\oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{2} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{2} \right) \right) \otimes (\mathbb{C}\varepsilon_{0,1} \oplus \mathbb{C}\varepsilon_{2,1}) \\ &\oplus \left(\text{Vir} \left(\frac{1}{2}, \frac{1}{16} \right) \otimes \widetilde{\text{Vir}} \left(\frac{1}{2}, \frac{1}{16} \right) \right) \otimes (\mathbb{C}\varepsilon_{1,1} \oplus \mathbb{C}\varepsilon_{1,-1})\end{aligned}\quad (3.6.140)$$

Now, in the $\text{Spin}(8)$ -description of the Ising CFT, we have

$$G_{\text{id}} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \{(\mathbf{o}, (0, 0, 0)), (\mathbf{o}, (1, 1, 2)), (\mathbf{s}, (0, 0, 2)), (\mathbf{s}, (1, 1, 0))\} . \quad (3.6.141)$$

We will write elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$ as $(\pm 1, \pm 1)$ so that under the above isomorphism

$$\begin{aligned}(\mathbf{o}, (0, 0, 0)) &\mapsto (1, 1), & (\mathbf{o}, (1, 1, 2)) &\mapsto (-1, -1), \\ (\mathbf{s}, (0, 0, 2)) &\mapsto (1, -1), & (\mathbf{s}, (1, 1, 0)) &\mapsto (-1, 1) .\end{aligned}\quad (3.6.142)$$

The action of G_{id} on $V_{(1,2,2)}(\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2))$ can be obtained by using the general fusion rule

$$L_{\text{su}(2)}(k, \ell_1) \boxtimes L_{\text{su}(2)}(k, \ell_2) \cong \bigoplus_{\ell=|\ell_1-\ell_2|}^{\min\{\ell_1+\ell_2, 2k-\ell_1-\ell_2\}} L_{\text{su}(2)}(k, \ell) . \quad (3.6.143)$$

We find that

$$\begin{aligned}(1, 1) \cdot (i, j, k) &= (i, j, k) , & (-1, 1) \cdot (i, j, k) &= (1-i, 1-j, k) , \\ (1, -1) \cdot (i, j, k) &= (i, j, 2-k) , & (-1, -1) \cdot (i, j, k) &= (1-i, 1-j, 2-k) .\end{aligned}\quad (3.6.144)$$

We find that

$$\mathcal{S}_{(i,j,k)} \cong \begin{cases} (1, 1), & k \neq 1, \\ \mathbb{Z}_2, & k = 1, \end{cases} \quad (3.6.145)$$

where $\mathbb{Z}_2 \cong \{(1, 1), (1, -1)\}$. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ -equivariant theory is thus given by

$$\begin{aligned}V_{(1,2,2)}(\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2))^{\mathbb{Z}_2 \times \mathbb{Z}_2} &\cong \bigoplus_{\substack{i,j \in \{0,1\} \\ k=0,2}} \mathbb{C}\varepsilon_{(i,j,k),(1,1)} \oplus \bigoplus_{i,j \in \{0,1\}, g \in \mathbb{Z}_2} \mathbb{C}\varepsilon_{(i,j,1),g} \\ &\cong \mathbb{C}^8 \oplus \mathbb{C}[\mathbb{Z}_2]^4 .\end{aligned}\quad (3.6.146)$$

The Hilbert space (3.6.42) is given by

$$\begin{aligned}\mathcal{H}_{G/H}^{\text{WZW}} &= \left| \text{Vir} \left(\frac{1}{2}, 0 \right) \right|^2 \otimes (\mathbb{C}\varepsilon_{(0,0,0),(1,1)} \oplus \mathbb{C}\varepsilon_{(1,1,2),(1,1)} \oplus \mathbb{C}\varepsilon_{(0,0,2),(1,1)} \oplus \mathbb{C}\varepsilon_{(1,1,0),(1,1)}) \\ &\oplus \left| \text{Vir} \left(\frac{1}{2}, \frac{1}{2} \right) \right|^2 \otimes (\mathbb{C}\varepsilon_{(0,0,0),(1,1)} \oplus \mathbb{C}\varepsilon_{(1,1,2),(1,1)} \oplus \mathbb{C}\varepsilon_{(0,0,2),(1,1)} \oplus \mathbb{C}\varepsilon_{(1,1,0),(1,1)}) \\ &\oplus \left| \text{Vir} \left(\frac{1}{2}, \frac{1}{16} \right) \right|^2 \otimes (\mathbb{C}\varepsilon_{(0,1,1),(1,1)} \oplus \mathbb{C}\varepsilon_{(0,1,1),(1,-1)} \oplus \mathbb{C}\varepsilon_{(1,0,1),(1,1)} \oplus \mathbb{C}\varepsilon_{(1,0,1),(1,-1)}) ,\end{aligned}\quad (3.6.147)$$

where we introduced the notation

$$|\mathrm{Vir}(c, h)|^2 := \mathrm{Vir}(c, h) \otimes \widetilde{\mathrm{Vir}}(c, h) . \quad (3.6.148)$$

Let us denote $G_{\mathrm{id}} \cong \mathbb{Z}_2 = \{\pm 1\}$. The equivariant $\mathrm{U}(1)/\mathrm{U}(1)$ theory is given by

$$V_k(\mathrm{U}(1))^{\mathbb{Z}_2} = \left\{ \begin{array}{ll} \bigoplus_{-k+1 \leq n \leq k} \mathbb{C} \varepsilon_{n,1} & k \text{ odd} \\ \bigoplus_{-k+1 \leq n \leq k} \mathbb{C} \varepsilon_{n,1} \oplus \mathbb{C} \varepsilon_{\frac{k}{2}, -1} & k \text{ even} \end{array} \right\} \cong \left\{ \begin{array}{ll} \mathbb{C}^{2k} & k \text{ odd} \\ \mathbb{C}^{2k-1} \oplus \mathbb{C}[\mathbb{Z}_2] & k \text{ even} \end{array} \right\} . \quad (3.6.149)$$

The G/H -WZW Hilbert space is given by

$$\mathcal{H}_{G/H}^{\mathrm{WZW}} \cong \bigoplus_{[(i,n)] \in \mathcal{E}/\sim} \left(L_{\mathfrak{su}(2)/\mathrm{u}(1)}(i, n) \otimes \widetilde{L_{\mathfrak{su}(2)/\mathrm{u}(1)}(i, n)} \right) \otimes (\mathbb{C} \varepsilon_{n,1} \oplus \mathbb{C} \varepsilon_{\pm(k-n), \pm 1}) , \quad (3.6.150)$$

where

$$\varepsilon_{\pm(k-n), \pm 1} = \begin{cases} \varepsilon_{\pm(k-n), 1} & k \text{ odd,} \\ \varepsilon_{\frac{k}{2}, -1} & k \text{ even, } n = \frac{k}{2} \end{cases} . \quad (3.6.151)$$

3.6.4 Hamiltonian Derivation Of $\mathrm{U}(1)/\mathrm{U}(1)$ TQFT

The Gaussian Model

Let us begin by discussing the compact boson on a circle of radius R , also called the *Gaussian model*.

The action is given by

$$S_{\mathrm{U}(1), R} = \frac{R^2}{4\pi} \int_{\Sigma} d\phi \wedge \star d\phi , \quad (3.6.152)$$

where $\phi : \Sigma \longrightarrow \mathrm{U}(1)$ with the identification $\phi \sim \phi + 2\pi$. Then, on the cylinder $\Sigma = \mathbb{R} \times S^1$ parametrized by (σ, t) , and writing $z = e^{-i\sigma + \tau}$ with $\tau = it$, we can expand ϕ as

$$\begin{aligned} \phi(z, \bar{z}) &= \phi_L(z) + \phi_R(\bar{z}) , \\ \phi_L(z) &= \frac{x_0}{2} - i \frac{p_L}{\sqrt{2}R} \ln z + i \sum_{n \neq 0} \frac{a_n}{n} z^{-n} , \\ \phi_R(\bar{z}) &= \frac{x_0}{2} - i \frac{p_R}{\sqrt{2}R} \ln \bar{z} + i \sum_{n \neq 0} \frac{\tilde{a}_n}{n} \bar{z}^{-n} . \end{aligned} \quad (3.6.153)$$

The periodicity $\phi \sim \phi + 2\pi$ results in the quantization:

$$p_L = \frac{1}{\sqrt{2}} \left(\frac{n}{R} + wR \right) , \quad p_R = \frac{1}{\sqrt{2}} \left(\frac{n}{R} - wR \right) , \quad n, w \in \mathbb{Z} . \quad (3.6.154)$$

We have the OPE

$$\phi(z, \bar{z}) \phi(w, \bar{w}) \sim -\ln |z - w|^2 . \quad (3.6.155)$$

Using this OPE, it is easy to see that the currents

$$\begin{aligned} J(z) &= i\partial_z \phi = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a_0 := \frac{p_L}{\sqrt{2}}, \\ \tilde{J}(z) &= -i\partial_{\bar{z}} \phi = -\sum_{n \in \mathbb{Z}} \tilde{a}_n \bar{z}^{-n-1}, \quad \tilde{a}_0 := \frac{p_R}{\sqrt{2}}, \end{aligned} \quad (3.6.156)$$

satisfy the $\widehat{\mathfrak{u}}(1)$ current algebra or equivalently the modes satisfy the Heisenberg algebra:

$$\begin{aligned} [a_m, a_n] &= m\delta_{m+n,0}, \\ [\tilde{a}_m, \tilde{a}_n] &= m\delta_{m+n,0}. \end{aligned} \quad (3.6.157)$$

The chiral algebra is the Heisenberg VOA

$$\mathcal{A} \cong S(\hat{\mathfrak{h}}^-), \quad (3.6.158)$$

where

$$\hat{\mathfrak{h}} = \text{Span}_{\mathbb{C}} \{a_n : n \in \mathbb{Z}\}, \quad \hat{\mathfrak{h}}^- = \text{Span}_{\mathbb{C}} \{a_n : n \in \mathbb{Z}_{<0}\}. \quad (3.6.159)$$

We also have the right moving copy of the chiral algebra

$$\widetilde{\mathcal{A}} \cong \widetilde{S(\hat{\mathfrak{h}}^-)}, \quad (3.6.160)$$

where $\widetilde{S(\hat{\mathfrak{h}}^-)}$ is spanned by $\tilde{a}_n$. The Heisenberg VOA has irreducible module corresponding to each $\beta \in \mathbb{R}$:

$$M(\beta) = S(\hat{\mathfrak{h}}^-) \otimes e^\beta, \quad (3.6.161)$$

where e^β is the highest weight vector annihilated by a_n for $n > 0$ and a_0 acts as

$$a_0 \cdot e^\beta = \beta e^\beta. \quad (3.6.162)$$

The Heisenberg VOA corresponds to $\beta = 0$ with the identification of e^0 as the vacuum vector. But only the modules

$$M(p_L(n, w)), \quad p_L(n, w) = \frac{1}{\sqrt{2}} \left(\frac{n}{R} + wR \right), \quad n, w \in \mathbb{Z}, \quad (3.6.163)$$

appear in the compact boson CFT. Similarly, only the irreducible modules

$$\widetilde{M}(p_R(n, w)), \quad p_R(n, w) = \frac{1}{\sqrt{2}} \left(\frac{n}{R} - wR \right), \quad n, w \in \mathbb{Z}, \quad (3.6.164)$$

of the right moving Heisenberg VOA appear in the compact boson CFT. The full theory is the gluing of the left moving sector $M(p_L(n, w))$ with the right moving sector $\widetilde{M}(p_R(n, w))$. The primary fields of the full theory are given by

$$V_{p_L, p_R}(z, \bar{z}) = {}_{\circ} e^{ip_L \phi_L(z) + ip_R \phi_R(\bar{z})} {}_{\circ}. \quad (3.6.165)$$

Clearly, this theory is irrational with infinitely many primary fields.

The Rational Gaussian Model.

The *rational Gaussian model* corresponds to the radius squared being a rational number:

$$R = \sqrt{\frac{p}{q}}, \quad p, q \in \mathbb{Z}_+, \quad (p, q) = 1. \quad (3.6.166)$$

In this case, if we put

$$k := pq, \quad (3.6.167)$$

then the primaries $V_{p_L,0}(z)$ and $V_{0,p_R}(\bar{z})$ with

$$p_L = \frac{1}{\sqrt{2}} \left(\frac{p}{R} + qR \right) = \sqrt{2k}, \quad p_R = \frac{1}{\sqrt{2}} \left(\frac{p}{R} - qR \right) = \sqrt{2k}, \quad (3.6.168)$$

have conformal dimension

$$\left(p_L^2 = \frac{2k}{2} = k, 0 \right), \quad (0, p_R^2 = k), \quad (3.6.169)$$

respectively. Moreover, noting that

$$\begin{aligned} \sqrt{2k}p_L &= \sqrt{pq} \frac{n}{\sqrt{p/q}} + w\sqrt{pq}\sqrt{p/q} = nq + wp \in \mathbb{Z}, \\ \sqrt{2k}p_R &= nq - wp \in 2\mathbb{Z}, \end{aligned} \quad (3.6.170)$$

the OPE of these primaries with all other primaries does not have branch cut:

$$V_{\sqrt{2k},0}(z)V_{p_L,p_R}(w,\bar{w}) \sim (z-w)^{\sqrt{2k}p_L}V_{p_L+\sqrt{2k},p_R}(w,\bar{w}), \quad (3.6.171)$$

$$V_{0,\sqrt{2k}}(\bar{z})V_{p_L,p_R}(w,\bar{w}) \sim (\bar{z}-\bar{w})^{\sqrt{2k}p_R}V_{p_L,p_R+\sqrt{2k}}(w,\bar{w}). \quad (3.6.172)$$

Thus the left-moving chiral algebra can be extended by the vertex operators

$$V_{p_L,0}(z), \quad p_L \in \sqrt{2k}\mathbb{Z}, \quad (3.6.173)$$

and the right-moving chiral algebra can be extended by

$$V_{0,p_R}(\bar{z}), \quad p_R \in \sqrt{2k}\mathbb{Z}. \quad (3.6.174)$$

The extended chiral algebra is the lattice VOA

$$\mathcal{A}_k \cong V_L, \quad L := \sqrt{2k}\mathbb{Z}a, \quad (3.6.175)$$

where a is a fixed vector with norm

$$\langle a, a \rangle = 1, \quad (3.6.176)$$

see Section 3.2.2 for the precise definition. This is also called the $\widehat{\mathfrak{u}}(1)_k$ affine VOA by abuse of terminology since the rescaled currents $\sqrt{k}J, \sqrt{k}\tilde{J}$ satisfy the affine $\widehat{\mathfrak{u}}(1)_k$ current algebra. Note that in the notation of Section 3.2.2, we can identify $a(-n) = a_n$ and the vertex operators

$$Y(a(-1)\mathbf{1}, z) = i\partial_z\phi(z). \quad (3.6.177)$$

The right-moving chiral algebra is another copy of the same lattice VOA which we will denote by

$$\widetilde{\mathcal{A}}_k \cong \widetilde{V}_L . \quad (3.6.178)$$

The irreducible modules of the lattice VOA V_L have been completely classified [Don93, LL12]. They are in 1-1 correspondence with elements of $L^\vee/L$. In the present case, we have

$$(\sqrt{2k}\mathbb{Z}a)^\vee/(\sqrt{2k}\mathbb{Z}a) \cong \left(\frac{1}{\sqrt{2k}}\mathbb{Z}a \right) / (\sqrt{2k}\mathbb{Z}a) \cong \mathbb{Z}_{2k} . \quad (3.6.179)$$

The irreducible modules are given by $V_{\beta+L}$, where

$$V_{\beta+L} := S(\hat{\mathfrak{h}}^-) \otimes \mathbb{C}[\beta + L] , \quad [\beta] \in L^\vee/L , \quad (3.6.180)$$

and the action of the vertex operators (3.2.41) is specified by defining³⁸

$$\begin{aligned} e_\gamma \cdot (u \otimes e^{\gamma'}) &= e^{\beta+\gamma'+\gamma} , \quad u \in S(\hat{\mathfrak{h}}^-) , \gamma, \gamma' \in L , \\ z^\gamma \cdot (u \otimes e^\beta) &= z^{\langle \beta, \gamma \rangle} (u \otimes e^\beta) , \quad \gamma \in L , u \in S(\hat{\mathfrak{h}}^-) . \end{aligned} \quad (3.6.181)$$

The full theory is obtained by gluing the left and right moving sectors so that the correlation functions are single-valued. This corresponds to a choice of an even Lorentzian lattice

$$\Gamma_k^{1,1} \subset \frac{1}{\sqrt{2k}}\mathbb{Z}a \oplus \frac{1}{\sqrt{2k}}\mathbb{Z}a , \quad (3.6.182)$$

with signature³⁹ $(1, 1)$. For each factorization $k = pq$ of k into relatively prime factors p, q , there is such a lattice

$$\Gamma_{k=pq}^{1,1} = \left\{ \left(\frac{(wp + nq)a}{\sqrt{2k}} , \frac{(-wp + nq)a}{\sqrt{2k}} \right) : n, w \in \mathbb{Z} \right\} . \quad (3.6.183)$$

This is consistent with the classification of modular invariants of $\widehat{\mathfrak{u}}(1)_k$ in [MS89b]. The full Hilbert space of the theory is given by

$$\mathcal{H}^{\widehat{\mathfrak{u}}(1)_k} = \bigoplus_{[(\alpha, \beta)] \in \Gamma_k^{1,1}/(L \oplus L)} V_{\alpha+L} \otimes \widetilde{V}_{\beta+L} . \quad (3.6.184)$$

Spectrum Of The $\mathbf{U(1)/U(1)}$ Theory

We now want to compute the Hilbert space of $\mathbf{U(1)}_k/\mathbf{U(1)}_k$ model. To do so, we shall impose the Gauss law constraint corresponding to the gauged symmetry. The immediate issue is that the conjugation action of $\mathbf{U(1)}$ on $\mathbf{U(1)}$ is trivial and hence the corresponding current vanishes. This was

³⁸Notice the absence of the cocycle ε in this action compared to (3.2.31). The reason is that we are allowed to pick the trivial central extension of $\mathbb{Z}a/\sqrt{2k}$ since the commutator map (3.2.30) is trivial on L .

³⁹This theory can also be understood as a Narain CFT based on the Lorentzian lattice $\Gamma_k^{1,1}$, see [SS23, SST24] for more details.

overcome in Section 3.5.1 by considering the gauged WZW model action (3.4.31) for generic compact group G, H and then restricting to $G = H = \mathrm{U}(1)$. Here we will impose the Gauss law constraints coming from the action (3.5.21). Let us parametrize the cylinder $S^1 \times \mathbb{R}$ by (σ, τ) as before. Then the action can be written as

$$\begin{aligned} S_{\mathrm{U}(1),k}(\phi, A) &= \frac{k}{8\pi} \int_{\Sigma} d\phi \wedge \star d\phi + \frac{k}{2\pi} \int_{\Sigma} d\phi \wedge A \\ &= \frac{k}{4\pi} \int_{\Sigma} d\sigma d\tau ((\partial_{\sigma}\phi)^2 - (\partial_{\tau}\phi)^2) + \frac{k}{\pi} \int_{\Sigma} d\sigma d\tau (\partial_{\sigma}\phi A_{\tau} - \partial_{\tau}\phi A_{\sigma}) . \end{aligned} \quad (3.6.185)$$

The Gauss law constraint at a fixed Euclidean $\tau = \tau_0$ is given by

$$G(\sigma, \tau_0)|\psi\rangle = 0 , \quad |\psi\rangle \in \mathcal{H}^{\widehat{\mathrm{U}}(1)_k} , \quad (3.6.186)$$

where

$$G(\sigma, \tau_0) := \left. \frac{\delta S_{\mathrm{U}(1),k}(\phi, A)}{\delta A_{\tau}} \right|_{A=0, \tau=\tau_0} . \quad (3.6.187)$$

The Gauss law constraint can be written as

$$\partial_{\sigma}\phi|\psi\rangle = 0 , \quad |\psi\rangle \in \mathcal{H}^{\widehat{\mathrm{U}}(1)_k} . \quad (3.6.188)$$

We will impose the Gauss law constraint at $\tau_0 = 0$. Using the relation (3.6.7) and the expansion (3.6.153), the Gauss law constraint can be written as

$$(\alpha_n - \tilde{\alpha}_{-n})|\psi\rangle = 0 , \quad |\psi\rangle \in \mathcal{H}^{\widehat{\mathrm{U}}(1)_k} , \quad n \in \mathbb{Z} . \quad (3.6.189)$$

For $n = 0$, we see that the constraint on highest weight state implies

$$(\alpha_0 - \tilde{\alpha}_0) \left(e^{\frac{(wp+nq)a}{\sqrt{2k}}} \otimes \tilde{e}^{\frac{(-wp+nq)a}{\sqrt{2k}}} \right) = 0 \implies w = 0 . \quad (3.6.190)$$

This kills the winding modes and the solutions only depends on the momentum modes $nq/\sqrt{2k} = n/\sqrt{2R}$. To find the exact solution, we notice that these are precisely the Ishibashi constraints corresponding to the Dirichlet boundary condition on the free Boson [RS13]. The solutions are given by [RS13]

$$|n/\sqrt{2R}\rangle\rangle = \exp \left(\sum_{r=1}^{\infty} \frac{1}{r} \alpha_{-r} \otimes \tilde{\alpha}_{-r} \right) \left(e^{\frac{na}{\sqrt{2R}}} \otimes \tilde{e}^{\frac{na}{\sqrt{2R}}} \right) , \quad n \in \mathbb{Z} . \quad (3.6.191)$$

We have only imposed the infinitesimal gauge invariance. As we have seen, the chiral algebra is extended to the lattice VOA V_L , thus there is more symmetry that we need to consider. We need to impose the constraints coming from all generators of the chiral algebra V_L . More precisely, in addition to the constraint (3.6.189), we must also impose the following constraint: assume the notation of Section 3.2.2, then

$$\left((e^{\beta})_n - (-1)^{\frac{\langle \beta, \beta \rangle}{2}} (\tilde{e}^{\beta})_{-n} \right) |\psi\rangle = 0 , \quad \beta \in L , \quad |\psi\rangle \in \mathcal{H}^{\widehat{\mathrm{U}}(1)_k} , \quad n \in \mathbb{Z} , \quad (3.6.192)$$

where

$$Y(e^\beta, z) = \sum_{n \in \mathbb{Z}} (e^\beta)_n z^{-n-1} . \quad (3.6.193)$$

In other words, we are trying to find solution to the Ishibashi constraints corresponding to the extended chiral algebra V_L . Note that the standard solutions to the Ishibashi constraints are known for diagonal gluing of the left and right moving chiral algebras, the solutions are precisely the Ishibashi states (or Cardy states). The rational Gaussian model at radius $R = \sqrt{p/q}$ is not a diagonal theory and hence we explicitly need to construct the solutions. We claim that the solutions to the Ishibashi constraints (3.6.189) and (3.6.192) are in one-to-one correspondence with the irreducible modules of V_L . To construct the solutions, let us first choose the following representatives of $\Gamma_k^{1,1}/(L \oplus L)$:

$$\Gamma_k^{1,1}/(L \oplus L) = \left\{ \left[\left(\frac{\ell a}{\sqrt{2k}}, \frac{\ell a}{\sqrt{2k}} \right) \right] : -k+1 \leq \ell \leq k \right\} . \quad (3.6.194)$$

Let

$$\tilde{\mathcal{U}} : \bigoplus_{\ell=-k+1}^k \tilde{V}_{\frac{\ell a}{\sqrt{2k}}+L} \longrightarrow \bigoplus_{\ell=-k+1}^k \tilde{V}_{\frac{\ell a}{\sqrt{2k}}+L} , \quad (3.6.195)$$

be an operator on the total antichiral sector acting as

$$\tilde{\mathcal{U}} \cdot (u \otimes \tilde{e}^\beta) = (-1)^{\frac{\langle \beta, \beta \rangle}{2}} (u \otimes \tilde{e}^\beta) , \quad u \in \widetilde{S(\mathfrak{h}^-)} , \beta \in L^\vee = \frac{1}{\sqrt{2k}} \mathbb{Z} a , \quad (3.6.196)$$

Then it is easy to see that $\tilde{\mathcal{U}}$ satisfies

$$\tilde{e}_\beta \tilde{\mathcal{U}} = (-1)^{\frac{\langle \beta, \beta \rangle}{2}} \tilde{\mathcal{U}} \tilde{e}_\beta , \quad \beta \in L = \sqrt{2k} \mathbb{Z} a . \quad (3.6.197)$$

Then the solutions are given by

$$|\ell/\sqrt{2k}\rangle := \sum_{m,n \in \mathbb{Z}} \exp \left(\sum_{r=1}^{\infty} \frac{1}{r} \alpha_{-r} \otimes \tilde{\alpha}_{-r} \right) \left(e^{\frac{(\ell+2mk)a}{\sqrt{2k}}} \otimes \tilde{\mathcal{U}} \tilde{e}^{\frac{(\ell+2nk)a}{\sqrt{2k}}} \right) , \quad -k+1 \leq \ell \leq k . \quad (3.6.198)$$

The Ishibashi constraint (3.6.189) is clearly satisfied. Noting the form (3.2.41) of the vertex operator $Y(e^\beta, z)$ for $\beta \in L = \sqrt{2k} \mathbb{Z} a$, we see that to prove (3.6.192), it suffices to prove that

$$(e_\beta - (-1)^{\frac{\langle \beta, \beta \rangle}{2}} \tilde{e}_\beta) \cdot |\ell/\sqrt{2k}\rangle = 0 . \quad (3.6.199)$$

To prove this, we note that e_β commutes with α_n for all $\alpha, \beta \in \sqrt{2k} \mathbb{Z}$. Then we have

$$e_\beta \cdot |\ell/\sqrt{2k}\rangle = \sum_{m,n \in \mathbb{Z}} \exp \left(\sum_{r=1}^{\infty} \frac{1}{r} \alpha_{-r} \otimes \tilde{\alpha}_{-r} \right) \left(e^{\frac{\ell a + 2k(ma + (\beta/\sqrt{2k}))}{\sqrt{2k}}} \otimes \tilde{\mathcal{U}} \tilde{e}^{\frac{(\ell+2nk)a}{\sqrt{2k}}} \right) . \quad (3.6.200)$$

Noting that $\beta a/\sqrt{2k} \in \mathbb{Z} a$, we can shift the summation variable m to get

$$e_\beta \cdot |\ell/\sqrt{2k}\rangle = |\ell/\sqrt{2k}\rangle . \quad (3.6.201)$$

Similarly, using (3.6.197), we have

$$(-1)^{\frac{\langle \beta, \beta \rangle}{2}} \tilde{e}^\beta \cdot |\ell/\sqrt{2k}\rangle = |\ell/\sqrt{2k}\rangle . \quad (3.6.202)$$

Thus $|\ell/\sqrt{2k}\rangle$ satisfies the Ishibashi constraints (3.6.189) and (3.6.192). We thus have

$$\mathcal{H}_{\mathrm{U}(1)/\mathrm{U}(1)}^{\mathrm{WZW}} = \mathrm{Span}_{\mathbb{C}}\{|\ell/\sqrt{2k}\rangle : -k+1 \leq \ell \leq k\} . \quad (3.6.203)$$

Thus we conclude that the $\mathrm{U}(1)_k/\mathrm{U}(1)_k$ WZW model is a TQFT whose corresponding Frobenius algebra is the Verlinde algebra of the rational conformal field theory based on the lattice VOA $V_{\sqrt{2k}\mathbb{Z}\alpha}$.

Chapter 4

Topological twisting of 4d $\mathcal{N} = 2$ field theories

The contents of this chapter is based on the paper [MSS24] written jointly with G.W. Moore and V. Saxena. In this chapter, we discuss what topological data must be provided to define topologically twisted partition functions of four-dimensional $\mathcal{N} = 2$ supersymmetric field theories. As discussed in the introduction, the original example of Donaldson-Witten theory depends only on the diffeomorphism type of the spacetime and 't Hooft fluxes (characteristic classes of background gerbe connections, a.k.a. “one-form symmetry connections.”) The example of $\mathcal{N} = 2^*$ theories, analyzed in [MM21], shows that, in general, the twisted partition functions depend on further topological data. We describe topological twisting for general four-dimensional $\mathcal{N} = 2$ theories and argue that the topological partition functions depend on (a): the diffeomorphism type of the spacetime, (b): the characteristic classes of background gerbe connections and (c): a “generalized spin-c structure,” a concept we introduce and define. The main ideas are illustrated with both Lagrangian theories and class $\mathcal{S}$ theories. In the case of class $\mathcal{S}$ theories of A_1 type, we note that the different S -duality orbits of a theory associated with a fixed UV curve $C_{g,n}$ can have different topological data.

4.1 Introduction

This chapter discusses topological twisting of $d = 4$ $\mathcal{N} = 2$ field theories. The first example of such a theory was discovered by Witten [Wit88], an example which had momentous implications. These “topological correlators” have been interesting to mathematicians because they provide invariants of smooth four-manifolds.¹ More precisely, they provide invariants of smooth four-manifolds equipped with extra topological data. In this chapter, we will clarify exactly what that extra topological data is.

In Donaldson-Witten theory (based on $SU(2)$ $\mathcal{N} = 2$ supersymmetric Yang-Mills theory), the topological correlators only depend on the orientation and the diffeomorphism type of $\mathbb{X}$ and a choice

¹Throughout this chapter, we work with “standard” four-manifolds: They are smooth, compact manifolds without boundary. They will be orientable and oriented, although a choice of orientation is an example of a “field,” as discussed below. We will denote a typical standard four-manifold by $\mathbb{X}$.

of 't Hooft flux for the gauge group. However, the example of $\mathcal{N} = 2^*$ theories shows that, in general, the correlators depend on further topological data. In this example, topological twisting requires the introduction of an “ultraviolet” spin^c structure [LMn97, MM21]. (This UV spin^c structure should not be confused with the IR spin^c structures which are used to express the answer derived from an IR viewpoint as in [MW97]). This raises the question:

Question 3. *How does one topologically twist an arbitrary $d = 4$ $\mathcal{N} = 2$ field theory, and what topological data do the correlators depend on?*

In this chapter, we present an answer to this question for all $d = 4$ $\mathcal{N} = 2$ theories and illustrates the general construction for renormalizable Lagrangian theories, and for some theories of class $\mathcal{S}$.

There are many viewpoints on what should be meant by the general notion of “topological twisting.” They all generalize the original example in [Wit88]. It would be interesting to relate the twistings we describe to the general viewpoint advocated in [ES18, ESW20, EGL24].

We will view *topological twisting* as a specification of background fields such that a supersymmetry, denoted $\mathcal{Q}$, remains unbroken. Since we do not work in a Hamiltonian framework, we view the supersymmetry as an odd vectorfield on the supermanifold of field configurations. We assume $\mathcal{Q}$ is a scalar since general manifolds $\mathbb{X}$ do not possess covariantly constant vectors, spinors, etc. $\mathcal{Q}$ must square to zero on physical operators (and, in the Hamiltonian formulation, on states). For example, in a gauge theory, “physical operators” are the gauge invariant operators. In general, there can be different “inequivalent” topological twistings of a given theory. For example, there are, famously, several inequivalent twistings of $\mathcal{N} = 4$ super Yang-Mills (SYM) [VW94] (which is a special case of an $\mathcal{N} = 2$ SYM), as we recall in Section 4.6.1. In our view, these are 3 inequivalent choices of background fields effecting a topological twisting.

The specification of background fields effecting a topological twisting will be given using the mathematical concept of *transfer of structure group*.² Briefly, given a continuous homomorphism $\varphi : G_1 \rightarrow G_2$ between topological groups, one can define a functor φ_* between categories (in fact, groupoids) of principal bundles with connection over $\mathbb{X}$ for the domain and codomain groups G_1 and G_2 , respectively. Thus, if (P_1, ∇_1) is a principal G_1 -bundle over $\mathbb{X}$ with connection ∇_1 , then $\varphi_*(P_1, \nabla_1)$ will be a principal G_2 -bundle over $\mathbb{X}$ with connection $\nabla_2 = \varphi_* \nabla_1$. We explain this concept with various examples in Appendix D.2.

In Section 4.2, we describe our general approach to topological twisting of a $d = 4$ $\mathcal{N} = 2$ field theory. We explain the implementation for any renormalizable Lagrangian theory in Section 4.6, and

²In an unpublished note, [Fre23], D. Freed interprets Witten’s topological twisting in the framework of the ideas of Cartan and Klein, a viewpoint close in spirit to that taken in this chapter.

for some class $\mathcal{S}$ theories in sections 4.8 and 4.9. We describe a twisting valid for generic masses (the background scalar vevs of flavor symmetry vectormultiplets). For special values of the background VM's other topological twistings will exist.

In order to sharpen the main question above, we organize the correlators into a “function” (more precisely formal series) defined on the $\mathcal{Q}$ -cohomology of the space of operators. This function will depend on continuous parameters and what we call the *topological data*. The topological data only depends on the (smooth) topology of $\mathbb{X}$.

In more technical terms, the $\mathcal{Q}$ -cohomology of the space of (extended) observables will be denoted $\mathcal{O}$. Choosing a basis O_i for the vector space $\mathcal{O}$ with dual basis x^i for $\mathcal{O}^\vee$, we denote the generating function of the correlators as:³

$$Z^{\text{tw}}(x) = \langle e^{\sum_i x^i O_i} \rangle. \quad (4.1.1)$$

We view this as a “function” (actually, formal power series) of $x \in \mathcal{O}^\vee$. As we just mentioned, $Z^{\text{tw}}(x)$ will also depend on continuous parameters. Examples of such parameters include the superconformal couplings $\tau_{uv} \in \mathcal{C}$, where $\mathcal{C}$ is the conformal manifold, and the mass parameters $m \in (\mathfrak{t}^f)^\vee$, where $\mathfrak{t}^f$ is the Lie algebra of the group of global “flavor” symmetries.⁴ Such parameters should be viewed as background fields in the theory, and we will be interested in the $\mathcal{Q}$ -invariant background parameters $\mathcal{F}^{\mathcal{Q}}$. The net result is that the generator of topological correlation functions is a function⁵

$$Z^{\text{tw}} : \mathcal{O}^\vee \times \mathcal{F}^{\mathcal{Q}} \rightarrow \mathbb{C}. \quad (4.1.2)$$

We will show that the function Z^{tw} depends on the topological data:⁶

1. The orientation and diffeomorphism type of $\mathbb{X}$.
2. 't Hooft fluxes (i.e., background connections for “discrete 1-form symmetries”)
3. Generalized spin^c structures.

and only depends on these data. No other choices of topological data are needed.

³The proper formulation of the generating series involves nontrivial issues related to contact terms [LNS97, MW97].

But these will not affect the present considerations.

⁴In Lagrangian theories, these can be viewed as equivariant parameters in the equivariant cohomology of moduli spaces of non-Abelian monopole equations.

⁵The domain could be a fibration of $\mathcal{O}^\vee$ over $\mathcal{F}^{\mathcal{Q}}$.

⁶We are assuming there is no BRST anomaly. (See the discussion near (4.2.19).) For manifolds with $b_2^+ = 1$, there is such an anomaly. The partition function does depend on the metric but only through the period point, i.e., the self-dual two-form J such that $J = \star J$ and $J^2 = 1$ (the sign of J is resolved by a choice of orientation). It is piecewise constant as a function of J , as has been analyzed in detail in special cases [MW97, MM21]. In the case $b_2^+ = 0$, one expects continuous metric dependence, but that dependence should be computable. Unfortunately, this has not been investigated in the literature. For $b_2^+ > 1$, these anomalies do not appear.

In Appendix D.3, we define the notion of a generalized spin^c structure. (We focus on four dimensions. The generalization to other dimensions is obvious.) It is a (mild) generalization of the notion of a spin^c structure. One replaces the $U(1)$ “factor” in the definition of the spin^c group by a general torus $\mathbb{T}$, and quotients by a finite Abelian subgroup of $\text{Spin}(4) \times \mathbb{T}$ that projects to the $\mathbb{Z}_2$ subgroup of $\text{Spin}(4)$ acting trivially on vectorial representations.

The general arguments of Section 4.2 apply to all $d = 4$ $\mathcal{N} = 2$ field theories, and thus support our main statement. Sections 4.3, 4.5, 4.6, 4.7, and 4.8 can be viewed as a check on the general reasoning in some important special cases. In Section 4.7 (which is largely a review of known results), we will show that the topological data for twisted Lagrangian theories indeed only consists of the above list of three. In Sections 4.8 and 4.9, we argue that the same is true of the class $\mathcal{S}$ theories we consider.

It is interesting to compare these considerations to those of the IR effective theory. Just the way global symmetries and their ’t Hooft anomalies are important RG invariants, we would propose that the topological data of twisted partition functions are likewise RG invariants and must be the same in the IR and the UV. After all, from the functorial viewpoint, the IR theory is simply the functor of the UV theory with the metric scaled to infinity. This invariance is tacitly assumed in derivations of topologically twisted partition functions from the IR viewpoint, but it seems worth stating explicitly. It is an interesting question whether it will prove a useful observation in other contexts.

The above paragraph raises an interesting question: Given the IR effective theory, how could one deduce the topological data needed to define the UV topological theory? Traditionally, the IR effective theory is presented in terms of a prepotential, or, more globally, in terms of special Kähler manifold. (See [Fre97] for a careful mathematical definition of special Kähler geometry.) We therefore ask: How could one deduce the topological data from this geometry? We offer the thought that one place to look is in the measure of the Coulomb branch integral (a.k.a. the “ u -plane integral”). One must use several facts about the topological data to check the single-valuedness of the Coulomb branch measure in the examples that have been studied in the literature [MW97, MM98, MMZ19, MM21, AFM22]. One may ask if, conversely, the single-valuedness of the Coulomb branch measure is sufficiently strong to determine the topological data. Posing this question in a precise way, to say nothing of answering it, is a subtle and interesting question, but one beyond the scope of the present work.

Another interesting direction for future research is whether one can discuss topological twisting of class $\mathcal{S}$ theories more thoroughly than we do here using a geometric approach based on branes in string/M-theory and supergravity. This quickly leads one into the theory of G_2 -manifolds and raises

many interesting questions.

Here is a brief summary of the remainder of the chapter: In Section 4.2, we describe our approach to topological twisting in very general terms. In Section 4.3, we begin an explication of this approach for Lagrangian field theories, first defining with some (but not complete) care the space of fields of the theories. The field content is usually described in terms of the metric, the vector- and hypermultiplets, the background gauge fields for flavor symmetries, and the background gauge fields for R-symmetries. But these are all related to each other by a choice of total structure group, which depends on a choice of Abelian subgroup of the center of the covering group. We stress the interconnection in Section 4.4, where we give cohomological conditions relating cohomology classes associated with the principal bundles associated with the above components. Section 4.5 contains a brief discussion of the distinction between dynamical and background fields in the context of Lagrangian theories. Section 4.6 gives a prescription for twisting arbitrary Lagrangian theories, and in Section 4.7, we review the standard arguments that the partition function is independent of the connections on the bundles associated with background fields (in particular, the metric). In Section 4.8, we turn to class $\mathcal{S}$ theories and how these can be topologically twisted. To keep the discussion under control, we only consider theories of A -type with simple and full punctures. Along the way, we noticed some interesting questions about Gaiotto gluing which do not seem to have been addressed in the literature. In Section 4.9, we focus on class $\mathcal{S}$ theories of type A_1 , which are also Lagrangian theories. We consider how the cohomological conditions of Section 4.4 behave under S -duality and show the expected result that they are the same for all theories in a fixed S -duality orbit. Curiously, a given theory based on an ultraviolet curve $C_{g,n}$ has several distinct S -duality orbits and hence has several different choices of topological twisting data, since in general, the data for the different S -duality orbits will be different.

In Appendix D.2, we review the notions of transfer and reduction of structure groups of bundles with connection. This simple mathematical concept is of crucial importance here. Then Appendix D.3 gives a careful definition of what we mean by a “generalized spin^c structure” together with some examples. On several occasions, we need to refer to superconformal multiplets and the superconformal index, so these well-known topics are briefly reviewed in Appendices D.5 and D.6. In Section 4.8, we make heavy use of the (Schur limit of the) superconformal index for trinion theories. However, one important class of trinion theories is free Lagrangian theories. Therefore in Appendix D.7, we check our statements using standard Lagrangian field theory. Finally, in Section 4.8, we mention an analog, for class $\mathcal{S}$, of the cohomological conditions of Section 4.4 for Lagrangian theories. Some examples of these are illustrated in Appendix D.8.

4.2 General Setup For Topological Twisting

Since we wish our discussion to apply to non-Lagrangian theories, it is very useful to adopt the “functorial” formulation of QFT going back to the work of Segal in 2d conformal field theory [Seg88a, Seg88b, Seg02] and Atiyah in topological field theory [Ati89].⁷ The reason is that these formulations make use of the answer to a path integral, without presupposing some representation in terms of dynamical fields governed by an action principle. For recent reviews, see [Fre13, Fre19]. In this approach, one defines a set of “fields” $\mathfrak{F}$ - to be thought of as background fields.⁸ One considers a bordism category $\text{Bord}_{\leq 4}^{\mathfrak{F}}$ of manifolds of dimension ≤ 4 equipped with such fields. Then a “theory” is a choice of $\mathfrak{F}$ and a monoidal functor

$$Z : \text{Bord}_{\leq 4}^{\mathfrak{F}} \rightarrow \text{VECT} , \quad (4.2.1)$$

where VECT is a category of vector spaces over $\mathbb{C}$.⁹ We stress that we regard the choice of fields, and hence the domain bordism category for Z as a choice that one makes in defining the theory.

Specializing now to the class of $d = 4$ $\mathcal{N} = 2$ field theories the most obvious choice for the fields, which we denote as $\tilde{\mathcal{F}}$, evaluated on a fixed manifold $\mathbb{X}$ includes the supermanifold of field configurations of $d = 4$ $\mathcal{N} = 2$ conformal supergravity coupled to $\mathcal{N} = 2$ vectormultiplets for the “flavor” symmetries.¹⁰ This viewpoint (which was important in [CMRS23]) is too general for our present purposes, so we will restrict attention to a submanifold where many of the fields in the conformal supergravity multiplet have been set to zero. For our purposes, the fields will include the orientation of $\mathbb{X}$ together with the groupoid of principal $\tilde{G}$ -bundles with connection for the Lie group

$$\tilde{G} = \text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times G^{\text{f}} , \quad (4.2.2)$$

where the group G^{f} is often called the “flavor” group. It is the global internal symmetry group that commutes with supersymmetry.

⁷Atiyah and Segal quote Feynman path integrals as their main motivation. However, as pointed out to us by Y. Tachikawa, the original papers of Schwinger [Sch48] and Tomonaga [Tom46] formulated relativistic field theory in terms of families of unitary operators $U(\sigma, \sigma')$ associated to spatial slices σ, σ' and satisfying the basic axioms of the functorial formulation. One important difference is that in the older works, the theories are formulated in Minkowski spacetime, hence on invertible bordisms with a globally defined time direction, and without Wick rotation.

⁸Technically, we should consider “fields” to be a sheaf on Man_4^{op} , as explained in [FT12, FH13].

⁹In modern parlance we are considering a two-step theory. Eventually one wishes to have a fully extended, or fully local theory, but such considerations are far beyond the scope of this thesis. Moreover, a fully rigorous formulation for the kinds of interacting field theories we will consider has not been formulated. Rather this is a topic of current research. See [KS21] for a significant contribution to this current research. We are being deliberately vague about the precise nature of the vector spaces. They will need to be infinite-dimensional.

¹⁰For general backgrounds involving $\mathcal{N} = 2$ conformal supergravity, it is natural to also consider couplings to background $\mathcal{N} = 2$ hypermultiplets, but we will not do so.

Let p_i , $i = 1, 2, 3$ denote the projection to the three factors of the group (4.2.2):

$$\begin{array}{ccccc} & & \tilde{G} & & \\ & \swarrow p_1 & \downarrow p_2 & \searrow p_3 & \\ \text{Spin}(4) & & \text{SU}(2)_R & & G^f \end{array} \quad (4.2.3)$$

Denote the principal $\tilde{G}$ -bundle with connection over $\mathbb{X}$ by $(\tilde{P}, \tilde{\nabla})$. The G^f component of the connection, that is,¹¹ the connection $p_{3,*}(\tilde{P}, \tilde{\nabla})$, should be enhanced to a full $d = 4$ $\mathcal{N} = 2$ vector-multiplet. The factor $\text{Spin}(4)$ is to be regarded as the spin covering of the $\text{SO}(4)$ structure group of the bundle of oriented frames of $T\mathbb{X}$, so $\pi_{\mathbb{X},*}(\tilde{P}) = \text{OFr}(T\mathbb{X})$, where $\pi_{\mathbb{X}}$ is a projection on the first factor followed by the standard quotient homomorphism $\text{Spin}(4) \rightarrow \text{SO}(4)$, and $\text{OFr}(T\mathbb{X})$ is the bundle of oriented frames on $T\mathbb{X}$. Thus we will begin with $d = 4$ $\mathcal{N} = 2$ theories viewed as a functor

$$\tilde{Z} : \text{Bord}_{\leq 4}^{\tilde{\mathcal{F}}} \rightarrow \text{VECT} . \quad (4.2.4)$$

Since this is a functor, $\tilde{Z}$ will define state spaces. On $\mathbb{R}^3$ with Euclidean metric and zero background fields for $\text{SU}(2)_R \times G^f$, the state space will be a unitary representation of the $d = 4$ $\mathcal{N} = 2$ Poincaré group. Related to this, the local operators of the theory should be in $d = 4$ $\mathcal{N} = 2$ Poincaré multiplets.

Note that the domain category of (4.2.4) only admits 4-manifolds $\mathbb{X}$ with a spin structure. The domain category of the theory can be enlarged. Put differently, the theory can be extended to include non-spin manifolds as follows. Consider a subgroup of the center of $\tilde{G}$, written $C \subset Z(\tilde{G})$. Define $\mathcal{F}$ to be the groupoid of principal $G := \tilde{G}/C$ bundles with connection. We say the theory “extends” to a theory

$$Z : \text{Bord}_{\leq 4}^{\mathcal{F}} \rightarrow \text{VECT} , \quad (4.2.5)$$

if the diagram

$$\begin{array}{ccc} \text{Bord}_{\leq 4}^{\tilde{\mathcal{F}}} & \xrightarrow{\tilde{Z}} & \text{VECT} \\ \pi_* \searrow & & \nearrow \textcolor{blue}{Z} \\ & \text{Bord}_{\leq 4}^{\mathcal{F}} & \end{array} \quad (4.2.6)$$

commutes (up to isomorphism of functors). Here $\pi_* : \text{Bord}_{\leq 4}^{\tilde{\mathcal{F}}} \rightarrow \text{Bord}_{\leq 4}^{\mathcal{F}}$ is transfer of structure group using the natural quotient homomorphism

$$\pi : \tilde{G} \rightarrow G .$$

A necessary and sufficient condition for this factorization is that the full functor $\tilde{Z}$ be C -invariant. If we just examine the partition function then we note that C acts as a group of automorphisms

¹¹We remind the reader that the transfer of structure group and connection is described in Appendix D.2.

on the principal $\tilde{G}$ -bundles with connection over $\mathbb{X}$. The action on the connection is via a center symmetry: Holonomies are multiplied by central elements. Therefore, it acts on the partition function $\tilde{Z}$ considered as a function of the bundle with connection. The condition is

$$c \cdot \tilde{Z} = \tilde{Z} \quad \forall c \in C. \quad (4.2.7)$$

There can be many choices of C . Each choice of C defines a different “theory.” A maximal choice is one such that all theories extend to that choice. If G^f is semisimple, the center is a finite Abelian group, so there exists a maximal subgroup.

Remark 4.2.1. An analogy might clarify our reasoning here. Suppose we are given the partition function of a theory of chiral fermions on $2n$ -dimensional Riemannian spin manifolds where n is an integer. But we are given this mathematical structure without being told what fields were used to compute that partition function. How could we deduce that the theory does not extend to non-spin manifolds? Thus, we are given, for every $2n$ -dimensional spin manifold a Hermitian line bundle over the space of Riemannian metrics on that manifold, and a section of that line bundle. (In fact, the line bundle is the determinant line bundle $\text{DET}(\not{D}^+)$ of the chiral Dirac operator with its Quillen metric, but we are not given that information because we do not know the fields.) We are given the information that the line bundle depends on a spin structure and we are given the action of the center symmetry. The center of the spin group acts as a group of automorphisms on the spin bundles, and this induces an action with character χ acting on sections of the line bundle. If the action is nontrivial then we cannot extend to nonspin manifolds. Indeed, for $\text{DET}(\not{D}^+)$, the action of -1 is given by $\chi(-1) = (-1)^{\text{Ind}(\not{D}^+)}$ and is nontrivial. In this way, we see that for a chiral fermion, the functor does not factorize through non-spin manifolds, even though we have not been informed of the nature of the fields. (A more elaborate argument applies to Dirac fermions.) The point of these remarks is that the same reasoning applies to theories where we do not know the fundamental fields, such as the non-Abelian 6d $(2, 0)$ theory or generic 4d $\mathcal{N} = 2$ theories of class $\mathcal{S}$.

Remark 4.2.2. It is important to remark that the invariance condition on the partition function is in principle less stringent than the invariance condition on the full functor. In this chapter, we are primarily concerned with the partition function on compact manifolds.

In the application to four-manifold invariants, we definitely wish to include manifolds $\mathbb{X}$ which are not spinnable. Therefore we wish to have principal G -bundles over an $\mathbb{X}$ which is nonspinnable. This implies the necessary condition:

$$p_1(C) = \langle (-1, -1) \rangle \subset Z(\text{Spin}(4)), \quad (4.2.8)$$

where we have used $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$.

Different theories will have different local operator content, but one supermultiplet that is present in all $d = 4$ $\mathcal{N} = 2$ local QFTs is the energy-momentum supermultiplet. See [CDI16a, CDI16b, Ebe20] for details and Appendix D.5 for a quick summary. In order for this multiplet to be invariant under C , we must also have

$$(p_1 \times p_2)(C) = \langle(-1, -1, -1)\rangle \subset Z(\text{Spin}(4) \times \text{SU}(2)_{\text{R}}) . \quad (4.2.9)$$

The original example [Wit88] had $G^{\text{f}} = 1$, and was based on a choice of background metric and R-symmetry gauge field obtained by transfer of structure group using a crucial homomorphism

$$\varphi^{\text{W}} : \text{SO}(4) \longrightarrow (\text{Spin}(4) \times \text{SU}(2)_{\text{R}}) / \langle(-1, -1, -1)\rangle . \quad (4.2.10)$$

To write it, consider $\text{SO}(4)$ as the quotient $\text{Spin}(4)/\mathbb{Z}_2$ where $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$ and

$$\mathbb{Z}_2 \cong \langle(-1, -1)\rangle \cong \text{diag}(\mathbb{Z}_2 \times \mathbb{Z}_2) \subset \mathbb{Z}_2 \times \mathbb{Z}_2 \cong Z(\text{Spin}(4)) . \quad (4.2.11)$$

Thus, an element of $\text{SO}(4)$ can be denoted as $[(u_1, u_2)]$ where square brackets indicate an equivalence class and $u_1, u_2 \in \text{SU}(2)$. Then we can write the homomorphism (4.2.10) as:

$$\varphi^{\text{W}}([(u_1, u_2)]) := [(u_1, u_2), u_2] , \quad (4.2.12)$$

where the superscript W stands for Witten and will be called the *Witten homomorphism* because it was essential in the original construction of [Wit88].

The specification of background fields will be done via transfer of structure group along a homomorphism

$$\varphi^{\text{tw}, \text{bck}} : \underbrace{(\text{Spin}(4) \times \mathbb{T}^{\text{f}})}_{G^{\text{tw}, \text{bck}}} / C^{\text{tw}, \text{bck}} \longrightarrow (\text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times G^{\text{f}}) / C , \quad (4.2.13)$$

where $C^{\text{tw}, \text{bck}}$ is a subgroup of the center of $\text{Spin}(4) \times \mathbb{T}^{\text{f}}$. If we take $\mathbb{T}^{\text{f}}$ to be a maximal torus of G^{f} then we can consider a homomorphism

$$\tilde{\varphi}^{\text{tw}, \text{bck}} : \text{Spin}(4) \times \mathbb{T}^{\text{f}} \longrightarrow (\text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times G^{\text{f}}) / C , \quad (4.2.14)$$

defined by

$$\tilde{\varphi}^{\text{tw}, \text{bck}}((u_1, u_2), \mu) = [(u_1, u_2), u_2, \mu] . \quad (4.2.15)$$

and then simply take $C^{\text{tw}, \text{bck}}$ to be the kernel of $\tilde{\varphi}^{\text{tw}, \text{bck}}$. This is sufficient to be able to define a twisting for all theories. (For special values of the masses, there will be special twistings.) Examples such as the case of $\mathcal{N} = 2^*$ theories teach us that the existence of a homomorphism to a quotient group such that $C \cdot \tilde{Z} = \tilde{Z}$ requires that, in general, C must intersect G^{f} nontrivially.

Other twisting homomorphisms are possible, but they are constrained by the requirement that the diagram

$$\begin{array}{ccc}
 G^{\text{tw,bck}} & \xrightarrow{\varphi^{\text{tw,bck}}} & \tilde{G}/C \\
 p_1 \downarrow & & \downarrow p_1 \times p_2 \\
 \text{Spin}(4)/\langle(-1, -1)\rangle & \xrightarrow{\varphi^{\text{W}}} & (\text{Spin}(4) \times \text{SU}(2)_{\text{R}})/\langle(-1, -1, -1)\rangle
 \end{array} \tag{4.2.16}$$

commutes. This is indeed satisfied by the construction above.

Note that if p_1 is the projection on the first factor of $G^{\text{tw,bck}}$ then we should impose

$$(p_1)_*(P^{\text{tw,bck}}, \nabla^{\text{tw,bck}}) \cong (\text{O}Fr(T\mathbb{X}), \nabla^{\text{LC}}), \tag{4.2.17}$$

where ∇^{LC} is the torsion-free (Levi-Civita) connection associated to a Riemannian metric on $\mathbb{X}$.

When $G^{\text{tw,bck}}$ is of the form given in (4.2.13) with $C^{\text{tw,bck}}$ a finite abelian subgroup of the center, we say that such a principal bundle $P^{\text{tw,bck}}$ with structure group $G^{\text{tw,bck}}$ defines a *generalized spin^c structure* on $\mathbb{X}$. Moreover, a connection $\nabla^{\text{tw,bck}}$ satisfying (4.2.17) is a *generalized spin^c connection*. We elaborate on this definition and give some examples in Appendix D.3. We will meet this condition again, in the context of Lagrangian theories in equation (4.3.11).

Returning to the $d = 4$ $\mathcal{N} = 2$ energy-momentum multiplet, the relation (see (D.5.8))

$$\{Q_{\alpha A}, G_{\mu\dot{\beta}}^B\} = \sigma_{\alpha\dot{\beta}}^\nu \delta_A^B T_{\mu\nu}, \tag{4.2.18}$$

implies that, in the presence of background fields obtained by transfer of structure group $\varphi^{\text{tw,bck}}$ satisfying (4.2.16), there will be a vectorfield $\mathcal{Q}$ on the super-manifold of field configurations such that¹²

$$T_{\mu\nu} = \{\mathcal{Q}, \Lambda_{\mu\nu}\}. \tag{4.2.19}$$

A standard argument, going back to [Wit88], shows that (4.2.19) leads to the “topological invariance” of Z . “Topological invariance” means independence of the choice of Riemannian metric. The reason is that the derivative of Z with respect to the metric $g_{\mu\nu}$ is given by Z with an insertion of $T_{\mu\nu}$. If there is no BRST anomaly, then insertions of $\mathcal{Q}$ -exact operators must vanish. (As noted above, there can be BRST anomalies, leading to metric dependence, when $b_2^+(\mathbb{X}) \leq 1$.)

We will refer to theories with domain $\text{Bord}_{\leq 4}^{\mathcal{F}}$ such that, for all manifolds and background fields in the domain, there is a $\mathcal{Q}$ -symmetry, as “twisted theories.”

Similar reasoning shows that a twisted theory has a partition function that is invariant under continuous deformation of the other components of the $G^{\text{tw,bck}}$ connection. Using the general structure of the conserved current multiplets (which couple to the flavor symmetry connections), the

¹²Note that (4.2.19) is obtained from the second equation of (D.5.8) because in our conventions, the scalar supercharge $\mathcal{Q}$ is obtained from the *right-handed* spinor supercharge of the $d = 4$ $\mathcal{N} = 2$ supersymmetry algebra. We are being slightly schematic here to avoid distracting notation.

general equation (see (D.5.6) and (D.5.7))

$$\delta_A^B \sigma_{\alpha\beta}^\mu J_\mu^a = \{ \mathcal{Q}_{A\alpha}, \tilde{\psi}_{\dot{\beta}}^{Ba} \} , \quad (4.2.20)$$

becomes, after topological twisting

$$J_\mu^a = \{ \mathcal{Q}, \Lambda_\mu^a \} . \quad (4.2.21)$$

Here, the subscript a is an adjoint index in the Lie algebra of the symmetry. Therefore, in the absence of a BRST anomaly, the derivative of the partition function with respect to the connection vanishes. Therefore, the twisted partition function only depends on the three pieces of topological data mentioned in the introduction.

Remark 4.2.3. *On the terminology “physical theory” vs. “twisted theory.”* We will use the (common) terminology of “twisted theory” and “physical theory” in our discussion below. Since we are regarding a “theory” as a functor $\mathbb{Z}$, part of the very definition of a “theory” involves a specification of the domain and codomain categories. As we have just mentioned, a “twisted theory” has a $\mathcal{Q}$ -symmetry for all objects in the domain category. By a “physical theory” we mean a theory where this is not true for all objects in the domain category, although whether these should be considered more “physical” is a debatable point, best left to philosophers.

Remark 4.2.4. In the interest of simplicity, our general discussion here has been limited to “zero-form symmetries.” It should be extended to include other generalized symmetries. We will indeed do this in the special case of Lagrangian theories in Sections 4.3 – 4.7.

4.3 Fields For Lagrangian Theories

In this section, we describe the fields that are needed to define the exponentiated action e^{-S} . We regard the exponentiated action on $\mathbb{X}$ as the partition function on $\mathbb{X}$ of an invertible field theory [FM04]. We restrict attention to general renormalizable $d = 4$ $\mathcal{N} = 2$ Lagrangian gauge theories. To define these, one makes the following choice of basic data:

1. $\tilde{G}^{\text{gauge}}$: a semisimple simply connected compact Lie group. For simplicity, we will assume it is connected. The gauge group G^{gauge} will be a finite quotient of $\tilde{G}^{\text{gauge}}$, as explained below.
2. $\rho : \tilde{G}^{\text{gauge}} \rightarrow \text{End}_{\mathbb{R}}(\mathcal{R})$: a quaternionic representation on a real vector space $\mathcal{R}$ equipped with a G^{gauge} -invariant positive inner product. The group of orthogonal hyperkähler isometries is denoted $\text{O}(\rho)$. This data defines a compact Lie group G^{f} – the *flavor group* – which is defined to be the commutant of $\rho(\tilde{G}^{\text{gauge}})$ in $\text{O}(\rho)$.

3. C^{phys} : an “admissible” subgroup of the center of

$$\tilde{G}^{\text{phys}} := \text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times G^{\text{f}} \times \tilde{G}^{\text{gauge}} . \quad (4.3.1)$$

The term “admissible” is defined by conditions (4.3.6) and (4.3.8) below. There are different choices of admissible groups C^{phys} , and they correspond to different theories.

The set of fields for the theory will include a principal bundle with connection over $\mathbb{X}$ for the structure group¹³

$$G^{\text{phys}} := (\text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times G^{\text{f}} \times \tilde{G}^{\text{gauge}}) / C^{\text{phys}} = \tilde{G}^{\text{phys}} / C^{\text{phys}} . \quad (4.3.2)$$

We denote the principal bundle with connection by

$$(P^{\text{phys}} \rightarrow \mathbb{X}, \nabla^{\text{phys}}) . \quad (4.3.3)$$

If we think of the invertible field theory (a.k.a. the exponentiated action e^{-S}) as a functor

$$Z : \text{Bord}_{\leq 4}^{\mathcal{F}} \rightarrow \text{VECT} , \quad (4.3.4)$$

then $\mathcal{F}(\mathbb{X})$ includes the groupoid of principal G^{phys} -bundles with connection.

The bundle with connection $(P^{\text{phys}} \rightarrow \mathbb{X}, \nabla^{\text{phys}})$ contains several pieces of physical data which are usually thought of as independent, but they are not completely independent because of the quotient by the finite group C^{phys} .

Let p_i , $i = 1, 2, 3, 4$ denote the projection to each of the factors of $\tilde{G}^{\text{phys}}$. We have a transfer of structure group for bundles for each of the image groups $p_i(G^{\text{phys}})$. We denote these by:

$$\begin{aligned} \text{Fr}(\mathbb{X}) &:= (p_1)_*(P^{\text{phys}}) , \\ P^{\text{R}} &:= (p_2)_*(P^{\text{phys}}) , \\ P^{\text{f}} &:= (p_3)_*(P^{\text{phys}}) , \\ P^{\text{gauge}} &:= (p_4)_*(P^{\text{phys}}) , \end{aligned} \quad (4.3.5)$$

and refer to them respectively as the frame bundle, R-symmetry bundle, flavor bundle, and gauge bundle.

We are now ready to define the notion that C^{phys} is *admissible* as follows:

1. We can rederive the condition (4.2.9) as follows. First of all, we have:

$$(p_1 \times p_2)(C^{\text{phys}}) \subset \langle (-1, -1, -1) \rangle \subset Z(\text{Spin}(4) \times \text{SU}(2)_{\text{R}}) \cong \mathbb{Z}_2^3 . \quad (4.3.6)$$

¹³Our discussion at this point has some tangential overlap with recent discussions [WWW18, HLL⁺22, BI23]. See also section 2.2 of [FH16].

The reason is that vectormultiplet fields must be singlets under the action of C^{phys} . They are in the adjoint representation of the gauge group and hence are singlets under the center of that group. Moreover, they are singlets under the action of the flavor group. Finally, we will want to identify $\text{Spin}(4)/p_1(C^{\text{phys}}) \cong \text{SO}(4)$ with the structure group of $\text{O}(\mathbb{X})$. If we wish the domain bordism category to include nonspin manifolds, then we must have that

$$(p_1 \times p_2)(C^{\text{phys}}) = \langle (-1, -1, -1) \rangle . \quad (4.3.7)$$

2. If there are hypermultiplets in the theory then C^{phys} must also be a subgroup of the group:

$$\begin{aligned} C_{\text{max}}^{\text{phys}} &:= \{(\zeta, \zeta, \zeta, \mu, g) \mid \zeta \mu \rho(g) = \mathbb{1}_{\text{End}_{\mathbb{R}}(\mathcal{R})}\} \\ &\subset Z(\text{Spin}(4)) \times Z(\text{SU}(2)_{\text{R}}) \times Z(G^{\text{f}}) \times Z(\tilde{G}^{\text{gauge}}) , \end{aligned} \quad (4.3.8)$$

where $\zeta \in \{\pm 1\} \cong \mathbb{Z}_2$ acts on $\mathcal{R}$ by scalar multiplication. The reason for this condition is that fields in the hypermultiplet representation ρ must be singlets under C^{phys} (and we have used condition (4.3.6)). Note that this forces C^{phys} to be a finite Abelian group. We will call $C_{\text{max}}^{\text{phys}}$ the *maximal admissible subgroup*.

We make a few remarks on the bundles appearing in (4.3.5):

- The gauge bundle P^{gauge} is a principal $G^{\text{gauge}} \cong \tilde{G}^{\text{gauge}}/C^{\text{grb}}$ bundle, where we define

$$C^{\text{grb}} := p_4(C^{\text{phys}}) . \quad (4.3.9)$$

The reason for this notation is explained in item 4 below.

- The bundle $\text{Fr}(\mathbb{X})$ is a principal $\text{SO}(4)$ bundle because

$$\text{Spin}(4)/p_1(C^{\text{phys}}) \cong \text{SO}(4) . \quad (4.3.10)$$

Moreover, if we use the transfer of structure group associated to the canonical homomorphism¹⁴

$p_1 : G^{\text{phys}} \rightarrow \text{SO}(4) \cong \text{Spin}(4)/\langle (-1, -1) \rangle$ then we require:

$$(p_1)_*(P^{\text{phys}} \rightarrow \mathbb{X}, \nabla^{\text{phys}}) \cong (\text{O}(\text{Fr}(T\mathbb{X}), \nabla^{\text{LC}}) . \quad (4.3.11)$$

- The R-symmetry bundle P^{R} is a principal $\text{SO}(3)_{\text{R}}$ bundle.
- The flavor bundle P^{f} is a principal $G^{\text{f}}/p_3(C^{\text{phys}})$ bundle.

Remark 4.3.1.

¹⁴We define $p_i(G^{\text{phys}}) := p_i(\tilde{G}^{\text{phys}})/p_i(C^{\text{phys}})$.

1. For generic hypermultiplet masses it is natural to replace G^f with a maximal torus, so that P^f is in fact a $\mathbb{T}^f/p_3(C^{\text{phys}})$ -bundle where $\mathbb{T}^f \subset G^f$ is a maximal torus. In this case, we have restricted the background fields to reducible G^f -connections and assumed a reduction of the structure group to its maximal torus.
2. The existence of P^{phys} implies a restriction on characteristic classes of the “frame,” “R-symmetry,” “flavor,” and “gauge” bundles derived from P^{phys} . See Section 4.4 below.

The exponentiated action of the theory e^{-S} , regarded as an invertible field theory, can be defined on a bordism category equipped with a set of fields $\mathcal{F}^{\text{phys}}$ that we informally write as:

$$\mathcal{F}^{\text{phys}} := \{\tau, m, (P^{\text{phys}} \rightarrow \mathbb{X}, \nabla^{\text{phys}}), \Phi^{\text{phys}}, b, \Phi^{\text{grav}}\}, \quad (4.3.12)$$

where

1. τ is a $\mathbb{C}$ -valued $\text{Ad}(G^{\text{gauge}} \times G^f)$ -invariant bilinear form on $\text{Lie}(G^{\text{gauge}} \times G^f)$ subject to the condition that the imaginary part is positive definite.¹⁵ τ enters the classical Lagrangian by defining the Yang-Mills couplings of the gauge theory. (τ will be replaced by τ_{uv} or Λ below in the quantum theory.)
2. $m \in \text{Lie}(G^f) \otimes \mathbb{C}$, subject to the condition that $[m, m^\dagger] = 0$. The field m enters the Lagrangian as $\langle m, \mu \rangle$ where μ is the hyperkähler moment map for G^f . One should view m as the background vectormultiplet scalars for G^f .
3. $\Phi^{\text{phys}} \in \Gamma(E_\rho)$ where E_ρ is an associated bundle to P^{phys} constructed using ρ and the rules of supersymmetry for constructing vectormultiplets and hypermultiplets. Its sections include fermions, and hence, we should view the space of fields on $\mathbb{X}$ as an infinite-dimensional supermanifold.¹⁶
4. The b -field is a flat C^{grb} -gerbe connection¹⁷ in a 5d topological field theory. In order to handle properly the conditions on how one sums over ’t Hooft fluxes, it is important to invoke the quiche picture of finite “categorical symmetry” in field theory. See [FMT22] and references therein. Example 3.15 of [FMT22] is particularly relevant, and an elaboration of the relevant issues can be found in [Fre22]. One must consider the 4d theory we discuss in this chapter

¹⁵This follows from unitarity of the theory evaluated on Minkowski space.

¹⁶We will see later that, for suitable background fields, there is a fermionic vectorfield on this supermanifold derived from a scalar supersymmetry with $\mathcal{Q}^2 = 0$. The space of fields that defines such an odd vectorfield is called the nilpotence variety. We will describe a subvariety of the nilpotence variety.

¹⁷A gerbe with connection is (up to isomorphism) an element of a suitable differential cohomology group of degree 3. For a recent pedagogical introduction, see [Hit99, MS25].

as a right boundary theory (on $\{1\} \times \mathbb{X}$) for a 5d π -finite topological field theory with target $B^2 C^{\text{grb}}$ on $[0, 1] \times \mathbb{X}$. There is a field in the right boundary theory which is an isomorphism of the restriction of b with the gerbe describing the obstruction to lifting $(p_4)_*(P^{\text{phys}})$ to a principal $\tilde{G}^{\text{gauge}}$ -bundle over $\mathbb{X}$. This is the origin of the notation C^{grb} introduced in equation (4.3.9) above. We denote the characteristic class by $\mu(b) \in H^2(\mathbb{X}; C^{\text{grb}})$. There is a topological boundary condition ρ (possibly with the insertion of topological defects) at the boundary $\{0\} \times \mathbb{X}$. Depending on the choice of ρ we might or might not sum over 't Hooft fluxes in computing physical quantities.¹⁸ We note that we see no obvious reason why one should not take b to be a gerbe for the full group C^{phys} and indeed this generalization will be needed in Appendix D.8. We have not investigated this generalization systematically, and doing so might be an interesting line of future enquiry. It might well streamline the presentation in this chapter.

5. Φ^{grav} is short hand for various “gravitational” fields. The Riemannian metric is already encoded in the connection ∇^{phys} , but we must also choose an orientation of $\mathbb{X}$, and a truly complete story would include all the fields of background supergravity, including the odd partners of the metric. We will not go into details here, although some relevant constructions can be found in [CMRS23].
6. Explicit formulae for e^{-S} as a function of these fields can be extracted from the standard references: [GGRS83, DF99, dWLVP85, FVP12].

Remark 4.3.2. It is sometimes said that, since the supersymmetric gauge theory contains fermions, it must be defined on a spin manifold. Then, after twisting, the twisted theory can also be defined on non-spin manifolds. From the viewpoint presented in this chapter, that reasoning is incorrect. In defining a theory, one must specify the domain bordism category. Different choices of group C^{phys} lead to different choices of the bordism category. For example, if we choose C^{phys} to be trivial, then the domain bordism category must contain only spin manifolds. However, if we choose C^{phys} so that $p_1 \times p_2$ in equation (4.3.6) is surjective, then the domain category might include all standard four-manifolds. This is just as true for the “physical theory” with general background fields as it is for the twisted theory defined below. (Related remarks can be found in section 2.4.2 of [CD18], and in [BI23].)

Remark 4.3.3. We have not been careful with the global structure of the space of fields in (4.3.12). In fact, the criterion (4.4.18) which connects the characteristic class of the (4d) boundary value

¹⁸The pullback of b to $\mathbb{X}$ itself can have automorphism, so there can be several inequivalent isomorphisms to the gauge theory gerbe on the right (physical) boundary. Therefore we should also sum over $H^1(\mathbb{X}; C^{\text{grb}})$.

of the (5d) gerbe b with a characteristic class of the flavor gauge fields suggests that there is a nontrivial fibration, and perhaps even a useful 2-group structure [CDI18]. We leave this for future investigation.

4.4 Cohomological Conditions For The Existence Of P^{phys}

In this section, we show that the existence of the bundle P^{phys} in equation (4.3.3) above implies cohomological conditions linking the background gravity, flavor, and gerbe data.

Let us assume that the G^{phys} -bundle $P^{\text{phys}} \rightarrow \mathbb{X}$ exists. We choose a good cover of $\mathbb{X}$, denoted by $\{U_p\}_{p \in \mathcal{P}}$, where $\mathcal{P}$ is some index set. Therefore, on double overlaps $U_{pq} := U_p \cap U_q$, we have transition functions,

$$t_{pq}^{\text{phys}} : U_{pq} \longrightarrow G^{\text{phys}}, \quad (4.4.1)$$

that satisfy the cocycle condition on triple overlaps $U_{pqr} := U_p \cap U_q \cap U_r$,

$$t_{pq}^{\text{phys}}(x)t_{qr}^{\text{phys}}(x)t_{rp}^{\text{phys}}(x) = \mathbb{1}_{G^{\text{phys}}}, \quad \forall x \in U_{pqr}. \quad (4.4.2)$$

We may choose the following transition functions for the bundles $(p_i)_*(P^{\text{phys}})$ of equation (4.3.5) on overlaps:

$$\begin{aligned} (\tilde{t}_{pq}^+, \tilde{t}_{pq}^-) : U_{pq} &\longrightarrow \text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2), \\ \tilde{t}_{pq}^{\text{R}} : U_{pq} &\longrightarrow \text{SU}(2)_{\text{R}}, \\ \tilde{t}_{pq}^{\text{f}} : U_{pq} &\longrightarrow \mathbb{T}^{\text{f}}, \\ \tilde{t}_{pq}^g : U_{pq} &\longrightarrow \tilde{G}^{\text{gauge}}, \end{aligned} \quad (4.4.3)$$

such that¹⁹

$$t_{pq}^{\text{phys}} = [(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{t}_{pq}^{\text{R}}, \tilde{t}_{pq}^{\text{f}}, \tilde{t}_{pq}^g)] . \quad (4.4.4)$$

Note that our choice is ambiguous by

$$(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{t}_{pq}^{\text{R}}, \tilde{t}_{pq}^{\text{f}}, \tilde{t}_{pq}^g) \sim (\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{t}_{pq}^{\text{R}}, \tilde{t}_{pq}^{\text{f}}, \tilde{t}_{pq}^g) \cdot \alpha, \quad (4.4.5)$$

where $\alpha \in C^{\text{phys}}$. Therefore, the functions (4.4.3) need not individually satisfy the cocycle conditions.

We can therefore define Čech-cocycles

$$\begin{aligned} \zeta_{pqr}^{\pm} &:= \tilde{t}_{pq}^{\pm} \tilde{t}_{qr}^{\pm} \tilde{t}_{rp}^{\pm}, \\ \zeta_{pqr}^{\text{R}} &:= \tilde{t}_{pq}^{\text{R}} \tilde{t}_{qr}^{\text{R}} \tilde{t}_{rp}^{\text{R}}, \\ \zeta_{pqr}^{\text{f}} &:= \tilde{t}_{pq}^{\text{f}} \tilde{t}_{qr}^{\text{f}} \tilde{t}_{rp}^{\text{f}}, \\ \zeta_{pqr}^g &:= \tilde{t}_{pq}^g \tilde{t}_{qr}^g \tilde{t}_{rp}^g. \end{aligned} \quad (4.4.6)$$

¹⁹The equivalence class $[\cdot]$ in (4.4.4) arises from the quotient by C^{phys} , see (4.3.2).

Then the cocycle condition (4.4.2) is equivalent to

$$\boxed{(\zeta_{pqr}^+, \zeta_{pqr}^-, \zeta_{pqr}^R, \zeta_{pqr}^f, \zeta_{pqr}^g) \in C^{\text{phys}}}. \quad (4.4.7)$$

From (4.3.6),

$$\zeta_{pqr}^- = \zeta_{pqr}^+ = \zeta_{pqr}^R. \quad (4.4.8)$$

We will also define $\zeta_{pqr}^s := \zeta_{pqr}^- = \zeta_{pqr}^+$ for the cocycle corresponding to the $\text{Spin}(4)$ factor of $\tilde{G}^{\text{phys}}$. To derive cohomological conditions from (4.4.7), we need to identify what each cocycle represents. By definition,

$$t_{pq}^g := [\tilde{t}_{pq}^g] : U_{pq} \longrightarrow \tilde{G}^{\text{gauge}}/C^{\text{grb}} \cong G^{\text{gauge}}, \quad (4.4.9)$$

represents a choice of transition functions on the gauge bundle $P^{\text{gauge}} \rightarrow \mathbb{X}$. The cocycle ζ_{pqr}^g represents the 't Hooft flux of the gauge bundle, i.e., the obstruction to lifting the structure group from $G^{\text{gauge}} \cong \tilde{G}^{\text{gauge}}/C^{\text{grb}}$ to $\tilde{G}^{\text{gauge}}$:

$$\mu(b) := [\zeta_{pqr}^g] \in H^2(\mathbb{X}; C^{\text{grb}}). \quad (4.4.10)$$

The requirement just below (4.3.6) implies that ζ_{pqr}^s represents the second Stiefel-Whitney class of $T\mathbb{X}$:

$$w_2(\mathbb{X}) := w_2(T\mathbb{X}) = [\zeta_{pqr}^s] \in H^2(\mathbb{X}; \mathbb{Z}_2), \quad (4.4.11)$$

which is the obstruction to $\mathbb{X}$ being spin. Similarly, ζ_{pqr}^R represents the obstruction to lifting the structure group of the R-symmetry bundle $P^R \rightarrow \mathbb{X}$ from $\text{SO}(3)_R$ to $\text{SU}(2)_R$:

$$w_2(P^R) := [\zeta_{pqr}^R] \in H^2(\mathbb{X}; \mathbb{Z}_2). \quad (4.4.12)$$

From (4.4.8), we immediately get the first cohomological condition, namely

$$\boxed{w_2(P^R) = w_2(\mathbb{X})}. \quad (4.4.13)$$

Whenever we have hypermultiplets in the theory, the condition (4.4.7) can be written using item 2 above as

$$\rho(\zeta_{pqr}^g) \zeta_{pqr}^s \zeta_{pqr}^f = 1 \iff \zeta_{pqr}^f = \rho(\zeta_{pqr}^g)^{-1} \zeta_{pqr}^s \in \mathbb{T}^f. \quad (4.4.14)$$

Note that (4.4.14) is equivalent to (4.4.7) when C^{phys} is the maximal admissible subgroup.²⁰

²⁰We emphasize that the condition (4.4.14) is not equivalent to (4.4.7) when C^{phys} is not the maximal admissible subgroup. The simplest counterexample is when $Z(\tilde{G}^{\text{gauge}})$ is a cyclic group with generator g . Consider

$$C^{\text{phys}} = \langle (-1, -1, -1, -\rho(g)^{-1}, g) \rangle.$$

In this case, C^{phys} is not the maximal admissible subgroup, and clearly,

$$p_4(C^{\text{phys}}) \cong Z(\tilde{G}^{\text{gauge}}), \quad (p_1 \times p_2)(C^{\text{phys}}) \cong \text{diag}(\mathbb{Z}_2^3).$$

We can turn the condition (4.4.14) into a cohomological restriction on the background fields as follows. Recall that we are choosing generic masses, so we are reducing the flavor symmetry to its maximal torus. Note that as a representation of $\tilde{G}^{\text{gauge}}$ over $\mathbb{C}$, there is an isotypical decomposition (here we choose a complex structure on $\mathcal{R}$):

$$\rho \cong \bigoplus_u \rho_u , \quad (4.4.15)$$

where each summand ρ_u is an irreducible representation. Also, note that the irreducible representations for $u \neq u'$ might be isomorphic. Choose an isomorphism:

$$\mathbb{T}^f \cong \prod_u \text{U}(1)_u . \quad (4.4.16)$$

We let $p_{4,u}$ denote the projection of $\tilde{G}^{\text{phys}}$ to the u^{th} component of (4.4.16).

Consider any irreducible subrepresentation ρ_u of ρ . Since C^{grb} is finite Abelian, Schur's Lemma²¹ implies that the image $\rho_u(C^{\text{grb}})$ in $\text{Aut}(\rho_u)$ is always a cyclic group acting by scalars. This image, together with -1 , generates a cyclic subgroup $\mathcal{Z}_u \subset \text{U}(1)$. Note there is a canonical injective homomorphism $\rho_s : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathcal{Z}_u$. Moreover, the condition (4.3.8) implies that there is an inclusion $\iota : p_{3,u}(C^{\text{phys}}) \hookrightarrow \mathcal{Z}_u$ and hence $p_{3,u}(C^{\text{phys}})$ is isomorphic to $\mathbb{Z}/n_u\mathbb{Z}$, thus defining a set of integers $\{n_u\}$.

Next, $(p_{3,u})_*(P^{\text{phys}}, \nabla^{\text{phys}}) = (P^{f,u}, \nabla^{f,u})$ is a well-defined principal $\text{U}(1)/p_{3,u}(C^{\text{phys}})$ -bundle over $\mathbb{X}$ with connection. We should think of $\nabla^{f,u}$ as the “background gauge fields for the flavor symmetry.” Let $c_1(u) \in H^2(\mathbb{X}; \mathbb{Z})$ be the first Chern class of the $\text{U}(1)$ -bundle obtained by transfer of structure group of $P^{f,u}$ along the isomorphism²²

$$\text{U}(1)/p_{3,u}(C^{\text{phys}}) \xrightarrow{\cong} \text{U}(1) . \quad (4.4.17)$$

Let $r_n : \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ denote reduction modulo n . The cocycle $[\zeta_{pqr}^f]$ is then a representative of $r_{n_u}(c_1(u))$ and is the obstruction to lifting $P^{f,u}$ to a $\text{U}(1)$ -bundle along the quotient map $\text{U}(1) \rightarrow$

Now, if g is not a square in $Z(\tilde{G}^{\text{gauge}})$, the element $(1, 1, 1, \rho(g)^{-1}, g) \in \mathbb{Z}_2^3 \times Z(G^f) \times Z(\tilde{G}^{\text{gauge}})$ satisfies (4.4.14) but is not an element of C^{phys} . More generally, whenever C^{phys} is a proper subgroup of the maximal admissible subgroup, (4.4.14) is not equivalent to (4.4.7).

²¹We are working over $\mathbb{C}$.

²²Note that this isomorphism is not the n_u^{th} -power map. More precisely, it is the usual isomorphism obtained from the short exact sequence

$$1 \longrightarrow p_{3,u}(C^{\text{phys}}) \xrightarrow{\iota} \text{U}(1) \xrightarrow{P_{n_u}} \text{U}(1) \longrightarrow 1 ,$$

where ι is the inclusion map and P_{n_u} is the n_u^{th} -power map.

$U(1)/p_{3,u}(C^{\text{phys}})$. The existence of P^{phys} implies the following cohomological restriction on background fields (“background 1-form symmetry connections”):²³

$$\boxed{\rho_u(\mu(b)) + \rho_s(w_2(\mathbb{X})) + \iota \circ r_{n_u}(c_1(u)) = 0 \in H^2(\mathbb{X}; \mathcal{Z}_u)} , \quad (4.4.18)$$

which must hold for all the irreducible subrepresentations of ρ .

We now show that (4.4.18) is equivalent to the existence of P^{phys} when C^{phys} is the maximal admissible subgroup and the flavor symmetry is $G^f = U(1)$. In the case that $G^f = U(1)^d$, we simply take a fiber product.

Consider now a principal $(\tilde{G}/C^{\text{grb}})$ -bundle P^{gauge} with 't Hooft flux $\mu(b) \in H^2(\mathbb{X}; C^{\text{grb}})$, a R -symmetry bundle P^R which is an $SO(3)_R$ -bundle with second Stiefel-Whitney class $w_2(P^R) \in H^2(\mathbb{X}; \mathbb{Z}_2)$ and a flavor symmetry bundle P^f which is a $U(1)/p_3(C^{\text{phys}})$ -bundle. The condition (4.3.8) implies that $p_3(C^{\text{phys}}) \hookrightarrow \mathcal{Z} \subset U(1)$ where $\mathcal{Z}$ is a subgroup generated by $\rho(C^{\text{grb}})$ and -1 . There is an isomorphism,

$$p_3(C^{\text{phys}}) \cong \mathbb{Z}/n\mathbb{Z} , \quad (4.4.19)$$

for some $n \in \mathbb{Z}_{>0}$. We also have the frame bundle $\text{Fr}(\mathbb{X})$ which is an $SO(4)$ -bundle with second Stiefel-Whitney class $w_2(\mathbb{X}) \in H^2(\mathbb{X}; \mathbb{Z}_2)$. Moreover, assume that the cohomological conditions (4.4.13) and (4.4.18) are satisfied. We now show that these bundles can be assembled into a G^{phys} -bundle $P^{\text{phys}} \rightarrow \mathbb{X}$. Indeed, take transition functions

$$\begin{aligned} t_{pq}^s &: U_{pq} \longrightarrow SO(4) , \\ t_{pq}^R &: U_{pq} \longrightarrow SO(3)_R , \\ t_{pq}^f &: U_{pq} \longrightarrow U(1)/p_3(C^{\text{phys}}) , \\ t_{pq}^g &: U_{pq} \longrightarrow \tilde{G}^{\text{gauge}}/C^{\text{grb}} , \end{aligned} \quad (4.4.20)$$

on $P^{\text{gauge}}, \text{Fr}(\mathbb{X}), P^R$ and P^f respectively, and consider local lifts – as defined in (4.4.3) – to

$$\begin{aligned} (\tilde{t}_{pq}^+, \tilde{t}_{pq}^-) &: U_{pq} \longrightarrow \text{Spin}(4) \cong SU(2) \times SU(2) , \\ \tilde{t}_{pq}^R &: U_{pq} \longrightarrow SU(2)_R , \\ \tilde{t}_{pq}^f &: U_{pq} \longrightarrow \mathbb{R} , \\ \tilde{t}_{pq}^g &: U_{pq} \longrightarrow \tilde{G}^{\text{gauge}} , \end{aligned} \quad (4.4.21)$$

where the lift $\tilde{t}_{pq}^f$ is along the homomorphism

$$\begin{aligned} \pi : \mathbb{R} &\longrightarrow U(1)/p_3(C^{\text{phys}}) \\ x &\longmapsto \left[\exp \left(\frac{2\pi i x}{n} \right) \right] , \end{aligned} \quad (4.4.22)$$

²³The second term in (4.4.18) is the image of $w_2(\mathbb{X})$ under the map $\rho_s : H^2(\mathbb{X}; \mathbb{Z}_2) \rightarrow H^2(\mathbb{X}; \mathcal{Z}_u)$ induced by the canonical homomorphism $\rho_s : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathcal{Z}_u$.

where n is given by (4.4.19). Thus

$$\left[\exp \left(\frac{2\pi i \tilde{t}_{pq}^f}{n} \right) \right] = t_{pq}^f . \quad (4.4.23)$$

The first Chern class $c_1(f) \in H^2(\mathbb{X}; \mathbb{Z})$ of the $U(1)$ -bundle obtained by transfer of structure group of P^f along the isomorphism $U(1)/p_3(C^{\text{phys}}) \cong U(1)$ (see footnote 22) is represented by the cocycle

$$\eta_{pqr}^f := \tilde{t}_{pq}^f + \tilde{t}_{qr}^f + \tilde{t}_{rp}^f : U_{pqr} \longrightarrow \mathbb{Z} . \quad (4.4.24)$$

Also, define

$$\zeta_{pqr}^f := \exp \left(\frac{2\pi i \eta_{pqr}^f}{n} \right) , \quad (4.4.25)$$

which is the cocycle representing the mod n reduction of $c_1(f)$. The cocycles for $(\tilde{t}_{pq}^-, \tilde{t}_{pq}^+)$, $\tilde{t}_{pq}^R$ and $\tilde{t}_{pq}^g$ are denoted by

$$\begin{aligned} (\zeta_{pqr}^+, \zeta_{pqr}^-) : U_{pqr} &\longrightarrow \text{diag}(\mathbb{Z}_2^2) , \\ \zeta_{pqr}^R : U_{pqr} &\longrightarrow \mathbb{Z}_2 , \\ \zeta_{pqr}^g : U_{pqr} &\longrightarrow C^{\text{grb}} = p_4(C^{\text{phys}}) , \end{aligned} \quad (4.4.26)$$

respectively. The condition (4.4.13) implies that, perhaps after a change of local trivialization,

$\zeta_{pqr}^R = \zeta_{pqr}^\pm = \zeta_{pqr}^s$. If we take

$$t_{pq}^{\text{phys}} = [(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{t}_{pq}^R, t_{pq}^f, \tilde{t}_{pq}^g)] , \quad (4.4.27)$$

it follows that

$$t_{pq}^{\text{phys}} t_{qr}^{\text{phys}} t_{rp}^{\text{phys}} = [(\zeta_{pqr}^+, \zeta_{pqr}^-, \zeta_{pqr}^R, \zeta_{pqr}^f, \zeta_{pqr}^g)] . \quad (4.4.28)$$

Note that ζ_{pqr}^g is valued in C^{grb} and $(\zeta_{pqr}^-, \zeta_{pqr}^+, \zeta_{pqr}^R)$ is valued in $(p_1 \times p_2)(C^{\text{phys}})$ and (4.4.18) implies that

$$\rho(\zeta_{pqr}^g) \zeta_{pqr}^s \zeta_{pqr}^f = 1 . \quad (4.4.29)$$

As noted above, for maximal admissible subgroup C^{phys} , this is equivalent to the condition

$$(\zeta_{pqr}^+, \zeta_{pqr}^-, \zeta_{pqr}^R, \zeta_{pqr}^f, \zeta_{pqr}^g) \in C^{\text{phys}} . \quad (4.4.30)$$

Thus t_{pq}^{phys} satisfies the cocycle condition for G^{phys} and hence can be used to construct a principal G^{phys} -bundle P^{phys} .

We now show that in specific examples, (4.4.18) reduces to familiar conditions known from the literature.

1. Pure $\mathcal{N} = 2$ Super Yang-Mills (SYM) Theory with gauge group G^{gauge} :

In this case, C^{phys} is constrained to be of the form

$$C^{\text{phys}} = \text{diag}(\mathbb{Z}_2^3) \times Z , \quad (4.4.31)$$

where Z is some subgroup of $Z(\tilde{G}^{\text{gauge}})$. The only condition that we get from the existence of P^{phys} is that

$$w_2(\mathbb{X}) = w_2(P^{\text{R}}) . \quad (4.4.32)$$

2. $\text{SU}(2)$ $\mathcal{N} = 2$ SYM with fundamental flavors:

For fundamental hypermultiplets,

$$\rho_u(\mu(b)) =: w_2(P^{\text{gauge}}) \in H^2(\mathbb{X}; \mathbb{Z}_2) . \quad (4.4.33)$$

If we choose not to “turn on the flavor bundle” so that $c_1(u) = 0$, (4.4.18) yields²⁴

$$w_2(P^{\text{gauge}}) = w_2(\mathbb{X}) = w_2(P^{\text{R}}) , \quad (4.4.34)$$

which was already noted in [MW97] for the twisted theory. If background $\text{U}(1)$ flavor gauge bundles are turned on, (4.4.18) reduces to

$$c_1(u) \equiv w_2(\mathbb{X}) + w_2(P^{\text{gauge}}) \pmod{2} , \quad (4.4.35)$$

which is precisely the condition proposed in [AFM22]. (The condition (4.4.32) still applies.) We stress again that – contrary to the viewpoint found in [AFM22, MW97] – these are conditions for the physical theory, not just specific to topologically twisted theories.

3. $\mathcal{N} = 2^*$ Theory:

Matter in the $\mathcal{N} = 2^*$ theory consists of a massive adjoint hypermultiplet. Since the center of the gauge group acts trivially on the adjoint representation, it follows that

$$\rho_u(\mu(b)) = 0 . \quad (4.4.36)$$

Thus, (4.4.18) yields

$$c_1(u) \equiv w_2(\mathbb{X}) \pmod{2} . \quad (4.4.37)$$

²⁴Note that we get the same condition if the physical theory corresponds to the choice

$$C^{\text{phys}} \cong \langle (-1, -1, -1, 1, -1) \rangle .$$

Indeed, in this case (4.4.14) implies that $\rho(\zeta_{pqr}^g) = \zeta_{pqr}^s$ which implies $w_2(P^{\text{gauge}}) = w_2(\mathbb{X})$.

Therefore, $c_1(u)$ can be identified with the characteristic class of a spin^c -structure²⁵ on $\mathbb{X}$. This condition was obtained in [LMn97, MM21]. (The condition (4.4.32) still applies.)

4. $\mathcal{N} = 4$ Super Yang-Mills Theory:

Consider the $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory. The field content consists of an $\mathcal{N} = 2$ vectormultiplet and a massless adjoint hypermultiplet. The theory has an $\text{SU}(4)_{\text{R}}$ R-symmetry. So,

$$\tilde{G}^{\text{phys}} = \text{Spin}(4) \times \text{SU}(4)_{\text{R}} \times \tilde{G}^{\text{gauge}} , \quad (4.4.38)$$

and now C^{phys} is a subgroup of the center of $\tilde{G}^{\text{phys}}$, which is

$$Z(\tilde{G}^{\text{phys}}) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times Z(\tilde{G}^{\text{gauge}}) . \quad (4.4.39)$$

The structure group of the physical theory is $G^{\text{phys}} = \tilde{G}^{\text{phys}}/C^{\text{phys}}$. The gauge, frame, and R-symmetry bundles can be obtained via pushforward by the pertinent projection maps as in the general discussion. We also have the identification (4.3.11).

The condition that the fields of the theory be singlets under C^{phys} along with (4.3.10) implies that C^{phys} be a subgroup of the maximal admissible group defined by²⁶

$$C_{\text{max}}^{\text{phys}} := \{(\zeta, \zeta, \zeta, g) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times Z(\tilde{G}^{\text{gauge}})\} , \quad (4.4.40)$$

where in the last component, ζ is understood to be the image of the inclusion $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$. Let us now derive the cohomological condition from the existence of the G^{phys} -bundle P^{phys} . Note that the structure group of the R-symmetry bundle P^{R} is $\text{SU}(4)_{\text{R}}/p_3(C^{\text{phys}}) \cong \text{SU}(4)_{\text{R}}/\mathbb{Z}_2$. Thus there is an associated obstruction $w_2(P^{\text{R}}) \in H^2(\mathbb{X}, \mathbb{Z}_2)$ to lifting this bundle to an $\text{SU}(4)_{\text{R}}$ -bundle. Following the same procedure as above, we find the cohomological condition

$$w_2(\mathbb{X}) = w_2(P^{\text{R}}) . \quad (4.4.41)$$

In Section 4.6.1, we will see that this theory admits three different twistings associated with three pairs $(G_i^{\text{tw}}, \varphi_i^{\text{tw}} : G_i^{\text{tw}} \rightarrow G^{\text{phys}})$ of twisting homomorphisms.

4.5 Dynamical And Background Fields In Lagrangian Theories

In order to define a quantum theory we need to choose a set of background fields $\mathcal{F}^{\text{bck}}$ and a fibration

$$\pi : \mathcal{F}^{\text{phys}} \rightarrow \mathcal{F}^{\text{bck}} . \quad (4.5.1)$$

²⁵Recall that a spin^c structure on $\text{Fr}(\mathbb{X})$ is a reduction of structure group of $\text{Fr}(\mathbb{X})$ from $\text{SO}(4)$ to $\text{Spin}^c(4) = (\text{Spin}(4) \times \text{U}(1))/\mathbb{Z}_2$ along the homomorphism $\text{Spin}^c(4) \rightarrow \text{Spin}(4)/\mathbb{Z}_2 \cong \text{SO}(4)$.

²⁶The fields in the $\mathcal{N} = 4$ vectormultiplet are: a $\tilde{G}$ -connection, an $\text{Ad}(\tilde{G})$ -valued Weyl fermion in the $\underline{4}$ of $\text{SU}(4)_{\text{R}}$ and an $\text{Ad}(\tilde{G})$ -valued scalar in the $\underline{6}$ of $\text{SU}(4)_{\text{R}}$. Thus, the invariance of the Weyl fermion under C^{phys} dictates the form of $C_{\text{max}}^{\text{phys}}$. Scalars are automatically singlets under $C_{\text{max}}^{\text{phys}}$.

The map π is not necessarily surjective. In principle, some background fields cannot be consistently coupled to the quantum theory. However, for simplicity, we will write subsequent formulae as if π were surjective. The *dynamical fields* are the fields in the fiber of π . Using the exponentiated action e^{-S} to define a (formal) measure, we (formally) integrate over the dynamical fields to produce functions of the background fields. Thus, the process of path integration defines a pushforward, so that the partition function $Z := (\pi)_*(e^{-S})$ defines a function on the space of background fields:

$$\begin{array}{ccc} \mathcal{F}^{\text{phys}} & \xrightarrow{e^{-S}} & \mathbb{C} \\ \pi_\Lambda \downarrow & & \\ \mathcal{F}^{\text{bck}} & \xrightarrow{Z} & \mathbb{C} . \end{array} \quad (4.5.2)$$

Note that the process of path integration usually introduces a scale Λ . We can regard Λ as one of the background fields, and if we wish to emphasize the scale dependence, we write $(\pi_\Lambda)_*$ for integration above background fields with fixed Λ . In the superconformal case, Λ can be replaced by the ultraviolet coupling τ_{uv} , which is also a background field, and the path integral will depend on τ_{uv} .

Now we choose our set of background fields to be

$$\mathcal{F}^{\text{bck}} := \{\Lambda, m, (P^{\text{bck}} \rightarrow \mathbb{X}, \nabla^{\text{bck}}), b, \Phi^{\text{grav}}\} , \quad (4.5.3)$$

where

$$(P^{\text{bck}} \rightarrow \mathbb{X}, \nabla^{\text{bck}}) := (p_1 \times p_2 \times p_3)_*(P^{\text{phys}} \rightarrow \mathbb{X}, \nabla^{\text{phys}}) , \quad (4.5.4)$$

is a principal bundle with connection over $\mathbb{X}$ with structure group

$$G^{\text{bck}} := (\text{Spin}(4) \times \text{SU}(2)_R \times G^{\text{f}}) / C^{\text{bck}} = (p_1 \times p_2 \times p_3)(G^{\text{phys}}) . \quad (4.5.5)$$

Note that C^{bck} is a subgroup of the center of $\text{Spin}(4) \times \text{SU}(2)_R \times G^{\text{f}}$ and note that the analog of (4.3.11) also holds for $\mathcal{F}^{\text{bck}}$. We might or might not wish to consider some subset of the gerbes b to be dynamical.²⁷ The choice of which subgroup of gerbes is dynamical leads to different gauge theories for different gauge groups $G^{\text{gauge}} \cong \tilde{G}^{\text{gauge}} / C^{\text{grb}}$ with the same simply connected cover $\tilde{G}^{\text{gauge}}$. For simplicity, we will consider b to be nondynamical. In general, the fibration π is not a direct product because the fiber of dynamical fields depends on the background fields. For example, the set of dynamical gauge fields depends on the background gerbe b . Moreover, there are consistency conditions on the background fields. See, for example, (4.4.18) above. See also Remark 4.2.2.

²⁷If one adopts the quiche picture [FMT22, Fre22] then the five-dimensional field b is always dynamical. Moreover, the isomorphism to the gauge theory gerbe is likewise dynamical. The distinction we discuss in this paragraph is then phrased in terms of choices one makes on the topological boundary.

It follows from the above discussion that the set of dynamical fields includes Φ^{phys} along with the connection on the $\tilde{G}^{\text{gauge}}/C^{\text{grb}}$ gauge bundle with connection obtained by transfer of structure group associated with the projection homomorphism p_1 .

In Section 4.6, we will explicitly describe one topological twisting of an arbitrary Lagrangian $d = 4$ $\mathcal{N} = 2$ theory (with generic masses) using a transfer of structure group based on a homomorphism $\varphi^{\text{tw}} : G^{\text{tw}} \rightarrow G^{\text{phys}}$, where

$$G^{\text{tw}} := (\text{Spin}(4) \times \mathbb{T}^f \times \tilde{G}^{\text{gauge}}) / C^{\text{tw}} , \quad (4.5.6)$$

where C^{tw} is a subgroup of the center of $\text{Spin}(4) \times \mathbb{T}^f \times \tilde{G}^{\text{gauge}}$. The key data that must be specified are the Abelian group C^{tw} and the homomorphism φ^{tw} . We will take C^{tw} to be a finite group. Moreover, we will require that the projection of C^{tw} to $Z(\text{Spin}(4))$ surjects to $\langle(-1, -1)\rangle \cong \mathbb{Z}_2$ so that we can have equations (4.5.8) and (4.2.17). For simplicity, we restrict the discussion to the case where $\tilde{G}^{\text{gauge}}$ is a simple group.

The fields of the twisted theory are in part determined by the bundle with connection $(P^{\text{tw}} \rightarrow \mathbb{X}, \nabla^{\text{tw}})$ for structure group G^{tw} , (which must satisfy a condition analogous to (4.3.11)). We can write the fields of the twisted theory informally as:

$$\mathcal{F}^{\text{tw}} := \{\tau, m, (P^{\text{tw}} \rightarrow \mathbb{X}, \nabla^{\text{tw}}), \Phi^{\text{tw}}, b\} , \quad (4.5.7)$$

where $\Phi^{\text{tw}} \in \Gamma(E^{\text{tw}})$ and $E^{\text{tw}} \rightarrow \mathbb{X}$ is a suitable bundle associated to $P^{\text{tw}} \rightarrow \mathbb{X}$, and

$$\varphi_*^{\text{tw}}(P^{\text{tw}} \rightarrow \mathbb{X}, \nabla^{\text{tw}}, \Phi^{\text{tw}}) = (P^{\text{phys}} \rightarrow \mathbb{X}, \nabla^{\text{phys}}, \Phi^{\text{phys}}) . \quad (4.5.8)$$

The choice of $G^{\text{tw,bck}}$ and $\varphi^{\text{tw,bck}}$ (cf. Section 4.2) is such that there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}^{\text{tw}} & \xrightarrow{\varphi_*^{\text{tw}}} & \mathcal{F}^{\text{phys}} & \xrightarrow{e^{-S}} & \mathbb{C} \\ \pi_\Lambda \downarrow & & \pi_\Lambda \downarrow & & \\ \mathcal{F}^{\text{tw,bck}} & \xrightarrow{\varphi_*^{\text{tw,bck}}} & \mathcal{F}^{\text{bck}} & \xrightarrow{Z} & \mathbb{C} , \end{array} \quad (4.5.9)$$

where

$$G^{\text{tw,bck}} := (\text{Spin}(4) \times \mathbb{T}^f) / C^{\text{tw,bck}} , \quad (4.5.10)$$

$$\mathcal{F}^{\text{tw,bck}} := \{m, (P^{\text{tw,bck}}, \nabla^{\text{tw,bck}}), b\} , \quad (4.5.11)$$

such that $Z \circ \varphi_*^{\text{tw,bck}}$ is locally constant as a function of $(P^{\text{tw,bck}}, \nabla^{\text{tw,bck}})$. Again, given the key choices of C^{tw} and φ^{tw} , the finite Abelian group $C^{\text{tw,bck}}$ and the homomorphism $\varphi_*^{\text{tw,bck}}$ are determined by the integration process associated to π_Λ . Once again, we note that in the superconformal case, $\mathcal{F}^{\text{tw,bck}}$ also includes the ultraviolet coupling, and the topologically twisted partition function will depend on it.

Imposing once again equation (4.3.11) we see that the partition function of the topologically twisted theory again depends on the generalized spin^c structure, and no other background field data, aside from the diffeomorphism type of $\mathbb{X}$ and the 't Hooft fluxes.

4.6 Twisting General Renormalizable Lagrangian 4d $\mathcal{N} = 2$ Theories

In this section, we will describe the choices C^{tw} and φ^{tw} for a general $d = 4$ $\mathcal{N} = 2$ gauge theory with a simple gauge group, which give us a topologically twisted version of the theory.

In order to construct the structure group G^{tw} of a topologically twisted theory we begin by defining $H := p_4(C^{\text{phys}}) \times \mathbb{Z}_2$ and we construct a homomorphism

$$\varphi_\rho^G : H \rightarrow \mathbb{T}^f , \quad (4.6.1)$$

where $\mathbb{T}^f$ is a maximal torus of the flavor group. To construct it, we start with the isotypical decomposition of ρ , as in Section 4.4,

$$\rho \cong \bigoplus_u \rho_u , \quad (4.6.2)$$

which gives us an isomorphism

$$\mathbb{T}^f \cong \prod_u \text{U}(1)_u . \quad (4.6.3)$$

As in Section 4.4, for each irreducible representation ρ_u ,

$$\langle \rho_u(p_4(C^{\text{phys}})), -1 \rangle \subset \text{Aut}(\rho_u) , \quad (4.6.4)$$

is a finite cyclic group isomorphic to $\mathbb{Z}/n_u\mathbb{Z}$ for some positive integer $n_u \in \mathbb{Z}_{>0}$. We consider this cyclic group to be the standard cyclic group of roots of unity in $\text{U}(1)$. Then we define

$$\varphi_\rho^G(g, \zeta) := \prod_u (\rho_u(g^{-1})\zeta) . \quad (4.6.5)$$

Now define

$$\tilde{G}^{\text{tw}} := \text{Spin}(4) \times \mathbb{T}^f \times \tilde{G}^{\text{gauge}} . \quad (4.6.6)$$

Writing a typical element of $\tilde{G}^{\text{tw}}$ as $((u_1, u_2), \mu, g)$, where $(u_1, u_2) \in \text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$, $\mu \in \mathbb{T}^f$ and $g \in \tilde{G}^{\text{gauge}}$, there is a crucial homomorphism based on the Witten homomorphism:

$$\begin{aligned} \varphi : \tilde{G}^{\text{tw}} &\longrightarrow \tilde{G}^{\text{phys}} , \\ \varphi((u_1, u_2), \mu, g) &:= ((u_1, u_2), u_2, \mu, g) , \end{aligned} \quad (4.6.7)$$

Now, define an injective homomorphism

$$\begin{aligned} \psi^{\text{tw}} : H &\longrightarrow Z(\tilde{G}^{\text{tw}}) \\ (g, \zeta) &\longmapsto \psi^{\text{tw}}(g, \zeta) := (\zeta, \zeta, \varphi_\rho^G(g, \zeta), g) , \end{aligned} \quad (4.6.8)$$

and observe that

$$\varphi(\psi^{\text{tw}}(H)) \subset C^{\text{phys}} , \quad (4.6.9)$$

when C^{phys} is the maximal admissible subgroup. In the notation of Section 4.3, we can take $C^{\text{tw}} = \psi^{\text{tw}}(H)$ when C^{phys} is the maximal admissible subgroup. In general, we will take:

$$C^{\text{tw}} = \{(\zeta, \zeta, \mu, g) : (\zeta, \zeta, \zeta, \mu, g) \in C^{\text{phys}}\} . \quad (4.6.10)$$

In any case, φ descends to a homomorphism

$$\bar{\varphi} : G^{\text{tw}} := \tilde{G}^{\text{tw}} / \psi^{\text{tw}}(H) \longrightarrow G^{\text{phys}} = \tilde{G}^{\text{phys}} / C^{\text{phys}} . \quad (4.6.11)$$

and in the notation of Section 4.3,

$$\varphi^{\text{tw}} = \bar{\varphi} . \quad (4.6.12)$$

Using the condition (4.4.18), one can show that the transition functions

$$t_{pq}^{\text{tw}} := [(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{t}_{pq}^f, \tilde{t}_{pq}^g)] : U_{pq} \longrightarrow G^{\text{tw}} = \frac{\text{Spin}(4) \times \mathbb{T}^f \times \tilde{G}^{\text{gauge}}}{\psi^{\text{tw}}(H)} , \quad (4.6.13)$$

where the various components are the components of t_{pq}^{phys} as in (4.4.4), satisfy the cocycle condition. Thus we can construct a principal G^{tw} -bundle P^{tw} with transition functions t_{pq}^{tw} . The physical bundle P^{phys} with connection is then obtained by transfer of structure group from the P^{tw} bundle with connection under the map induced by the group homomorphism $\bar{\varphi} : G^{\text{tw}} \rightarrow G^{\text{phys}}$.

4.6.1 Three Inequivalent Twistings of 4d $\mathcal{N} = 4$ Super Yang-Mills Theory

In this section, we describe the three inequivalent twistings of 4d, $\mathcal{N} = 4$ super Yang-Mills theory discussed in [Yam88, VW94, Mar95, DPW20], using the transfer of structure group. The three twisting homomorphisms are based on the three homomorphisms $\text{Spin}(4) \rightarrow \text{SU}(4)$ discussed in [VW94]:

1. The first homomorphism $\text{Spin}(4) \rightarrow \text{SU}(4)$ is chosen such that the **4** of $\text{SU}(4)$ decomposes to $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ under pullback. This corresponds to the Geometric Langlands (GL) twist [KW06], originally discussed in [Yam88, Mar95].
2. The second homomorphism $\text{Spin}(4) \rightarrow \text{SU}(4)$ is chosen such that the **4** of $\text{SU}(4)$ decomposes to $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1})$ under pullback. This corresponds to the usual Donaldson-Witten (DW) twist, thinking of the $\mathcal{N} = 4$ theory as an $\mathcal{N} = 2$ theory [Wit88].
3. The third homomorphism $\text{Spin}(4) \rightarrow \text{SU}(4)$ is chosen such that the **4** of $\text{SU}(4)$ decomposes to $(\mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2})$ under pullback. This corresponds to the Vafa-Witten (VW) twist [VW94].

We now describe the twisting group G^{tw} and twisting homomorphisms $\varphi^{\text{tw}} : G^{\text{tw}} \rightarrow G^{\text{phys}}$ in each case. The notation in this section follows Example 4.4 above.

1. Define

$$\begin{aligned}\tilde{G}_1^{\text{tw}} &= \text{Spin}(4) \times \text{U}(1)_R \times \tilde{G}^{\text{gauge}} , \\ C_1^{\text{tw}} &= \{(\zeta, \zeta, 1, g) : (\zeta, \zeta, \zeta, g) \in C^{\text{phys}}\} .\end{aligned}\tag{4.6.14}$$

The twisting group is $G_1^{\text{tw}} = \tilde{G}_1^{\text{tw}} / C_1^{\text{tw}}$. The twisting homomorphism is given by

$$\begin{aligned}\varphi_1^{\text{GL}} : \tilde{G}_1^{\text{tw}} &\longrightarrow \tilde{G}^{\text{phys}} \\ ((u_1, u_2), e^{i\theta}, g) &\longmapsto ((u_1, u_2), \text{diag}(e^{i\theta} u_1, e^{-i\theta} u_2), g) ,\end{aligned}\tag{4.6.15}$$

where $(u_1, u_2) \in \text{SU}(2) \times \text{SU}(2) \cong \text{Spin}(4)$. Then φ_1^{GL} descends to a homomorphism

$$\overline{\varphi}_1^{\text{GL}} : G_1^{\text{tw}} \longrightarrow G^{\text{phys}} .\tag{4.6.16}$$

The $\text{U}(1)_R$ factor in $\tilde{G}^{\text{tw}}$ reflects the fact that a $\text{U}(1)$ subgroup of $\text{SU}(4)$ commutes with the image of the first homomorphism $\text{Spin}(4) \longrightarrow \text{SU}(4)$ and hence is a residual R-symmetry of the twisted theory.

2. Define

$$\begin{aligned}\tilde{G}_2^{\text{tw}} &= \text{Spin}(4) \times \text{U}(2)_R \times \tilde{G}^{\text{gauge}} , \\ C_2^{\text{tw}} &= \{(a, a, 1, g) : (a, a, a, g) \in C^{\text{phys}}\} .\end{aligned}\tag{4.6.17}$$

The twisting group is $G_2^{\text{tw}} = \tilde{G}_2^{\text{tw}} / C_2^{\text{tw}}$. The homomorphism is given by

$$\begin{aligned}\varphi_2^{\text{DW}} : \tilde{G}_2^{\text{tw}} &\longrightarrow \tilde{G}^{\text{phys}} \\ ((u_1, u_2), u, g) &\longmapsto ((u_1, u_2), \text{diag}(\frac{u_2}{\sqrt{\det u}}, u), g) .\end{aligned}\tag{4.6.18}$$

Then φ_2^{DW} descends to a homomorphism

$$\overline{\varphi}_2^{\text{DW}} : G_2^{\text{tw}} \longrightarrow G^{\text{phys}} .\tag{4.6.19}$$

The $\text{U}(2)_R$ factor in $\tilde{G}^{\text{tw}}$ reflects residual R-symmetry of the twisted theory.

3. Define

$$\begin{aligned}\tilde{G}_3^{\text{tw}} &= \text{Spin}(4) \times \text{SU}(2)_R \times \tilde{G}^{\text{gauge}} , \\ C_3^{\text{tw}} &= \{(\zeta, \zeta, 1, g) : (\zeta, \zeta, \zeta, g) \in C^{\text{phys}}\} .\end{aligned}\tag{4.6.20}$$

The twisting group is $G_3^{\text{tw}} = \tilde{G}_3^{\text{tw}} / C_3^{\text{tw}}$. To define the twisting homomorphism, introduce the map

$$\begin{aligned}f : \text{SU}(2) \times \text{SU}(2)_R &\longrightarrow \text{SU}(4)_R \\ \left(u, \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}\right) &\longmapsto \begin{pmatrix} \alpha u & \beta u \\ \bar{\beta} u & \bar{\alpha} u \end{pmatrix} .\end{aligned}\tag{4.6.21}$$

It is easy to check that f is a well-defined homomorphism. We define the twisting homomorphism by

$$\begin{aligned} \varphi_3^{\text{VW}} : \tilde{G}_3^{\text{tw}} &\longrightarrow \tilde{G}^{\text{phys}} \\ ((u_1, u_2), u, g) &\longmapsto ((u_1, u_2), f(u_2, u), g) . \end{aligned} \quad (4.6.22)$$

Clearly, φ_3^{VW} descends to a homomorphism

$$\bar{\varphi}_3^{\text{VW}} : G_3^{\text{tw}} \longrightarrow G^{\text{phys}} . \quad (4.6.23)$$

Again, the twisted theory has residual $\text{SU}(2)_R$ R-symmetry.

4.7 Independence Of Topological Correlators Under Continuous Deformation Of Background Connection

In equation (4.2.16) et. seq., we gave a general argument that backgrounds defined via transfer of structure group satisfying (4.2.16) will have partition functions that are invariant under continuous deformations of the background connections. In this section, we review the more traditional argument that applies to Lagrangian field theories. We begin by spelling out the sublocus of topologically twisted fields and the twisted action in more detail.

The rules of $d = 4$ $\mathcal{N} = 2$ supersymmetry dictate that Φ^{phys} be a section of the direct sum of G^{phys} -vector bundles associated to the representations listed in Tables 4.1 and 4.2.

Description	Representation under G^{phys}
scalar	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \text{adj})$
scalar	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \text{adj})$
Weyl spinor	$(\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}, \text{adj})$
Weyl spinor	$(\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1}, \text{adj})$
Auxiliary field	$(\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}, \text{adj})$

Table 4.1: Matter fields of an untwisted off-shell G^{gauge} vectormultiplet. The scalars are not related by complex conjugation.

Description	Representation under G^{phys}
Scalar	$(\mathbf{1}, \mathbf{1}, \mathbf{2}, \text{def}, \rho)$
Weyl spinor	$(\mathbf{2}, \mathbf{1}, \mathbf{1}, \text{def}, \rho)$
Weyl spinor	$(\mathbf{1}, \mathbf{2}, \mathbf{1}, \text{def}, \rho)$

Table 4.2: Fields of an untwisted on-shell hypermultiplet. Note that ρ is a quaternionic representation of $\widetilde{G}$ and def denotes the defining representation of $\mathbb{T}^f$. The condition that the tuple in the second column is a representation of G^{phys} fixes the charges of various hypermultiplet fields under the $U(1)_u$ factor of $\mathbb{T}^f$ to be integral multiples of $\frac{1}{n_u}$.

The G^{phys} -connections ∇^{phys} form a torsor for $\Omega^1(P^{\text{phys}}; \text{ad } P^{\text{phys}})$. Only the G^{gauge} -connection is dynamical. The other components of the G^{phys} -connection correspond to background fields. Topological twisting then picks out a sublocus of those background connections consistent with the transfer of structure group (4.5.8) as will be explained below.

To determine the sublocus of topologically twisted matter fields, we pull back the representations in Tables 4.1 and 4.2 to representations of G^{tw} by precomposing with φ^{tw} . By a simple group theory argument, we obtain the irreducible representations of G^{tw} listed in Tables 4.3 and 4.4. We choose a complex structure on the quaternionic representation ρ , so that

$$\rho \cong \varrho \oplus \bar{\varrho}, \quad (4.7.1)$$

where $\bar{\varrho}$ is the complex conjugate representation of ϱ . Various matter fields in Table 4.2 can then be decomposed under the decomposition (4.7.1). The fields $M_{\dot{\alpha}}$ and $\widetilde{M}_{\dot{\alpha}}$ are the pullbacks of the hypermultiplet scalars decomposed into two $\mathfrak{su}(2)_{\mathbb{R}}$ -doublet scalars. In particular, $M_{\dot{\alpha}}$ and $\widetilde{M}_{\dot{\alpha}}$ are related by complex conjugation.

Description	Local chart component	Grassmann parity	Rep. under G^{tw}
Scalar	$\phi \in \Omega^0(\mathbb{X}; \text{ad } P^{\text{phys}})$	even	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \text{adj})$
Scalar	$\widetilde{\phi} \in \Omega^0(\mathbb{X}; \text{ad } P^{\text{phys}})$	even	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \text{adj})$
1-form gaugino	$\psi_{\mu} \in \Pi \Omega^1(\mathbb{X}; \text{ad } P^{\text{phys}})$	odd	$(\mathbf{2}, \mathbf{2}, \mathbf{1}, \text{adj})$
0-form gaugino	$\eta \in \Pi \Omega^0(\mathbb{X}; \text{ad } P^{\text{phys}})$	odd	$(\mathbf{1}, \mathbf{1}, \mathbf{1}, \text{adj})$
SD 2-form gaugino	$\chi_{\mu\nu} \in \Pi \Omega_g^{2,+}(\mathbb{X}; \text{ad } P^{\text{phys}})$	odd	$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \text{adj})$
SD 2-form auxiliary field	$D_{\mu\nu} \in \Omega_g^{2,+}(\mathbb{X}; \text{ad } P^{\text{phys}})$	even	$(\mathbf{1}, \mathbf{3}, \mathbf{1}, \text{adj})$

Table 4.3: Matter fields of the twisted off-shell $\mathcal{N} = 2$ vectormultiplet.

Description	Local chart component	Grassmann parity	Rep. under G^{tw}
Monopole field	$M_{\dot{\alpha}} \in \Gamma(S^- \otimes E^{\text{tw}})$	even	$(\mathbf{1}, \mathbf{2}, \text{def}, \varrho)$
Conjugate monopole field	$\widetilde{M}^{\dot{\alpha}} := (M_{\dot{\alpha}})^{\dagger} \in \Gamma(S^- \otimes \overline{E}^{\text{tw}})$	even	$(\mathbf{1}, \mathbf{2}, \overline{\text{def}}, \overline{\varrho})$
Weyl fermion	$\mu_{\alpha} \in \Pi \Gamma(S^+ \otimes E^{\text{tw}})$	odd	$(\mathbf{2}, \mathbf{1}, \text{def}, \varrho)$
Weyl fermion	$\widetilde{\lambda}_{\dot{\alpha}} \in \Pi \Gamma(S^- \otimes E^{\text{tw}})$	odd	$(\mathbf{1}, \mathbf{2}, \text{def}, \varrho)$
Weyl fermion	$\widetilde{\mu}_{\dot{\alpha}} \in \Pi \Gamma(S^- \otimes \overline{E}^{\text{tw}})$	odd	$(\mathbf{1}, \mathbf{2}, \overline{\text{def}}, \overline{\varrho})$
Weyl fermion	$\lambda_{\alpha} \in \Pi \Gamma(S^+ \otimes \overline{E}^{\text{tw}})$	odd	$(\mathbf{2}, \mathbf{1}, \overline{\text{def}}, \overline{\varrho})$
Auxiliary Weyl spinor	$h_{\alpha} \in \Gamma(S^+ \otimes E^{\text{tw}})$	even	$(\mathbf{2}, \mathbf{1}, \text{def}, \varrho)$
Auxiliary Weyl spinor	$\widetilde{h}_{\alpha} \in \Gamma(S^+ \otimes \overline{E}^{\text{tw}})$	even	$(\mathbf{2}, \mathbf{1}, \overline{\text{def}}, \overline{\varrho})$

Table 4.4: Fields of the twisted off-shell $\mathcal{N} = 2$ hypermultiplet.

These fields are collectively denoted by Φ^{tw} . Finally there is a G^{tw} -connection ∇^{tw} on P^{tw} satisfying the requirement analogous to (4.3.11). The transfer of structure group from P^{tw} to P^{phys} along the homomorphism $\varphi^{\text{tw}} : G^{\text{tw}} \rightarrow G^{\text{phys}}$ then determines the connection $\varphi_*^{\text{tw}}(\nabla^{\text{tw}})$ on P^{phys} . The background fields in the physical theory are determined by the pushforward (4.5.8). The connections that enter the action are given by pushforwards of $\varphi_*^{\text{tw}}(\nabla^{\text{tw}})$ via the transfer of structure group of P^{phys} to P^{gauge} , $\text{Fr}(\mathbb{X})$, P^{R} and $P^{\text{f},u}$. These connections are listed in Table 4.5.

Connection	Bundle	Local chart component	Dynamical/Background
$(p_1)_*(\varphi_*^{\text{tw}}(\nabla^{\text{tw}}))$	$\text{Fr}(\mathbb{X})$	ω_{μ}	Background
$(p_2)_*(\varphi_*^{\text{tw}}(\nabla^{\text{tw}}))$	P^{R}	$(\omega_+)_{\mu}$	Background
$(p_{3,u})_*(\varphi_*^{\text{tw}}(\nabla^{\text{tw}}))$	$P^{\text{f},u}$	$\mathcal{A}_{\mu}^u$	Background
$(p_4)_*(\varphi_*^{\text{tw}}(\nabla^{\text{tw}}))$	P^{gauge}	A_{μ}	Dynamical

Table 4.5: Various dynamical and background connections in the physical theory obtained by pushforward of ∇^{tw} , see Section 4.4 for the definition of various bundles. Here ω_+ denotes the self-dual part of the Levi-Civita connection.

Note that the requirement analogous to (4.3.11) for $(P^{\text{tw}}, \nabla^{\text{tw}})$ forces the connection ω_{μ} on $\text{Fr}(\mathbb{X})$ to be the LC connection ω^{LC} and the twisting homomorphism φ^{tw} forces the connection on the R-symmetry bundle P^{R} to be the SD part ω_+^{LC} of the LC connection.

The supercharges are in the representation $(\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$ of G^{phys} . They can be pulled back to representations of G^{tw} . Again, simple group theory arguments show that there is a

supercharge $\mathcal{Q}$ singlet under G^{tw} . The supersymmetry algebra implies that

$$\mathcal{Q}^2 = -\frac{1}{2}Z , \quad (4.7.2)$$

where Z is the generator of the central charge in the physical theory (the action of which on fields is given in Table 4.6; fields of the vectormultiplet are neutral under Z). Using the action of supercharges in the physical theory, we obtain the action of $\mathcal{Q}$ on Φ^{tw} and on the connections in Table 4.5.

One should think of $\mathcal{Q}$ as an odd vectorfield on the supermanifold of all fields, background, and dynamical. There is a sub-supermanifold where the background fermions are set to zero, and we will write down the action of $\mathcal{Q}$ on this submanifold. There is no off-shell formulation of an untwisted $\mathcal{N} = 2$ hypermultiplet with a finite number of auxiliary fields, but there exists one for the twisted hypermultiplet [KR88].²⁸ One introduces spinorial, Grassmann-even auxiliary fields h^α and $\tilde{h}^\alpha$ [KR88, LMn05, HPP95]. The $\mathcal{Q}$ -action on fields is:

$$\begin{aligned} [\mathcal{Q}, \phi] &= 0 , & [\mathcal{Q}, \tilde{\phi}] &= \eta , & \{\mathcal{Q}, \eta\} &= [\phi, \tilde{\phi}] , \\ [\mathcal{Q}, A_\mu] &= \psi_\mu , & \{\mathcal{Q}, \psi_\mu\} &= -D_\mu \phi , & [\mathcal{Q}, \mathcal{A}_\mu^u] &= 0 , \\ \{\mathcal{Q}, \chi_{\mu\nu}\} &= F_{\mu\nu}^+ - D_{\mu\nu} , & [\mathcal{Q}, D_{\mu\nu}] &= 2(D_{[\mu} \psi_{\nu]})^+ - [\phi, \chi_{\mu\nu}] , & [\mathcal{Q}, \omega_\mu] &= 0 , \\ [\mathcal{Q}, M_{\dot{\alpha}}] &= \tilde{\mu}_{\dot{\alpha}} , & [\mathcal{Q}, \tilde{M}_{\dot{\alpha}}] &= \tilde{\lambda}_{\dot{\alpha}} , & & (4.7.3) \\ \{\mathcal{Q}, \tilde{\mu}_{\dot{\alpha}}\} &= m M_{\dot{\alpha}} - \phi M_{\dot{\alpha}} , & \{\mathcal{Q}, \tilde{\lambda}_{\dot{\alpha}}\} &= -m \tilde{M}_{\dot{\alpha}} + \tilde{M}_{\dot{\alpha}} \phi , \\ \{\mathcal{Q}, \lambda_\alpha\} &= h_\alpha , & \{\mathcal{Q}, \mu_\alpha\} &= \tilde{h}_\alpha , \\ [\mathcal{Q}, h_\alpha] &= m \lambda_\alpha - \phi \lambda_\alpha , & [\mathcal{Q}, \tilde{h}_\alpha] &= -m \mu_\alpha + \mu_\alpha \phi . \end{aligned}$$

The central charge acts as multiplication by $\pm m$ on the hypermultiplet fields. The ± 1 factors in the action of $Z, \bar{Z}$ on various hypermultiplet fields are shown in Table 4.6.

²⁸The fact that the untwisted off-shell $\mathcal{N} = 2$ hypermultiplet requires infinitely many auxiliary fields is perhaps most elegantly seen in the formalism of projective superspace [LR90, LR08]: here, the hypermultiplet superfield is an infinite series in a twistorial coordinate $\zeta \in \mathbb{CP}^1$, with infinitely many coefficients corresponding to the infinitely many auxiliary fields. The off-shell twisted $\mathcal{N} = 2$ hypermultiplet with a finite number of auxiliary fields was given in [KR88], the authors having arrived at it by explicit construction. It would be interesting to understand this from a projective superspace viewpoint, but we leave this for future work. Finally, we note that among the two presentations of the off-shell twisted $\mathcal{N} = 2$ hypermultiplet transformations in the book [LMn05], their equations (5.51) do not close off-shell: in their language, the supersymmetry variation δ satisfies $\delta^2 \varphi \not\sim \varphi$ and, in fact, $\delta^4 \varphi \sim \varphi$ for a typical hypermultiplet field φ . Their second presentation, namely equations (5.56) in [LMn05], does close off-shell, and that is what we use.

Field	Central charge $(Z, \bar{Z})$
M	$(-1, 1)$
$\widetilde{M}$	$(1, -1)$
λ	$(-1, 1)$
$\widetilde{\lambda}$	$(1, -1)$
μ	$(1, -1)$
$\widetilde{\mu}$	$(-1, 1)$
h	$(-1, 1)$
$\widetilde{h}$	$(1, -1)$

Table 4.6: The action of $(Z, \bar{Z})$ on various fields is given by the signs multiplying m .

From (D.4.151), $\mathcal{Q}$ squares to a gauge transformation plus a central charge transformation:

$$\mathcal{Q}^2 = -\frac{1}{2}\delta_m^{\text{central-charge}} + \delta_\phi^{\text{gauge}} , \quad (4.7.4)$$

where the subscript m is the parameter (a mass) for the central charge transformation.

To formulate an action for the twisted theory, we begin with the action for the physical theory with fields Φ^{phys} , τ , m and ∇^{phys} and then specialize the connection ∇^{phys} to the pushforward of ∇^{tw} and express Φ^{phys} in terms of Φ^{tw} . Here A_μ and $\mathcal{A}_\mu^u$ are YM and U(1) connections respectively, while ω_μ , $(\omega_+)_\mu$ are gravitational connections. A sensible way to formulate the action is in terms of a $(G^{\text{gauge}} \times \mathbb{T}^f)$ -gauge theory with matter coupled to background $\mathcal{N} = 2$ conformal supergravity (where the $\mathbb{T}^f$ vectormultiplet is treated as a nondynamical background vectormultiplet), with fermions in the supergravity multiplet and $\mathbb{T}^f$ -vectormultiplet set to zero (to ensure a $\mathcal{Q}$ -exact energy-momentum tensor). Note that $\mathcal{A}_\mu^u$ is a $\text{U}(1)/p_{3,u}(C^{\text{phys}})$ -connection, and $p_{3,u}(C^{\text{phys}}) \cong \mathbb{Z}/n_u\mathbb{Z}$. Thus,

$$\mathcal{A}_\mu^u = \frac{1}{n_u} B_\mu^u , \quad (4.7.5)$$

where B_μ^u is a U(1)-connection in the $\text{U}(1)_u$ -factor of the $\mathbb{T}^f$ -vectormultiplet.

Furthermore, the scalars in the $\mathbb{T}^f$ -vectormultiplets are set equal to the bare mass parameters: since $m \in \text{Lie}(G^f) \otimes \mathbb{C}$ and G^f breaks to $\mathbb{T}^f$, m can be identified with $\text{rank}(\mathbb{T}^f)$ complex parameters m_u , for $u = 1, \dots, \text{rank}(\mathbb{T}^f)$. Let $\phi_u, \widetilde{\phi}_u$ for $u = 1, \dots, \text{rank}(\mathbb{T}^f)$ denote the scalars in the $\mathbb{T}^f$ -vectormultiplets. Then we set²⁹

$$\phi_u = -m_u, \quad \widetilde{\phi}_u = \frac{m_u}{2} , \quad u = 1, \dots, \text{rank}(\mathbb{T}^f) . \quad (4.7.6)$$

²⁹The rescalings are chosen to make contact with the vectormultiplet action of [CMRS23].

Finally choose generators $\{T^a\}_{a=1}^{\dim(G^{\text{gauge}})}$ for $\text{Lie}(G^{\text{gauge}})$ and $\{T^u\}_{u=1}^{\text{rank}(\mathbb{T}^f)}$ and denote the components of the bilinear form τ in this basis by³⁰ τ_{ab}, τ_{uv} . Also, let

$$T_\rho^a = T_\varrho^a \oplus T_{\bar{\varrho}}^a, \quad (4.7.7)$$

be the representation matrix of the generators $\{T^a\}_{a=1}^{\dim(G^{\text{gauge}})}$ of $\text{Lie}(G^{\text{gauge}})$ under ρ . The action can be written using the results of [dWLVP85, KR88, CMRS23]. It is given by³¹

$$\mathcal{S}_{\mathcal{N}=2+\text{bck}}^{\text{twisted}} = \mathcal{S}_{\text{vec}}^{\text{twisted}} + \mathcal{S}_{\text{hyp}}^{\text{twisted}} + \mathcal{S}^{\text{top}}, \quad (4.7.8)$$

where

$$\begin{aligned} \mathcal{S}_{\text{vec}}^{\text{twisted}} = \frac{1}{4\pi} \int \text{vol}(g) \text{Im } \tau_{ab} & \left[\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,b} - \frac{1}{2} D_{\mu\nu}^a D^{\mu\nu,b} - 2\chi_{\mu\nu}^a D^\mu \psi^{\nu,b} + \frac{1}{2} \chi_{\mu\nu}^a [\phi, \chi^{\mu\nu}]^b \right. \\ & \left. - \eta^a D_\mu \psi^{\mu,b} + \tilde{\phi}^a D_\mu D^\mu \phi^b - \frac{1}{2} \phi^a [\eta, \eta]^b - \frac{1}{2} [\phi, \tilde{\phi}]^a [\phi, \tilde{\phi}]^b - \tilde{\phi}^a [\psi_\mu, \psi^\mu]^b \right], \end{aligned} \quad (4.7.9)$$

$$\begin{aligned} \mathcal{S}_{\text{hyp}}^{\text{twisted}} = \int \text{vol}(g) & \left[-\tilde{h}^\alpha \sigma_{\alpha\dot{\alpha}}^\rho D_\rho M^{\dot{\alpha}} + D_\rho \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\rho h^\alpha - \tilde{h}^\alpha h_\alpha - m^2 \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} + m \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} \right. \\ & - 2m \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} + \tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\rho\sigma})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\rho\sigma} + \eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\rho\sigma} \tilde{\mu}^{\dot{\beta}} \chi_{\rho\sigma} \\ & - \eta \tilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} + \lambda^\alpha \sigma_{\alpha\dot{\beta}}^\mu D_\mu \tilde{\lambda}^{\dot{\beta}} - \tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^\rho)^{\dot{\alpha}\alpha} \lambda_\alpha \psi_\rho + \mu^\alpha \sigma_{\alpha\dot{\beta}}^\rho M_{\dot{\beta}} \psi_\rho \\ & + \mu^\alpha \sigma_{\alpha\dot{\beta}}^\rho D_\rho \tilde{\mu}^{\dot{\beta}} + \mu^\alpha \phi \lambda_\alpha - 2\tilde{\lambda}_{\dot{\alpha}} \tilde{\phi} \tilde{\mu}^{\dot{\alpha}} - m \mu^\alpha \lambda_\alpha - m \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} \\ & \left. + \tilde{M}_{\dot{\alpha}} \{\phi, \tilde{\phi}\} M^{\dot{\alpha}} - \tilde{M}_{(\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} + \tilde{M}_{(\dot{\alpha}} D^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} \right], \end{aligned} \quad (4.7.10)$$

$$\mathcal{S}^{\text{top}} := \frac{i}{32\pi} \int_{\mathbb{X}} \tau_{ab} (F^a \wedge F^b) + \frac{i}{32\pi} \int_{\mathbb{X}} \tau_{uv} (F^u \wedge F^v). \quad (4.7.11)$$

Here F and F^u are the field strengths of the dynamical gauge field A_μ and the background $\text{U}(1)_u$ -connection B_μ^u , and the covariant derivatives are given by³²

$$D_\mu \tilde{\phi} = \partial_\mu \tilde{\phi} + [A_\mu, \tilde{\phi}], \quad (4.7.12)$$

$$D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi], \quad (4.7.13)$$

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu - \Gamma_{\mu\nu}^\rho \psi_\rho + [A_\mu, \psi_\nu], \quad (4.7.14)$$

$$D_\mu M_u^{\dot{\alpha}} = \partial_\mu M_u^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{\underline{a}\underline{b}} (\tilde{\sigma}_{\underline{a}\underline{b}})^{\dot{\alpha}}_{\dot{\beta}} M_u^{\dot{\beta}} + A_\mu^a T_\varrho^a M_u^{\dot{\alpha}} + \frac{q_u}{n_u} B_\mu^u M_u^{\dot{\alpha}}, \quad (4.7.15)$$

$$D_\mu \tilde{M}^{\dot{\alpha}u} = \partial_\mu \tilde{M}^{\dot{\alpha}u} - \frac{1}{2} \omega_\mu^{\underline{a}\underline{b}} (\tilde{\sigma}_{\underline{a}\underline{b}})^{\dot{\alpha}}_{\dot{\beta}} \tilde{M}^{\dot{\beta}u} + A_\mu^a T_\varrho^a \tilde{M}^{\dot{\alpha}u} - \frac{q_u}{n_u} B_\mu^u \tilde{M}^{\dot{\alpha}u}, \quad (4.7.16)$$

$$D_\mu \tilde{\lambda}^{\dot{\alpha}u} = \partial_\mu \tilde{\lambda}^{\dot{\alpha}u} - \frac{1}{2} \omega_\mu^{\underline{a}\underline{b}} (\tilde{\sigma}_{\underline{a}\underline{b}})^{\dot{\alpha}}_{\dot{\beta}} \tilde{\lambda}^{\dot{\beta}u} + A_\mu^a T_\varrho^a \tilde{\lambda}^{\dot{\alpha}u} + \frac{q_u}{n_u} B_\mu^u \tilde{\lambda}^{\dot{\alpha}u}, \quad (4.7.17)$$

$$D_\rho \tilde{\mu}_u^{\dot{\alpha}} = \partial_\rho \tilde{\mu}_u^{\dot{\alpha}} - \frac{1}{2} \omega_\rho^{\underline{a}\underline{b}} (\tilde{\sigma}_{\underline{a}\underline{b}})^{\dot{\alpha}}_{\dot{\beta}} \tilde{\mu}_u^{\dot{\beta}} + A_\rho^a T_\varrho^a \tilde{\mu}_u^{\dot{\alpha}} - \frac{q_u}{n_u} B_\rho^u \tilde{\mu}_u^{\dot{\alpha}}, \quad (4.7.18)$$

³⁰Note that there are no cross terms because of $\text{Ad}(G^{\text{gauge}} \times G^f)$ -invariance of τ .

³¹Our conventions for differential forms are the same as [CMRS23], but for spinor index contractions and self-duality differ mildly: we contract spinor indices using the North-West-South-West convention, and we write a self-dual tensor $\chi_{\mu\nu}$ in dotted spinor indices as the symmetric object $\chi_{(\dot{\alpha}\dot{\beta})}$.

³²There is no sum over the repeated index u in the covariant derivatives.

where ω_μ^{ab} is the spin connection. The factor of $\frac{q_u}{n_u}$ in the covariant derivatives above comes from the charge $q_u \in \mathbb{Z}$ of hypermultiplet fields under $U(1)_u$. Using the relation (4.7.5) we see that these covariant derivatives are local expressions for ∇^{tw} on suitable associated G^{tw} -vector bundles. Moreover, since F^u is the curvature for the background $U(1)_u$ -connection B_μ^u , we have

$$\left[\frac{F^u}{2\pi} \right] = c_1(u) , \quad u = 1, \dots, \text{rank}(\mathbb{T}^f) , \quad (4.7.19)$$

where $c_1(u)$ is as in (4.4.18). It turns out that the action is $\mathcal{Q}$ -exact up to topological terms:

$$\mathcal{S}_{\mathcal{N}=2+\text{bck}}^{\text{twisted}} = \{ \mathcal{Q}, \mathcal{V} \} + \mathcal{S}^{\text{top}} , \quad (4.7.20)$$

where

$$\mathcal{V} := \mathcal{V}_{\text{vec}} + \mathcal{V}_{\text{hyp}} , \quad (4.7.21)$$

$$\mathcal{V}_{\text{vec}} := \frac{1}{4\pi} \int_{\mathbb{X}} \text{vol}(g) \, \text{Im} \, \tau_{ab} \left(\frac{1}{4} \chi^{\mu\nu,a} (F_{\mu\nu}^+ + D_{\mu\nu})^b - \frac{1}{4} \eta^a [\phi, \tilde{\phi}]^b + \frac{1}{2} \psi^{\mu,a} D_\mu \tilde{\phi}^b \right) , \quad (4.7.22)$$

$$\begin{aligned} \mathcal{V}_{\text{hyp}} := \int_{\mathbb{X}} \text{vol}(g) & \left(D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha + D_\mu M^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \mu^\alpha + \frac{1}{2} m \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \frac{1}{2} \mu_\alpha h^\alpha - \frac{1}{2} m \tilde{M}^{\dot{\alpha}} \tilde{\mu}_{\dot{\alpha}} \right. \\ & \left. + \frac{1}{2} \tilde{h}_\alpha \lambda^\alpha + \frac{1}{2} \left(\tilde{M}^{\dot{\alpha}} \tilde{\phi} \tilde{\mu}_{\dot{\alpha}} + \tilde{\lambda}^{\dot{\alpha}} \tilde{\phi} M_{\dot{\alpha}} \right) - \tilde{M}^{(\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta})} \right) . \end{aligned} \quad (4.7.23)$$

Since $\mathcal{V}$ and $\mathcal{S}^{\text{top}}$ are gauge invariant and invariant under the action of central charge (see Table 4.6), it follows that on a closed 4-manifold,

$$\{ \mathcal{Q}, \mathcal{S}_{\mathcal{N}=2+\text{bck}}^{\text{twisted}} \} = 0 . \quad (4.7.24)$$

The response currents to variations of the background metric and background $U(1)$ connections are $\mathcal{Q}$ -exact as a result of $\mathcal{Q}$ -exactness of the action. Thus the twisted partition function and topological correlators are formally locally constant as a function of the background connections.

4.7.1 Non-Abelian Monopole Equations

Given that the action is $\mathcal{Q}$ -exact up to a topological term, one can show that the path integral localizes to supersymmetric backgrounds i.e., solutions of $\mathcal{Q}$ -fixed equations with fermions set to zero. From (D.4.151), we see that supersymmetric configurations correspond to the solutions of the following $\mathcal{Q}$ -fixed equations:

$$\left. \begin{aligned} D_{\dot{\alpha}\dot{\beta}}^a &= F_{\dot{\alpha}\dot{\beta}}^a , \\ h_{\alpha u} &= 0 , \\ \tilde{h}_\alpha^u &= 0 , \end{aligned} \right\} \quad \begin{aligned} a &= 1, \dots, \dim(G^{\text{gauge}}) , \\ u &= 1, \dots, \text{rank}(\mathbb{T}^f) . \end{aligned} \quad (4.7.25)$$

Setting the auxiliary fields of the dynamical vectormultiplets and hypermultiplets to their equations of motion

$$\left. \begin{aligned} D_{\dot{\alpha}\dot{\beta}}^a &= 2\widetilde{M}_{(\dot{\alpha}}^u T_{\dot{\beta})u}^a M_{\dot{\beta}}^{\dot{\alpha}} , \\ h_{\alpha v} &= \sigma_{\alpha\dot{\alpha}}^\mu D_\mu M_v^{\dot{\alpha}} , \\ \widetilde{h}_\alpha^v &= -D_\mu \widetilde{M}^{\dot{\alpha}v} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu , \end{aligned} \right\} \begin{aligned} &u \text{ summed over} , \\ &v = 1, \dots, \text{rank}(\mathbb{T}^f) , \end{aligned} \quad (4.7.26)$$

yields

$$\left. \begin{aligned} F_{\dot{\alpha}\dot{\beta}}^a - 2\widetilde{M}_{(\dot{\alpha}}^u T_{\dot{\beta})u}^a M_{\dot{\beta}}^{\dot{\alpha}} &= 0 , \\ \sigma_{\alpha\dot{\alpha}}^\mu D_\mu M_v^{\dot{\alpha}} &= 0 , \\ \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \widetilde{M}^{\dot{\alpha}v} &= 0 , \end{aligned} \right\} \begin{aligned} &u \text{ summed over} , \\ &v = 1, \dots, \text{rank}(\mathbb{T}^f) . \end{aligned} \quad (4.7.27)$$

These $\mathcal{Q}$ -fixed equations are called *non-Abelian monopole equations* [LMn95, HPP95, LNS97, PT95, Wit99] and are generalizations of both the usual ASD instanton equations and the “monopole” or “Seiberg-Witten” equations [Wit94]. We can write these equations in coordinate-free notation as follows: the monopole field M is a section of the associated G^{tw} -vector bundle associated with the representation $(\mathbf{1}, \mathbf{2}, \text{def}, \varrho)$. Denote this vector bundle by E^{mon} . The conjugate monopole field $\widetilde{M}$ is a section of the associated G^{tw} -vector bundle associated to the representation $(\mathbf{1}, \mathbf{2}, \overline{\text{def}}, \overline{\varrho})$. Denote this vector bundle by $\overline{E}^{\text{mon}}$. On every local chart (U, x^μ) on $\mathbb{X}$, there is a natural bilinear map:

$$\begin{aligned} \mu : \Gamma(\overline{E}^{\text{mon}}) \times \Gamma(E^{\text{mon}}) &\longrightarrow \Omega^{2,+}(U; \text{ad } P^{\text{gauge}}) \\ (\widetilde{M}, M) &\longmapsto \frac{1}{2!} \left(\frac{1}{2} e_\mu^a e_\nu^b \widetilde{\sigma}_{\underline{a}\underline{b}}^{\dot{\alpha}\dot{\beta}} \widetilde{M}_{(\dot{\alpha}}^u T_{\dot{\beta})u}^a M_{\dot{\beta}}^{\dot{\alpha}} T^a \right) dx^\mu \wedge dx^\nu , \end{aligned} \quad (4.7.28)$$

where we have chosen a local section $s : U \rightarrow P^{\text{tw}}$ to write

$$\widetilde{M} = [(s, \widetilde{M}_{\dot{\alpha}m}^u)] , \quad M = [(s, M_{\dot{\alpha}u}^{\bar{u}})] , \quad (4.7.29)$$

where m and $\bar{n}$ are indices for the representations ϱ and $\overline{\varrho}$ respectively. We have chosen a basis for the orthonormal frame bundle with vielbein e_μ^a in the local chart. Finally as noted before, D_μ acting on $\widetilde{M}$ and M is the Dirac operator coupled to $\overline{E}^{\text{mon}}$ and E^{mon} respectively. The monopole equations can then be written as

$$F^+((p_4)_*(\nabla^{\text{tw}})) = 2\mu(\widetilde{M}, M) , \quad \not{D}_{E^{\text{mon}}} M = 0 , \quad \not{D}_{\overline{E}^{\text{mon}}} \widetilde{M} = 0 , \quad (4.7.30)$$

where $F^+((p_4)_*(\nabla^{\text{tw}}))$ is the SD part of the curvature of the G^{gauge} -connection $(p_4)_*(\nabla^{\text{tw}})$ on P^{gauge} and $\not{D} = \widetilde{\sigma}^\mu D_\mu$. This allows us to generalize the non-Abelian monopole equations to any compact, simple gauge group.

The path integral for topological correlators and the partition function formally localizes to the moduli space of non-Abelian monopole equations. After performing the path integral, we obtain topological invariants which depend on the first Chern classes $c_1(u)$ and the background fields

$\rho_u(\mu(b))$ for “1-form symmetries” constrained according to (4.4.18). This topological dependence on the background fields arises from the topological terms $\mathcal{S}^{\text{top}}$ in the action.

In conclusion, after integrating out the dynamical fields, we obtain a twisted partition function depending only on the data 1–3.

4.8 Topological Twisting Of General Class $\mathcal{S}$ Theories

We now turn to class $\mathcal{S}$ theories. In general, we cannot apply the general discussion of Sections 4.3–4.7 because in general class $\mathcal{S}$ theories are non-Lagrangian. (Theories of type A_1 are Lagrangian, and this is a notable and important exception.)

4.8.1 Pants Decompositions

As noted in [GMN09, Gai09], weak coupling regimes are determined by a pants decomposition of the ultraviolet curve $C_{g,n}$ used to define the theory. Here g is the genus and n is the number of punctures.

Recall that a pants decomposition of a stable Riemann surface $C_{g,n}$ [Hat] is a maximal collection of pairwise disjoint simple closed curves (also called *cutting curves* or *cuffs*).³³ For a given connected surface $C_{g,n}$, any pants decomposition will consist of $(2g - 2 + n)$ pairs of pants.³⁴ Moreover, any pants decomposition consists of $(3g - 3 + n)$ internal cutting curves and n external cutting curves. We denote a typical pants decomposition by a list of cutting curves:

$$\mathfrak{p} = \{c_1, \dots, c_{3g-3+2n}\} . \quad (4.8.1)$$

The complement of these curves in $C_{g,n}$ is a collection of $(2g - 2 + n)$ pairs of pants³⁵ (also called *trinions*).

In these regimes, we can express the theory as a gauge theory, coupled to possibly non-Lagrangian theories associated with the trinions. The key point is that we need only study general properties of the partition functions of these (possibly non-Lagrangian) trinion theories in the presence of

³³A simple closed curve on $\mathbb{X}$ is a map $c : [0, 1] \rightarrow \mathbb{X}$ that is closed, i.e., $c(0) = c(1)$, and also injective as a map $[0, 1] \rightarrow \mathbb{X}$. Following [MM21], we will refer to cutting curves surrounding the punctures as *external cutting curves*, and cutting curves c_i associated with gauge groups as *internal cutting curves*. In the mathematical literature, only the internal cutting curves are counted as *cuffs*.

³⁴The Euler characteristic $\chi(C_{g,n}) = 2 - 2g - n$ of a stable Riemann surface is negative, this excludes $C_{0,0}$, $C_{0,1}$, $C_{0,2}$, and $C_{1,0}$. Strictly speaking, one should distinguish between boundary components and punctures (the latter being marked points in the interior of the closure of the surface). Here the n “punctures” are obtained by removing n small open disks from the surface.

³⁵A pair of pants (hereafter abbreviated simply to “pants”) is a surface homeomorphic to $C_{0,3}$.

background fields. This is all that is needed to discuss the analogues of the cohomological conditions (4.4.18) and the twisting procedure.

4.8.2 Trinion Theories

The trinion theory³⁶ $T[\mathfrak{g}, D_{p_1}, D_{p_2}, D_{p_3}]$ is a relative field theory. We would like to gauge a collection of trinion theories to produce a class $\mathcal{S}$ theory, and we would like the latter to be an absolute 4d theory that can be defined on all standard four-manifolds. We will make the conservative assumption that this requires each component trinion theory to be an absolute theory definable on all standard four-manifolds.³⁷

It is thought that the trinion theory can be defined as an absolute 4d theory once some extra data is specified [Tac13a]. For the trinion theory, it is argued in [Tac13a] that the only extra data is a choice of global form of the symmetry group. (Equivalently, a choice of global form of the group is used to define the Hitchin system.) We simply denote this extra data as $\mathcal{L}$ so the absolute theory is denoted $T[\mathfrak{g}, D_{p_1}, D_{p_2}, D_{p_3}; \mathcal{L}]$.

For our considerations, we need to know the global symmetry group of the theory, not just its Lie algebra. The group form of the symmetry group of class $\mathcal{S}$ theories in general has been studied in [Tac13a, ADZGEH20, Bha21, BHSN21, BGHSN22, GEHLS21, DZGE22, HLL⁺22, G23].³⁸ The viewpoint presented in this chapter, based on remarks from Section 4.2, differs slightly from the above references. We regard the global symmetry as a choice based on the choice of domain category in the functorial formulation of field theory.

To each puncture p we associate a global symmetry algebra $\mathfrak{g}_p^{\text{fl}}$ and we denote by $\widetilde{G}_p$ a Lie group with this algebra which is a product of a simply connected semi-simple Lie group and a torus. (Specification of the torus requires further discussion, not found here.) We can then define a theory $T[\mathfrak{g}, D_{p_1}, D_{p_2}, D_{p_3}; \mathcal{L}]$ with symmetry group of the form

$$\widetilde{G}_T = \text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times \widetilde{G}_{p_1} \times \widetilde{G}_{p_2} \times \widetilde{G}_{p_3} . \quad (4.8.2)$$

Since, as we have mentioned, the theories $T[\mathfrak{g}, D_{p_1}, D_{p_2}, D_{p_3}; \mathcal{L}]$ are in general non-Lagrangian it is

³⁶As noted in Section 4.9, the trinion theory is a class $\mathcal{S}$ theory on a three-punctured sphere. The notation $T[\mathfrak{g}, D_{p_1}, D_{p_2}, D_{p_3}]$ specifies the gauge algebra of the 6d theory and the puncture data $D_{p_1}, D_{p_2}, D_{p_3}$ at the three punctures. The data at a regular puncture p includes a reductive Lie algebra $\mathfrak{g}_p$ and a homomorphism $\rho : \mathfrak{su}(2) \rightarrow \mathfrak{g}_p$, among other things. The flavor symmetry algebra $\mathfrak{f}_p^{\text{fl}}$ at puncture p is given by the centralizer of the image of ρ in $\mathfrak{g}_p$. See [CDT12, Tac13b, Tac15], for a more complete discussion of the punctures in class $\mathcal{S}$ theories.

³⁷It would be interesting if one could relax this conservative assumption by gauging the relative trinion theories to produce an absolute theory.

³⁸See also [BDZH22, BDZHK22] for some recent work on generalized global symmetries of class $\mathcal{S}$ theories.

very useful to adopt the functorial viewpoint of what is meant by a “theory” as in Section 4.2. Thus, we take a “theory” to be a functor

$$\tilde{Z}_T : \text{Bord}_{\leq 4}^{\tilde{\mathcal{F}}} \rightarrow \text{VECT} , \quad (4.8.3)$$

where $\tilde{\mathcal{F}}$ is a set of background fields that includes the groupoid of principal $\tilde{G}_T$ bundles with connection. We identify the $\text{Spin}(4)$ factor with the covering group of the tangent bundle of $\mathbb{X}$, and hence the bordism category $\text{Bord}_{\leq 4}^{\tilde{\mathcal{F}}}$ only includes spin manifolds. The background fields $\tilde{\mathcal{F}}$ should include the full set of vectormultiplets for the flavor symmetry, so the partition function on $\mathbb{X}$ in the presence of background fields for $\tilde{G}_T$ can be written informally as:

$$Z_T[\mathbb{X}, \omega_\mu^{\frac{a}{b}}, A_\mu^R, \text{VM}_{p_1}, \text{VM}_{p_2}, \text{VM}_{p_3}] , \quad (4.8.4)$$

where VM_p is the full set of fields of the standard $\mathcal{N} = 2$ vectormultiplet.

The symmetry group of the trinion theory can be taken to be a quotient³⁹

$$G_T := \tilde{G}_T / C_T , \quad (4.8.5)$$

where C_T is a subgroup of the center of the numerator. As in (4.2.6) above we require that the functor $\tilde{Z}$ factors through

$$Z_T : \text{Bord}_4^{\mathcal{F}} \rightarrow \text{VECT} . \quad (4.8.6)$$

For purposes of topological twisting, we would like this bordism category to include non-spin manifolds. Therefore, the subgroup C_T should satisfy (4.2.9). Applying the reasoning of Section 4.2 we arrive at:

$$C_T \subset C_T^{\max} := \{(\zeta, \zeta, \zeta, \mu) : (\zeta, \zeta, \zeta, \mu) \cdot \tilde{Z}_T = \tilde{Z}_T\} \subset Z(\tilde{G}_T) , \quad (4.8.7)$$

where $\mu \in Z(\widetilde{G_{p_1}} \times \widetilde{G_{p_2}} \times \widetilde{G_{p_3}})$.

Very little is known about the trinion theories. Therefore we will restrict attention to the top level of the functor, i.e., the partition function. Fortunately, one example of a partition function is known, it is the superconformal index⁴⁰ [Rom05, KMMR05, GPRR09, GRRY11a, GRRY11b, Tac15] which is $\tilde{Z}_T$ evaluated on $S^3 \times S^1$ with nonbounding spin structure on S^1 , coupled to a $\tilde{G}_T$ bundle with connections specialized so as to give an index. We can work out the action of C_T on these. Given our present ignorance, we will simply postulate that if C_T acts trivially on the superconformal

³⁹It has been noted in [Tac13a, Bha21] that the *flavor* symmetry group of the theory can be a quotient group.

⁴⁰See Appendix D.6 for a brief review.

index it acts trivially on $\tilde{Z}$ and therefore $\tilde{Z}$ factorizes.⁴¹

The full superconformal index is not known in closed analytic form for generic trinion theories, but a simplification known as the Schur index is known (see Appendix D.6). The Schur index for a $d = 4$ $\mathcal{N} = 2$ theory (on $S^3 \times S^1$) has the trace formula

$$\mathcal{I}_{\text{Schur}}(q, \mathbf{g}) := \text{Tr}_{\mathcal{H}(S^3)} (-1)^F q^{\Delta - \frac{R}{2}} \mathbf{g} , \quad (4.8.8)$$

where $\mathcal{H}(S^3)$ is the Hilbert space of the theory on S^3 , F is the fermion number, Δ is the dilatation generator, R is the Cartan generator of $\text{SU}(2)_R$, and $\mathbf{g} \in G^f$ is the fugacity for the flavor symmetry (understood as an operator via its action on $\mathcal{H}(S^3)$). Only the operators with quantum numbers satisfying

$$\Delta = R + \frac{j_1 + j_2}{2} , \quad (4.8.9)$$

contribute to the Schur index (these are the so-called Schur operators [GRRY11a, GRRY11b, RR14, RR16]), where $(j_1, j_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ in (4.8.9) are the eigenvalues of the Cartan generators of the $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$ isometry group of S^3 .⁴² Thus the Schur index (4.8.8) reduces to

$$\mathcal{I}_{\text{Schur}}(q, \mathbf{g}) = \text{Tr}_{\mathcal{H}(S^3)} (-1)^F q^{\frac{R+j_1+j_2}{2}} \mathbf{g} . \quad (4.8.10)$$

The fugacities are holonomies for background gauge fields. Thus an element $(\zeta, \zeta, \zeta, \mu) \in Z(\tilde{G}_T)$ acts on the holonomies $q^{1/2}$ and $\mathbf{g}$ for the background gauge fields: the holonomies simply get multiplied by ζ and μ . From the exponent of $q^{1/2}$ in (4.8.10), it is clear that the center acts on the fugacities as

$$(q^{\frac{1}{2}}, \mathbf{g}) \mapsto (\zeta q^{\frac{1}{2}}, \mu \mathbf{g}) . \quad (4.8.11)$$

In this way, we can use the action of the center on the Schur index to determine C_T^{max} . We now work out two explicit examples.

⁴¹In fact, although this assumption seems to be standard in the literature, it has a fundamental flaw. It is well known that the action of a symmetry group on operators $O \rightarrow UOU^\dagger$ differs slightly from that on states $|\psi\rangle \rightarrow U|\psi\rangle$. Put more formally, the group that acts faithfully on the Hilbert space is in general a central extension of the group that acts faithfully on the operator algebra. This happens quite frequently in quantum theory. Sure enough, in the case of class $\mathcal{S}$ theories, L. Hollands and A. Neitzke [HN16] discovered that the $\mathbb{Z}_3$ central subgroup of E_6 acts nontrivially on the spaces of BPS states on $\mathbb{R}^3$ in the Minahan-Nemeschansky theory. They show that the Hilbert space of BPS states has a nontrivial isotypical decomposition for the $\mathbb{Z}_3$ action, where the degeneracy spaces for all three irreps are infinite-dimensional, and has a nontrivial BPS index. (The IR description of the degeneracy spaces has a $\mathbb{Z} \oplus \mathbb{Z}$ grading from an IR $U(1) \times U(1)$ global symmetry.) It is conceivable that, when working on compact manifolds, only the adjoint group of E_6 acts faithfully on the Hilbert space, but we are not aware of any evidence for or against this. We will nevertheless proceed to determine C_T^{max} from the superconformal index, *faute de mieux*, bearing in mind that our results might need to be refined by a central extension.

⁴²We use the same symbols for the operators and their eigenvalues.

Example 4.8.1. The simplest example is that of the trinion theory of class $\mathcal{S}$ of type A_{N-1} with two “full” punctures and one “simple” puncture, so the global symmetry Lie algebra is $\mathfrak{su}(N) \oplus \mathfrak{su}(N) \oplus \mathfrak{u}(1)$. In this case

$$\tilde{G}_T = \text{Spin}(4) \times \text{SU}(2)_R \times \text{SU}(N) \times \text{SU}(N) \times \text{U}(1) . \quad (4.8.12)$$

This is a Lagrangian theory expressible in terms of free hypermultiplets and background vector-multiplets. We refer the reader to Appendix D.7 for our notation and relevant details of the action and the partition function. Here we just make use of the superconformal index. The Schur index of the trinion theory with two “full” punctures and one “simple” puncture is given by [GRRY11a, Tac15]

$$\mathcal{I}_{\text{Schur}}^{\text{ffs}}(q; \mathbf{a}, \mathbf{b}, \alpha) = \prod_{u,i=1}^N \prod_{n \geq 0} \left[1 - q^{n+\frac{1}{2}} \left(\frac{\alpha a_i}{b_u} \right) \right]^{-1} \cdot \left[1 - q^{n+\frac{1}{2}} \left(\frac{b_u}{\alpha a_i} \right) \right]^{-1} , \quad (4.8.13)$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{T}(\text{SU}(N))$ are the fugacities for the Cartan of the two $\text{SU}(N)$ flavor symmetries, parametrized as $\mathbf{a} = \text{diag}(a_1, \dots, a_N)$ and $\mathbf{b} = \text{diag}(b_1, \dots, b_N)$, and $\alpha \in \text{U}(1)$ is the flavor fugacity for $\text{U}(1)$. Under the action of the central element

$$(\zeta, \zeta, \zeta, \omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \xi) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_N \times \mathbb{Z}_N \times \text{U}(1) , \quad (4.8.14)$$

where $\omega := e^{\frac{2\pi i}{N}}$ and $k_1, k_2 \in \{0, \dots, N-1\}$, the fugacities transform as in (4.8.11):

$$q^{\frac{1}{2}} \mapsto \zeta q^{\frac{1}{2}} , \quad (\mathbf{a}, \mathbf{b}, \alpha) \mapsto (\omega^{k_1} \mathbf{a}, \omega^{k_2} \mathbf{b}, \alpha \xi) . \quad (4.8.15)$$

From the explicit character expansions of (4.8.13), we find that the Schur index is invariant under (4.8.15) if and only if

$$\zeta \omega^{k_1} \omega^{-k_2} \xi = 1 , \quad (4.8.16)$$

and hence:

$$C_T^{\text{max}} = \{(\zeta, \zeta, \zeta, \omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \zeta \omega^{-k_1} \omega^{k_2}) \in Z(\tilde{G}_T)\} . \quad (4.8.17)$$

Clearly,

$$C_T^{\text{max}} = \langle (-1, -1, -1, \mathbf{1}_N, \mathbf{1}_N, -1), (1, 1, 1, \omega \mathbf{1}_N, \mathbf{1}_N, \omega^{-1}), (1, 1, 1, \mathbf{1}_N, \omega \mathbf{1}_N, \omega) \rangle . \quad (4.8.18)$$

There are different subgroups of C_T^{max} satisfying (4.8.7). They define different theories.

Topological Twisting Of The ffs Theory.

To define a topological twisting we can take $\tilde{G}^{\text{tw}} := \text{Spin}(4) \times \mathbb{T}^f$ with a homomorphism $\tilde{G}^{\text{tw}} \rightarrow \tilde{G}_T$ given by $(u_1, u_2, \mu) \mapsto ((u_1, u_2), u_2, \mu)$ which will descend to a homomorphism $G^{\text{tw}} \rightarrow G_T$ for groups C_T described above, where $G^{\text{tw}} \cong \tilde{G}^{\text{tw}} / C^{\text{tw}}$ and we can take C^{tw} to be the kernel of the

homomorphism from $\widetilde{G}^{\text{tw}}$. Via pullback under the homomorphism $G^{\text{tw}} \rightarrow G_T$, the $\text{SU}(2)_R$ -connection is identified with the self-dual part of the spin connection.

As explained in Section 4.7, the bosonic scalars of the hypermultiplet become chiral spinors upon twisting. For details of the untwisted and twisted actions of the ffs theory and our notation, see Appendix D.7. The Gaussian path integral over the twisted bosonic spinor fields yields the product of a topological invariant of the background fields and an inverse of the determinant of the square of the coupled Dirac operator. In particular, due to $\mathcal{Q}$ -symmetry, the determinants cancel:

$$\int [\mathcal{D}\text{HM}] e^{-S_{\text{hyp}}^{\text{twisted}}} = \left(\text{Det} \left(\not{D}_{\mathcal{V} \otimes \overline{\mathcal{V}}}^+ \not{D}_{\mathcal{V} \otimes \overline{\mathcal{V}}}^- \right) \right)^{-1} \text{Det}(\text{i} \not{D}_{\mathcal{V} \otimes \overline{\mathcal{V}}}^+) \text{Det}(-\text{i} \not{D}_{\mathcal{V} \otimes \overline{\mathcal{V}}}^-) = 1, \quad (4.8.19)$$

and the partition function is simply given by a topological invariant of the background $\text{SU}(N) \times \text{SU}(N) \times \text{U}(1)$ bundle with connection:

$$Z_T^{\text{twisted}} = \exp \left(-\frac{\text{i}}{16\pi} \int_{\mathbb{X}} \tau_{ab} (F^a \wedge F^b) \right). \quad (4.8.20)$$

Example 4.8.2. We also consider the A_{N-1} type trinion theory with 3 “full” punctures (fff) for which⁴³

$$\widetilde{G}_T = \text{Spin}(4) \times \text{SU}(2)_R \times \text{SU}(N) \times \text{SU}(N) \times \text{SU}(N). \quad (4.8.21)$$

The only further information known about the operator content comes from the superconformal index [Rom05, KMMR05] (see Appendix D.6 for a review). The Schur index of the T_N theory has the following form [GRRY11a, Tac15]:

$$\mathcal{I}_{\text{Schur}}^{T_N}(q; \mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{K(\mathbf{a})K(\mathbf{b})K(\mathbf{c})}{K_0} \sum_{\lambda} \frac{\chi_{\lambda}(\mathbf{a})\chi_{\lambda}(\mathbf{b})\chi_{\lambda}(\mathbf{c})}{\chi_{\lambda}(\mathbf{q}^{\rho})}. \quad (4.8.22)$$

Here, the sum is over representations of the global flavor symmetries, for which the flavor symmetry elements are

$$(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}(\text{SU}(N) \times \text{SU}(N) \times \text{SU}(N)), \quad (4.8.23)$$

where $\mathbb{T}$ denotes the standard maximal torus and χ_{λ} is the character of the representation λ . Also,

$$\mathbf{q}^{\rho} := \text{diag}(q^{(N-1)/2}, q^{(N-3)/2}, \dots, q^{(1-N)/2}) \in \text{SU}(N). \quad (4.8.24)$$

The prefactor in (4.8.22) is the contribution of the flavor multiplets to the index for $\text{SU}(N) \times \text{SU}(N) \times \text{SU}(N)$, where for any $\mathbf{a} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$,

$$K(\mathbf{a}) := \prod_{n \geq 0} \left[(1 - q^{n+1})^{1-N} \prod_{i \neq j} \left(1 - q^{n+1} \frac{a_i}{a_j} \right)^{-1} \right], \quad (4.8.25)$$

$$K_0 := \prod_{d=2}^N \prod_{n \geq 0} (1 - q^{d+n}). \quad (4.8.26)$$

⁴³There are some issues about whether one should use $\text{SU}(N)$ or $\text{U}(N)$. We will not attempt to address them here.

and $\mathbf{a} := \text{diag}(\mathbf{a}_1, \dots, \mathbf{a}_N) \in \mathbb{T}(\text{SU}(N))$.

The element

$$(\zeta, \zeta, \zeta, \omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \omega^{k_3} \mathbf{1}_N) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N , \quad (4.8.27)$$

where $k_1, k_2, k_3 \in \{0, \dots, N-1\}$, acts on the fugacities as in (4.8.11), which implies

$$\mathbf{q}^\rho \mapsto \zeta^{N-1} \mathbf{q}^\rho , \quad (\mathbf{a}, \mathbf{b}, \mathbf{c}) \mapsto (\omega^{k_1} \mathbf{a}, \omega^{k_2} \mathbf{b}, \omega^{k_3} \mathbf{c}) . \quad (4.8.28)$$

Consider the term in the Schur index corresponding to a representation λ of N -ality $N(\lambda)$. From the expression of the character χ_λ in terms of Schur polynomials, we infer that

$$\frac{\chi_\lambda(\mathbf{a})\chi_\lambda(\mathbf{b})\chi_\lambda(\mathbf{c})}{\chi_\lambda(\mathbf{q}^\rho)} \mapsto (\zeta^{N-1} \omega^{k_1} \omega^{k_2} \omega^{k_3})^{N(\lambda)} \frac{\chi_\lambda(\mathbf{a})\chi_\lambda(\mathbf{b})\chi_\lambda(\mathbf{c})}{\chi_\lambda(\mathbf{q}^\rho)} . \quad (4.8.29)$$

Note that the prefactor $K(\mathbf{a})K(\mathbf{b})K(\mathbf{c})/K_0$ is invariant under translation by the center. Since representations of all N -alities appear in the sum, we conclude that

$$C_T^{\max} = \begin{cases} \{(\zeta, \zeta, \zeta, \omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \omega^{k_3} \mathbf{1}_N) \in Z(\tilde{G}_T) : \zeta \omega^{k_1} \omega^{k_2} \omega^{k_3} = 1\} & \text{for } N \text{ even} , \\ \{(\zeta, \zeta, \zeta, \omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \omega^{k_3} \mathbf{1}_N) \in Z(\tilde{G}_T) : \omega^{k_1} \omega^{k_2} \omega^{k_3} = 1\} & \text{for } N \text{ odd} . \end{cases} \quad (4.8.30)$$

We now return to the general trinion theory. Since the theory is supersymmetric, its partition function Z_T should satisfy a set of supersymmetric Ward identities analogous to those of the partition function of a hypermultiplet in the presence of external fields. In particular, upon choosing a homomorphism

$$\varphi : G^{\text{tw}, \text{bck}} \rightarrow \tilde{G}_T / C_T , \quad (4.8.31)$$

such that the pullback of the supersymmetry operators contains a singlet we obtain a function of the topologically twisted (background) VM's and the metric which is – moreover – $\mathcal{Q}$ -closed, and therefore metric independent. We take

$$G^{\text{tw}, \text{bck}} = (\text{Spin}(4) \times \mathbb{T}^f) / C^{\text{tw}, \text{bck}} , \quad (4.8.32)$$

and as in equation (4.2.16) we require:

$$\begin{array}{ccc} G^{\text{tw}, \text{bck}} & \xrightarrow{\varphi^{\text{tw}, \text{bck}}} & \tilde{G}_T / C_T \\ p_1 \downarrow & & \downarrow p_1 \times p_2 \\ \text{Spin}(4) / \langle (-1, -1) \rangle & \xrightarrow{\varphi^W} & (\text{Spin}(4) \times \text{SU}(2)_R) / \langle (-1, -1, -1) \rangle \end{array} \quad (4.8.33)$$

The description of $G^{\text{tw}, \text{bck}}$ and $\varphi^{\text{tw}, \text{bck}}$ is now very similar to that of the Lagrangian case. For example, for the A_{N-1} fff trinion theory, we can take

$$G^{\text{tw}, \text{bck}} = (\text{Spin}(4) \times \mathbb{T}(\text{SU}(N) \times \text{SU}(N) \times \text{SU}(N))) / C , \quad (4.8.34)$$

$$C = \{(\zeta, \zeta, \omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \omega^{k_3} \mathbf{1}_N) \mid \zeta^{N-1} \omega^{k_1+k_2+k_3} = 1\} , \quad (4.8.35)$$

and

$$\varphi^{\text{tw}, \text{bck}}([(u_1, u_2), \mu]) = [(u_1, u_2), u_2, \mu] . \quad (4.8.36)$$

4.8.3 Gaiotto Gluing

We are now ready to discuss the general class $\mathcal{S}$ theory. In a weak-coupling domain, these can be assembled from trinion theories and vectormultiplets by the Gaiotto gluing conjecture [Gai09]. Recall that this states the following:

Consider two class $\mathcal{S}$ theories $T[\mathfrak{g}, C^{(1)}, D^{(1)}]$ and $T[\mathfrak{g}, C^{(2)}, D^{(2)}]$ on (stable) Riemann surfaces $C^{(1)}$ and $C^{(2)}$, with full punctures D_{p_1} in theory one, D_{p_2} in theory two (see footnote 36). Then, since $\tilde{G}_{p_1} = \tilde{G}_{p_2}$, we can gauge the diagonal subgroup $\tilde{G}$ of $\tilde{G}_{p_1} \times \tilde{G}_{p_2}$ with coupling constant τ . (When gluing full punctures, the beta function vanishes, so the coupling is scale-independent.) Denote the new gauged theory by

$$T[\mathfrak{g}, C^{(1)}, D^{(1)}] \times_{\tau} T[\mathfrak{g}, C^{(2)}, D^{(2)}] . \quad (4.8.37)$$

On the other hand, we can glue the Riemann surfaces $C^{(1)}$ to $C^{(2)}$ with disks around p_1 and p_2 using the plumbing fixture $z_1 z_2 = q$. Denote the resulting Riemann surface $C^{(1)} \times_{p_1, p_2, q} C^{(2)}$. The Gaiotto gluing conjecture states that

$$T[\mathfrak{g}, C^{(1)}, D^{(1)}] \times_{\tau} T[\mathfrak{g}, C^{(2)}, D^{(2)}] = T[\mathfrak{g}, C^{(1)} \times_{p_1, p_2, q} C^{(2)}, D^{(1,2)}] , \quad (4.8.38)$$

where $q = e^{2\pi i \tau}$ and $D^{(1,2)}$ is the disjoint union of the punctures that were not glued.

Remark 4.8.1. One may wonder if one could gauge a quotient of $\tilde{G}_p$ in the gluing procedure. Moreover, one may ask how the data making the theory associated with the glued surface absolute is related to such data for the two factors. These are interesting and nontrivial questions we must leave for another time.

4.8.4 Expressing The Theory As A Gauge Theory

Now consider a pants decomposition of $C_{g,n}$ into a set $\mathcal{T}$ of trinions. We let p_i denote the external punctures, where i runs over the set of external punctures. The internal punctures come in unordered pairs $\{p_{\alpha,1}, p_{\alpha,2}\}$ with $\tilde{G}_{p_{\alpha,1}} \cong \tilde{G}_{p_{\alpha,2}}$ where α runs over the set of internal cutting curves. Consider first the product of all the trinion theories associated to $T \in \mathcal{T}$. This will have a global symmetry group based on

$$\tilde{G}_{\mathcal{T}} = \text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times \prod_i \tilde{G}_{p_i} \times \prod_{\alpha} \left(\tilde{G}_{p_{\alpha,1}} \times \tilde{G}_{p_{\alpha,2}} \right) . \quad (4.8.39)$$

The possible theories we can construct from this product of trinion theories will have global symmetry $\tilde{G}_{\mathcal{T}}/C_{\mathcal{T}}$. Again, if we wish our domain category to include non-spin manifolds, we use the considerations of section 4.2 to conclude that

$$(p_1 \times p_2)(C_{\mathcal{T}}) = \langle (-1, -1, -1) \rangle \subset Z(\mathrm{Spin}(4) \times \mathrm{SU}(2)_{\mathrm{R}}) . \quad (4.8.40)$$

as well as

$$C_{\mathcal{T}} \subset C_{\mathcal{T}}^{\max} := \{(\zeta, \zeta, \zeta, z) \mid (\zeta, \zeta, \zeta, z) \cdot \tilde{Z}_T = \tilde{Z}_T \quad \forall T \in \mathcal{T}\} . \quad (4.8.41)$$

where z summarizes the component in the factors other than $\mathrm{Spin}(4) \times \mathrm{SU}(2)_{\mathrm{R}}$.

Example 4.8.3. Suppose, for simplicity, that all the punctures are “full” punctures. For a given trinion $T \in \mathcal{T}$ let $R_T(z)$ be the N^{th} root of unity obtained by taking the product of the $\mathrm{SU}(N)$ factors in z pertaining to T . Then we require that $\zeta R_T(z) = 1$ for all $T \in \mathcal{T}$ when N is even and $R_T(z) = 1$ for all $T \in \mathcal{T}$ when N is odd.

Now we wish to gauge the diagonal flavor symmetry associated with the cutting curves. For each α consider the homomorphism⁴⁴

$$\psi_{\mathrm{glue}} : \prod_{\alpha} \mathrm{SU}(N)_{\alpha} \rightarrow \tilde{G}_{\mathcal{T}} , \quad (4.8.42)$$

defined by

$$\psi_{\mathrm{glue}} : \prod_{\alpha} g_{\alpha} \longmapsto \left(\mathbb{1}_{\mathrm{Spin}(4)}, \mathbb{1}_{\mathrm{SU}(2)_{\mathrm{R}}}, \prod_i \mathbb{1}_{\tilde{G}_i}, \prod_{\alpha} (g_{\alpha}, g_{\alpha}) \right) . \quad (4.8.43)$$

Let $\pi : \tilde{G}_{\mathcal{T}} \rightarrow \tilde{G}_{\mathcal{T}}/C'_{\mathcal{T}}$ be the projection where

$$C'_{\mathcal{T}} = \left[\mathrm{Spin}(4) \times \mathrm{SU}(2)_{\mathrm{R}} \times \prod_i \tilde{G}_{p_i} \times \prod_{\alpha} \left(\Delta(\tilde{G}_{p_{\alpha}}) \right) \right] \cap C_{\mathcal{T}} , \quad (4.8.44)$$

where $\Delta(G) \subset G \times G$ is the diagonal subgroup. Then $\pi \circ \psi_{\mathrm{glue}}$ will have a nontrivial kernel $K \subset Z(\prod_{\alpha} \mathrm{SU}(N)_{\alpha})$. The gauge group associated with the cutting curves is

$$\left(\prod_{\alpha} \mathrm{SU}(N)_{\alpha} \right) / K . \quad (4.8.45)$$

This determines what ’t Hooft fluxes can be turned on, and what ’t Hooft fluxes must be summed over when computing physical quantities.

The analog of G^{phys} from Section 4.3 is now

$$G_{\mathrm{S}} = \left(\mathrm{Spin}(4) \times \mathrm{SU}(2)_{\mathrm{R}} \times \prod_i \tilde{G}_{p_i} \times \prod_{\alpha} \mathrm{SU}(N)_{\alpha} \right) / C''_{\mathcal{T}} . \quad (4.8.46)$$

(where $C''_{\mathcal{T}}$ is the preimage of $C'_{\mathcal{T}}$ under the product over α of the map $\Delta : \mathrm{SU}(N) \rightarrow \Delta(\mathrm{SU}(N))$).

⁴⁴Other choices of gauge group are possible [CD10]. We restrict to the simplest case.

Remark 4.8.2.

1. As a simple example consider the gluing of two A_{N-1} trinion theories with fff punctures to form a genus two surface. Choosing $C = C_T^{\max}$ the gauge group of the theory is

$$(\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}(N)) / \{(\omega^{k_1} \mathbf{1}_N, \omega^{k_2} \mathbf{1}_N, \omega^{k_3} \mathbf{1}_N) | \omega^{k_1+k_2+k_3} = 1\} . \quad (4.8.47)$$

Note that K is not a direct product of the form $\prod_{\alpha} K_{\alpha}$.

2. The existence of a principal G_S bundle $P_S \rightarrow \mathbb{X}$ in terms of cohomology classes of the bundles associated with transfer of structure group for the projections on the factors will be based on conditions analogous to the conditions (4.4.18) we found above. Some explicit examples are given in Appendix D.8.
3. It is an interesting question whether all the theories constructed above for different choices of C have six-dimensional descriptions, and what the extra data are in those descriptions. This question becomes highly relevant when considering the action of S -duality. In section 4.9 we will assume that all the theories have a 6d origin, and therefore the mapping class groups permute them as an S -duality group.

Now we turn to the partition function of the theory $\mathcal{S}$ on a manifold $\mathbb{X}$. It will have the general form:

$$Z_S[\omega_{\mu}^{\frac{a}{b}}, A_{\mu}^R, \mathrm{VM}_i] = \int \prod_{c_{\alpha}} [d\mathrm{VM}_{c_{\alpha}}] e^{-S_{\mathrm{VM}_{c_{\alpha}}}} \prod_T Z_T[\omega_{\mu}^{\frac{a}{b}}, A_{\mu}^R, \mathrm{VM}_{p_{T,1}}, \mathrm{VM}_{p_{T,2}}, \mathrm{VM}_{p_{T,3}}] , \quad (4.8.48)$$

where the product over c_{α} is over the internal cutting curves and the product over T is a product over the trinions, and the vectormultiplet path integrals are for the gauge group (4.8.45). We choose the external VM_i so that the mass terms are generic, the external fermions are zero, and the structure group is reduced to the maximal torus. However, for the internal cutting curves c_{α} one must maintain the full VM in writing Z_T in the integrand of the path integral.

4.8.5 Topological Twisting

Having framed the discussion as above, topological twisting proceeds in a completely analogous fashion to what we have described in previous sections. We choose a homomorphism $\varphi^{\mathrm{tw}, \mathrm{bck}}$ so that

$$\begin{array}{ccc} G^{\mathrm{tw}} & \xrightarrow{\varphi^{\mathrm{tw}, \mathrm{bck}}} & G_S \\ p_1 \downarrow & & \downarrow p_1 \times p_2 \\ \mathrm{Spin}(4)/\langle(-1, -1)\rangle & \xrightarrow{\varphi^{\mathrm{W}}} & (\mathrm{Spin}(4) \times \mathrm{SU}(2)_{\mathrm{R}})/\langle(-1, -1, -1)\rangle . \end{array} \quad (4.8.49)$$

Much as in the previous discussion we can take

$$G^{\text{tw}} = \left(\text{Spin}(4) \times \mathbb{T} \left(\prod_i \tilde{G}_i \right) \times \left(\prod_\alpha \text{SU}(N)_\alpha \right) \right) / C_S^{\text{tw}} . \quad (4.8.50)$$

To find C_S^{tw} we begin with

$$\tilde{G}^{\text{tw}} = \text{Spin}(4) \times \mathbb{T} \left(\prod_i \tilde{G}_i \right) \times \left(\prod_\alpha \text{SU}(N)_\alpha \right) , \quad (4.8.51)$$

and consider $\tilde{\varphi}^{\text{tw}} : \tilde{G}^{\text{tw}} \rightarrow G_S$ defined by

$$\tilde{\varphi}^{\text{tw}}((u_1, u_2), \mu, g) := [(u_1, u_2), u_2, \mu, g] . \quad (4.8.52)$$

The homomorphism has a kernel, which we can take to be C_S^{tw} .

Upon topological twisting the actions $S_{\text{VM}_{c_\alpha}}$ will be precisely of the form equation (4.7.9). After doing the path integral over the internal vectormultiplets the path integral will depend on background fields. From this, we derive $G^{\text{tw}, \text{bck}}$. The path integral will be invariant under continuous deformation of connection on the $G^{\text{tw}, \text{bck}}$ -bundle and therefore will only depend on the topological data outlined in items 1–3.

4.9 Twisting $\mathfrak{a}_1$ Class $\mathcal{S}$ Theories And S-duality

The ideas presented so far can be used to twist $\mathfrak{a}_1$ class $\mathcal{S}$ theories. The cohomological conditions for the existence of P^{phys} were presented in [MM21]. Since these theories are Lagrangian theories we can rederive the conditions of [MM21] using the discussion in Section 4.4.⁴⁵ Then, in Section 4.9.2 we will study how these conditions change under S -duality, using the fact that these are also class $\mathcal{S}$ theories.

4.9.1 Cohomological Conditions For The $\mathfrak{a}_1$ Class $\mathcal{S}$ Theory

The weak coupling descriptions of $T[\mathfrak{a}_1, C_{g,n}]$ corresponding to different pants decompositions of $C_{g,n}$ were studied in [Gai09, GMN10, DMO09]. The gauge algebra of the theory is $\mathfrak{g}^{\text{gauge}} = \text{Lie}(G^{\text{gauge}}) = \mathfrak{su}(2)^{3g-3+n}$, and as before, $\tilde{G}^{\text{phys}} = \text{Spin}(4) \times \text{SU}(2)_R \times G^f \times \tilde{G}^{\text{gauge}}$, where

$$\tilde{G}^{\text{gauge}} \cong \text{SU}(2)^{3g-3+n} , \quad n = \text{rank}(G^f) . \quad (4.9.1)$$

We will assume masses are generic and there is a reduction of flavor structure group to its maximal torus so we choose the domain category so that $G^f \cong \text{U}(1)^n$. Note that

$$Z(\tilde{G}^{\text{phys}}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{U}(1)^n \times \mathbb{Z}_2^{3g-3+n} . \quad (4.9.2)$$

⁴⁵In [MM21] these conditions were cited as conditions for topological twisting. One of the points of this chapter is that the cohomological conditions for the existence of P^{phys} should be distinguished from topological twisting.

The different global forms of the physical theory are then characterized by choices of $C^{\text{phys}} \subset Z(\tilde{G}^{\text{phys}})$, subject to the requirements discussed in Section 4.3. There are several choices of C^{phys} (see, for example, [GMN10, Section 2.3.1] for a related discussion).

We then identify

$$\tilde{G}^{\text{gauge}} = \prod_{\substack{c_\alpha \\ \text{internal}}} \text{SU}(2)_{c_\alpha}, \quad G^{\text{f}} = \prod_{\substack{c_i \\ \text{external}}} \text{U}(1)_{c_i}. \quad (4.9.3)$$

We define the projection maps,

$$p_{4,c_\alpha} : \tilde{G}^{\text{phys}} \longrightarrow \text{SU}(2)_{c_\alpha}, \quad p_{3,c_i} : \tilde{G}^{\text{phys}} \longrightarrow \text{U}(1)_{c_i}, \quad (4.9.4)$$

to the c_α^{th} and c_i^{th} factors respectively. The preliminary discussion so far has been for the physical $\mathfrak{a}_1$ class $\mathcal{S}$ theory on $\mathbb{R}^4$ (after a Wick rotation from $\mathbb{R}^{1,3}$). Next, we seek cohomological conditions – guided by the discussion of Section 4.4 – from the existence of the bundle $P^{\text{phys}} \rightarrow \mathbb{X}$, for the $\mathfrak{a}_1$ class $\mathcal{S}$ theory on the 4-manifold $\mathbb{X}$.

As discussed above, we choose C^{phys} so that the field theory functor factors through a functor from a bordism category that includes non-spin manifolds. As before, this leads to

$$w_2(\mathbb{X}) = w_2(P^{\text{R}}). \quad (4.9.5)$$

To derive additional conditions, we first note that $p_4(C^{\text{phys}}) \cong C^{\text{grb}} \subset \mathbb{Z}_2^{3g-3+n}$, so the characteristic class $\mu(b) \in H^2(\mathbb{X}; C^{\text{grb}})$ of the C^{grb} -gerbe b can then be projected to a tuple of characteristic classes $\mu(b)_{c_\alpha} \in H^2(\mathbb{X}; \mathbb{Z}_2)$, one for each internal cutting curve c_α , by projecting C^{phys} onto the c_α^{th} factor using (4.9.4).⁴⁶ Note that, depending on our choice of C^{phys} , which determines C^{grb} , it might happen that $\mu(b)_{c_\alpha}$ vanishes. We also recall remark 4 above: one should include a topological boundary condition ρ for the 5d gerbe b which determines which 't Hooft fluxes are to be summed over.

The matter content of the theory is a collection of half-hypermultiplets Φ_{c_1, c_2, c_3} labeled by unordered triples of cutting curves of the pants decomposition.⁴⁷ Let us begin by considering conditions associated with $\Phi_{c_i c_j c_\alpha}$, containing one internal and two external punctures. Introduce the notation

$$P^{\text{f}, c_i} = (p_{3, c_i})_*(P^{\text{phys}}). \quad (4.9.6)$$

⁴⁶Explicitly, let ϖ be the map $\varpi : H^2(\mathbb{X}; C^{\text{grb}}) \rightarrow H^2(\mathbb{X}; \mathbb{Z}_2^{3g-3+n})$ induced by the inclusion $C^{\text{grb}} \hookrightarrow \mathbb{Z}_2^{3g-3+n}$.

Therefore $\varpi(\mu(b))$ is a $(3g-3+n)$ -tuple of classes in $H^2(\mathbb{X}; \mathbb{Z}_2)$. Then p_{4, c_α} induces a natural projection onto the c_α^{th} factor of this tuple, which we denote by $\mu(b)_{c_\alpha} \in H^2(\mathbb{X}; \mathbb{Z}_2)$. If $\mu(b) \in \ker \varpi$, then the corresponding $\mu(b)_{c_\alpha} = 0$ for all c_α .

⁴⁷Therefore, the scalar fields in Φ_{c_1, c_2, c_3} are in a real subspace of the $2_{c_1} \otimes 2_{c_2} \otimes 2_{c_3} \otimes 2_{\text{SU}(2)_R}$ giving a total of 16 independent real scalar fields.

Then P^{f,c_i} is a $U(1)/p_{3,c_i}(C^{\text{phys}})$ -bundle. Let $c_1(c_i)$ denote the first Chern class of the $U(1)$ -bundle obtained by transfer of structure group of P^{f,c_i} along the isomorphism (see footnote 22) $U(1)/p_{3,c_i}(C^{\text{phys}}) \xrightarrow{\cong} U(1)$. Now, item 2 of Section 4.4 (or (4.4.18)) directly yields the conditions

$$\begin{aligned} c_1(c_i) + \mu(b)_{c_\alpha} + w_2(\mathbb{X}) &= 0 \bmod 2, \\ c_1(c_j) + \mu(b)_{c_\alpha} + w_2(\mathbb{X}) &= 0 \bmod 2. \end{aligned} \tag{4.9.7}$$

Next, we consider trinions $\Phi_{c_i, c_\alpha, c_\beta}$ bounded by one external cutting curve c_i and two internal cutting curves c_α, c_β . Following Section 4.4, we obtain

$$c_1(c_i) + \mu(b)_{c_\alpha} + \mu(b)_{c_\beta} + w_2(\mathbb{X}) = 0 \bmod 2. \tag{4.9.8}$$

Finally for trinions $\Phi_{c_\alpha c_\beta c_\gamma}$ bounded by three internal cutting curves $c_\alpha, c_\beta, c_\gamma$, the matter $\Phi_{c_\alpha c_\beta c_\gamma}$ is a half-hypermultiplet in the $\mathbf{2}_\alpha \otimes \mathbf{2}_\beta \otimes \mathbf{2}_\gamma$ of the $SU(2)_{c_\alpha} \times SU(2)_{c_\beta} \times SU(2)_{c_\gamma}$ factor of $\tilde{G}$. The discussion in Section 4.4 leads to the cohomological condition

$$\mu(b)_{c_\alpha} + \mu(b)_{c_\beta} + \mu(b)_{c_\gamma} + w_2(\mathbb{X}) = 0 \bmod 2. \tag{4.9.9}$$

4.9.2 Transformation Of Cohomological Conditions Under S-duality

We next analyze the effect of S-duality transformations on these cohomological conditions. Recall that the S-duality group of the theory is the mapping class group $MCG(C_{g,n})$ and acts on the space of gauge couplings of the theory (identified with the Teichmüller space of $C_{g,n}$, see [Wit97, GMN09, Gai09]). $MCG(C_{g,n})$, being the group of isotopy classes of orientation-preserving diffeomorphisms of $C_{g,n}$, acts on the set of simple closed curves on $C_{g,n}$, and thus also acts on the (infinite) set of all pants decompositions of $C_{g,n}$. The set of equivalence classes of pants decompositions modulo the action of the mapping class group $MCG(C_{g,n})$ is finite.⁴⁸ Thus, there are finitely many S-duality orbits of theories based on $C_{g,n}$. We will show that there is a natural transformation of the cohomological conditions of theories in the same S-duality orbit. To do so, we introduce the *graph* of a pants decomposition of $C_{g,n}$.

Given a pants decomposition $\mathbf{p}$, we define an undirected graph $G(\mathbf{p})$ of $\mathbf{p}$ with vertices as the pairs of pants *and* external cutting curves in $\mathbf{p}$, such that two pairs of pants are connected by an edge in $G(\mathbf{p})$ if they are bounded by a common internal cutting curve. Further, an external cutting curve (a vertex in $G(\mathbf{p})$) is connected by an edge in $G(\mathbf{p})$ to that pair of pants which it bounds (also

⁴⁸Note that $MCG(C_{g,n})$ is isomorphic to the automorphism group of the *pants graph* or *pants complex* of $C_{g,n}$, see [Mar04] for a proof. This is the pants complex of Hatcher and Thurston [HT80] with the inclusion of 2-cells by Hatcher, Lochak, and Schneps [HLS00]. The resulting pants complex is connected and simply connected. $MCG(C_{g,n})$ acts co-compactly on this complex [Wol09].

a vertex in $G(\mathbf{p})$). Thus $G(\mathbf{p})$ has $2g - 2 + n$ trivalent vertices, n univalent vertices and $3g - 3 + 2n$ edges. An illustrative example is shown in Figure 4.1.



Figure 4.1: The graph of a pants decomposition of $C_{2,2}$. Figure adapted from [Wol09].

Two pants decompositions $\mathbf{p}, \mathbf{p}'$ are in the same $\text{MCG}(C_{g,n})$ -orbit if and only if the corresponding graphs $G(\mathbf{p}), G(\mathbf{p}')$ are isomorphic [Wol09].⁴⁹

We can now describe the cohomological conditions for the theory defined by the pants decomposition $\mathbf{p}$ in terms of the graph $G(\mathbf{p})$ as follows: we begin by labeling the vertices and edges in $G(\mathbf{p})$ as

$$V(G(\mathbf{p})) := \{t_1, \dots, t_{2g-2+n}, u_1, \dots, u_n\}, \quad (4.9.10)$$

$$E(G(\mathbf{p})) := \{e_1, \dots, e_{3g-3+n}, \tilde{e}_1, \dots, \tilde{e}_n\}, \quad (4.9.11)$$

where t and u indicate trivalent and univalent vertices respectively, whereas the edge e_α connects two trivalent vertices and $\tilde{e}_i$ connects a univalent vertex with a trivalent vertex.

In Section 4.9.1, we encountered maps $c_i \mapsto c_1(c_i) \in H^2(\mathbb{X}; \mathbb{Z})$ and $c_\alpha \mapsto \mu(b)_{c_\alpha} \in H^2(\mathbb{X}; \mathbb{Z}_2)$. By design, the set of external cutting curves $\{c_i\}_{i=1}^n$ is bijectively mapped to the subset $\{\tilde{e}_i\}_{i=1}^n$ of edges of $G(\mathbf{p})$. Likewise, the set of internal cutting curves $\{c_\alpha\}_{\alpha=1}^{3g-3+n}$ also gets bijectively mapped to the subset $\{e_\alpha\}_{\alpha=1}^{3g-3+n}$. Note, however, that in each case, there might be several bijections due to different choices of indexing. We simply pick one, so that we have one-to-one map between elements of the sets $\{c_i\}_{i=1}^n$ and $\{\tilde{e}_j\}_{j=1}^n$, and between elements of the sets $\{c_\alpha\}_{\alpha=1}^{3g-3+n}$ and $\{e_\beta\}_{\beta=1}^{3g-3+n}$.

⁴⁹Two graphs $G(\mathbf{p})$ and $G(\mathbf{p}')$ are said to be isomorphic if there exists a bijection between their vertex sets $f : V(G(\mathbf{p})) \rightarrow V(G(\mathbf{p}'))$ such that any two vertices $v_1, v_2 \in V(G(\mathbf{p}))$ of $G(\mathbf{p})$ are adjacent in $G(\mathbf{p})$ if and only if their images $f(v_1)$ and $f(v_2)$ are adjacent in $G(\mathbf{p}')$. When such an isomorphism exists between $G(\mathbf{p})$ and $G(\mathbf{p}')$, we will frequently write it as $f : G(\mathbf{p}) \rightarrow G(\mathbf{p}')$, with the understanding that it is an isomorphism between the corresponding vertex sets.

Then, with this choice of map, we can associate fluxes to the edges in $G(\mathbf{p})$:

$$\tilde{e}_i \mapsto c_1(\tilde{e}_i) \in H^2(\mathbb{X}; \mathbb{Z}) \quad \text{for } i = 1, \dots, n, \quad (4.9.12)$$

$$e_\alpha \mapsto \mu(b)_{e_\alpha} \in H^2(\mathbb{X}; \mathbb{Z}_2) \quad \text{for } \alpha = 1, \dots, 3g - 3 + n, \quad (4.9.13)$$

with the proviso that $\mu(b)_{e_\alpha} = 0$ if e_α is a loop (i.e., an edge connecting one vertex to itself). Then, the cohomological conditions described in the preceding section can be recast into equations at each trivalent vertex. If a trivalent vertex is connected by three edges $e_\alpha, e_\beta, e_\gamma$ to three other trivalent vertices, then we impose the condition

$$\mu(b)_{e_\alpha} + \mu(b)_{e_\beta} + \mu(b)_{e_\gamma} + w_2(\mathbb{X}) = 0. \quad (4.9.14)$$

If a trivalent vertex is connected by edges $e_\alpha, e_\beta, \tilde{e}_i$ to two trivalent and one univalent vertex, then we have the cohomological condition

$$c_1(\tilde{e}_i) + \mu(b)_{e_\alpha} + \mu(b)_{e_\beta} + w_2(\mathbb{X}) = 0 \bmod 2. \quad (4.9.15)$$

Finally, if a trivalent vertex is connected by edges $e_\alpha, \tilde{e}_i, \tilde{e}_j$ to a trivalent and two univalent vertices, then

$$c_1(\tilde{e}_i) + \mu(b)_{e_\alpha} + w_2(\mathbb{X}) = 0 \bmod 2, \quad (4.9.16)$$

$$c_1(\tilde{e}_j) + \mu(b)_{e_\alpha} + w_2(\mathbb{X}) = 0 \bmod 2. \quad (4.9.17)$$

Now consider two pants decompositions $\mathbf{p}, \mathbf{p}'$ with graphs $G(\mathbf{p}), G(\mathbf{p}')$ in the same $\text{MCG}(\Sigma_{g,n})$ -orbit. Let the vertices and edges of $G(\mathbf{p}')$ be labeled as

$$V(G(\mathbf{p}')) := \{t'_1, \dots, t'_{2g-2+n}, u'_1, \dots, u'_n\}, \quad (4.9.18)$$

$$E(G(\mathbf{p}')) := \{e'_1, \dots, e'_{3g-3+n}, \tilde{e}'_1, \dots, \tilde{e}'_n\}. \quad (4.9.19)$$

We associate fluxes to the edges as in (4.9.12) and (4.9.13).

Recall that for an undirected graph $G(\mathbf{p})$, its line graph $L(G(\mathbf{p}))$ is constructed in the following way: to each edge in $G(\mathbf{p})$, we assign a vertex in $L(G(\mathbf{p}))$ ⁵⁰ and for every two edges in $G(\mathbf{p})$ emanating from a common vertex, we assign an edge between their corresponding vertices in $L(G(\mathbf{p}))$. Since $G(\mathbf{p}) \cong G(\mathbf{p}')$, by the Whitney graph isomorphism theorem [Har69, Theorem 8.3], their line graphs are also isomorphic:

$$L(G(\mathbf{p})) \cong L(G(\mathbf{p}')). \quad (4.9.20)$$

⁵⁰Therefore, $V(L(G(\mathbf{p}))) \cong E(G(\mathbf{p}))$.

Moreover, there is a one-to-one correspondence between graph isomorphisms $G(\mathbf{p}) \cong G(\mathbf{p}')$ and the corresponding line graph isomorphisms $L(G(\mathbf{p})) \cong L(G(\mathbf{p}'))$.⁵¹ The vertices of the graphs $L(G(\mathbf{p}))$ and $L(G(\mathbf{p}'))$ are respectively, $E(G(\mathbf{p}))$ and $E(G(\mathbf{p}'))$. Choose an isomorphism $\mathbf{F} : E(G(\mathbf{p})) \rightarrow E(G(\mathbf{p}'))$ and let $f : G(\mathbf{p}) \rightarrow G(\mathbf{p}')$ denote the graph isomorphism corresponding to $\mathbf{F}$.

Let us denote an edge in $G(\mathbf{p})$ connecting two vertices u and v by uv . By arguments outlined in [Har69, Theorem 8.3], we have

$$\mathbf{F}(uv) = f(u)f(v) . \quad (4.9.21)$$

In particular, if e_α , e_β , and e_γ are edges in $G(\mathbf{p})$ connecting a vertex u to three other *trivalent* vertices, then $\mathbf{F}(e_\alpha)$, $\mathbf{F}(e_\beta)$, $\mathbf{F}(e_\gamma)$ are three edges in $G(\mathbf{p}')$ connecting $f(u)$ to three other *trivalent* vertices in $G(\mathbf{p}')$. Thus by (4.9.14), the cohomological condition at the trivalent vertex $f(u)$ in $G(\mathbf{p}')$ is

$$\mu(b')_{\mathbf{F}(e_\alpha)} + \mu(b')_{\mathbf{F}(e_\beta)} + \mu(b')_{\mathbf{F}(e_\gamma)} + w_2(\mathbb{X}) = 0 , \quad (4.9.22)$$

where b' is the gerbe for the theory defined by the pants decomposition $\mathbf{p}'$. Arguing along the same lines, we conclude that the cohomological conditions for the theory defined by $\mathbf{p}'$ are given by applying the map

$$\tilde{e}_i \mapsto \mathbf{F}(\tilde{e}_i) , \quad e_\alpha \mapsto \mathbf{F}(e_\alpha) , \quad (4.9.23)$$

to (4.9.14), (4.9.15), (4.9.16) and (4.9.17). Thus, we see that at least in this class of examples, S -dual theories with different Lagrangians require precisely the same cohomological conditions as needed for the existence of P^{phys} .

We emphasize that in general the cohomological conditions for two Lagrangian theories which are not in the same S -duality orbit are different and *not* related by a nice transformation. Consider, for example, the $\mathfrak{a}_1$ class $\mathcal{S}$ theory on a 3 punctured torus $C_{1,3}$. Consider the two pants decompositions of $C_{1,3}$ shown in Figure 4.2 below.

⁵¹The Whitney graph isomorphism theorem implies an equivalence between isomorphisms of graphs and isomorphisms of corresponding line graphs with the following exception: the graphs K_3 (the complete graph on three vertices, a.k.a. the three-vertex triangle) and $K_{1,3}$ (a complete bipartite graph, a.k.a. the four-vertex claw) are non-isomorphic but their line graphs are isomorphic. (See [Har69, Ch. 8] for a discussion of this point.) However, this does not concern us for two reasons: first of all, we require only the forward implication of the theorem which says that *if* two graphs are isomorphic, *then* their corresponding line graphs are isomorphic, and second, the graphs K_3 and $K_{1,3}$ cannot arise from any pants decomposition.

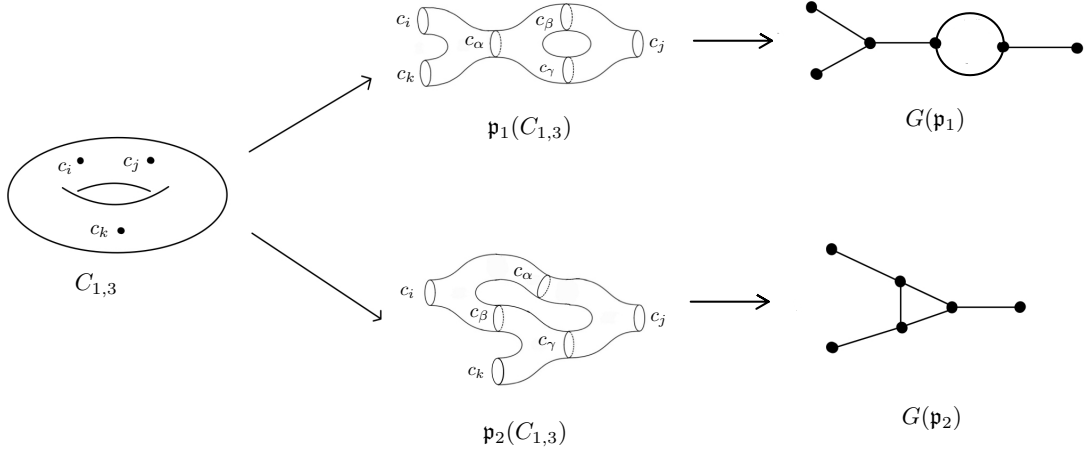


Figure 4.2: Two $\text{MCG}(C_{1,3})$ -inequivalent pants decompositions of $C_{1,3}$ and their associated pants graphs.

The pants graphs are non-isomorphic as can be easily checked using their adjacency matrix and hence their corresponding Lagrangian theories are *not* related by S -duality. Let us now look at the cohomological conditions we get for the two theories using the general prescription described above. There are three pairs of pants in both pants decompositions. With the labeling of cutting curves in Figure 4.2, we obtain the following cohomological conditions for the two theories:

1. For the pants decomposition $\mathfrak{p}_1(C_{1,3})$ we get

$$\begin{aligned}
 c_1(c_i) + \mu(b)_{c_\alpha} + w_2(\mathbb{X}) &= 0 \bmod 2, \\
 c_1(c_k) + \mu(b)_{c_\alpha} + w_2(\mathbb{X}) &= 0 \bmod 2, \\
 \mu(b)_{c_\alpha} + \mu(b)_{c_\beta} + \mu(b)_{c_\gamma} + w_2(\mathbb{X}) &= 0 \bmod 2, \\
 c_1(c_j) + \mu(b)_{c_\beta} + \mu(b)_{c_\gamma} + w_2(\mathbb{X}) &= 0 \bmod 2.
 \end{aligned} \tag{4.9.24}$$

2. For the pants decomposition $\mathfrak{p}_2(C_{1,3})$ we get

$$\begin{aligned}
 c_1(c_i) + \mu(b)_{c_\alpha} + \mu(b)_{c_\beta} + w_2(\mathbb{X}) &= 0 \bmod 2, \\
 c_1(c_k) + \mu(b)_{c_\beta} + \mu(b)_{c_\gamma} + w_2(\mathbb{X}) &= 0 \bmod 2, \\
 c_1(c_j) + \mu(b)_{c_\alpha} + \mu(b)_{c_\gamma} + w_2(\mathbb{X}) &= 0 \bmod 2.
 \end{aligned} \tag{4.9.25}$$

We see that in the second case, we have no restriction on background gerbes since we can always choose a suitable $c_1(c_{i,j,k})$ while in the first case, the background gerbes are restricted. Thus the cohomological conditions for the two theories are different.

To twist these theories, we need to provide C^{tw} and a homomorphism $\varphi^{\text{tw}} : G^{\text{tw}} \rightarrow G^{\text{phys}}$ satisfying (4.2.16). Since these theories are Lagrangian, we simply choose C^{tw} as in (4.6.10) above, and define

$$\begin{aligned} \varphi^{\text{tw}} : G^{\text{tw}} &\longrightarrow G^{\text{phys}} \\ [((u_1, u_2), \mu, g)] &\longmapsto [((u_1, u_2), u_2, \mu, g)]. \end{aligned} \tag{4.9.26}$$

The theories in different S -duality orbits correspond to different topological field theories.

Appendix A

Appendix for Chapter 2

A.1 Codes And Lattices

Linear codes and their associated lattices play a central role in our discussion of the superconformal symmetry in the Monster module. We collect the definitions and some preliminary results about codes and lattices in this section.

A.1.1 Binary Linear Codes

Definition A.1.1. Let $\mathbb{F}_2$ be the field with two elements 0, 1.

1. A *binary linear code* $\mathcal{C}$ of length n is a vector subspace of $\mathbb{F}_2^n$. If the dimension of $\mathcal{C}$ is k then it is called an $[n, k]$ binary code. The elements of the code are called *codewords*.
2. Two binary linear codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be *equivalent* if there exists a permutation matrix P such that $\mathcal{C}_1 P = \mathcal{C}_2$.
3. A $k \times n$ matrix with rows being the basis vectors of $\mathcal{C}$ is called a generator matrix of $\mathcal{C}$.
4. The inner product of codewords $w, w' \in \mathcal{C}$ is defined as

$$w \cdot w' = \sum_{i=1}^n w_i \cdot w'_i \pmod{2}. \quad (\text{A.1.1})$$

where $w = (w_1, \dots, w_n)$, $w' = (w'_1, \dots, w'_n)$. The code *dual* to $\mathcal{C}$ is denoted by $\mathcal{C}^\perp$ and is defined as

$$\mathcal{C}^\perp := \{w \in \mathbb{F}_2^n : w \cdot w' = 0 \ \forall \ w' \in \mathcal{C}\}. \quad (\text{A.1.2})$$

The code is called *self-orthogonal* if $\mathcal{C}^\perp \subseteq \mathcal{C}$ and *self-dual* if $\mathcal{C}^\perp = \mathcal{C}$.

5. The (*Hamming*) *weight* of a codeword w is defined as the number of nonzero components in the codeword and is denoted by $\text{wt}(w)$. The code is called *even* and *doubly even* if the weight of every codeword is divisible by 2 and 4 respectively.

6. The *Hamming distance* $d_H(w, w')$ of two codewords w, w' is defined as the number of places in which the codewords differ. More precisely, $d_H(w, w') = \text{wt}(w - w')$. The (*Hamming*) *distance* of the code is defined

$$d_H(\mathcal{C}) := \min_{\substack{w, w' \in \mathcal{C} \\ w \neq w'}} d_H(w, w'). \quad (\text{A.1.3})$$

With the notation $d_H := d_H(\mathcal{C})$, we call $\mathcal{C}$ an $[n, k, d_H]$ code.

7. The *automorphism group* $\text{Aut}(\mathcal{C})$ of the code $\mathcal{C}$ is a subset of the permutation group S_n which fixes the code as a set.

By standard row echelon form algorithm, it is easy to see that for any $[n, k]$ code $\mathcal{C}$, there is an equivalent code with generator matrix G of the form

$$G = [\mathbb{1}_k, A] \quad (\text{A.1.4})$$

where $\mathbb{1}_k$ is the $k \times k$ identity matrix and A is some $k \times (n - k)$ matrix. The dual code $\mathcal{C}^\perp$ is an $[n, n - k]$ code with generator matrix $[-A^{\text{tr}}, \mathbb{1}_{n-k}]$. In particular, if $\mathcal{C}$ is self-dual then $n - k = k$ and $n = 2k$. We will be interested in $[24, 12]$ codes. The following theorem gives the classification of self-dual, doubly even $[24, 12]$ codes.

Theorem A.1.1. [PS75] *There are exactly 9 inequivalent self-dual, doubly even $[24, 12]$ codes. Up to equivalence, there is a unique $[24, 12, 8]$ code called the Golay code denoted by $\mathcal{G}$.*

Golay Code

The Golay code is of particular interest to us due to its application in the construction of Leech lattice. It is determined by a generator matrix of the form

$$\begin{pmatrix} 1000 & 0000 & 0000 & 1010 & 0011 & 1011 \\ 0100 & 0000 & 0000 & 1101 & 0001 & 1101 \\ 0010 & 0000 & 0000 & 0110 & 1000 & 1111 \\ 0001 & 0000 & 0000 & 1011 & 0100 & 0111 \\ 0000 & 1000 & 0000 & 1101 & 1010 & 0011 \\ 0000 & 0100 & 0000 & 1110 & 1101 & 0001 \\ 0000 & 0010 & 0000 & 0111 & 0110 & 1001 \\ 0000 & 0001 & 0000 & 0011 & 1011 & 0101 \\ 0000 & 0000 & 1000 & 0001 & 1101 & 1011 \\ 0000 & 0000 & 0100 & 1000 & 1110 & 1101 \\ 0000 & 0000 & 0010 & 0100 & 0111 & 0111 \\ 0000 & 0000 & 0001 & 1111 & 1111 & 1110 \end{pmatrix} \quad (\text{A.1.5})$$

The following results can be proved using the standard constructions of the Golay code. All codewords have weights of 0, 8, 12, 16, or 24. Codewords of weight 8 are called *octads* and codewords of weight 12 are called *dodecads*.

Proposition A.1.1. [CS99] *The following is true about the Golay code:*

1. *There are 759 octads and 2576 dodecads in the Golay code.*
2. *Two octads intersect in 0, 2, or 4 coordinates and an octad and a dodecad intersect at 2, 4, or 6 coordinates.*

This proposition also implies that there are 759 weight 16 codewords obtained by adding $\underline{1} = (11 \dots 1) \in \mathcal{G}$ to the octads totalling to $2 + (2 \times 759) + 2576 = 2^{12}$ codewords.

A.1.2 $\mathbb{Z}_4$ -Codes

We now recall the definition and some properties of $\mathbb{Z}_4$ -codes (see [Car05] for details).

Definition A.1.2. Let $\mathbb{Z}_4$ be the ring with elements 0,1,2,3 with modulo 4 addition and multiplication as the operations.

1. A $\mathbb{Z}_4$ -code $\mathcal{C}$ of length n is a submodule of $\mathbb{Z}_4^n$. The elements of the code are called codewords.
2. A matrix whose rows are a minimal spanning set for the code is called a *generator matrix*.
3. The *Euclidean weights* of the elements 0, 1, 2 and $3 = -1$ of $\mathbb{Z}_4$ are ordinary integers 0, 1, 4 and 1, respectively, and the Euclidean weight of a codeword is the sum of the Euclidean weights of its components. The minimum Euclidean weight d_E of $\mathcal{C}$ is the smallest Euclidean weight among all non-zero codewords of $\mathcal{C}$.
4. The (Hamming) weight $\text{wt}(x)$ of a codeword $x \in \mathcal{C}$ is the number of non-zero coordinates of x . The Hamming distance of two codewords is $d_H(x, y) = \text{wt}(x - y)$ and the Hamming distance $d_H(\mathcal{C})$ of the code is defined as before as

$$d_H(\mathcal{C}) = \min_{\substack{x, y \in \mathcal{C} \\ x \neq y}} d_H(x, y). \quad (\text{A.1.6})$$

5. An invertible monomial matrix M over the ring $\mathbb{Z}_4$ is an invertible matrix of the form $M = PD$ where P is a permutation matrix and D is a diagonal matrix with ± 1 on the diagonal. Two codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are *equivalent* if there exists an invertible monomial matrix M such that $\mathcal{C}_1 M = \mathcal{C}_2$ where $\mathcal{C}_1 M = \{xM : x \in \mathcal{C}_1\}$.

6. Self-orthogonality and self-duality of the code is defined as before with the same inner product now calculated modulo 4. Self-dual codes with the property that all Euclidean weights are divisible by 8 are called Type II.

7. The *residue code* $\text{Res}(\mathcal{C})$ and the *torsion code* $\text{Tor}(\mathcal{C})$ are binary linear codes defined by the exact sequence

$$0 \longrightarrow \text{Tor}(\mathcal{C}) \xrightarrow{f} \mathcal{C} \xrightarrow{g} \text{Res}(\mathcal{C}) \longrightarrow 0 \quad (\text{A.1.7})$$

obtained from the componentwise exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0 \quad (\text{A.1.8})$$

where $f : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ maps $0 \rightarrow 0, 1 \rightarrow 2$ and $g : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2$ maps $0, 1, 2, 3 \in \mathbb{Z}_4$ to $0, 1, 0, 1 \in \mathbb{Z}_2$ respectively. Explicitly, the residue and torsion codes are given by

$$\text{Res}(\mathcal{C}) = \{x \pmod{2} : x \in \mathcal{C}\}. \quad (\text{A.1.9})$$

and

$$\text{Tor}(\mathcal{C}) = \{x \pmod{2} : x \in \mathbb{Z}_4^n : 2x \in \mathcal{C}\}. \quad (\text{A.1.10})$$

We have the following theorem:

Theorem A.1.2. [BSBM97] *Let $\mathcal{C}$ be a Type II $\mathbb{Z}_4$ -code of length n and minimum Euclidean weight $d_E(\mathcal{C})$. Then*

$$d_E(\mathcal{C}) \leq 8\lfloor n/24 \rfloor + 8. \quad (\text{A.1.11})$$

Definition A.1.3. If equality holds in Theorem A.1.2, then $\mathcal{C}$ is called an *extremal code*. Such extremal codes will play a role in our explicit constructions of superconformal currents.

There is no known classification of extremal $\mathbb{Z}_4$ codes. In section 2.5.3, we will relate the construction of superconformal currents to certain codes. Three of the codes that will be relevant are described below:

Pseudo Golay codes: One set of $\mathbb{Z}_4$ codes which concern us here are those whose residue code is the Golay code. Following [Rai99], we call such codes *pseudo Golay codes*. There are exactly 13 pseudo Golay codes [Rai99]. The generator matrix for two of these pseudo Golay code is given

below. The rest of the 11 generator matrices can be found in [Rai99].

$$\begin{pmatrix} 1000 & 0000 & 0000 & 1300 & 2111 & 0123 \\ 0100 & 0000 & 0000 & 1210 & 2300 & 3111 \\ 0010 & 0000 & 0000 & 3311 & 2330 & 1200 \\ 0001 & 0000 & 0000 & 0331 & 1233 & 0120 \\ 0000 & 1000 & 0000 & 0033 & 1123 & 3012 \\ 0000 & 0100 & 0000 & 2023 & 3312 & 1123 \\ 0000 & 0010 & 0000 & 1012 & 1231 & 1223 \\ 0000 & 0001 & 0000 & 1311 & 0023 & 0233 \\ 0000 & 0000 & 1000 & 1301 & 3302 & 2130 \\ 0000 & 0000 & 0100 & 0130 & 1330 & 2213 \\ 0000 & 0000 & 0010 & 1223 & 2033 & 3332 \\ 0000 & 0000 & 0001 & 2102 & 3003 & 1111 \end{pmatrix} \begin{pmatrix} 1000 & 0000 & 0000 & 2202 & 2313 & 3311 \\ 0100 & 0000 & 0000 & 2233 & 3301 & 1001 \\ 0010 & 0000 & 0000 & 1010 & 3203 & 3312 \\ 0001 & 0000 & 0000 & 0101 & 3220 & 3331 \\ 0000 & 1000 & 0000 & 3223 & 3013 & 2121 \\ 0000 & 0100 & 0000 & 1113 & 3210 & 1020 \\ 0000 & 0010 & 0000 & 1302 & 3130 & 2310 \\ 0000 & 0001 & 0000 & 2112 & 3231 & 0211 \\ 0000 & 0000 & 1000 & 3121 & 2323 & 1320 \\ 0000 & 0000 & 0100 & 1231 & 0113 & 2212 \\ 0000 & 0000 & 0010 & 3332 & 0102 & 1013 \\ 0000 & 0000 & 0001 & 2311 & 0132 & 2121 \end{pmatrix}$$

Quaternary Golay code: Another important example of a $\mathbb{Z}_4$ code is a code denoted $\widehat{\mathcal{Q}}_4$ in [BSC06]. It has the property that $\text{Res}(\widehat{\mathcal{Q}}_4) \cong \text{Tor}(\widehat{\mathcal{Q}}_4) \cong \mathcal{G}$. Thus it fits in an exact sequence

$$0 \longrightarrow \mathcal{G} \xrightarrow{f} \widehat{\mathcal{Q}}_4 \xrightarrow{g} \mathcal{G} \longrightarrow 0. \quad (\text{A.1.12})$$

Let $n_0(x), n_{\pm 1}(x), n_2(x)$ denote the number of entries in a codeword $x \in \widehat{\mathcal{Q}}_4$ with $0, \pm 1, 2 \in \mathbb{Z}/4\mathbb{Z}$, respectively. Since $\text{Res}(\widehat{\mathcal{Q}}_4) = \mathcal{G}$, $n_{+1}(x) + n_{-1}(x) = 0, 8, 12, 16, 24$ for every $x \in \widehat{\mathcal{Q}}_4$. The number of codewords of each type is listed in Table A.1.

Remark A.1.1. The codewords in $\widehat{\mathcal{Q}}_4$ with $n_{\pm 1} = 12$ have ± 1 s supported on a dodecad in the Golay code. The reason is that the mod 2 reduction of $\widehat{\mathcal{Q}}_4$ is the Golay code and 0, 2 reduce to 0 mod 2.

In Theorem 2.5.6 of Subsection 2.5.3 the following result will be useful in the construction of a nontrivial supercurrent, see Subsection 2.5.3 for the details of the construction.

Lemma A.1.1. *Let $w \in \mathcal{G}$ be a dodecad and $\widehat{\mathcal{Q}}_w \subset \widehat{\mathcal{Q}}_4$ be the preimage of the mod 2 reduction map from $\widehat{\mathcal{Q}}_4$ to $\mathcal{G}$. Then for any choice of three out of the 12 remaining positions, there is a codeword $x \in \widehat{\mathcal{Q}}_w$ with 2 at those three positions and 0 at the remaining 9 positions.*

Proof. Let $x \in \widehat{\mathcal{Q}}_w$. Then ± 1 s in x are supported at w . At the other 12 positions, we can either have 0 or 2. From Table A.1, we see that the number of codewords with 12 ± 1 s, 9 0s and 3 2s is $220 \cdot 2576$. Moreover, since there are exactly 2576 dodecads in the Golay code, from Table A.1 we see that, if we fix the position of the 12 ± 1 s at w then there are a total of 220 codewords in $\widehat{\mathcal{Q}}_4$ with

9 0s and 3 2s at the other 12 positions. Since there are exactly $\binom{12}{3} = 220$ choices for the positions of 2, there is a codeword in $\widehat{\mathcal{Q}}_w$ with ± 1 s at w and every possible position of the three 2s. $\square$

$n_{\pm 1} = 0$	$n_{\pm 1} = 8$	$n_{\pm 1} = 12$	$n_{\pm 1} = 16$	$n_{\pm 1} = 24$
$(0; 0; 24) = 1$	$(2; 8; 14) = 2 \cdot 8 \cdot 759$	$(1; 12; 11) = 2 \cdot 12 \cdot 2576$	$(0; 16; 8) = 2 \cdot 16 \cdot 759$	$(0; 24; 0) = 2^{12}$
$(24; 0; 0) = 1$	$(14; 8; 2) = 2 \cdot 8 \cdot 759$	$(11; 12; 1) = 2 \cdot 12 \cdot 2576$	$(8; 16; 0) = 2 \cdot 16 \cdot 759$	Total 2^{12}
$(8; 0; 16) = 759$	$(4; 8; 12) = 2 \cdot 112 \cdot 759$	$(3; 12; 9) = 2 \cdot 220 \cdot 2576$	$(2; 16; 6) = 2 \cdot 448 \cdot 759$	
$(16; 0; 8) = 759$	$(12; 8; 4) = 2 \cdot 112 \cdot 759$	$(9; 12; 3) = 2 \cdot 220 \cdot 2576$	$(2; 16; 6) = 2 \cdot 448 \cdot 759$	
$(12; 0; 12) = 2576$	$(6; 8; 12) = 2 \cdot 504 \cdot 759$	$(5; 12; 7) = 2 \cdot 792 \cdot 2576$	$(4; 16; 4) = 2240 \cdot 759$	
Total 2^{12}	$(12; 8; 6) = 2 \cdot 504 \cdot 759$ $(8; 8; 8) = 1600 \cdot 759$ Total $759 \cdot 2^{12}$	$(7; 12; 5) = 2 \cdot 792 \cdot 2576$ Total $2576 \cdot 2^{12}$	Total $759 \cdot 2^{12}$	

Table A.1: Number of codewords of each type in $\hat{\mathcal{Q}}_4$. The entries $(a; b; c) = N$ in each column means that there are N codewords of type $(n_0; n_{\pm 1}; n_2)$. The factor of 2 in the first few rows of columns 2, 3 and 4 arises from the fact that in those cases for every codeword with $n_{+1}(x) = a, n_{-1}(x) = b$ there is another codeword with $n_{+1}(x) = b, n_{-1}(x) = a$.

Harada-Kitazume code: One other $\mathbb{Z}_4$ code that we will be interested in was constructed in [HK00]. The generator matrix for the code is shown below.

$$\begin{pmatrix} 2222 & 0000 & 0000 & 0000 & 0000 & 0000 \\ 0022 & 2200 & 0000 & 0000 & 0000 & 0000 \\ 0000 & 0022 & 2020 & 0000 & 0000 & 0000 \\ 0000 & 0000 & 0202 & 2020 & 0000 & 0000 \\ 0000 & 0000 & 0000 & 0202 & 2002 & 0000 \\ 2020 & 2020 & 0000 & 0000 & 0000 & 0000 \\ 0000 & 0220 & 2200 & 0000 & 0000 & 0000 \\ 0000 & 0000 & 2002 & 2002 & 0000 & 0000 \\ 0000 & 0000 & 0000 & 0022 & 2020 & 0000 \\ 2000 & 2000 & 2000 & 2000 & 2000 & 2000 \\ 1111 & 1111 & 2000 & 2000 & 0000 & 0000 \\ 2000 & 1111 & 1111 & 0000 & 2000 & 0000 \\ 0000 & 0000 & 1111 & 1111 & 2000 & 2000 \\ 2000 & 0000 & 2000 & 1111 & 1111 & 0000 \\ 2000 & 2000 & 0000 & 0000 & 1111 & 1111 \\ 3012 & 1010 & 1001 & 1001 & 1100 & 1100 \\ 3201 & 1001 & 1100 & 1100 & 1010 & 1010 \end{pmatrix}$$

It is an extremal Type II code. This code also gives rise to a supercurrent, see Section 2.5.3, specifically Remark 2.5.3.

A.1.3 Lattice Construction From Codes

Codes can be used to construct lattices. In this section we describe the construction of lattices from binary and $\mathbb{Z}_4$ codes. We begin by defining Euclidean lattices.

Definition A.1.4. Let $\mathbb{R}^n$ be the n -dimensional Euclidean space with inner product $x \cdot y = x_1y_1 + \dots + x_ny_n$ for $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$.

1. An n -dimensional *Euclidean lattice* Λ is a free $\mathbb{Z}$ -module spanned by n vectors $e_j \in \mathbb{R}^n, 1 \leq j \leq n$ linearly independent in $\mathbb{R}^n$, i.e.

$$\Lambda = \left\{ \sum_{j=1}^n n_j e_j : n_j \in \mathbb{Z} \right\}.$$

2. The *norm* of a vector $x \in \Lambda$ is $\|x\| = \sqrt{x^2} = \sqrt{x \cdot x}$.

3. The *dual lattice* Λ^* is defined as $\Lambda^* = \{y : x \cdot y \in \mathbb{Z} \ \forall x \in \Lambda\}$ (which is obviously a lattice). The lattice Λ is said to be *unimodular* if $\|\Lambda\|^2 \equiv \det(e_i \cdot e_j) = 1$.
4. The lattice Λ is said to be *integral* if $\Lambda \subseteq \Lambda^*$, i.e. $x \cdot y \in \mathbb{Z}$ for all $x, y \in \Lambda$ and *self-dual* if $\Lambda = \Lambda^*$.
5. The lattice Λ is said to be *even* if $\|x\|^2$ is even for all $x \in \Lambda$. The set $\Lambda(\sqrt{2}) \equiv \{x \in \Lambda : \|x\|^2 = 2\}$ is called the *root system*¹ of Λ .
6. Two n -dimensional lattices Λ_1, Λ_2 are said to be isomorphic if there exists a norm-preserving $\mathbb{Z}$ -module isomorphism between them.
7. An automorphism of the lattice is a rotation $R \in SO(n)$ which fixes the lattice. The group of all automorphisms (the group operation being composition) is called the automorphism group of Λ and denoted by $\text{Aut}(\Lambda)$.

It is easy to see that a lattice Λ is self-dual if and only if it is integral and unimodular since $\|\Lambda^*\| = \|\Lambda\|^{-1}$. Moreover it is a well known result that the dimension of an even self-dual lattice has to be a multiple of 8. Even self-dual lattices in 24 dimensions have been completely classified up to isomorphism. There are 24 inequivalent such lattices called the *Neimeier lattices* classified by their root system. There is a unique lattice with empty root system called the Leech lattice. We denote the Leech lattice by Λ_L . An explicit generator matrix for the Leech lattice is given below: the rows

¹In general, a root is a primitive vector that generates a reflection isometry of the lattice. For even lattices, the roots can have different norms. For even, unimodular lattices, the norm-2 vectors are all the roots.

of the matrix below generates the Leech lattice under integer linear combinations.

$$\frac{1}{2\sqrt{2}} \begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ -3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

We will give a more conceptual description of the Leech lattice in terms of the Golay code in the next section.

Lattice construction from binary codes: We can construct Euclidean lattice from a code. For an r -dimensional binary code $\mathcal{C}$, define

$$\Lambda(\mathcal{C}) := \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}^r \quad (\text{A.1.13})$$

where $\mathcal{C}$ is now considered as a subset of $\mathbb{Z}^r$ via a map such as²

$$\begin{aligned}\mathbb{F}_2 &\hookrightarrow \mathbb{Z} \\ 0 &\mapsto 0, \quad 1 \mapsto 1.\end{aligned}\tag{A.1.14}$$

Proposition A.1.2. *The lattice $\Lambda(\mathcal{C})$ is integral if and only if $\mathcal{C} \subset \mathcal{C}^\perp$ and is even if and only if $\mathcal{C}$ is doubly-even. It is self-dual if and only if $\mathcal{C}$ is self-dual. Unimodularity of $\Lambda(\mathcal{C})$ is equivalent to $\dim \mathcal{C} = \frac{1}{2}r$.*

We can also construct another lattice by “twisting” this lattice as follows: define the subspaces $\Lambda_a(\mathcal{C})$, $a = 0, 1, 2, 3$ as follows:

$$\begin{aligned}\Lambda_0(\mathcal{C}) &:= \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}_+^r, \\ \Lambda_1(\mathcal{C}) &:= \frac{1}{\sqrt{2}}\mathcal{C} + \sqrt{2}\mathbb{Z}_-^r, \\ \Lambda_2(\mathcal{C}) &:= \frac{1}{\sqrt{2}}\mathcal{C} + \frac{1}{2\sqrt{2}}\mathbf{1} + \sqrt{2}\mathbb{Z}_{(-)^{r+1}}^r, \\ \Lambda_3(\mathcal{C}) &:= \frac{1}{\sqrt{2}}\mathcal{C} + \frac{1}{2\sqrt{2}}\mathbf{1} + \sqrt{2}\mathbb{Z}_{(-)^r}^r,\end{aligned}\tag{A.1.15}$$

where

$$\begin{aligned}\mathbb{Z}_+^r &:= \{\mathbf{x} \in \mathbb{Z}^r : \mathbf{x}^2 \in 2\mathbb{Z}\}, \\ \mathbb{Z}_-^r &:= \{\mathbf{x} \in \mathbb{Z}^r : \mathbf{x}^2 \in 2\mathbb{Z} + 1\},\end{aligned}\tag{A.1.16}$$

and $\mathbf{1}$ is a vector with all components 1. Clearly $\Lambda(\mathcal{C}) = \Lambda_0(\mathcal{C}) \cup \Lambda_1(\mathcal{C})$. We define the twisted lattice by

$$\tilde{\Lambda}(\mathcal{C}) := \Lambda_0(\mathcal{C}) \cup \Lambda_3(\mathcal{C}).\tag{A.1.17}$$

The twisted construction is particularly important for our case because of the following theorem.

Theorem A.1.3. [CS99] *Let $\mathcal{G}$ be the binary Golay code. Then*

$$\tilde{\Lambda}(\mathcal{G}) \cong \Lambda_L.\tag{A.1.18}$$

This enables us to list all possible vectors of norm 4, 6, 8 in the Leech lattice. This is listed in [CS99, Table 4.13, Chapter 4]. Finally one can construct lattices from $\mathbb{Z}_4$ codes as described in the theorem:

Theorem A.1.4. [BSC06, Theorem 4.1] *Let $\mathcal{C}$ be a self-dual $\mathbb{Z}_4$ -code of length n with minimum Euclidean weight d_E . Then the lattice*

$$A(\mathcal{C}) = \frac{1}{2} \{\lambda \in \mathbb{Z}^n : \lambda \equiv x \pmod{4} \text{ for some } x \in \mathcal{C}\}\tag{A.1.19}$$

is an n -dimensional unimodular lattice with the minimum norm squared $\min\{4, d_E/4\}$. Moreover, if $\mathcal{C}$ is Type II, then the lattice $A(\mathcal{C})$ is even and unimodular.

²This map is just an injective function. In particular it is not a group or ring homomorphism.

A.2 Locality Of The Twisted Vertex Operators

We will closely follow [DGM90, Section 7,8] to establish the locality relation (2.5.40). The following lemma is proved similar to the proof of (2.4.21). We record the proof for completeness.

Lemma A.2.1. *For any untwisted state ψ , we have*

$$w^{L_0^T} V_T(\psi, z) w^{-L_0^T} = w^{h_\psi} V_T(\psi, zw). \quad (\text{A.2.1})$$

Proof. Using (2.2.18), we have

$$w^{L_0^T} V_T(\psi, z) w^{-L_0^T} \chi = w^{-h_\chi} w^{L_0^T} e^{-zL_{-1}^T} W(\chi, -z) \psi,$$

where χ is a general twisted quasi-primary state. We now use the Baker-Campbell-Hausdorff formula (2.4.20) to write

$$w^{L_0^T} e^{zL_{-1}^T} = e^{L_0^T \ln w} e^{zL_{-1}^T}.$$

Using the fact that $[L_0^T, L_{-1}^T] = L_{-1}^T$, we see that the above theorem implies

$$e^{L_0^T \ln w} e^{zL_{-1}^T} = e^{\exp(\ln w) L_{-1}^T} w^{L_0^T}.$$

Thus we get

$$w^{L_0^T} e^{zL_{-1}^T} = e^{zwL_{-1}^T} w^{L_0^T}. \quad (\text{A.2.2})$$

This implies

$$w^{L_0^T} V_T(\psi, z) w^{-L_0^T} \chi = w^{-h_\chi} e^{-zwL_{-1}^T} w^{L_0^T} W(\chi, -z) \psi. \quad (\text{A.2.3})$$

Next using [DGM94, eq. 2.18] for the twisted vertex operators $\tilde{V}(\chi, z)$, we get

$$w^{L_0^T} W(\chi, z) w^{-L_0^T} = w^{h_\chi} W(\chi, zw) \quad (\text{A.2.4})$$

Thus we get

$$w^{L_0^T} V_T(\psi, z) w^{-L_0^T} \chi = w^{h_\psi} e^{-zwL_{-1}^T} W(\chi, -zw) \psi = w^{h_\psi} V_T(\psi, zw) \chi. \quad (\text{A.2.5})$$

□

A.2.1 Locality Relation $\overline{W}(\chi, z) W(\chi', w) = (-1)^{|\chi||\chi'|} \overline{W}(\chi', w) W(\chi, z)$

For quasi-primary states $\chi, \chi' \in \mathcal{H}_T^-(\Lambda_L)$, following the steps in [DGM90, Section 7], we see that the required locality relation is equivalent to proving

$$\langle \overline{\chi}' | V_T(\phi, -z) | \chi \rangle z^{2h_\chi} = \left\langle \overline{\chi} \left| V_T \left(e^{L_1/z} \phi, z \right) \right| \chi' \right\rangle z^{2h_{\chi'}}, \quad (\text{A.2.6})$$

where ϕ is an arbitrary untwisted state. Now observe that

$$\begin{aligned} V_T \left(e^{L_1/z} \phi, z \right)^\dagger &= \left(V_T \left(e^{L_1/z} z^{2L_0} \phi, z \right) z^{-2h_\phi} \right)^\dagger \\ &= V_T \left(e^{L_1/z} z^{2L_0} \phi, z \right)^\dagger (z^*)^{-2h_\phi} \\ &= V_T \left(\bar{\phi}, 1/z^* \right) (z^*)^{-2h_\phi} \end{aligned}$$

which implies that

$$V_T \left(\bar{\phi}, 1/z^* \right)^\dagger z^{-2h_\phi} = V_T \left(e^{L_1/z} \phi, z \right).$$

So the right hand side of (A.2.6) becomes (see eq. (2.4.13) for notation)

$$\begin{aligned} \left\langle \bar{\chi} \left| V_T \left(\bar{\phi}, 1/z^* \right)^\dagger \right| \chi' \right\rangle z^{2h_{\chi'} - 2h_\phi} &= \left\langle \chi' \left| V_T \left(\bar{\phi}, 1/z^* \right) \right| \bar{\chi} \right\rangle^* z^{2h_{\chi'} - 2h_\phi} \\ &= \left(\left\langle \chi' \left| (z^{-L_0})^\dagger (z^*)^{L_0} V_T \left(\bar{\phi}, 1/z^* \right) (z^*)^{-L_0} \right| \bar{\chi} \right\rangle (z^*)^{h_\chi} \right)^* z^{2h_{\chi'} - 2h_\phi} \\ &= \left((z^*)^{-h_{\chi'}} \left\langle \chi' \left| (z^*)^{L_0} V_T \left(\bar{\phi}, 1/z^* \right) (z^*)^{-L_0} \right| \bar{\chi} \right\rangle (z^*)^{h_\chi} \right)^* z^{2h_{\chi'} - 2h_\phi} \\ &= \left\langle \chi' \left| V_T \left(\bar{\phi}, 1 \right) \right| \bar{\chi} \right\rangle^* z^{h_\chi + h_{\chi'} - h_\phi} \\ &= (f_{\bar{\chi}' \bar{\phi} \bar{\chi}})^* z^{h_\chi + h_{\chi'} - h_\phi}, \end{aligned}$$

the left hand side of (A.2.6) becomes

$$\begin{aligned} \left\langle \bar{\chi}' \left| V_T(\phi, -z) \right| \chi \right\rangle z^{2h_\chi} &= \left\langle \bar{\chi}' \left| (-1/z)^{-L_0} (-1/z)^{L_0} V_T(\phi, -z) (-1/z)^{-L_0} (-1/z)^{L_0} \right| \chi \right\rangle z^{2h_\chi} \\ &= (-1/z)^{-h_{\chi'}} \left\langle \bar{\chi}' \left| V_T(\phi, 1) \right| \chi \right\rangle (-1/z)^{h_\phi} (-1/z)^{h_\chi} z^{2h_\chi} \\ &= (-1)^{2h_\chi} (-z)^{h_{\chi'} + h_\chi - h_\phi} f_{\chi' \phi \chi} \\ &= (-1)^{h_{\chi'} + h_\chi + h_\phi} z^{h_\chi + h_{\chi'} - h_\phi} f_{\chi' \phi \chi}, \end{aligned} \tag{A.2.7}$$

where we have used $(-z)^{-k} z^{2k} = (-z)^k$ for $k \in \frac{1}{2}\mathbb{Z}$ which is allowed³ in the principal branch of square root. So the locality relation is equivalent to proving

$$(f_{\bar{\chi}' \bar{\phi} \bar{\chi}})^* = (-1)^{|\chi||\chi'|} (-1)^{h_{\chi'} + h_\chi + h_\phi} f_{\chi' \phi \chi}. \tag{A.2.8}$$

For $\chi, \chi' \in \mathcal{H}_T^+(\Lambda)$ (the sign is + in this case), (A.2.8) is proved in [DGM90]. We prove it for $\chi, \chi' \in \mathcal{H}_T^-(\Lambda)$. Following the calculations on [DGM90, Page 547], we have

$$\begin{aligned} \langle \chi | V_T(\phi, 1) | \chi' \rangle^* &= \langle M(\chi) | V_T(\phi, 1) | M(\chi') \rangle \\ &= (-1)^{h_\phi} \langle M(\chi) | \theta V_T(\bar{\phi}, 1) \theta | M(\chi') \rangle \\ &= (-1)^{h_\phi} \left\langle M(\chi) \left| \theta e^{i\pi L_0^T} e^{-i\pi L_0^T} V_T(\bar{\phi}, 1) e^{i\pi L_0^T} e^{-i\pi L_0^T} \theta \right| M(\chi') \right\rangle \\ &= (-1)^{h_\phi} \left\langle \bar{\chi} \left| e^{-i\pi L_0^T} V_T(\bar{\phi}, 1) e^{i\pi L_0^T} \right| \bar{\chi}' \right\rangle \\ &= (-1)^{h_\phi} e^{-i\pi h_\chi} e^{i\pi h_{\chi'}} \langle \bar{\chi} | V_T(\bar{\phi}, 1) | \bar{\chi}' \rangle \\ &= -(-1)^{h_\chi + h_\phi + h_{\chi'}} \langle \bar{\chi} | V_T(\bar{\phi}, 1) | \bar{\chi}' \rangle, \end{aligned}$$

³Since if z is in the upper half-plane then $-z$ is in the lower half-plane and vice-versa.

where we used the fact that $h_\chi \in \mathbb{Z} + \frac{1}{2}$. In the last step we used the definition of the conjugation operation as in (2.5.41). Thus we have established the locality relation.

A.2.2 Locality Relation $W(\chi, z)\overline{W}(\chi', w) = (-1)^{|\chi||\chi'|}W(\chi', w)\overline{W}(\chi, z)$

This is the hard one! We begin by proving a lemma.

Lemma A.2.2. *We can choose a zero-momentum untwisted state ψ such that $\langle \mu | V_T(\psi, w)$ and equivalently $V_T(\psi, w) | \varrho \rangle$ is an arbitrary twisted state.*

Proof. We take

$$\psi = e^{(1/w)L_1} w^{2L_0} \bar{\psi}', \quad w = 1/\tilde{w}^*.$$

By Remark 2.2.3, we have

$$V_T(\psi, w) = V_T(\psi', \tilde{w})^\dagger$$

where we have defined

$$\psi' = e^{-\Delta(\tilde{w})} \tilde{\psi}$$

where

$$\tilde{\psi} = \left(\prod_{a=1}^M a_{-n_a}^{j_a} \right) |0\rangle$$

and

$$\Delta(z) = \frac{1}{2} \sum_{\substack{n, m \geq 0 \\ m+n > 0}} \binom{-\frac{1}{2}}{m} \binom{\frac{1}{2}}{n} \frac{(-z)^{-m-n}}{m+n} a_m \cdot a_n. \quad (\text{A.2.9})$$

We then have

$$V_T(\psi, w) = V_T^0(\tilde{\psi}, \tilde{w})^\dagger$$

We Laurent expand the vertex operator

$$V_T^0(\tilde{\psi}, \tilde{w}) = \sum_n (V_T^0)_n(\tilde{\psi}) \tilde{w}^{-n-h_{\tilde{\psi}}}$$

We see that the mode $(V_T^0)_{-h_{\tilde{\psi}}+M/2}$ creates twisted states containing M oscillators with conformal weight $r/16 + h_{\tilde{\psi}} - M/2$ by acting on twisted ground states. Next observe that the number of linearly independent states of conformal weight $r/16 + h_{\tilde{\psi}} - M/2$ obtained by acting M oscillators on a twisted ground state is the number of ways we can write $h_{\tilde{\psi}} - M/2$ as sum of M positive strictly half-integers. Let this number be $p_M^{\mathbb{Z}+\frac{1}{2}}(h_{\tilde{\psi}} - M/2)$. On the other hand the number of linearly independent untwisted states with conformal weight $h_{\tilde{\psi}}$ obtained by acting M oscillators on $|0\rangle$ is the number of ways we can write $h_{\tilde{\psi}}$ as sum of M positive integers. Let this number be $p_M^{\mathbb{Z}}(h_{\tilde{\psi}})$. We want to show that

$$p_M^{\mathbb{Z}+\frac{1}{2}}(h_{\tilde{\psi}} - M/2) = p_M^{\mathbb{Z}}(h_{\tilde{\psi}}). \quad (\text{A.2.10})$$

Indeed if

$$h_{\tilde{\psi}} = \sum_{a=1}^M n_a, \quad n_a \in \mathbb{N} \quad (\text{A.2.11})$$

then we get a partition of $h_{\tilde{\psi}} - M/2$ into positive strictly half-integers: $p_M^{\mathbb{Z}+\frac{1}{2}}(h_{\tilde{\psi}} - M/2)$ since

$$h_{\tilde{\psi}} - \frac{M}{2} = \sum_{a=1}^M \left(n_a - \frac{1}{2} \right). \quad (\text{A.2.12})$$

Thus

$$p_M^{\mathbb{Z}+\frac{1}{2}}(h_{\tilde{\psi}} - M/2) \geq p_M^{\mathbb{Z}}(h_{\tilde{\psi}}). \quad (\text{A.2.13})$$

Similarly

$$p_M^{\mathbb{Z}}(h_{\tilde{\psi}}) \geq p_M^{\mathbb{Z}+\frac{1}{2}}(h_{\tilde{\psi}} - M/2). \quad (\text{A.2.14})$$

Thus we can get any twisted state by acting with suitable linear combinations of the modes of $V_T^0(\tilde{\psi}, \tilde{w})$. $\square$

Lemma A.2.3. *The relation $\langle \mu | W(\chi, z) V(\psi, w) \overline{W}(\varphi, \zeta) | \varrho \rangle = -\langle \mu | W(\varphi, \zeta) V(\psi, w) \overline{W}(\chi, z) | \varrho \rangle$ for $\mu, \chi, \varphi, \varrho$ twisted ground states is equivalent to the locality relation $W(\chi, z) \overline{W}(\varphi, w) = -W(\varphi, w) \overline{W}(\chi, z)$ for twisted ground states χ, φ .*

Proof. Using the locality relation of $V_T W$ and $V \overline{W}$, we have that

$$\langle \mu | W(\chi, z) V(\psi, w) \overline{W}(\varphi, \zeta) | \varrho \rangle = \langle \mu | V_T(\psi, w) W(\chi, z) \overline{W}(\varphi, \zeta) | \varrho \rangle.$$

But since $\langle \mu | V_T(\psi, w)$ can be taken to be a general twisted state, we see that

$$\begin{aligned} \langle \mu | W(\chi, z) V(\psi, w) \overline{W}(\varphi, \zeta) | \varrho \rangle &= -\langle \mu | W(\varphi, \zeta) V(\psi, w) \overline{W}(\chi, z) | \varrho \rangle \\ \implies \langle \mu | V_T(\psi, w) W(\chi, z) \overline{W}(\varphi, \zeta) | \varrho \rangle &= -\langle \mu | V_T(\psi, w) W(\varphi, \zeta) \overline{W}(\chi, z) | \varrho \rangle \\ \implies W(\chi, z) \overline{W}(\varphi, \zeta) | \varrho \rangle &= -W(\varphi, \zeta) \overline{W}(\chi, z) | \varrho \rangle \end{aligned}$$

Next, for an arbitrary untwisted state ψ , we have from above

$$\begin{aligned} V_T(\psi, w) W(\chi, z) \overline{W}(\varphi, \zeta) | \varrho \rangle &= -V_T(\psi, w) W(\varphi, \zeta) \overline{W}(\chi, z) | \varrho \rangle \\ \implies W(\chi, z) \overline{W}(\varphi, \zeta) V_T(\psi, w) | \varrho \rangle &= -W(\varphi, \zeta) \overline{W}(\chi, z) V_T(\psi, w) | \varrho \rangle \end{aligned}$$

Since $V_T(\psi, w) | \varrho \rangle$ can be taken to be general twisted state, we get the required statement. Here χ and φ are still twisted ground states. $\square$

Theorem A.2.1. *For twisted ground states χ and φ , the locality relation*

$$W(\chi, z) \overline{W}(\varphi, w) = -W(\varphi, w) \overline{W}(\chi, z) \quad (\text{A.2.15})$$

holds.

Proof. In view of Lemma A.2.3 and Lemma A.2.2, we prove that

$$\langle \mu | W(\chi, z) V(\psi, w) \overline{W}(\varphi, \zeta) | \varrho \rangle = - \langle \mu | W(\varphi, \zeta) V(\psi, w) \overline{W}(\chi, z) | \varrho \rangle, \quad (\text{A.2.16})$$

for twisted ground states $\mu, \chi, \varphi, \varrho$ and a general zero momentum untwisted state ψ . Then by Lemma A.2.3, the required locality relation follows. We will closely follow [DGM90, Section 9] for this proof. We begin by simplifying the left hand side. From [DGM90, eq. 4.2] we have

$$W(\chi, z) \psi = e^{zL_1^T} \sum_{\lambda \in \Lambda_L} \langle \lambda | : e^{B(z)} e^{A(z)} : | \psi \rangle e_\lambda \chi \quad (\text{A.2.17})$$

where $A(z)$ is as in (2.3.45) and

$$B(z) := B_+(z) + B_-(z) \quad (\text{A.2.18})$$

with

$$B_\pm(z) := - \sum_{\substack{n \geq 0 \\ s \in (\mathbb{Z} + \frac{1}{2})_\pm}} \binom{-s}{n} \frac{(-z)^{-n-s}}{s} a_n \cdot c_s, \quad (\text{A.2.19})$$

where $(\mathbb{Z} + \frac{1}{2})_\pm$ denotes positive and negative half-integers respectively. Note that $\langle \lambda | : e^{B(z)} e^{A(z)} : | \psi \rangle$ is an operator on the twisted Hilbert space, and that it is normal ordered and acting on a twisted ground state. Since c_s annihilates twisted ground states for $s > 0$ we see that normal ordering gives

$$W(\chi, z) \psi = e^{zL_1^T} \sum_{\lambda \in \Lambda_L} \langle \lambda | : e^{B_-(z)} e^{A(z)} : | \psi \rangle e_\lambda \chi. \quad (\text{A.2.20})$$

Next by (2.2.25) we have

$$\overline{W}(\varphi, \zeta) | \varrho \rangle = \zeta^{-2h_\varphi} W(\overline{\varphi}, 1/\zeta^*)^\dagger | \varrho \rangle \quad (\text{A.2.21})$$

where we used the fact that L_1^T annihilates the twisted ground state. Replacing ψ by $V(\psi, w) \overline{W}(\varphi, \zeta) | \varrho \rangle$ in (A.2.17) we obtain

$$\begin{aligned} \langle \mu | W(\chi, z) V(\psi, w) \overline{W}(\varphi, \zeta) | \varrho \rangle &= \sum_{\lambda \in \Lambda_L} \left\langle \mu \left| e^{zL_1^T} \left\langle \lambda \left| e^{B_-(z)} e^{A(z)} V(\psi, w) \overline{W}(\varphi, \zeta) \right| \varrho \right\rangle e_\lambda \right| \chi \right\rangle \\ &= \sum_{\lambda \in \Lambda_L} \left\langle \lambda \left| \left\langle e^{B_-(z)} e^{zL_1^T} \mu \left| e_\lambda \right| \chi \right\rangle e^{A(z)} V(\psi, w) W(\overline{\varphi}, 1/\zeta^*)^\dagger \right| \varrho \right\rangle \zeta^{-2h_\varphi} \\ &= \sum_{\lambda \in \Lambda_L} \langle \mu | e_\lambda | \chi \rangle \left\langle \lambda \left| e^{A(z)} V(\psi, w) W(\overline{\varphi}, 1/\zeta^*)^\dagger \right| \varrho \right\rangle \zeta^{-2h_\varphi}, \end{aligned} \quad (\text{A.2.22})$$

where we used the fact that $B_-(z)^\dagger$ and L_1^T annihilates the twisted ground state μ . Again using (A.2.17) we have

$$\begin{aligned} W(\overline{\varphi}, 1/\zeta^*) \left(e^{A(z)} V(\psi, w) \right)^\dagger | \lambda \rangle &= e^{L_1^T / \zeta^*} \sum_{\lambda' \in \Lambda_L} \left\langle \lambda' \left| e^{B_-(1/\zeta^*)} e^{A(1/\zeta^*)} \left(e^{A(z)} V(\psi, w) \right)^\dagger \right| \lambda \right\rangle e_{\lambda'} | \overline{\varphi} \rangle \\ &= e^{L_1^T / \zeta^*} \left\langle \lambda \left| e^{B_-(1/\zeta^*)} e^{A(1/\zeta^*)} \left(e^{A(z)} V(\psi, w) \right)^\dagger \right| \lambda \right\rangle e_\lambda | \overline{\varphi} \rangle \end{aligned} \quad (\text{A.2.23})$$

where we used the fact that $\langle \lambda | \mathcal{O} | \lambda' \rangle = 0$ if $\lambda \neq \lambda'$ for any operator $\mathcal{O}$ which does not contain $e^{q(\lambda' - \lambda)}$ (see (2.3.11)). Thus we have

$$\left\langle \lambda \left| e^{A(z)} V(\psi, w) W(\bar{\varphi}, 1/\zeta^*)^\dagger \right| \lambda \right\rangle = \left\langle \lambda \left| e^{A(z)} V(\psi, w) e^{\bar{A}(\zeta)} e^{B_-(1/\zeta^*)^\dagger} \right| \lambda \right\rangle \langle \bar{\varphi} | e_\lambda^\dagger e^{L_1^T / \zeta^*}, \quad (\text{A.2.24})$$

where

$$\bar{A}(z) = A(1/z^*)^\dagger. \quad (\text{A.2.25})$$

Substituting eq. (A.2.24) into eq. (A.2.22) and using the fact that L_1^T and $B_-(1/\zeta^*)^\dagger$ annihilates twisted ground states, we get in the region $|z| > |w| > |\zeta|$

$$\langle \mu | W(\chi, z) V(\psi, w) \bar{W}(\varphi, \zeta) | \varrho \rangle = \sum_{\lambda \in \Lambda_L} \langle \mu | e_\lambda | \chi \rangle \left\langle \lambda \left| e^{A(z)} V(\psi, w) e^{\bar{A}(\zeta)} \right| \lambda \right\rangle \langle \bar{\varphi} | e_\lambda^\dagger | \varrho \rangle \zeta^{-2h_\varphi}. \quad (\text{A.2.26})$$

Let

$$\Delta(-z) = \frac{1}{2} \sum_{\substack{n, m \geq 0 \\ m+n > 0}} \binom{-\frac{1}{2}}{m} \binom{\frac{1}{2}}{n} \frac{(-z)^{-m-n}}{m+n} a_m \cdot a_n. \quad (\text{A.2.27})$$

Then we have

$$\begin{aligned} \left\langle \lambda \left| e^{A(z)} V(\psi, w) e^{\bar{A}(\zeta)} \right| \lambda \right\rangle &= \left\langle \lambda \left| e^{-\frac{1}{2}p^2 \ln(-4z)} e^{\Delta(-z)} V(\psi, w) e^{\bar{\Delta}(-\zeta)} e^{-\frac{1}{2}p^2 \ln(-4/\zeta)} \right| \lambda \right\rangle \\ &= (-4z)^{-\frac{\lambda^2}{2}} (-4/\zeta)^{-\frac{\lambda^2}{2}} \left\langle \lambda \left| e^{\Delta(-z)} V(\psi, w) e^{\bar{\Delta}(-\zeta)} \right| \lambda \right\rangle \\ &= \left(-4\sqrt{z/\zeta} \right)^{-\lambda^2} \left\langle \lambda \left| e^{\Delta(-z)} \left(\sqrt{z\zeta} \right)^{L_0} \left(\sqrt{z\zeta} \right)^{-L_0} V(\psi, w) \left(\sqrt{z\zeta} \right)^{L_0} \left(\sqrt{z\zeta} \right)^{-L_0} e^{\bar{\Delta}(-\zeta)} \right| \lambda \right\rangle \\ &= (-4u^{-1})^{-\lambda^2} (z\zeta)^{-\frac{1}{2}h_\psi} \left\langle \lambda \left| e^{\Delta(-z)} \left(\sqrt{z\zeta} \right)^{L_0} V(\psi, y) \left(\sqrt{z\zeta} \right)^{-L_0} e^{\bar{\Delta}(-\zeta)} \right| \lambda \right\rangle \end{aligned} \quad (\text{A.2.28})$$

where we defined new variables

$$u = \sqrt{\frac{\zeta}{z}}, \quad y = \frac{w}{\sqrt{z\zeta}} \quad (\text{A.2.29})$$

and used (2.4.19). Now using the translation property

$$w^{L_0} a_n w^{-L_0} = w^{-n} a_n, \quad (\text{A.2.30})$$

we see that

$$\left(\sqrt{z\zeta} \right)^{-L_0} \Delta(-z) \left(\sqrt{z\zeta} \right)^{L_0} = \Delta(-u^{-1}). \quad (\text{A.2.31})$$

Similarly using $a_n^\dagger = a_{-n}$ we have

$$\left(\sqrt{z\zeta} \right)^{-L_0} \bar{\Delta}(-\zeta) \left(\sqrt{z\zeta} \right)^{L_0} = \bar{\Delta}(-u), \quad (\text{A.2.32})$$

So we get

$$\begin{aligned} &\left\langle \lambda \left| e^{A(z)} V(\psi, w) e^{\bar{A}(\zeta)} \right| \lambda \right\rangle \\ &= (-4u^{-1})^{-\lambda^2} (z\zeta)^{-\frac{1}{2}h_\psi} \left\langle \lambda \left| \left(\sqrt{z\zeta} \right)^{L_0} e^{\Delta(-u^{-1})} V(\psi, y) e^{\bar{\Delta}(-u)} \left(\sqrt{z\zeta} \right)^{-L_0} \right| \lambda \right\rangle \\ &= (-4u^{-1})^{-\lambda^2} (z\zeta)^{-\frac{1}{2}h_\psi} \left\langle \lambda \left| e^{\Delta(-u^{-1})} V(\psi, y) e^{\bar{\Delta}(-u)} \right| \lambda \right\rangle \end{aligned} \quad (\text{A.2.33})$$

where we used the fact that

$$\langle \lambda | \left(\sqrt{z\zeta} \right)^{L_0} = \langle \lambda | \left(\left(\sqrt{z^* \zeta^*} \right)^{L_0} \right)^\dagger = \left(\sqrt{z\zeta} \right)^{\frac{\lambda^2}{2}} \langle \lambda |. \quad (\text{A.2.34})$$

Note that (A.2.33) is valid in the region $|u| < |y| < |u|^{-1}$ and $|u| < 1$. The next step is to simplify (A.2.33). For that we express the vertex operator $V(\psi, y)$ as exponential. Suppose

$$\psi = \left(\prod_{a=1}^M a_{-m_a}^{j_a} \right) |0\rangle. \quad (\text{A.2.35})$$

Then

$$\begin{aligned} V(\psi, y) &=: \prod_{a=1}^M \frac{i}{(m_a - 1)!} \frac{d^{m_a} X^{j_a}}{dy^{m_a}} : \\ &=: \prod_{a=1}^M \mathcal{D}^{(m_a)} \sum_{n \in \mathbb{Z}} a_n^{j_a} y^{-n-1} : \end{aligned} \quad (\text{A.2.36})$$

where

$$\mathcal{D}^{(m_a)} := \frac{1}{(m_a - 1)!} \frac{d^{m_a-1}}{dy^{m_a-1}}, \quad (\text{A.2.37})$$

and $p^j = a_0^j$. One can now write this as an exponential. To do this we introduce the variables $\rho_a^{j_a}$ for $1 \leq a \leq M$ and $1 \leq j_a \leq 24$. If we make a fixed choice of M -tuple $(j_1, \dots, j_M)$ then, relative to this choice we define

$$\frac{\partial}{\partial \rho} := \frac{\partial}{\partial \rho_1^{j_1}} \cdots \frac{\partial}{\partial \rho_M^{j_M}} = \prod_{a=1}^M \frac{\partial}{\partial \rho_1^{j_a}}. \quad (\text{A.2.38})$$

We do not explicitly indicate the M -tuple $(j_1, \dots, j_M)$ in the notation $\frac{\partial}{\partial \rho}$ to avoid cluttering the notation. Next introduce vectors

$$\begin{aligned} (\rho_n^<)^k &= \sum_{a=1}^M \sqrt{n} \rho_a^k \mathcal{D}^{(m_a)} y^{n-1} = \sum_{a=1}^M \sqrt{n} \binom{n-1}{m_a-1} \rho_a^k y^{n-m_a}, \\ (\rho_n^>)^k &= \sum_{a=1}^M \sqrt{n} \rho_a^k \mathcal{D}^{(m_a)} y^{-n-1} = \sum_{a=1}^M \sqrt{n} \binom{-n-1}{m_a-1} \rho_a^k y^{-n-m_a}, \\ (\rho_0)^k &= \sum_{a=1}^M \rho_a^k \mathcal{D}^{(m_a)} y^{-1} = \sum_{a=1}^M \binom{-1}{m_a-1} \rho_a^k y^{-m_a}. \end{aligned} \quad (\text{A.2.39})$$

and then define

$$\rho^< a^\dagger = \sum_{n>0} \frac{\rho_n^< \cdot a_{-n}}{\sqrt{n}}, \quad \rho^> a = \sum_{n>0} \frac{\rho_n^> \cdot a_n}{\sqrt{n}}. \quad (\text{A.2.40})$$

Then we have

$$V(\psi, y) = \frac{\partial}{\partial \rho} \left(e^{\rho^< a^\dagger} e^{p \cdot \rho_0} e^{\rho^> a} \right) \Big|_{\{\rho_a^k=0\}}. \quad (\text{A.2.41})$$

Next we have

$$\Delta(-u^{-1}) = \sum_{n>0} \binom{-\frac{1}{2}}{n} \frac{(-u^{-1})^{-n}}{n} a_n \cdot p + \frac{1}{2} \sum_{n, m>0} \binom{-\frac{1}{2}}{m} \binom{\frac{1}{2}}{n} \frac{(-u^{-1})^{-m-n}}{m+n} a_m \cdot a_n. \quad (\text{A.2.42})$$

Defining

$$T_{mn} = \frac{\sqrt{nm}}{n+m} \nu_n \nu_m, \quad \nu_n = (-u)^n \binom{-\frac{1}{2}}{n}, \quad (\text{A.2.43})$$

we can write

$$\Delta(-u^{-1}) = \tilde{p}a + \frac{1}{2}aTa \quad (\text{A.2.44})$$

where

$$\tilde{p}_n^j = \frac{p^j \nu_n}{\sqrt{n}}, \quad \tilde{p}a = \sum_{n>0} \tilde{p}_n \cdot \frac{a_n}{\sqrt{n}}, \quad aTa = \sum_{m,n>0} T_{mn} \frac{a_m}{\sqrt{m}} \cdot \frac{a_n}{\sqrt{n}} \quad (\text{A.2.45})$$

Thus we have

$$\begin{aligned} \left\langle \lambda \left| e^{A(z)} V(\psi, w) e^{\bar{A}(\zeta)} \right| \lambda \right\rangle &= (-4u^{-1})^{-\lambda^2} (z\zeta)^{-\frac{1}{2}h_\psi} \frac{\partial}{\partial \rho} \left\langle \lambda \left| \exp \left(\tilde{p}a + \frac{1}{2}aTa \right) \left(e^{\rho^< a^\dagger} e^{\rho \cdot \rho_0} e^{\rho^> a} \right) \right. \right. \\ &\quad \times \left. \exp \left(\tilde{p}a^\dagger + \frac{1}{2}a^\dagger Ta^\dagger \right) \right| \lambda \right\rangle \Big|_{\{\rho_a^k=0\}} \\ &= (-4u^{-1})^{-\lambda^2} (z\zeta)^{-\frac{1}{2}h_\psi} \frac{\partial}{\partial \rho} e^{\lambda \cdot \rho_0} \left\langle \lambda \left| \exp \left(\tilde{\lambda}a + \frac{1}{2}aTa \right) \left(e^{\rho^< a^\dagger} e^{\rho^> a} \right) \right. \right. \\ &\quad \times \left. \exp \left(\tilde{\lambda}a^\dagger + \frac{1}{2}a^\dagger Ta^\dagger \right) \right| \lambda \right\rangle \Big|_{\{\rho_a^k=0\}} \end{aligned} \quad (\text{A.2.46})$$

where $\tilde{\lambda}a$ is defined in exactly the same way as $\tilde{p}a$ with p^j replaced by λ^j . Now noting that $e^{\lambda \cdot q} |0\rangle = |\lambda\rangle$ and q commutes with all other oscillators a_n , we have

$$\begin{aligned} \left\langle \lambda \left| e^{A(z)} V(\psi, w) e^{\bar{A}(\zeta)} \right| \lambda \right\rangle &= (-4u^{-1})^{-\lambda^2} (z\zeta)^{-\frac{1}{2}h_\psi} \frac{\partial}{\partial \rho} e^{\lambda \cdot \rho_0} \left\langle 0 \left| \exp \left(\tilde{\lambda}a + \frac{1}{2}aTa \right) \left(e^{\rho^< a^\dagger} e^{\rho^> a} \right) \right. \right. \\ &\quad \times \left. \exp \left(\tilde{\lambda}a^\dagger + \frac{1}{2}a^\dagger Ta^\dagger \right) \right| 0 \right\rangle \Big|_{\{\rho_a^k=0\}} \end{aligned} \quad (\text{A.2.47})$$

Normal ordering the oscillators, we can write

$$\left\langle 0 \left| \exp \left(\tilde{\lambda}a + \frac{1}{2}aTa \right) \left(e^{\rho^< a^\dagger} e^{\rho^> a} \right) \exp \left(\tilde{\lambda}a^\dagger + \frac{1}{2}a^\dagger Ta^\dagger \right) \right| 0 \right\rangle = \left\langle 0 \left| e^{\tilde{\lambda}a} e^{\frac{1}{2}aTa} e^{\rho^< a^\dagger} e^{\rho^> a} e^{\tilde{\lambda}a^\dagger} e^{\frac{1}{2}a^\dagger Ta^\dagger} \right| 0 \right\rangle \quad (\text{A.2.48})$$

The vacuum expectation value on the right hand side can be evaluated using coherent state techniques and has been done in [DGM90, Appendix D]. We get

$$\begin{aligned} \left\langle 0 \left| e^{\tilde{\lambda}a} e^{\frac{1}{2}aTa} e^{\rho^< a^\dagger} e^{\rho^> a} e^{\frac{1}{2}a^\dagger Ta^\dagger} e^{\tilde{\lambda}a^\dagger} \right| 0 \right\rangle \\ = \det(1 - T^2)^{-\frac{1}{2}r} \exp \left(\tilde{\lambda}(1 - T)^{-1} \tilde{\lambda} \right) \exp \left(\tilde{\lambda}(1 - T)^{-1} (\rho^< + \rho^>) \right) e^{h(u, y)} \end{aligned} \quad (\text{A.2.49})$$

where

$$h(u, y) = \frac{1}{2} \left[\rho^< T (1 - T^2)^{-1} T \rho^> + \rho^> T (1 - T^2)^{-1} T \rho^< + \rho^< (1 - T^2)^{-1} T \rho^< + \rho^> (1 - T^2)^{-1} T \rho^> \right]. \quad (\text{A.2.50})$$

The determinant can be evaluated using group theory. The result is [DGM90, eq. (9.16)]

$$\det(1 - T^2)^{-\frac{1}{2}r} = (1 - u^2)^{-r/8} \Theta_3(q^2)^{-r}, \quad (\text{A.2.51})$$

where

$$q = \exp \left\{ -\pi \frac{K(\sqrt{1 - u^2})}{K(u)} \right\}, \quad (\text{A.2.52})$$

where $K(u)$ is a complete elliptic integral given by

$$K(u) = \frac{1}{2} \int_0^1 dx x^{-1/2} (1 - x)^{-1/2} (1 - u^2 x)^{-1/2} = \frac{1}{2} \pi \Theta_3(q^2)^2, \quad (\text{A.2.53})$$

Here $\Theta_3(q)$ is the Jacobi theta function [DLMF, Section 20.9] given by

$$\Theta_3(q) = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} = \prod_{n=1}^{\infty} (1 - q^n) \left(1 + q^{n-\frac{1}{2}} \right)^2. \quad (\text{A.2.54})$$

Because we are working with $|\zeta| < |z|$ we have $|u| < 1$. Also we have [DGM90, eq. (9.20)]

$$\exp \left(\tilde{\lambda} (1 - T)^{-1} \tilde{\lambda} \right) = (4u^{-1})^{\lambda^2} q^{\frac{1}{2}\lambda^2}. \quad (\text{A.2.55})$$

Next define

$$\lambda \cdot t(u, y) = \lambda \cdot \rho_0 + \tilde{\lambda} (1 - T)^{-1} (\rho^< + \rho^>). \quad (\text{A.2.56})$$

Then we can easily see that

$$\left\langle \lambda \left| e^{A(z)} V(\psi, \omega) e^{\tilde{A}(\zeta)} \right| \lambda \right\rangle = (z\zeta)^{-\frac{1}{2}h_\psi} \frac{\partial}{\partial \rho} (1 - u^2)^{-r/8} \Theta_3(q^2)^{-r} q^{\frac{1}{2}\lambda^2} e^{\lambda \cdot t(u, y)} e^{h(u, y)} \Big|_{\{\rho_a^k\}=0}. \quad (\text{A.2.57})$$

This implies

$$\begin{aligned} \langle \mu | W(\chi, z) V(\psi, w) \overline{W}(\varphi, \zeta) | \rho \rangle &= \zeta^{-2h_\varphi} (z\zeta)^{-\frac{1}{2}h_\psi} \sum_{\lambda \in \Lambda_L} \langle \mu | e_\lambda | \chi \rangle \left\langle \overline{\varphi} \left| e_\lambda^\dagger \right| \varrho \right\rangle \\ &\times (1 - u^2)^{-r/8} \Theta_3(q^2)^{-r} q^{\frac{1}{2}\lambda^2} \frac{\partial}{\partial \rho} e^{\lambda \cdot t(u, y)} e^{h(u, y)} \Big|_{\{\rho_a^k\}=0}. \end{aligned} \quad (\text{A.2.58})$$

We have kept the r explicit in above formula but at the end we need to plug in $r = 24$. We now need to show that the right hand side of (A.2.58) is invariant under $\varphi \leftrightarrow \chi$ and $z \leftrightarrow \zeta$ up to the factor $(-1)^{|\chi||\varphi|} = -1$.

To facilitate the exchange of φ and χ we proceed as follows. We begin with the relation [DGM90, Appendix C]

$$\sum_{\gamma \in \Gamma} \gamma_{ab}^\dagger \gamma_{cd} = 2^{12} \delta_{ad} \delta_{bc} \quad (\text{A.2.59})$$

to transform the inner products in (A.2.26). Choose a basis for $\mathcal{S}$ and let us denote the column vectors for the states $\mu, \chi, \varphi, \varrho$ by the same symbol. We then have

$$\begin{aligned}
\langle \mu | e_\lambda | \chi \rangle \langle \bar{\varphi} | e_\lambda^\dagger | \varrho \rangle &= (e^{-i\pi h_\varphi})^* \langle \mu | e_\lambda | \chi \rangle \langle \theta M \varphi^* | e_\lambda^\dagger | \varrho \rangle \\
&= -e^{i\pi h_\varphi} (\mu^\dagger \gamma_\lambda \chi) (\varphi^T M^* \gamma_\lambda^\dagger \varrho) \\
&= -2^{-12} e^{i\pi h_\varphi} \sum_{\gamma_\nu \in \Gamma} (\mu^\dagger \gamma_\nu^\dagger \gamma_\nu \gamma_\lambda \chi) (\varphi^T M^* \gamma_\lambda^\dagger \varrho) \\
&= -2^{-12} e^{i\pi h_\varphi} \sum_{\gamma_\nu \in \Gamma} (\mu^\dagger \gamma_\nu^\dagger \gamma_\nu \gamma_\lambda \chi) ((M^* \gamma_\lambda^\dagger \varrho)^T \varphi) \\
&= -2^{-12} e^{i\pi h_\varphi} \sum_{\gamma_\nu \in \Gamma} (\mu^\dagger \gamma_\nu^\dagger \varphi) ((\varrho^T M^* \gamma_\lambda \gamma_\nu \gamma_\lambda \chi)) \\
&= -2^{-12} e^{i\pi h_\varphi} \sum_{\gamma_\nu \in \Gamma} (\mu^\dagger \gamma_\nu^\dagger \varphi) ((\varrho^T M^* \gamma_\lambda \gamma_\nu \gamma_\lambda \chi)) \\
&= 2^{-12} e^{i\pi h_\varphi} \sum_{\gamma_\nu \in \Gamma} (\mu^\dagger \gamma_\nu^\dagger \varphi) ((\theta M^* \varrho)^T \gamma_\lambda \gamma_\nu \gamma_\lambda \chi) \\
&= 2^{-12} e^{i\pi(h_\varphi - h_\varrho)} (-1)^{\frac{\lambda^2}{2}} \sum_{e_\nu \in \Gamma(\Lambda_L)} (-1)^{\lambda \cdot \nu} \langle \mu | e_\nu^\dagger | \varphi \rangle \langle \bar{\varrho} | e_\nu | \chi \rangle.
\end{aligned} \tag{A.2.60}$$

Thus we can write

$$\begin{aligned}
\langle \mu | W(\chi, z) V(\psi, w) \bar{W}(\varphi, \zeta) | \varrho \rangle &= \zeta^{-2h_\varphi} (z\zeta)^{-\frac{1}{2}h_\psi} e^{i\pi(h_\varphi - h_\varrho)} 2^{-12} \sum_{e_\nu \in \Gamma(\Lambda_L)} \langle \mu | e_\nu^\dagger | \varphi \rangle \langle \bar{\varrho} | e_\nu | \chi \rangle \\
&\quad \times (1 - u^2)^{-r/8} \Theta_3(q^2)^{-r} \frac{\partial}{\partial \rho} e^{h(u, y)} \Gamma(q, \nu) \Big|_{\{\rho_a^k\}=0}
\end{aligned} \tag{A.2.61}$$

where

$$\Gamma(q, \nu) = \sum_{\lambda \in \Lambda_L} (-q)^{\frac{1}{2}\lambda^2} (-1)^{\lambda \cdot \nu} e^{\lambda \cdot t(u, y)}. \tag{A.2.62}$$

We now need to exchange z and ζ , which is equivalent to changing $u \rightarrow 1/u$. Of course, this requires analytic continuation since $|1/u| > 1$. Under $u \rightarrow 1/u$, q does not transform simply. So we change variables to

$$q' = \exp \left(-\pi \frac{K(u)}{K(\sqrt{1-u^2})} \right). \tag{A.2.63}$$

This is because $q' \rightarrow -q'$ under $u \rightarrow 1/u$ [DGM90, eq. (9.27)]. To change the variables from q to q' we need to express $\Gamma(q, \nu)$ in terms of q' . Note that if we write $q = e^{2\pi i \tau}$ and $q' = e^{2\pi i \tau'}$ then $\tau' = -1/4\tau$. One can now use the Poisson summation formula and get [DGM90, eq. (9.31)]

$$\Gamma(q, \nu) = (-1)^{\frac{1}{2}\nu^2} \left(\frac{\ln q'}{i\pi} \right)^{\frac{1}{2}r} \sum_{\lambda \in \Lambda_L} (-q')^{\frac{1}{2}\lambda^2} (-1)^{\lambda \cdot \nu} (q')^{\lambda \cdot t(u, y)/i\pi} (q')^{-t(u, y)^2/2\pi^2}. \tag{A.2.64}$$

Next we have the transformation of the theta function given by

$$\Theta_3(-1/\tau) = \sqrt{-i\tau} \Theta_3(\tau). \tag{A.2.65}$$

where $\Theta_3(\tau) := \Theta_3(q)$ with $q = e^{2\pi i\tau}$. Using this transformation, it is easy to see that

$$\Theta_3(q'^2) = \Theta_3(2\tau') = \Theta_3(-2/4\tau) = \sqrt{-2i\tau} \Theta_3(2\tau) = \sqrt{\frac{i}{2\tau'}} \Theta(q^2) = \Theta(q^2) \left(-\frac{\ln q'}{\pi}\right)^{-\frac{1}{2}} \quad (\text{A.2.66})$$

where we used the fact that $q = e^{2\pi i\tau} \rightarrow q' = e^{2\pi i\tau'}$ under $\tau \rightarrow \tau' = -1/4\tau$. This gives us

$$\Theta_3(q^2) = \Theta_3(q'^2) \left(-\frac{\ln q'}{\pi}\right)^{\frac{1}{2}} \quad (\text{A.2.67})$$

At this point we can return to the algebraic computations needed to exchange φ and χ . Doing a computation similar to (A.2.60), we get

$$2^{-12}(-1)^{\frac{1}{2}\lambda^2} \sum_{e_\nu \in \Gamma(\Lambda_L)} \langle \bar{\varphi} | e_\nu^\dagger | \bar{\mu} \rangle \langle \bar{\varrho} | e_\nu | \chi \rangle (-1)^{\lambda \cdot \nu} = \langle \bar{\varphi} | e_\lambda | \chi \rangle \langle \mu | e_\lambda^\dagger | \varrho \rangle e^{-i\pi(h_\mu - h_\varrho)} \quad (\text{A.2.68})$$

To use this relation, we need to relate $\langle \bar{\varphi} | e_\nu^\dagger | \bar{\mu} \rangle$ with $\langle \mu | e_\nu^\dagger | \varphi \rangle$. Indeed we have

$$\begin{aligned} \langle \bar{\varphi} | e_\nu^\dagger | \bar{\mu} \rangle &= (e^{-i\pi h_\varphi})^* e^{-i\pi h_\mu} (M\varphi^*)^\dagger \gamma_\nu^\dagger M\mu^* \\ &= e^{i\pi(h_\varphi - h_\mu)} \varphi^T M^* \gamma_\nu^\dagger M\mu^* \quad (M^* \gamma_\nu^\dagger M = \gamma_\nu^T) \\ &= e^{i\pi(h_\varphi - h_\mu)} (-1)^{\frac{1}{2}\nu^2} \varphi^T (\gamma_\nu^T)^{-1} \mu^* \\ &= e^{i\pi(h_\varphi - h_\mu)} (-1)^{\frac{1}{2}\nu^2} (\gamma_\nu^\dagger \varphi)^T \mu^* \\ &= e^{i\pi(h_\varphi - h_\mu)} (-1)^{\frac{1}{2}\nu^2} \langle \mu | e_\nu^\dagger | \varphi \rangle \end{aligned} \quad (\text{A.2.69})$$

where we used the fact that e_ν is unitary.

Putting these facts together we can express our matrix element in the form:

$$\begin{aligned} \langle \mu | W(\chi, z) V(\psi, w) \bar{W}(\varphi, \zeta) | \varrho \rangle &= \zeta^{-2h_\varphi} (z\zeta)^{-\frac{1}{2}h_\psi} \frac{\partial}{\partial \rho} \sum_{\lambda \in \Lambda_L} \langle \bar{\varphi} | e_\lambda | \chi \rangle \langle \mu | e_\lambda^\dagger | \varrho \rangle (q')^{\frac{1}{2}\lambda^2} (q')^{\lambda \cdot t(u, y)/i\pi} \\ &\quad \times i^{\frac{1}{2}r} (1 - u^2)^{-r/8} \Theta_3(q'^2)^{-r} e^{h(u, y)} (q')^{-t(u, y)^2/2\pi^2} \Big|_{\{\rho_a^k=0\}} \end{aligned} \quad (\text{A.2.70})$$

We now show that the right hand side of (A.2.70) is invariant under $\varphi \leftrightarrow \chi$ and $z \leftrightarrow \zeta$ up to the factor $(-1)^{|\chi||\phi|} = -1$. Under this, $u \leftrightarrow 1/u$ and $q' \leftrightarrow -q'$. To check invariance under exchange we need the identities:

$$\begin{aligned} \langle \bar{\varphi} | e_\lambda | \chi \rangle &= \langle \bar{\chi} | e_\lambda | \varphi \rangle e^{i\pi(h_\chi + h_\varphi)} (-1)^{\frac{1}{2}\lambda^2} \\ (1 - u^{-2})^{-r/8} \Theta_3(-q'^2)^{-r} &= (1 - u^2)^{-r/8} \Theta_3(q'^2)^{-r} (-1)^{r/8} (\zeta/z)^{-r/8} \\ \Theta_3(-q'^2) &= u^{1/2} \Theta_3(q'^2) \end{aligned} \quad (\text{A.2.71})$$

We also note that $e^{h(u, y)} (q')^{-t(u, y)^2/2\pi^2}$ and $(q')^{\lambda \cdot t(u, y)/i\pi}$ are invariant under $u \leftrightarrow 1/u$ and $q' \leftrightarrow -q'$ [DGM90, Section 9, 10] and $r = 24$. Putting all these together we finally arrive at the desired identity:

$$\langle \mu | W(\chi, z) V(\psi, w) \bar{W}(\varphi, \zeta) | \varrho \rangle = -\langle \mu | W(\varphi, \zeta) V(\psi, w) \bar{W}(\chi, z) | \varrho \rangle. \quad (\text{A.2.72})$$

It remains to prove the identities (A.2.71). The last relation in eq. (A.2.71) is proved using the fact that $\Theta_3(-q'^2) = \Theta_4(q'^2)$ where

$$\Theta_4(q') = \sum_{n \in \mathbb{Z}} (-1)^n q'^{\frac{n^2}{2}} = \prod_{n=1}^{\infty} (1 - q'^n) \left(1 - q'^{n-\frac{1}{2}}\right)^2 \quad (\text{A.2.73})$$

and [DLMF]

$$\Theta_4(q'(u)^2) = u^{\frac{1}{2}} \Theta_3(q'(u)^2), \quad (\text{A.2.74})$$

where $q'(u)$ is given by eq. (A.2.63). The middle relation in (A.2.71) follows from the usual modular transformation rules. The first relation in eq. (A.2.71) is similar to, but different from one proven in [DGM90], because their context is slightly different. We prove the first relation as follows:

$$\begin{aligned} \langle \bar{\varphi} | e_{\lambda} | \chi \rangle &= (e^{-i\pi h_{\varphi}})^* \langle \theta M(\varphi) | e_{\lambda} | \chi \rangle \\ &= -e^{i\pi h_{\varphi}} \varphi^T M^* \gamma_{\lambda} \chi \\ &= -e^{i\pi h_{\varphi}} \varphi^T \gamma_{\lambda}^* M^* \chi \quad (M \gamma_{\lambda}^* = \gamma_{\lambda} M \text{ and } M^* = M^{-1}) \\ &= e^{i\pi(h_{\varphi} - h_{\chi})} \varphi^T \gamma_{\lambda}^* (e^{-i\pi h_{\chi}} \theta M \chi^*)^* \\ &= e^{i\pi(h_{\varphi} + h_{\chi})} \varphi^T \gamma_{\lambda}^* \bar{\chi}^* \\ &= e^{i\pi(h_{\varphi} - h_{\chi})} \varphi^T \gamma_{\lambda}^T (\gamma_{\lambda}^{-1})^T \gamma_{\lambda}^* \bar{\chi}^* \\ &= e^{i\pi(h_{\varphi} - h_{\chi})} (\gamma_{\lambda} \varphi)^T (\gamma_{\lambda}^*)^2 \bar{\chi}^* \\ &= e^{i\pi(h_{\varphi} - h_{\chi})} (-1)^{\frac{\lambda^2}{2}} (\gamma_{\lambda} \varphi)^T \bar{\chi}^* \\ &= e^{i\pi(h_{\varphi} - h_{\chi})} (-1)^{\frac{\lambda^2}{2}} \langle \bar{\chi} | e_{\lambda} | \varphi \rangle, \end{aligned} \quad (\text{A.2.75})$$

where we used the properties of the gamma matrices and the fact that they are unitary. $\square$

To complete the proof of locality, we need to promote the states χ and φ to arbitrary fermionic states in $\mathcal{H}_T^-(\Lambda_L)$.

Lemma A.2.4. *The locality relation $W(\chi, z) \bar{W}(\varphi, w) = -W(\varphi, w) \bar{W}(\chi, z)$ for arbitrary fermionic twisted states χ, φ holds if and only if*

$$W(\chi, z) \bar{W}(\varphi, \zeta) = -W(\varphi, \zeta) \bar{W}(\chi, z) \quad (\text{A.2.76})$$

for χ, φ twisted ground states.

Proof. Direct implication is obvious. We prove the converse. Since we already have the locality of $\tilde{V}(\psi, z) \tilde{V}(\chi, w) = \tilde{V}(\chi, w) \tilde{V}(\psi, z)$ for twisted state χ and untwisted state ψ , we have the duality relation

$$\tilde{V}(\psi, z) \tilde{V}(\chi, w) = \tilde{V}(\tilde{V}(\psi, z - w) \chi, w). \quad (\text{A.2.77})$$

Note that by Proposition 2.5.1, the right hand side of from (A.2.77) has square root singularity when $\chi \in \mathcal{H}_T^-(\Lambda_L)$ and $\psi \in \mathcal{H}^-(\Lambda_L)$. Fortunately we do not have to use (A.2.77) for $\psi \in \mathcal{H}^-(\Lambda_L)$. To prove this lemma, we only need $\psi \in \mathcal{H}^+(\Lambda_L)$. Using explicit matrix forms of $\tilde{V}$ (2.5.17) and (2.5.18), we get

$$V(\psi, z)\overline{W}(\chi, w) = \overline{W}(V_T(\psi, z - w)\chi, w)$$

$$V_T(\psi, z)W(\chi, w) = W(V_T(\psi, z - w)\chi, w)$$

Acting on $W(\chi, z)\overline{W}(\varphi, \zeta)|\varrho\rangle$ by $V_T(\psi, w)$ with $\psi \in \mathcal{H}^+(\Lambda_L)$ and using the above relations and Lemma A.2.2, we can generalise χ to arbitrary fermionic twisted state on both sides of (A.2.76). This is because $V_T(\psi, z)$ maps fermions to bosons and vice-versa in the twisted sector if $\theta\psi = -\psi$ and preserve the parity if $\theta\psi = \psi$. Similarly acting with $V_T(\psi, w)$ on $W(\chi, z)\overline{W}(\varphi, \zeta)\varrho$, we can generalise φ to arbitrary fermionic twisted state on both sides of (A.2.76). The statement of the lemma now follows. $\square$

Appendix B

Modular tensor category from rational conformal field theory

In this appendix, we explain the rigorous construction of a modular tensor category from a 2d rational conformal field theory (RCFT) using the theory of vertex operator algebras. The foundational paper on the subject is the work by Moore-Seiberg where they derived polynomial equations for braiding and fusion matrices defined using conformal blocks [MS88]. These equations were then used to prove the Verlinde conjecture, a foundational result in 2d rational conformal field theory [Ver88]. A more systematic treatment was then presented by Moore-Seiberg in [MS89c, MS89a]. The physics motivated discussion in Moore-Seiberg was then formalized into categorical language by Turaev [Tur92] to define a *modular tensor category*¹. The rigorous construction of the modular tensor category from an RCFT was described in a series of works by Huang-Lepowsky [HL95, Hua95a, HL93, HL94] and Huang² [HKJL15, Hua05].

The goal of this appendix is to summarize the construction of modular tensor category from a vertex operator algebra and sketch the proof of the Verlinde formula. This appendix will also serve the purpose of introducing several terms used in Chapter 2 and Chapter 3.

B.1 Vertex operator algebra

The algebraic theory of RCFTs described by Moore-Seiberg in [MS89c, MS89a] can be given a rigorous mathematical treatment using the theory of vertex operator algebras, invented originally to prove the Monstrous moonshine conjecture. To describe vertex operator of 2d CFT rigorously, mathematicians use formal calculus. We begin by discussing some basic definitions in formal calculus.

¹The name modular tensor category was suggested by Igor Frenkel.

²The construction of rigid braided tensor category from affine Lie algebras was first presented by Kazhdan-Lusztig in [KL91, KL93a, KL93b, KL94a, KL94b] but its generalization to arbitrary rational VOA was first presented by Huang.

B.1.1 Formal calculus

The symbols $x, x_0, x_1, \dots$ will denote commuting formal variables. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a vector space V , we set

$$\begin{aligned} V[x] &= \left\{ \sum_{n \in \mathbb{N}_0} v_n x^n : v_n \in V, \text{ all but finitely many } v_n = 0 \right\} , \\ V[x, x^{-1}] &= \left\{ \sum_{n \in \mathbb{Z}} v_n x^n : v_n \in V, \text{ all but finitely many } v_n = 0 \right\} , \\ V[[x]] &= \left\{ \sum_{n \in \mathbb{N}_0} v_n x^n : v_n \in V \right\} , \\ V[[x, x^{-1}]] &= \left\{ \sum_{n \in \mathbb{Z}} v_n x^n : v_n \in V \right\} , \\ V\{x\} &= \left\{ \sum_{n \in \mathbb{Q}} v_n x^n : v_n \in V \right\} . \end{aligned} \tag{B.1.1}$$

If we define the operator $e^{x \frac{d}{dx}}$ by the expansion

$$e^{x_0 \frac{d}{dx}} = \sum_{n=0}^{\infty} \frac{x_0^n}{n!} \frac{d^n}{dx^n} , \tag{B.1.2}$$

then one can show that the formal version of Taylor's theorem holds:

$$e^{x_0 \frac{d}{dx}} f(x) = f(x + x_0) , \quad f \in V[[x, x^{-1}]] . \tag{B.1.3}$$

For any $s \in \mathbb{C}$, we will define the formal series

$$(x_1 + x_2)^s = \sum_{n=0}^{\infty} \binom{s}{n} x_1^{s-n} x_2^n , \tag{B.1.4}$$

where

$$\binom{s}{n} = \frac{s(s-1) \cdots (s-n+1)}{n!} . \tag{B.1.5}$$

Note that the convention is to always have the second variable raised to positive integral powers. In particular,

$$(x_1 + x_2)^s \neq (x_2 + x_1)^s , \quad \text{for } s \in \mathbb{C} \setminus \mathbb{N}_0 . \tag{B.1.6}$$

A basic function which will feature prominently is the “delta function”

$$\delta(x) := \sum_{n \in \mathbb{Z}} x^n \in \mathbb{C}[x, x^{-1}] . \tag{B.1.7}$$

The delta function satisfies the property

$$f(x)\delta(x) = f(1)\delta(x) \quad \text{for } f(x) \in \mathbb{C}[x, x^{-1}] . \tag{B.1.8}$$

We will often use the three-variable delta function

$$x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) = \sum_{n \in \mathbb{Z}} \frac{(x_1 - x_2)^n}{x_0^{n+1}} = \sum_{m \in \mathbb{N}_0, n \in \mathbb{Z}} (-1)^m \binom{n}{m} x_0^{-n-1} x_1^{n-m} x_2^m . \quad (\text{B.1.9})$$

The following identities can be verified by formal manipulations:

$$\begin{aligned} x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \\ x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) . \end{aligned} \quad (\text{B.1.10})$$

Limits of formal expression are defined in a purely algebraic sense. For example

$$\lim_{x \rightarrow 0} f(x) = v_0 , \quad \text{for } f(x) = \sum_{n=0}^{\infty} v_n x^n \in V[[x]] . \quad (\text{B.1.11})$$

Formal residue is defined as

$$\text{Res}_x \left(\sum_{n \in \mathbb{Z}} v_n x^n \right) := v_{-1} . \quad (\text{B.1.12})$$

For example

$$\text{Res}_{x_1} x_1^{-1} \delta \left(\frac{x_1 + x_0}{x_1} \right) = 1 . \quad (\text{B.1.13})$$

We will usually work with $\mathbb{C}$ -graded vector spaces

$$V = \bigoplus_{n \in \mathbb{C}} V_{(n)} . \quad (\text{B.1.14})$$

We will often denote this direct sum by

$$V = \coprod_{n \in \mathbb{C}} V_{(n)} . \quad (\text{B.1.15})$$

We also work with direct products

$$\overline{V} = \prod_{n \in \mathbb{C}} V_{(n)} . \quad (\text{B.1.16})$$

$\overline{V}$ is also called the *completion* or the *algebraic extension* of V . The main difference between these two notations is that in direct sum ($\coprod$), we only take finite sums of homogeneous vectors while in direct product ($\prod$), we are allowed to take formal infinite sums.

B.1.2 Definition and consequences

Definition B.1.1. A vertex operator superalgebra (SVOA) is a tuple $(V, Y_V, \mathbf{1}, \omega, c)$, where

- V is a $\frac{1}{2}\mathbb{Z}$ -graded complex vector space

$$V = \coprod_{n \in \frac{1}{2}\mathbb{Z}} V_{(n)} , \quad (\text{B.1.17})$$

and V is also $\mathbb{Z}_2$ -graded $V = V_{\bar{0}} \oplus V_{\bar{1}}$,

- Y_V is the vertex operator map

$$Y_V(\bullet, x) : V \longrightarrow \text{End}(V)[[x^{\frac{1}{2}}, x^{-\frac{1}{2}}]] ,$$

$$u \longmapsto Y_V(u, x) = \sum_{n \in \frac{1}{2}\mathbb{Z}} u_n x^{-n-1} , \quad (\text{B.1.18})$$

such that we have $u_n v \in V_{i+\bar{j}}$ when $u \in V_i$ and $v \in V_{\bar{j}}$

- $\mathbf{1} \in V_{(0)}$ is the *vacuum vector*,
- $\omega \in V_{(2)}$ is the *conformal vector*.
- $c \in \mathbb{C}$ is the *central charge*.

This data satisfies the following axioms:

1. Creation property:

$$Y_V(\mathbf{1}, x) = \mathbb{1}_V , \quad \lim_{x \rightarrow 0} Y_V(v, x) \mathbf{1} = v . \quad (\text{B.1.19})$$

Equivalently,

$$\begin{aligned} \mathbf{1}_n &= 0, \quad n \neq -1, \quad \mathbf{1}_{-1} = \mathbb{1}_V , \\ v_n \cdot \mathbf{1} &= 0, \quad n > -1, \quad v_{-1} \cdot \mathbf{1} = v . \end{aligned} \quad (\text{B.1.20})$$

2. Grading restriction property: $V_{(n)} = 0$ for $n < 0$ and $\dim(V_{(n)}) < \infty$.
3. Virasoro property: Define the operators $L_n \in \text{End}(V)$ by³

$$Y_V(\omega, x) = \sum_{n \in \mathbb{Z}} L_n x^{-n-2} . \quad (\text{B.1.21})$$

Then L_n satisfy the Virasoro algebra with central charge c :

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} . \quad (\text{B.1.22})$$

4. L_0 property: L_0 is the grading operator on V :

$$L_0 v = n v , \quad \text{for all } v \in V_{(n)} . \quad (\text{B.1.23})$$

The eigenvalues of L_0 are called *weights* or *conformal dimension*. The vertex operators satisfies

$$\frac{d}{dx} Y_V(v, x) = Y_V(L_{-1} v, x) . \quad (\text{B.1.24})$$

³It is conventional in Mathematics literature to denote the Virasoro generators by $L(m)$. We will use the two notations interchangeably.

5. Jacobi identity: For any $u, v \in V$, we have

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_V(u, x_1) Y_V(v, x_2) - (-1)^{|u||v|} x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_V(v, x_2) Y_V(u, x_1) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_V(Y_V(u, x_0)v, x_2) , \end{aligned} \quad (\text{B.1.25})$$

where $|u|$ is the $\mathbb{Z}_2$ -degree of u .

We will often denote an SVOA by (V, Y_V) or simply by V . An SVOA is called a VOA if

- $V_{\bar{1}} = 0$ and $V_{(n+\frac{1}{2})} = 0$ for $n \in \mathbb{Z}$,
- the vertex operator map is a formal series in integral powers of x , i.e., for $u \in V$,

$$Y_V(u, x) = \sum_{n \in \mathbb{Z}} u_n x^{-n-1} \in \text{End}(V)[[x, x^{-1}]] . \quad (\text{B.1.26})$$

Remark B.1.1. 1. A vertex operator algebra is to be understood as the chiral algebra of a 2d CFT.

2. Using Jacobi identity, we can prove the following properties:

$$Y_V(v, x) \mathbf{1} = e^{xL_{-1}} v , \quad (\text{B.1.27})$$

$$Y_V(e^{x_0 L_{-1}} v, x) = e^{x_0 \frac{d}{dx}} Y_V(v, x) = Y_V(v, x + x_0) , \quad (\text{B.1.28})$$

$$[L_{-1}, Y_V(v, x)] = Y_V(L_{-1}v, x) , \quad (\text{B.1.29})$$

$$[L_0, Y_V(v, x)] = Y_V(L_0v, x) + x Y_V(L_{-1}v, x) , \quad (\text{B.1.30})$$

$$[L_1, Y_V(v, x)] = Y_V(L_1v, x) + 2x Y_V(L_0v, x) + x^2 Y_V(L_{-1}v, x) . \quad (\text{B.1.31})$$

This is the action of $\text{SL}(2, \mathbb{C})$, spanned by $\{L_0, L_{\pm 1}\}$, on the vertex operators. More generally, if $v \in V_{(h)}$ is such that $L_n v = 0$ for all $n > 0$, then

$$[L_n, Y_V(v, x)] = \left(x^{n+1} \frac{d}{dx} + h(n+1)x^n \right) Y_V(v, x) . \quad (\text{B.1.32})$$

This is equivalent to the OPE of the stress tensor with a primary field of conformal dimension h .

3. The Jacobi identity can be written in terms of commutators of $v_n \in \text{End}(V)$. This is how Borcherds originally defined vertex algebra [Bor86b].

From now on, we will restrict to VOAs but almost all of the discussion goes through for SVOAs.

Jacobi identity as commutativity and associativity

Most of the axioms of vertex operator algebra is familiar from physical theory of 2d CFT except for the Jacobi identity. As we will now discuss, the Jacobi identity captures the locality of generators of chiral algebras. We will give a brief review of the results of [FHL93b] which shows that the Jacobi identity is equivalent to the “commutativity” and “associativity” of vertex operators which are more familiar from physics point of view.

Define the *restricted dual*

$$V' := \coprod_{n \in \mathbb{Z}} V_{(n)}^\vee, \quad (\text{B.1.33})$$

where $V_{(n)}^\vee := \text{Hom}(V_{(n)}, \mathbb{C})$ is the dual of $V_{(n)}$. Then there is a natural pairing of V, V' , which we will denote by $\langle \cdot, \cdot \rangle$. By grading restriction and creation property, it is clear that for $u, v \in V$ and $v' \in V'$, we have

$$\langle v', Y_V(u, x)v \rangle \in \mathbb{C}[x, x^{-1}]. \quad (\text{B.1.34})$$

We have the following theorem.

Theorem B.1.1. *Let (V, Y_V) be a VOA. Let $v, v_1, v_2 \in V$ and $v' \in V'$. The the following is true:*

1. (Rationality of products) *The formal series $\langle v', Y_V(v_1, x_1)Y_V(v_2, x_2)v \rangle$, which has only finitely many negative powers of x_2 and only finitely many positive powers of x_1 , takes the form*

$$\langle v', Y_V(v_1, x_1)Y_V(v_2, x_2)v \rangle = \iota_{12}f(x_1, x_2), \quad (\text{B.1.35})$$

where f is of the form

$$f(x_1, x_2) = \frac{g(x_1, x_2)}{x_1^r x_2^s (x_1 - x_2)^t} \quad (\text{B.1.36})$$

for some $g \in \mathbb{C}[x_1, x_2]$, $r, s, t \in \mathbb{Z}$ and the notation $\iota_{12}f$ means that we expand f in positive integral powers of x_2 .

2. (Commutativity) *We have*

$$\langle v', Y_V(v_2, x_2)Y_V(v_1, x_1)v \rangle = \iota_{21}f(x_1, x_2), \quad (\text{B.1.37})$$

where $\iota_{21}f$ means that we expand f in positive integral powers of x_1 .

3. (Rationality of iterates) *The formal series $\langle v', Y_V(Y_V(v_1, x_0)v_2, x_2)v \rangle$, which only has finitely many negative powers of x_0 and only finitely many positive powers of x_2 , has the form*

$$\langle v', Y_V(Y_V(v_1, x_0)v_2, x_2)v \rangle = \iota_{20}h(x_0, x_2), \quad (\text{B.1.38})$$

where h is of the form

$$h(x_0, x_2) = \frac{k(x_0, x_2)}{x_0^r x_2^s (x_0 + x_2)^t} , \quad (\text{B.1.39})$$

for some $k \in \mathbb{C}[x_0, x_2]$, $r, s, t \in \mathbb{Z}$. Moreover, we also have

$$\langle v', Y_V(v_1, x_0 + x_2) Y_V(v_2, x_2) v \rangle = \iota_{02} h(x_0, x_2) . \quad (\text{B.1.40})$$

Remark B.1.2. Commutativity means that the formal series

$$\langle v', Y_V(v_1, x_1) Y_V(v_2, x_2) v \rangle \quad \text{and} \quad \langle v', Y_V(v_2, x_2) Y_V(v_1, x_1) v \rangle \quad (\text{B.1.41})$$

are the expansion of the *same* rational function f expanded with positive integral powers of x_2, x_1 respectively. Associativity means that the formal series

$$\langle v', Y_V(Y_V(v_1, x_0) v_2, x_2) v \rangle \quad \text{and} \quad \langle v', Y_V(v_1, x_0 + x_2) Y_V(v_2, x_2) v \rangle \quad (\text{B.1.42})$$

are the expansion of the *same* rational function expanded with positive integral powers of x_0, x_2 respectively.

When we replace the formal variable x_0, x_1, x_2 by nonzero complex numbers, then we get the familiar notion of locality of vertex operators.

Proposition B.1.1. *The Laurent series $\langle v', Y_V(v_1, z_1) Y_V(v_2, z_2) v \rangle$ and $\langle v', Y_V(v_2, z_2) Y_V(v_1, z_1) v \rangle$ converge to a common rational function in the domains*

$$|z_1| > |z_2| > 0 \quad \text{and} \quad |z_2| > |z_1| > 0 , \quad (\text{B.1.43})$$

respectively. Moreover, this function has poles only at $z_1, z_2 = 0$ and $z_1 = z_2$. The Laurent series $\langle v', Y_V(v_1, z_1) Y_V(v_2, z_2) v \rangle$ and $\langle v', Y_V(Y_V(v_1, z_1 - z_2) v_2, z_2) v \rangle$ converge to a common rational function in the domains

$$|z_1| > |z_2| > 0 \quad \text{and} \quad |z_2| > |z_1 - z_2| > 0 , \quad (\text{B.1.44})$$

respectively and in the common domain

$$|z_1| > |z_2| > |z_1 - z_2| > 0 , \quad (\text{B.1.45})$$

these two series converge to the common series.

The associativity of vertex operators becomes the operator product expansion from physics. More precisely, from Proposition B.1.1, inside correlation function, the identity

$$Y_V(u, z_1) Y_V(v, z_2) = Y_V(Y_V(u, z_1 - z_2) v, z_2) = \sum_{n \in \mathbb{Z}} Y(u_n \cdot v, z_2) (z_1 - z_2)^{-n-1} , \quad (\text{B.1.46})$$

holds in the domain $|z_1| > |z_2| > |z_1 - z_2| > 0$. This is precisely the operator product expansion. In physics, we only care about the $n \geq -1$ terms.

Theorem B.1.2. *Associativity and commutativity of vertex operators is equivalent to the Jacobi identity. In particular, in the definition of a VOA, we could replace the Jacobi identity by the following Locality axiom: For $u_1, u_2, v \in V$ and $v' \in V'$ and complex numbers z_1, z_2 , the series*

$$\begin{aligned} &\langle v', Y_V(u_1, z_1) Y_V(u_2, z_2) v \rangle \\ &\langle v', Y_V(u_2, z_2) Y_V(u_1, z_1) v \rangle \\ &\langle v', Y_V(Y_V(u_1, z_1 - z_2) u_2, z_2) v \rangle \end{aligned} \quad (\text{B.1.47})$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0, |z_2| > |z_1| > 0, |z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$.

Definition B.1.2. A VOA homomorphism between $(V_1, Y_{V_1}, \mathbf{1}_1, \omega_1, c_1)$ and $(V_2, Y_{V_2}, \mathbf{1}_2, \omega_2, c_2)$ is a linear map $f : V_1 \rightarrow V_2$ such that

$$\begin{aligned} f(Y_{V_1}(u, x)v) &= Y_{V_2}(f(u), x)f(v), \quad u, v \in V_1, \\ f(\mathbf{1}_1) &= \mathbf{1}_2, \quad f(\omega_1) = \omega_2. \end{aligned} \quad (\text{B.1.48})$$

A VOA isomorphism is a bijective homomorphism, a VOA endomorphism is a homomorphism from V to itself and a VOA automorphism is a bijective endomorphism. We say that V_1 is a subalgebra of V_2 if there is an injective homomorphism $f : V_1 \rightarrow V_2$.

The tensor product of two VOAs is defined as follows: given VOAs $V^1, \dots, V^n$, the vector space

$$V = V^1 \otimes V^2 \cdots \otimes V^n, \quad (\text{B.1.49})$$

is equipped with the vertex operator map

$$Y_V(v^1 \otimes \cdots \otimes v^n, x) = Y_{V^1}(v^1, x) \otimes \cdots \otimes Y_{V^n}(v^n, x), \quad v^i \in V^i, \quad (\text{B.1.50})$$

vacuum vector

$$\mathbf{1} = \mathbf{1}_1 \otimes \cdots \otimes \mathbf{1}_n, \quad (\text{B.1.51})$$

and conformal vector

$$\omega = \sum_{i=1}^n \mathbf{1}_1 \otimes \cdots \otimes \omega_i \otimes \cdots \otimes \mathbf{1}_n. \quad (\text{B.1.52})$$

(V, Y_V) can then be proved to be a VOA with central charge $c = c_1 + \cdots + c_n$.

B.1.3 Modules of a VOA

We now define modules of a VOA.

Definition B.1.3. Let (V, Y_V) be a VOA. A V -module is a $\mathbb{C}$ -graded vector space

$$W = \coprod_{n \in \mathbb{C}} W_{(n)} , \quad (\text{B.1.53})$$

equipped with the module vertex operator map

$$\begin{aligned} Y_W(\bullet, x) : V &\longrightarrow \text{End}(W)[[x, x^{-1}]] , \\ v &\longmapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} , \end{aligned} \quad (\text{B.1.54})$$

satisfying

1. Identity property: $Y_W(\mathbf{1}, x) = \mathbb{1}_W$.
2. Grading restriction property: $W_{(n)} = 0$ for sufficiently small $\text{Re}(n)$, $\dim(W_{(n)}) < \infty$ and $v_n \cdot w = 0$ for sufficiently large n .
3. Virasoro property: Define the operators $L_n \in \text{End}(W)$ by

$$Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} L_n x^{-n-2} . \quad (\text{B.1.55})$$

Then L_n satisfy the Virasoro algebra with central charge c :

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} . \quad (\text{B.1.56})$$

4. L_0 property: L_0 is the grading operator on W :

$$L_0 w = n w , \quad \text{for all } w \in W_{(n)} . \quad (\text{B.1.57})$$

Moreover the vertex operators satisfies

$$\frac{d}{dx} Y_W(v, x) = Y_W(L_{-1} v, x) . \quad (\text{B.1.58})$$

5. Jacobi identity: For any $u, v \in V$, we have

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2) Y_W(u, x_1) \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y_V(u, x_0) v, x_2) . \end{aligned} \quad (\text{B.1.59})$$

If we relax [item 2](#) to $v_n w = 0$ for all $v \in V, w \in W$ for sufficiently large n and remove [item 3](#) and [item 4](#), then we call (W, Y_W) a *weak* V -module.

A V -module homomorphism between V -modules (W_1, Y_{W_1}) and (W_2, Y_{W_2}) is a linear map $f : W_1 \longrightarrow W_2$ such that

$$f(Y_{W_1}(v, x)w) = Y_{W_2}(v, x)f(w) , \quad v \in V, \quad w \in W_1 . \quad (\text{B.1.60})$$

Isomorphisms (equivalences), automorphisms, submodules, direct sums, irreducible (simple) modules, completely reducible (semisimple) modules etc., are defined in the usual way.

A VOA V is called *simple* if it is *simple* as a module for itself. W will be called simple or irreducible V -module if it does not contain any nontrivial submodule.

Theorem B.1.3. *A tensor product of VOAs $V^1 \otimes \cdots \otimes V^n$ is simple if and only if each V^i is simple. A simple $(V^1 \otimes \cdots \otimes V^n)$ -module W with the property that, the eigenvalues of L_0^i on the lowest weight space of W are rational, is a tensor product of simple V^i -modules.*

Contragredient module

Let (W, Y_W) be a V -module. Define the restricted dual

$$W' = \coprod_{n \in \mathbb{C}} W'_{(n)} . \quad (\text{B.1.61})$$

W' has the obvious $\mathbb{C}$ -grading. Define the *adjoint vertex operators* $Y_{W'}(v, x)$ by the linear map

$$\begin{aligned} Y_{W'}(\bullet, x) : V &\longrightarrow \text{End}(W')[[x, x^{-1}]] , \\ v &\longmapsto Y_{W'}(v, x) = \sum_{n \in \mathbb{Z}} v'_n x^{-n-1} , \end{aligned} \quad (\text{B.1.62})$$

defined by

$$\langle Y_{W'}(v, x)w', w \rangle = \langle w', Y_W(e^{xL_1}(-x^{-2})^{L_0}v, x^{-1})w \rangle , \quad v \in V, w \in W, w' \in W' . \quad (\text{B.1.63})$$

Theorem B.1.4. *Let (W, Y_W) be a V -module. Then $(W', Y_{W'})$ is also a V -module.*

The proof of this theorem is by direct computations and the details can be found in [FHL93b]. If we expand

$$Y_{W'}(\omega, x) = \sum_{n \in \mathbb{Z}} L'_n x^{-n-2} , \quad (\text{B.1.64})$$

then we have

$$\langle L'_n w', w \rangle = \langle w', L_{-n} w \rangle , \quad w \in W, w' \in W', n \in \mathbb{Z} . \quad (\text{B.1.65})$$

This is the usual Belavin-Polyakov-Zamolodchikov BPZ conjugation.

Here are some properties of contragredient modules.

Proposition B.1.2. *Let (W, Y_W) be a V -module and $(W', Y_{W'})$ be the contragredient module.*

1. $(W'', Y_{W''})$ is naturally isomorphic to (W, Y_W) .
2. $(W', Y_{W'})$ is irreducible if and only if (W, Y_W) is irreducible.
3. $(W', Y_{W'})$ is isomorphic to (W, Y_W) if and only if there exists a nondegenerate bilinear form $(\cdot, \cdot)$ on W . Moreover for $W = V$, this bilinear form is symmetric.

A VOA V is called *selfdual* if $V \cong V'$ as V -modules.

B.1.4 Intertwining operators

Definition B.1.4. Let (V, Y_V) be a VOA and let $(W_i, Y_{W_i}), i = 1, 2, 3$ be V -modules. An *intertwining operator* of type $\binom{W_3}{W_1 W_2}$ is a linear map

$$\begin{aligned} \mathcal{Y}(\bullet, x) : W_1 &\longrightarrow \text{Hom}(W_2, W_3)\{x\} , \\ w_1 &\longmapsto \mathcal{Y}(w_1, x) = \sum_{n \in \mathbb{C}} (w_1)_n x^{-n-1} , \end{aligned} \tag{B.1.66}$$

satisfying the following properties:

1. $(w_1)_n \cdot w_2 = 0$ for all $w_2 \in W_2$ for sufficiently large $\text{Re}(n)$.
2. Jacobi identity: for all $v \in V, w_1 \in W_1, w_2 \in W_2$

$$\begin{aligned} x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_{W_3}(v, x_1) \mathcal{Y}(w_1, x_2) w_2 - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \mathcal{Y}(w_1, x_2) Y_{W_2}(v, x_1) w_2 \\ = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y_{W_1}(v, x_0) w_1, x_2) w_2 . \end{aligned} \tag{B.1.67}$$

3. L_{-1} property:

$$\frac{d}{dx} \mathcal{Y}(w_1, x) = \mathcal{Y}(L_{-1} w_1, x) , \quad w_1 \in W_1 . \tag{B.1.68}$$

The vertex operator map Y_V is an example of an intertwining operator of type $\binom{V}{V V}$ and the module vertex operator Y_W is an example of an intertwining operator of type $\binom{W}{V W}$.

The space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ forms a vector space, which we denote by $\mathcal{V}_{W_1 W_2}^{W_3}$. The dimension of these spaces are called *fusion rules*:

$$\mathcal{N}_{W_1 W_2}^{W_3} := \dim \mathcal{V}_{W_1 W_2}^{W_3} . \tag{B.1.69}$$

From the discussion above

$$\mathcal{N}_{VV}^V \geq 1 , \quad \mathcal{N}_{VW}^W \geq 1 , \tag{B.1.70}$$

for a nonzero V -module W . Let us now discuss the *skew-symmetry map*. Let $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$ be a nonzero intertwining operator. For $p \in \mathbb{Z}$ and $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$, define the operator

$$\begin{aligned} \Omega_p(\mathcal{Y}) : W_2 \otimes W_1 &\longrightarrow W_3\{x\} \\ \Omega_p(\mathcal{Y})(w_2, x_0)w_1 &:= e^{x_0 L_{-1}} \mathcal{Y}(w_1, x)w_2 \Big|_{x^n = e^{\pi i(2p+1)n} x_0^n, n \in \mathbb{C}} \quad , \quad w_1 \in W_1, w_2 \in W_2 . \end{aligned} \quad (\text{B.1.71})$$

Then we have the following theorem from [HL93].

Theorem B.1.5. *For $p \in \mathbb{Z}$, the operator $\Omega_p(\mathcal{Y})$ is an intertwining operator of type $\left(\begin{smallmatrix} W_3 \\ W_2 W_1 \end{smallmatrix}\right)$ and $\Omega_{-p-1} \circ \Omega_p = \Omega_p \circ \Omega_{-p-1} = \mathbb{1}$. In particular, $\Omega_p : \mathcal{V}_{W_1 W_2}^{W_3} \longrightarrow \mathcal{V}_{W_2 W_1}^{W_3}$ is an isomorphism and*

$$\mathcal{N}_{W_1 W_2}^{W_3} = \mathcal{N}_{W_2 W_1}^{W_3} . \quad (\text{B.1.72})$$

One can also define the *contragredient* of an intertwining operator as follows: given an intertwining operator $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$ and $r \in \mathbb{Z}$, we define the contragredient of $\mathcal{Y}$, denoted by $A_r(\mathcal{Y})$, by

$$\begin{aligned} A_r(\mathcal{Y}) : W_1 \otimes W'_3 &\longrightarrow W'_2\{x\} \\ w_1 \otimes w'_3 &\longmapsto A_r(\mathcal{Y})(w_1, x)w'_3 , \end{aligned} \quad (\text{B.1.73})$$

where $A_r(\mathcal{Y})(w_1, x)w'_3$ is defined by

$$\langle A_r(\mathcal{Y})(w_1, x)w'_3, w_2 \rangle = \left\langle w'_3, \mathcal{Y} \left(e^{x L_1} e^{(2r+1)\pi i L_0} x^{-2L_0} w_1, x^{-1} \right) w_2 \right\rangle . \quad (\text{B.1.74})$$

For $W_1 = V$ and $W_2 = W_3 = W$, the operator $A_r(Y_W)$ agrees with the contragredient vertex operator $Y_{W'}$ defined in (B.1.63). We have the following theorem from [HL93].

Theorem B.1.6. *The contragredient operator $A_r(\mathcal{Y})$ of an intertwining operator $\mathcal{Y}$ of type $\left(\begin{smallmatrix} W_3 \\ W_1 W_2 \end{smallmatrix}\right)$ is an intertwining operator of type $\left(\begin{smallmatrix} W'_2 \\ W_1 W'_3 \end{smallmatrix}\right)$. Moreover,*

$$A_{-r-1} \circ A_r = A_r \circ A_{-r-1} = \mathbb{1} . \quad (\text{B.1.75})$$

In particular, the map $\mathcal{Y} \mapsto A_r(\mathcal{Y})$ defines a linear isomorphism

$$A_r : \mathcal{V}_{W_1 W_2}^{W_3} \longrightarrow \mathcal{V}_{W_1 W'_3}^{W'_2} , \quad (\text{B.1.76})$$

and we have

$$\mathcal{N}_{W_1 W_2}^{W_3} = \mathcal{N}_{W_1 W'_3}^{W'_2} . \quad (\text{B.1.77})$$

B.2 Tensor product of modules of a VOA

Let V be a VOA and W_1, W_2 two V -modules. We would like to define the tensor product of modules $W_1 \boxtimes W_2$ such that it is also a V -module. The usual tensor product of vector spaces $W_1 \otimes W_2$ does

not work: the usual module vertex operator is

$$Y_{W_1 \otimes W_2}(v, x) = Y_{W_1}(v, x) \otimes \mathbf{1} + \mathbf{1} \otimes Y_{W_2}(v, x) , \quad (\text{B.2.1})$$

so that the Virasoro generators act as

$$L_n \otimes \mathbf{1} + \mathbf{1} \otimes L_n , \quad (\text{B.2.2})$$

which satisfies the Virasoro algebra but with central charge $2c$. A solution to this problem based on physical reasoning was given by Moore-Seiberg in [MS89c, MS89a]. In a series of papers, Huang-Lepowsky [HL95, Hua95a, HL93, HL94] rigourised the Moore-Seiberg definition of tensor product. Huang later used this tensor product to prove that the category of modules of a *rational* VOA is a modular tensor category [HKJL15, Hua05]. In this appendix, we will review the Huang-Lepowsky construction and the construction of the modular tensor category.

We begin by defining $P(z)$ -tensor product of modules.

Definition B.2.1. For $z \in \mathbb{C}^\times$, the $P(z)$ -product of two V -modules W_1, W_2 is a pair $(W_3, I_{P(z)})$ where W_3 is a V -module and ⁴ $I_{P(z)} : W_1 \otimes W_2 \longrightarrow \overline{W}_3$ is an *intertwining map*, i.e.,

$$I_{P(z)}(w_1 \otimes w_2) := \mathcal{Y}(w_1, z)w_2 , \quad w_1 \otimes w_2 \in W_1 \otimes W_2 , \quad (\text{B.2.5})$$

for some intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$. The $P(z)$ -tensor product of W_1, W_2 is then defined by the following universal property: the $P(z)$ -tensor product of W_1, W_2 , denoted by $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ is a $P(z)$ -product such that for any other $P(z)$ -product $(W_3, I_{P(z)})$, there exists a *unique* V -module map $f : W_1 \boxtimes_{P(z)} W_2 \longrightarrow W_3$, with the unique extension ⁵ $\bar{f} : \overline{W_1 \boxtimes_{P(z)} W_2} \longrightarrow \overline{W}_3$, such that $I_{P(z)} = \bar{f} \circ \boxtimes_{P(z)}$, i.e., the following diagram commutes:

⁴Recall that $\overline{W}_3$ denotes the algebraic completion of W_3 defined by

$$\overline{W}_3 = \prod_{n \in \mathbb{C}} (W_3)_{(n)} . \quad (\text{B.2.3})$$

We use the principal branch of logarithm

$$\log z = \log |z| + i \arg(z) , \quad 0 \leq \arg(z) < 2\pi , \quad (\text{B.2.4})$$

to define rational powers of z as $z^{-n-1} = e^{-(n+1) \log z}$.

⁵Given a V -module map $f : W_1 \rightarrow W_2$, its extension $\bar{f} : \overline{W}_1 \rightarrow \overline{W}_2$ is defined as

$$\bar{f} \left(\sum_{n \in \mathbb{C}} w_{(n)} \right) = \sum_{n \in \mathbb{C}} f(w_{(n)}) , \quad (\text{B.2.6})$$

where $w_{(n)} \in (W_1)_{(n)}$. This extension is unique since V -module maps preserve the grading.

$$\begin{array}{ccc}
\overline{W_1 \boxtimes_{P(z)} W_2} & \xrightarrow{\bar{f}} & \overline{W_3} \\
& \nwarrow \boxtimes_{P(z)} & \uparrow I \\
& & W_1 \otimes W_2
\end{array}$$

Remark B.2.1. By standard arguments, because of the universal property the $P(z)$ -tensor product of two V -modules, if it exists, is unique up to unique isomorphism.

The rigorous construction of $P(z)$ -tensor product of V -modules is a laborious task. We will not go in full details but summarize the construction.

The strategy is to find a subspace $W_1 \boxtimes W_2 \subset (W_1 \otimes W_2)^\vee$ which gives the tensor product. Let $\mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3}$. For a fixed $z \in \mathbb{C}^\times$, we can construct a linear map

$$\begin{aligned}
\lambda_{\mathcal{Y}}^z : W_3' &\longrightarrow (W_1 \otimes W_2)^\vee \\
w_3' &\longmapsto \lambda_{\mathcal{Y}}^z(w_3') ,
\end{aligned} \tag{B.2.7}$$

where

$$\lambda_{\mathcal{Y}}^z(w_3')(w_1 \otimes w_2) := \langle w_3', \mathcal{Y}(w_1, z)w_2 \rangle . \tag{B.2.8}$$

We then define

$$W_1 \boxtimes_{P(z)} W_2 := \text{Span}_{\mathbb{C}} \left\{ \lambda_{\mathcal{Y}}^z(w_3') : W_3 \text{ a } V\text{-module}, w_3' \in W_3', \mathcal{Y} \in \mathcal{V}_{W_1 W_2}^{W_3} \right\} . \tag{B.2.9}$$

Thus the map $\mathcal{Y} \mapsto \lambda_{\mathcal{Y}}^z$ is a linear map from $\mathcal{V}_{W_1 W_2}^{W_3}$ and $W_1 \boxtimes_{P(z)} W_2$. The grading on $W_1 \boxtimes_{P(z)} W_2$ is given by declaring

$$\text{wt } \lambda_{\mathcal{Y}}^z(w_3') = \text{wt}(w_3') . \tag{B.2.10}$$

We can then write

$$W_1 \boxtimes_{P(z)} W_2 = \coprod_{n \in \mathbb{C}} (W_1 \boxtimes_{P(z)} W_2)_{(n)} . \tag{B.2.11}$$

We now want to construct a module vertex operator

$$\begin{aligned}
Y_{P(z)} : V \otimes (W_1 \boxtimes_{P(z)} W_2) &\longrightarrow (W_1 \boxtimes_{P(z)} W_2)\{x\} , \\
u \otimes \lambda_{\mathcal{Y}}^z(w_3') &\longmapsto Y_{P(z)}(u, x) \lambda_{\mathcal{Y}}^z(w_3') .
\end{aligned} \tag{B.2.12}$$

Since

$$Y_{P(z)}(u, x) \lambda_{\mathcal{Y}}^z(w_3') \in (W_1 \otimes W_2)^\vee \{x\} , \tag{B.2.13}$$

we define its action on $w_1 \otimes w_2 \in W_1 \otimes W_2$ by

$$(Y_{P(z)}(u, x) \lambda_{\mathcal{Y}}^z(w'_3)) (w_1 \otimes w_2) = \langle Y_{W'_3}(u, x) w'_3, \mathcal{Y}(w_1, x) w_2 \rangle, \quad (\text{B.2.14})$$

where $(W'_3, Y_{W'_3})$ is the contragredient of (W_3, Y_{W_3}) . We would like the pair $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ to be a V -module. This is not true in general, we need to impose some restrictions on the VOA. The appropriate condition comes from RCFTs:

1. There are only finitely many irreducible V -modules.
2. All V -modules are completely reducible, i.e., isomorphic to the direct sum of finitely many irreducible V -modules
3. All fusion rules are finite.

Such VOAs have been called *rational* in [Hua95a]⁶. Under these assumptions $(W_1 \boxtimes_{P(z)} W_2, Y_{P(z)})$ is a V -module. We now define

$$W_1 \boxtimes_{P(z)} W_2 := (W_1 \boxtimes_{P(z)} W_2)' . \quad (\text{B.2.15})$$

This gives us a V -module. To construct the intertwining map $\boxtimes_{P(z)}$ such that $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ satisfies the universal property of the $P(z)$ -tensor product, we will need the following proposition.

Proposition B.2.1. *Let V be a rational VOA and W_1, W_2, W_3 be three V -modules. Then there is a linear isomorphism of the space $\text{Hom}_V(W'_3, W_1 \boxtimes_{P(z)} W_2)$ of V -module maps from W'_3 to $W_1 \boxtimes_{P(z)} W_2$ and the space $\mathcal{V}_{W_1 W_2}^{W_3}$ of intertwining operators of type $\binom{W_3}{W_1 W_2}$.*

Sketch of Proof. The map from $\mathcal{V}_{W_1 W_2}^{W_3}$ to $\text{Hom}_V(W'_3, W_1 \boxtimes_{P(z)} W_2)$ is given by $\mathcal{Y} \mapsto \lambda_{\mathcal{Y}}^z$ defined in (B.2.7). The fact that this map is a V -module map is obvious from the definition (B.2.14). The construction of the inverse map is slightly more involved. Given $J \in \text{Hom}_V(W'_3, W_1 \boxtimes_{P(z)} W_2)$, we define $\mathcal{Y}_J^z : W_1 \otimes W_2 \longrightarrow W_3$ by defining its pairing with W'_3 as follows:

$$\langle w'_3, \mathcal{Y}_J^z(w_1 \otimes w_2) \rangle = J(w'_3)(w_1 \otimes w_2) . \quad (\text{B.2.16})$$

We then define the map

$$\begin{aligned} \mathcal{Y}_J : W_1 \otimes W_2 &\longrightarrow W_3\{x\} \\ w_1 \otimes w_2 &\longmapsto \mathcal{Y}_J(w_1, x) w_2 := x^{L_0} z^{-L_0} \mathcal{Y}_J^z(x^{-L_0} z^{L_0} w_1 \otimes x^{-L_0} z^{L_0} w_2) . \end{aligned} \quad (\text{B.2.17})$$

⁶The definition of rationality in [Hua05] is that it admits an $\mathbb{N}_0$ -grading compatible with V -action. Such modules have been called *admissible* V -modules in [ABD02]. V is called a rational VOA if every admissible V -module is isomorphic to direct sum of simple admissible V -modules [ABD02].

One then shows that $\mathcal{Y}_J \in \mathcal{V}_{W_1 W_2}^{W_3}$ and is the inverse to the map $\mathcal{Y} \mapsto \lambda_{\mathcal{Y}}^z$. $\square$

Now consider the identity map

$$\mathbb{1}_{P(z)} : W_1 \boxtimes_{P(z)} W_2 \longrightarrow W_1 \boxtimes_{P(z)} W_2 . \quad (\text{B.2.18})$$

By Proposition B.2.1, and the natural identification $W_3'' \cong W_3$ for any V -module W_3 , there is a unique intertwining operator

$$\mathcal{Y}_{W_1 \boxtimes_{P(z)} W_2} := \mathcal{Y}_{1_{P(z)}} \quad (\text{B.2.19})$$

of type $\binom{W_1 \boxtimes_{P(z)} W_2}{W_1 W_2}$. We then define the $P(z)$ -product $\boxtimes_{P(z)}$ by

$$\begin{aligned} \boxtimes_{P(z)} : W_1 \otimes W_2 &\longrightarrow \overline{W_1 \boxtimes_{P(z)} W_2} \\ w_1 \otimes w_2 &\longmapsto w_1 \boxtimes_{P(z)} w_2 := \boxtimes_{P(z)}(w_1 \otimes w_2) := \mathcal{Y}_{W_1 \boxtimes_{P(z)} W_2}(w_1, z)w_2 . \end{aligned} \quad (\text{B.2.20})$$

Theorem B.2.1. *Let V be a rational VOA and W_1, W_2 be two V -modules. Then the pair $(W_1 \boxtimes_{P(z)} W_2, \boxtimes_{P(z)})$ defined in (B.2.15) and (B.2.20) is the $P(z)$ -tensor product of W_1, W_2 .*

Let $f_1 : W_1 \rightarrow W_3$ and $f_2 : W_2 \rightarrow W_4$ be V -module homomorphisms. We now define the $P(z)$ -tensor product

$$f_1 \boxtimes_{P(z)} f_2 : W_1 \boxtimes_{P(z)} W_2 \longrightarrow W_3 \boxtimes_{P(z)} W_4 , \quad (\text{B.2.21})$$

of these homomorphisms. Let us first define the extension

$$\overline{f_1 \boxtimes_{P(z)} f_2} : \overline{W_1 \boxtimes_{P(z)} W_2} \longrightarrow \overline{W_3 \boxtimes_{P(z)} W_4} , \quad (\text{B.2.22})$$

by

$$\overline{f_1 \boxtimes_{P(z)} f_2}(w_1 \boxtimes_{P(z)} w_2) = f_1(w_1) \boxtimes_{P(z)} f_2(w_2) = \mathcal{Y}_{W_3 \boxtimes_{P(z)} W_4}(f_1(w_1), z)f_2(w_2) . \quad (\text{B.2.23})$$

Let us expand

$$\mathcal{Y}_{W_1 \boxtimes_{P(z)} W_2}(w_1, x)w_2 = \sum_{n \in \mathbb{C}} \left((\mathcal{Y}_{W_1 \boxtimes_{P(z)} W_2})_{(n)}(w_1) \cdot w_2 \right) x^{-n-1} . \quad (\text{B.2.24})$$

One can show that

$$W_1 \boxtimes_{P(z)} W_2 = \text{Span}_{\mathbb{C}} \{ (\mathcal{Y}_{W_1 \boxtimes_{P(z)} W_2})_{(n)}(w_1) \cdot w_2 : n \in \mathbb{C}, w_1 \in W_1, w_2 \in W_2 \} . \quad (\text{B.2.25})$$

Then, we define

$$f_1 \boxtimes_{P(z)} f_2 \left((\mathcal{Y}_{W_1 \boxtimes_{P(z)} W_2})_{(n)}(w_1) \cdot w_2 \right) := \left((\mathcal{Y}_{W_3 \boxtimes_{P(z)} W_4})_{(n)}(f_1(w_1)) \cdot f_2(w_2) \right) , \quad (\text{B.2.26})$$

and then extend linearly. One can show that this is a V -module homomorphism.

B.2.1 The category of VOA modules

Let V be a VOA. Then the set of all V -modules as objects and V -module homomorphisms as morphisms is a category. Let us denote this category by $\mathcal{C}_V$. Given that we have constructed tensor product of V -modules in the previous section, we would like to conclude that $\mathcal{C}_V$ is a tensor (also called monoidal) category. This is indeed true and in fact something more is true: under certain conditions on V , $\mathcal{C}_V$ is a modular tensor category. Let us briefly discuss the conditions on V :

1. V is CFT type i.e., $V_{(n)} = 0$ for $n < 0$, $V_{(0)} = \mathbb{C}\mathbf{1}$ and V is selfdual $V \cong V'$.
2. V is regular⁷, i.e., every $\mathbb{N}_0$ -gradable weak V -module is completely reducible, i.e., isomorphic to the direct sum of finitely many irreducible V -modules.
3. V is C_2 -cofinite⁸, i.e.,

$$\dim V/C_2(V) < \infty , \quad (\text{B.2.27})$$

where

$$C_2(V) = \text{Span}_{\mathbb{C}}\{u_{-2} \cdot v : u, v \in V\} . \quad (\text{B.2.28})$$

Following [SST24], we will call VOAs satisfying Conditions [item 1](#), [item 2](#) and [item 3](#) as “nice” VOAs. It was shown in [ABD02, Li99] that in the presence of Condition [item 1](#), Condition [item 2](#) and [item 3](#) is equivalent to regularity of V , i.e., every weak V -module is completely reducible. It was shown in [Li99, DLM97, Zhu95] that these conditions imply the conditions for rationality used to construct the $P(z)$ -tensor product.

Before we discuss the modular and tensor structure on $\mathcal{C}_V$, let us briefly recall what is a modular tensor category, we refer the reader to [Tur92, BK01] for more details.

A modular tensor category $\mathcal{C}$ is equipped with the following structures:

1. Tensor bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, with *unit object* $\mathbf{1}_{\mathcal{C}}$, *associator isomorphism*

$$\alpha_{W_1, W_2, W_3} : W_1 \boxtimes (W_2 \boxtimes W_3) \longrightarrow (W_1 \boxtimes W_2) \boxtimes W_3 , \quad (\text{B.2.29})$$

left and right unit isomorphism

$$\ell_W : \mathbf{1}_{\mathcal{C}} \boxtimes W \longrightarrow W , \quad r_W : W \boxtimes \mathbf{1}_{\mathcal{C}} \longrightarrow W , \quad (\text{B.2.30})$$

such that the following diagrams

⁷The notion of regular VOA was introduced in [ABD02]

⁸ C_2 -cofiniteness was defined in [Zhu95].

$$\begin{array}{ccc}
& W_1 \boxtimes (W_2 \boxtimes (W_3 \boxtimes W_4)) & \\
\alpha_{W_1, W_2, W_3 \boxtimes W_4} \swarrow & & \searrow \mathbb{1}_{W_1} \boxtimes \alpha_{W_2, W_3, W_4} \\
(W_1 \boxtimes W_2) \boxtimes (W_3 \boxtimes W_4) & & W_1 \boxtimes ((W_2 \boxtimes W_3) \boxtimes W_4) \\
\downarrow \alpha_{W_1 \boxtimes W_2, W_3, W_4} & & \downarrow \alpha_{W_1, W_2 \boxtimes W_3, W_4} \\
((W_1 \boxtimes W_2) \boxtimes W_3) \boxtimes W_4 & \xrightarrow{\alpha_{W_1, W_2, W_3} \boxtimes \mathbb{1}_{W_4}} & (W_1 \boxtimes (W_2 \boxtimes W_3)) \boxtimes W_4
\end{array}$$

Figure B.1: Pentagon identity for tensor category

$$\begin{array}{ccc}
(W_1 \boxtimes \mathbf{1}_C) \boxtimes W_2 & \xrightarrow{\alpha_{W_1, \mathbf{1}_C, W_2}} & W_1 \boxtimes (\mathbf{1}_C \boxtimes W_2) \\
& \searrow r_{W_1} \boxtimes \mathbb{1}_{W_2} & \swarrow \mathbb{1}_{W_1} \boxtimes \ell_{W_2} \\
& W_1 \boxtimes W_2 &
\end{array}$$

Figure B.2: Triangle identity for tensor category

commute and the morphisms $\ell_{\mathbf{1}_C} : \mathbf{1}_C \boxtimes \mathbf{1}_C \rightarrow \mathbf{1}_C$ and $r_{\mathbf{1}_C} : \mathbf{1}_C \boxtimes \mathbf{1}_C \rightarrow \mathbf{1}_C$ are equal. The category $\mathcal{C}$ equipped with these structures is called *tensor* or *monoidal* category.

2. A braiding isomorphism $c_{W_1, W_2} : W_1 \boxtimes W_2 \rightarrow W_2 \boxtimes W_1$ such that the hexagon and triangle diagrams below commute.

$$\begin{array}{ccccc}
& (W_1 \boxtimes W_2) \boxtimes W_3 & & & \\
c_{W_1, W_2} \boxtimes \mathbb{1}_{W_3} \swarrow & & \searrow \alpha_{W_1, W_2, W_3}^{-1} & & \\
(W_2 \boxtimes W_1) \boxtimes W_3 & & W_1 \boxtimes (W_2 \boxtimes W_3) & & \\
\downarrow \alpha_{W_2, W_1, W_3}^{-1} & & \downarrow c_{W_1, W_2} \boxtimes \mathbb{1}_{W_3} & & \\
W_2 \boxtimes (W_1 \boxtimes W_3) & & (W_2 \boxtimes W_3) \boxtimes W_1 & & \\
& \searrow \mathbb{1}_{W_2} \boxtimes c_{W_1, W_3} & \swarrow \alpha_{W_2, W_3, W_1}^{-1} & & \\
& W_2 \boxtimes (W_3 \boxtimes W_1) & & &
\end{array}$$

Figure B.3: Hexagon diagram for braided monoidal category

$$\begin{array}{ccc}
W \otimes \mathbf{1}_C & & \\
\downarrow c_{W, \mathbf{1}_C} & \searrow r_W & \\
\mathbf{1}_C \boxtimes W & \nearrow \ell_W & W
\end{array}$$

Figure B.4: Triangle diagram for braided monoidal category

This structure makes $\mathcal{C}$ a *braided tensor category*.

3. Rigidity axioms: for every object W , there is a (*right*) *dual object* $W^\vee$ along with morphisms

$$e_W : W^\vee \boxtimes W \longrightarrow \mathbf{1}_C, \quad i_W : \mathbf{1}_C \longrightarrow W \boxtimes W^\vee, \quad (\text{B.2.31})$$

called *annihilation* and *creation morphisms* such that the diagram below is commutative.

$$\begin{array}{ccccc}
W & \xrightarrow{\ell_W^{-1}} & \mathbf{1}_C \boxtimes W & \xrightarrow{i_W \boxtimes \mathbf{1}_W} & (W \boxtimes W^\vee) \boxtimes W \\
\uparrow \mathbf{1}_W & & & & \downarrow \alpha_{W, W^\vee, W}^{-1} \\
W & \xleftarrow{r_W} & W \boxtimes \mathbf{1}_C & \xleftarrow{\mathbf{1}_W \boxtimes e_W} & W \boxtimes (W^\vee \boxtimes W) \\
\\
W^\vee & \xrightarrow{r_{W^\vee}^{-1}} & W^\vee \boxtimes \mathbf{1}_C & \xrightarrow{\mathbf{1}_{W^\vee} \boxtimes i_W} & W^\vee \boxtimes (W \boxtimes W^\vee) \\
\uparrow \mathbf{1}_{W^\vee} & & & & \downarrow \alpha_{W^\vee, W, W^\vee} \\
W^\vee & \xleftarrow{\ell_{W^\vee}} & \mathbf{1}_C \boxtimes W^\vee & \xleftarrow{e_W \boxtimes \mathbf{1}_{W^\vee}} & (W^\vee \boxtimes W) \boxtimes W^\vee
\end{array}$$

Figure B.5: Coherence condition for right creation and annihilation morphism

Similarly, *left duals*, denoted by ${}^\vee W$ and corresponding annihilation, creation morphisms

$$\tilde{e}_W : W \boxtimes {}^\vee W \longrightarrow \mathbf{1}_C, \quad \tilde{i}_W : \mathbf{1}_C \longrightarrow {}^\vee W \boxtimes W, \quad (\text{B.2.32})$$

such that the diagrams below commute.

$$\begin{array}{ccccc}
W & \xrightarrow{r_W^{-1}} & \mathbf{1}_C \boxtimes W & \xrightarrow{\mathbf{1}_W \boxtimes \tilde{i}_W} & W \boxtimes (W^\vee \boxtimes W) \\
\uparrow \mathbf{1}_W & & & & \downarrow \alpha_{W, W^\vee, W} \\
W & \xleftarrow{\ell_W} & W \boxtimes \mathbf{1}_C & \xleftarrow{\tilde{e}_W \boxtimes \mathbf{1}_W} & (W \boxtimes W^\vee) \boxtimes W \\
\\
W^\vee & \xrightarrow{\ell_{W^\vee}^{-1}} & W^\vee \boxtimes \mathbf{1}_C & \xrightarrow{\tilde{i}_W \boxtimes \mathbf{1}_W} & (W^\vee \boxtimes W) \boxtimes W^\vee \\
\uparrow \mathbf{1}_{W^\vee} & & & & \downarrow \alpha_{W^\vee, W, W^\vee}^{-1} \\
W^\vee & \xleftarrow{r_{W^\vee}} & \mathbf{1}_C \boxtimes W^\vee & \xleftarrow{\mathbf{1}_{W^\vee} \boxtimes \tilde{e}_W} & W^\vee \boxtimes (W \boxtimes W^\vee)
\end{array}$$

Figure B.6: Coherence condition for left creation and annihilation morphism

We must also have $\mathbf{1}_C^\vee \cong {}^\vee \mathbf{1}_C \cong \mathbf{1}_C$. A braided tensor category satisfying rigidity axioms is called a *rigid braided tensor category*.

4. Linearity and Semisimplicity: The space of morphisms $\text{Hom}_C(W_1, W_2)$ are vector spaces⁹ and $\mathcal{C}$ is endowed with the structure of direct sum and that every object is isomorphic to the direct sum of a finite number of *simple objects*¹⁰.
5. Finiteness: $\mathcal{C}$ must have finitely many isomorphism classes of simple objects and the unit object $\mathbf{1}_C$ must be one of them. Moreover $\text{Hom}_C(\mathbf{1}_C, \mathbf{1}_C)$ must be one-dimensional ($\mathbb{C}$ for us). These structures define a *rigid braided fusion category*.
6. Ribbon structure: For every object W , there is an isomorphism $\theta_W : W \longrightarrow W$, called the *twist map*, satisfying the *balancing axioms*:

$$\begin{aligned}
\theta_{W_1 \boxtimes W_2} &= c_{W_2, W_1} \circ c_{W_1, W_2} \circ (\theta_{W_1} \boxtimes \theta_{W_2}) , \\
\theta_{W^\vee} &= (\theta_W)^\vee , \\
\theta_{\mathbf{1}_C} &= \mathbf{1}_{\mathbf{1}_C} ,
\end{aligned} \tag{B.2.33}$$

where for a morphism $f \in \text{Hom}_C(W_1, W_2)$, the dual $f^\vee \in \text{Hom}_C(W_2^\vee, W_1^\vee)$ is defined as

$$f^\vee := (e_{W_2} \boxtimes \mathbf{1}_{W_1^\vee}) \circ (\mathbf{1}_{W_2^\vee} \boxtimes f \boxtimes \mathbf{1}_{W_1^\vee}) \circ (\mathbf{1}_{W_2^\vee} \boxtimes i_{W_1}) . \tag{B.2.34}$$

⁹We will work with complex vector spaces.

¹⁰Linearity involves other technical conditions which allow us to define *subobjects* of an object. Simple objects are those objects whose only subobjects are the *zero object* and itself. We will not go into the details, the reader is referred to [Tur94, BK01] for details.

This is the additional data for a *ribbon category*. In a ribbon category, the left and right duals coincide.

7. Nondegeneracy: Let $\{W^a\}_{a \in \mathcal{A}}$ be the isomorphism classes of simple objects with $W^0 = \mathbf{1}_C$. Then the S -matrix¹¹ defined by

$$\begin{aligned} S_{ab} := & \ell_{\mathbf{1}_C} \circ (e_{W^a} \boxtimes e_{(W^b)^\vee}) \circ (\mathbf{1}_{(W^a)^\vee} \boxtimes c_{W^a W^b} \boxtimes \mathbf{1}_{(W^b)^\vee}) \\ & \circ (\mathbf{1}_{(W^a)^\vee} \boxtimes c_{W^a W^b} \boxtimes \mathbf{1}_{(W^b)^\vee}) \circ (i_{(W^a)^\vee} \boxtimes i_{W^a}) \circ \ell_{\mathbf{1}_C}^{-1}, \end{aligned} \quad (\text{B.2.35})$$

where $\mathbf{1}_{W^a} : W^a \rightarrow W^a$ is the identity morphism, is invertible. This condition on a rigid braided ribbon fusion category defines a modular tensor category (MTC). The number of isomorphism classes of simple objects is called the *rank* of the MTC.

We now briefly summarize how $\mathcal{C}_V$ is endowed with these structures. Throughout, we will assume that V is a rational VOA, although many of the results below can be proved under much weaker assumptions [Hua04, HKJL15, Hua05]. Firstly, it is clear that $\mathcal{C}_V$ is a linear category. We define the tensor product on $\mathcal{C}_V$ by declaring

$$W_1 \boxtimes W_2 := W_1 \boxtimes_{P(1)} W_2, \quad (\text{B.2.36})$$

i.e., the $P(z)$ -tensor product at $z = 1$. The construction of associativity and left, right isomorphisms is slightly more involved.

Theorem B.2.2. *Let $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Then there exists a unique V -module isomorphism*

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1 - z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \rightarrow (W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3, \quad (\text{B.2.37})$$

called *associativity isomorphism*, such that for every $w_1 \in W_1, w_2 \in W_2$ and $w_3 \in W_3$, the extension

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1 - z_2)}} : \overline{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)} \rightarrow \overline{(W_1 \boxtimes_{P(z_1 - z_2)} W_2) \boxtimes_{P(z_2)} W_3}, \quad (\text{B.2.38})$$

satisfies

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1 - z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) = (w_1 \boxtimes_{P(z_1 - z_2)} w_2) \boxtimes_{P(z_2)} w_3. \quad (\text{B.2.39})$$

Sketch of Proof. The construction of the associativity isomorphism uses the universal property of

¹¹The map $S_{a,b} \in \text{Hom}(\mathbf{1}_C, \mathbf{1}_C) \cong \mathbb{C}$ and hence is a complex number.

the $P(z)$ -tensor product. Let us introduce the notation

$$\begin{aligned}
W_4 &:= (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \\
W_6 &:= W_1 \boxtimes_{P(z_1-z_2)} W_2 \\
\mathcal{Y}_3 &:= \mathcal{Y}_{W_6 \boxtimes_{P(z_2)} W_3} \\
\mathcal{Y}_4 &:= \mathcal{Y}_{W_1 \boxtimes_{P(z_1-z_2)} W_2} \\
\mathcal{Y}^2 &:= \mathcal{Y}_{W_2 \boxtimes_{P(z_2)} W_3} \\
\mathcal{Y}^1 &:= \mathcal{Y}_{W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)}
\end{aligned} \tag{B.2.40}$$

Under the assumption of rationality of V , one can show that there exists a V -module W_5 and intertwining operator $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of the type $\binom{W_4}{W_1 W_5}$ and $\binom{W_5}{W_2 W_3}$ respectively, such that for all $w'_4 \in W'_4, w_1 \in W_1, w_2 \in W_2$ and $w_3 \in W_3$ we have [?] ¹²

$$\begin{aligned}
\langle w'_4, (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \rangle &= \langle w'_4, \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3 \rangle \\
&= \langle w'_4, \mathcal{Y}_1(w_1, z_1) \mathcal{Y}_2(w_2, z_2)w_3 \rangle.
\end{aligned} \tag{B.2.41}$$

Let $I_2 = \mathcal{Y}_2(\bullet, z_2)$, then (W_5, I_2) is a $P(z_2)$ -product. By the universal property of the $P(z_2)$ -tensor product $(W_2 \boxtimes_{P(z_2)} W_3, \boxtimes_{P(z_2)})$, there exists a unique V -module map $f : W_2 \boxtimes_{P(z_2)} W_3 \rightarrow W_5$, such that $I_2 = \bar{f} \circ \boxtimes_{P(z_2)}$, i.e.,

$$I_2(w_2 \otimes w_3) = \mathcal{Y}_2(w_2, z_2)w_3 = \bar{f}(w_2 \boxtimes_{P(z_2)} w_3) = \bar{f}(\mathcal{Y}^2(w_2, z_2)w_3), \tag{B.2.42}$$

Now let us denote $I_1 := \mathcal{Y}_1(\bullet, z_1)f(\bullet)$. It can be shown that I_1 is an intertwining map of type $\binom{W_4}{W_1 W_2 \boxtimes_{P(z_2)} W_3}$, and the pair (W_4, I_1) is an $P(z_1)$ -product. Then, by the universal property of the $P(z_1)$ -tensor product $(W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3), \boxtimes_{P(z_1)})$, there exists a unique V -module map

$$\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1-z_2)} : W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) \longrightarrow (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \quad (= W_4) \tag{B.2.43}$$

such that $I_1 = \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1-z_2)}} \circ \boxtimes_{P(z_1)}$. Then we have

$$\overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1-z_2)}}(w_1 \boxtimes_{P(z_1)} w) = I_1(w_1 \otimes w) = \mathcal{Y}_1(w_1, z_1)f(w) \tag{B.2.44}$$

for $w_1 \in W_1$ and $w \in W_2 \boxtimes_{P(z_2)} W_3$. Some calculations then show that

$$\left\langle w'_4, \overline{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1-z_2)}}(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) \right\rangle = \langle w'_4, (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3 \rangle \tag{B.2.45}$$

for $w'_4 \in W'_4, w_1 \in W_1, w_2 \in W_2$ and $w_3 \in W_3$. The same construction gives a V -module map from $(W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3$ to $W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3)$ which is the inverse of $\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1-z_2)}$. $\square$

This theorem give us a z -dependent associativity isomorphism. But we want to construct associativity isomorphism for the $P(1)$ -tensor product. This is achieved by the *parallel transport map*.

¹²In Physics, this is the statement about OPE of primary fields.

Theorem B.2.3. Let $z_1, z_2 \in \mathbb{C}^\times$ and $\gamma : [0, 1] \rightarrow \mathbb{C}^\times$ be paths in $\mathbb{C}^\times$ from z_1 to z_2 . Then there exists a unique V -module isomorphism

$$\mathcal{T}_{[\gamma]} : W_1 \boxtimes_{P(z_1)} W_2 \longrightarrow W_1 \boxtimes_{P(z_2)} W_2 , \quad (\text{B.2.46})$$

called the parallel transport isomorphism associated to the homotopy class $[\gamma]$ such that the extension

$$\overline{\mathcal{T}_{[\gamma]}} : \overline{W_1 \boxtimes_{P(z_1)} W_2} \longrightarrow \overline{W_1 \boxtimes_{P(z_2)} W_2} , \quad (\text{B.2.47})$$

satisfies the following property: For $w_1 \in W_1$ and $w_2 \in W_2$, we have

$$\overline{\mathcal{T}_{[\gamma]}}(w_1 \boxtimes_{P(z_1)} w_2) = \mathcal{Y}_{W_1 \boxtimes_{P(z_2)} W_2}(w_1, x) w_2 \Big|_{x^n = e^{n\ell_p(z_1)}} , \quad (\text{B.2.48})$$

where $\mathcal{Y}_{W_1 \boxtimes_{P(z_2)} W_2}$ is the intertwining operator of type $\left(\begin{smallmatrix} W_1 \boxtimes_{P(z_2)} W_2 \\ W_1 \quad W_2 \end{smallmatrix} \right)$ such that

$$\bullet \boxtimes_{P(z_2)} \bullet = \mathcal{Y}_{W_1 \boxtimes_{P(z_2)} W_2}(\bullet, z_2) \bullet , \quad (\text{B.2.49})$$

and

$$\ell_p(z_1) := \log |z_1| + i \text{Arg}(z_1) + 2\pi i p , \quad (\text{B.2.50})$$

is the value of logarithm of z_1 obtained by analytically extending the value $\log z_2$ along the path γ .

Sketch of Proof. We again use the universal property. Let

$$I(z_1)(w_1 \otimes w_2) := \mathcal{Y}(w_1, x) w_2 \Big|_{x^n = e^{n\ell_p(z_1)}} \in W_1 \boxtimes_{P(z_2)} W_2 . \quad (\text{B.2.51})$$

Then $(W_1 \boxtimes_{P(z_2)} W_2, I)$ is a $P(z_1)$ -product. By the universal property, there exists a module map

$$\mathcal{T}_{[\gamma], [\tilde{\gamma}]} : W_1 \boxtimes_{P(z_1)} W_2 \longrightarrow W_1 \boxtimes_{P(z_2)} W_2 , \quad (\text{B.2.52})$$

such that the diagram

$$\begin{array}{ccc} \overline{W_1 \boxtimes_{P(z_1)} W_2} & \xrightarrow{\overline{\mathcal{T}_{[\gamma]}}} & \overline{W_1 \boxtimes_{P(z_2)} W_2} \\ & \nwarrow \boxtimes_{P(z_1)} & \uparrow I \\ & & W_1 \otimes W_2 \end{array}$$

Using the universal property of $W_1 \boxtimes_{P(z_2)} W_2$, we get the inverse map

$$\mathcal{T}_{[\gamma^{-1}]} : W_1 \boxtimes_{P(z_2)} W_2 \longrightarrow W_1 \boxtimes_{P(z_1)} W_2 , \quad (\text{B.2.53})$$

where γ^{-1} is the curve γ travelling in the reverse direction. One can show that

$$\mathcal{T}_{[\gamma^{-1}]} \circ \mathcal{T}_{[\gamma]} = \mathbb{1}_{W_1 \boxtimes_{P(z_1)} W_2}, \quad \mathcal{T}_{[\gamma], [\tilde{\gamma}]} \circ \mathcal{T}_{[\gamma^{-1}]} = \mathbb{1}_{W_1 \boxtimes_{P(z_2)} W_2} . \quad (\text{B.2.54})$$

□

Using the parallel transport map, we can construct the associativity isomorphism as follows: let $z_1, z_2 \in \mathbb{C}$ with $|z_1| > |z_2| > |z_1 - z_2| > 0$ and let $\gamma_i : [0, 1] \rightarrow \mathbb{C}, i = 1, 2, 3, 4$ be curves connecting

$$\begin{aligned} \gamma_1 : 1 \rightarrow z_1, \quad \gamma_2 : 1 \rightarrow z_2, \\ \gamma_3 : z_1 - z_2 \rightarrow 1, \quad \gamma_4 : z_2 \rightarrow 1. \end{aligned} \quad (\text{B.2.55})$$

Then

$$\alpha_{W_1, W_2, W_3} : W_1 \boxtimes (W_2 \boxtimes W_3) \longrightarrow (W_1 \boxtimes W_2) \boxtimes W_3 \quad (\text{B.2.56})$$

is defined by the diagram below.

$$\begin{array}{ccc} W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(1)} W_3) & \xrightarrow{\alpha_{W_1, W_2, W_3}} & (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(1)} W_3 \\ \downarrow \mathbb{1}_{W_1} \boxtimes_{P(1)} \mathcal{T}_{[\gamma_2]} & & \uparrow \mathcal{T}_{[\gamma_4]} \\ W_1 \boxtimes_{P(1)} (W_2 \boxtimes_{P(z_2)} W_3) & & (W_1 \boxtimes_{P(1)} W_2) \boxtimes_{P(z_2)} W_3 \\ \downarrow \mathcal{T}_{[\gamma_1]} & & \uparrow \mathcal{T}_{[\gamma_3]} \boxtimes_{P(z_2)} \mathbb{1}_{W_3} \\ W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \xrightarrow{\mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_1-z_2)}} & (W_1 \boxtimes_{P(z_1-z_2)} W_2) \boxtimes_{P(z_2)} W_3 \end{array}$$

The pentagon identity Figure B.1 is proved by first proving the commutativity of the diagram below

$$\begin{array}{ccc} & W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4)) & \\ \swarrow \mathcal{A}_{P(z_1), P(z_2)}^{P(z_2), P(z_{12})} & & \searrow \mathbb{1}_{W_1} \boxtimes_{P(z_1)} \mathcal{A}_{P(z_2), P(z_3)}^{P(z_3), P(z_{23})} \\ (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} (W_3 \boxtimes_{P(z_3)} W_4) & & W_1 \boxtimes_{P(z_1)} ((W_2 \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4) \\ \downarrow \mathcal{A}_{P(z_2), P(z_3)}^{P(z_3), P(z_{23})} & & \downarrow \mathcal{A}_{P(z_1), P(z_3)}^{P(z_3), P(z_{13})} \\ ((W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_{23})} W_3) \boxtimes_{P(z_3)} W_4 & \xrightarrow{\mathcal{A}_{P(z_{13}), P(z_{23})}^{P(z_{23}), P(z_{12})} \boxtimes_{P(z_3)} \mathbb{1}} & (W_1 \boxtimes_{P(z_{13})} (W_2 \boxtimes_{P(z_{23})} W_3)) \boxtimes_{P(z_3)} W_4 \end{array}$$

Figure B.7: Pentagon identity for associativity isomorphism for $P(z)$ -tensor product. We use the notation $z_{ij} = z_i - z_j$.

Diagram B.7 commutes because the associativity isomorphism maps tensor product elements to tensor product elements

$$(w_1 \boxtimes_{P(z_1)} (w_2 \boxtimes_{P(z_2)} w_3)) \longmapsto (w_1 \boxtimes_{P(z_1-z_2)} w_2) \boxtimes_{P(z_2)} w_3. \quad (\text{B.2.57})$$

To prove the pentagon diagram [B.1](#), we use the parallel transport map. In particular the rectangle diagram shown below commutes.

$$\begin{array}{ccc}
 W_1 \boxtimes_{P(z_1)} (W_2 \boxtimes_{P(z_2)} W_3) & \longrightarrow & (W_1 \boxtimes_{P(z_{12})} W_2) \boxtimes_{P(z_2)} W_3 \\
 \uparrow (\mathbb{1}_{W_1} \boxtimes_{P(z_1)} \mathcal{T}_{\gamma_2}) \circ \mathcal{T}_{\gamma_1} & & \uparrow \mathcal{T}_{\gamma_3}^{-1} \circ (\mathcal{T}_{\gamma_4}^{-1} \boxtimes_{P(z_2)} \mathbb{1}_{W_3}) \\
 W_1 \boxtimes (W_2 \boxtimes W_3) & \longrightarrow & (W_1 \boxtimes W_2) \boxtimes W_3
 \end{array}$$

Thus the outer rectangles in [Figure B.8](#) commutes and hence by commutativity of [Figure B.7](#), we get the pentagon identity for the associativity isomorphism. This is the recurring structure of proofs for coherence conditions for all the isomorphisms we construct and hence we will omit them from now on. We will now give the construction of other isomorphisms.



The unit object of $\mathcal{C}_V$ is the VOA V itself. The left unitor isomorphism

$$\ell_W : V \boxtimes W \longrightarrow W , \quad (\text{B.2.58})$$

is constructed as follows: let Y_W be the module vertex operator. It is an intertwining operator of type $\binom{W}{V W}$. Then

$$\begin{aligned} I_{P(z)} : V \otimes W &\longrightarrow \overline{W} \\ u \otimes w &\longmapsto I_{P(z)}(u \otimes w) := Y_W(u, z)w , \end{aligned} \quad (\text{B.2.59})$$

is a $P(z)$ -product. So by universal property, we have a V -module map $\ell_{W,z} : V \boxtimes_{P(z)} W \longrightarrow W$ satisfying the property

$$\overline{\ell_{W,z}}(u \boxtimes_{P(z)} w) = I_{P(z)}(u \otimes w) = Y_W(u, z)w , \quad u \in V, w \in W . \quad (\text{B.2.60})$$

The left unitor is then defined as

$$\ell_W := \ell_{W,1} . \quad (\text{B.2.61})$$

It satisfies the property

$$\ell_W(\mathbf{1} \boxtimes w) = w , \quad w \in W , \quad (\text{B.2.62})$$

where $\mathbf{1}$ is the vacuum vector. To define the right unitor $r_W : W \boxtimes V \longrightarrow W$, we consider the unique V -module map

$$r_{W,z} : W \boxtimes_{P(z)} V \longrightarrow W , \quad (\text{B.2.63})$$

obtained using the universal property from the $P(z)$ -product $(W, \Omega_0(Y_W))$, where Ω_p is the skew symmetry map defined in (B.1.71). It satisfies

$$\overline{r_{W,z}}(w \boxtimes_{P(z)} u) = \Omega_0(Y_W)(w, z)u , \quad u \in V, w \in W . \quad (\text{B.2.64})$$

We then define

$$r_W := r_{W,1} . \quad (\text{B.2.65})$$

It satisfies

$$\overline{r_{W,w}}(w \boxtimes_{P(z)} \mathbf{1}) = e^{zL-1}w , \quad w \in W . \quad (\text{B.2.66})$$

One can show that these maps satisfy the triangle Diagram B.2. This makes $\mathcal{C}_V$ a tensor category.

Let us now define the braiding map. Let γ be a curve from 1 to -1 in the closed upper half plane with 0 removed traveling in the counterclockwise direction. Let I be the $P(1)$ intertwining map of type $\binom{W_2 \boxtimes_{P(1)} W_1}{W_1 W_2}$ defined by

$$I(w_1 \otimes w_2) = e^{L-1} \overline{\mathcal{T}_{[\gamma]}}(w_2 \boxtimes_{P(-1)} w_1) , \quad w_1 \in W_1, \quad w_2 \in W_2 . \quad (\text{B.2.67})$$

Then the braiding map is the unique module map

$$c_{W_1, W_2} : W_1 \boxtimes W_2 \rightarrow W_2 \boxtimes W_1 \quad (\text{B.2.68})$$

such that

$$I(w_1 \otimes w_2) = \overline{c_{W_1, W_2}}(w_1 \boxtimes_{P(1)} w_2) . \quad (\text{B.2.69})$$

It is characterized by

$$\overline{c_{W_1, W_2}}(w_1 \boxtimes_{P(1)} w_2) = e^{L-1} \overline{\mathcal{T}_{[\gamma]}}(w_2 \boxtimes_{P(-1)} w_1) , \quad w_1 \in W_1, \quad w_2 \in W_2 . \quad (\text{B.2.70})$$

Again using the parallel transport map, one can show that the braiding map satisfies the hexagon Diagram B.3.

The category $\mathcal{C}_V$ is clearly semisimple by assumption, the simple objects being the irreducible V -modules. We will label the *finitely many* isomorphism classes of irreducible V -modules by the index set $\mathcal{I}_V$ and denote a representative from isomorphism classes of irreducible modules by $\{W^a\}_{a \in \mathcal{I}_V}$ with the convention that $W^0 = V$. Since $V' \cong V$, we will identify $0' = 0$. Also V -modules for V satisfying Conditions item 1, item 2 and item 3 are graded by rational numbers [DLM97]¹³. In fact, we have

$$W^a = \coprod_{n \in h_a + \mathbb{Z}_{\geq 0}} W_{(n)}^a . \quad (\text{B.2.71})$$

We will label the contragredient of W^a by $W^{a'}$. We will denote the space of intertwining operators of type $\binom{W^c}{W^a W^b}$ by $\mathcal{V}_{ab}^c$ and the corresponding fusion rule by $\mathcal{N}_{ab}^c$. A basis of $\mathcal{V}_{ab}^c$ will be denoted by $\mathcal{Y}_{ab;i}^c, i = 1, \dots, \mathcal{N}_{ab}^c$. We will choose

$$\mathcal{Y}_{0a;1}^a = Y_{W^a} \quad (\text{B.2.72})$$

and $\mathcal{Y}_{a0;1}^a$ to be the intertwining operator defined as

$$\mathcal{Y}_{a0;1}^a = \Omega_{-1}(\mathcal{Y}_{0a;1}^a) , \quad (\text{B.2.73})$$

¹³The first proof of rationality of conformal dimensions and central charge under the assumption of modular invariance and finiteness of number of irreducible modules was given by Anderson and Moore in [AM88].

where Ω_p is the skew-symmetry map defined in (B.1.71). We also choose the intertwining operator $\mathcal{Y}_{aa';1}^0$ to be

$$(u, \mathcal{Y}_{aa';1}^0(w_a, x)w'_a) = \langle \mathcal{Y}_{a0;1}^a(e^{xL_1}x^{-2L_0}w_a, x^{-1})u, w'_a \rangle, \quad u \in V, \quad w_a \in W^a, \quad w'_a \in (W^a)', \quad (\text{B.2.74})$$

where $(\cdot, \cdot)$ is the symmetric bilinear form on V which exists by Proposition B.1.2 since $V \cong V'$. We have the following theorem.

Theorem B.2.4. *We have*

$$W^a \boxtimes W^b \cong \bigoplus_{c \in \mathcal{I}_V} \mathcal{N}_{ab}^c W^c. \quad (\text{B.2.75})$$

Proof. Since $\mathcal{C}_V$ is a semisimple category, we have

$$W^a \boxtimes W^b \cong \bigoplus_{c \in \mathcal{I}_V} \mathcal{M}_{ab}^c \otimes W^c, \quad (\text{B.2.76})$$

where

$$\mathcal{M}_{ab}^c = \text{Hom}_V(W^a \boxtimes W^b, W^c), \quad (\text{B.2.77})$$

are the multiplicity spaces. By the universal property of $P(z)$ -tensor product, we get a natural isomorphism between the space $\mathcal{V}_{W^a W^b}^{W^c}$ of intertwining operators of type $\binom{W^c}{W^a W^b}$ and $\mathcal{M}_{ab}^c$. Indeed, given an intertwining operator $\mathcal{Y} \in \mathcal{V}_{W^a W^b}^{W^c}$,

$$I(w_a \otimes w_b) = \mathcal{Y}(w_a, z)w_b, \quad w_a \in W^a, \quad w_b \in W^b, \quad (\text{B.2.78})$$

is a $P(z)$ -intertwining map and hence by universal property, there exists a unique V -module map $J_{\mathcal{Y}} : W^a \boxtimes_{P(z)} W^b \rightarrow W^c$ satisfying

$$J_{\mathcal{Y}}(w_a \boxtimes_{P(z)} w_b) = \mathcal{Y}(w_a, z)w_b, \quad w_a \in W^a, \quad w_b \in W^b. \quad (\text{B.2.79})$$

One can show that the map $\mathcal{Y} \rightarrow J_{\mathcal{Y}}$ is an isomorphism. Thus, we get

$$\text{Hom}_V(W^a \boxtimes W^b, W^c) \cong \mathcal{V}_{W^a W^b}^{W^c}. \quad (\text{B.2.80})$$

This implies the statement. $\square$

We now describe the rigidity axioms. We take the left and right dual of a V -module (W, Y_W) to be the contragredient module $(W', Y_{W'})$. We now give the construction of the creation and

annihilation morphisms. Since our tensor category is semisimple, to prove the rigidity, we will only discuss irreducible modules. For $a \in \mathcal{I}_V$ and $z \in \mathbb{C}^\times$, the operator

$$I_{P(z)} : (W^a)' \otimes W^a \longrightarrow \bar{V} \quad (B.2.81)$$

$$w'_a \otimes w_a \longmapsto I(w'_a \otimes w_a) = \mathcal{Y}_{a'a;1}^0(w'_a, z)w_a ,$$

where $\mathcal{Y}_{a'a;1}^0$ is defined using (B.2.74). Then by universal property, we have the unique V -module map

$$\hat{e}_{a;z} : (W^a)' \boxtimes_{P(z)} W^a \longrightarrow V , \quad (B.2.82)$$

satisfying

$$\overline{\hat{e}_{a;z}}(w'_a \boxtimes_{P(z)} w_a) = \mathcal{Y}_{a'a;1}^0(w'_a, z)w_a . \quad (B.2.83)$$

Define the map

$$\hat{e}_a := \hat{e}_{a;1} : (W^a)' \boxtimes W^a \longrightarrow V . \quad (B.2.84)$$

This map will be related to the annihilation morphism.

We now construct another map which will give us the annihilation morphism. By universal property, for any fixed basis $\{\mathcal{Y}_{aa';i}^b\}$ of $\mathcal{V}_{aa'}^b$, there exists an isomorphism

$$f_{a;z} : W^a \boxtimes_{P(z)} (W^a)' \longrightarrow \bigoplus_{b \in \mathcal{I}_V} \mathcal{N}_{aa'}^b W^b , \quad (B.2.85)$$

such that

$$\overline{f_{a;z}}(w_a \boxtimes_{P(z)} w'_a) = \mathcal{Y}(w_a, z)w'_a , \quad (B.2.86)$$

where

$$\mathcal{Y} = \sum_{b \in \mathcal{I}_V} \sum_{i=1}^{\mathcal{N}_{aa'}^b} \mathcal{Y}_{aa';i}^b , \quad (B.2.87)$$

is an intertwining operator of type $\left(\begin{smallmatrix} \oplus_{b \in \mathcal{I}_V} \mathcal{N}_{aa'}^b W^b \\ W^a (W^a)' \end{smallmatrix} \right)$. We have

$$\mathcal{N}_{0a}^b = \delta_a^b . \quad (B.2.88)$$

By Theorem B.1.5 and Theorem B.1.6 we have

$$\mathcal{N}_{0a}^b = \mathcal{N}_{a0}^b = \mathcal{N}_{ab'}^0 = \delta_a^b . \quad (B.2.89)$$

Since $\mathcal{N}_{aa'}^0 = 1$, there is an injective map

$$g_{a;z} : V \longrightarrow \bigoplus_{b \in \mathcal{I}_V} \mathcal{N}_{aa'}^b W^b , \quad (B.2.90)$$

such that

$$g_{a;z}(\mathbf{1}) = \mathbf{1} \in V = W^0 \subset \bigoplus_{b \in \mathcal{I}_V} \mathcal{N}_{aa'}^b W^b . \quad (\text{B.2.91})$$

Define

$$i_{a;z} := f_{a;z}^{-1} \circ g_{a;z} : V \longrightarrow W^a \boxtimes_{P(z)} (W^a)' . \quad (\text{B.2.92})$$

One can show that the definition of $i_{a;z}$ is independent of the choice of basis of $\mathcal{V}_{aa'}^b$. The creation morphism is then defined as

$$i_a := i_{a;1} : V \longrightarrow W^a \boxtimes (W^a)' . \quad (\text{B.2.93})$$

We can now compute the following composition of maps

$$\lambda_a := r_a \circ (\mathbb{1}_{W^a} \boxtimes \hat{e}_a) \circ \alpha_{W^a, (W^a)', W^a}^{-1} \circ (i_a \boxtimes \mathbb{1}_{W^a}) \circ \ell_a^{-1} , \quad (\text{B.2.94})$$

which appears in the coherence condition Diagram B.5. Here $\ell_a := \ell_{W^a}$, $r_a := r_{W^a}$. Since this $\lambda_a \in \text{Hom}_V(W^a, W^a)$ and W^a is an irreducible V -module, $\lambda_a \in \mathbb{C}$. A tedious computation performed in [Hua05] shows that $\lambda_a \neq 0$. Thus, we can define

$$e_a := \frac{1}{\lambda_a} \hat{e}_a . \quad (\text{B.2.95})$$

Then clearly, e_a, i_a satisfies the coherence condition in Diagram B.5. e^a is thus the annihilation morphism for $\mathcal{C}_V$.

We now construct the creation and annihilation morphisms for left duals. Using universal property of $W^a \boxtimes_{P(z)} (W^a)'$, we get the unique V -module map

$$\hat{\hat{e}}_{a;z} : W^a \boxtimes_{P(z)} (W^a)' \longrightarrow V , \quad (\text{B.2.96})$$

such that

$$\overline{\hat{\hat{e}}_{a;z}}(w_a \boxtimes_{P(z)} w'_a) = \mathcal{Y}_{aa';1}^0(w_a, z) w'_a , \quad w_a \in W^a , \quad w'_a \in (W^a)' . \quad (\text{B.2.97})$$

We also introduce the notation

$$\hat{\hat{e}}_a := \hat{\hat{e}}_{a;1} . \quad (\text{B.2.98})$$

By universal property, for any fixed basis $\{\mathcal{Y}_{a'a';i}^b\}$ of $\mathcal{V}_{a'a'}^b$, there exists an isomorphism

$$\tilde{f}_{a;z} : (W^a)' \boxtimes_{P(z)} W^a \longrightarrow \bigoplus_{b \in \mathcal{I}_V} \mathcal{N}_{a'a'}^b W^b , \quad (\text{B.2.99})$$

such that

$$\overline{\tilde{f}_{a;z}}(w'_a \boxtimes_{P(z)} w_a) = \mathcal{Y}(w'_a, z) w_a , \quad (\text{B.2.100})$$

where

$$\mathcal{Y} = \sum_{b \in \mathcal{I}_V} \sum_{i=1}^{\mathcal{N}_{a'a}^b} \mathcal{Y}_{a'a;i}^b , \quad (\text{B.2.101})$$

is an intertwining operator of type $(\oplus_{b \in \mathcal{I}_V} \mathcal{N}_{a'a}^b W^b_{(W^a)'})$. Since $\mathcal{N}_{a'a}^0 = 1$, there is a unique injective V -module map

$$\tilde{g}_{a;z} : V \longrightarrow \bigoplus_{b \in \mathcal{I}_V} \mathcal{N}_{a'a}^b W^b , \quad (\text{B.2.102})$$

such that

$$\tilde{g}_{a;z}(\mathbf{1}) = \mathbf{1} \in V = W^0 \subset \bigoplus_{b \in \mathcal{I}_V} \mathcal{N}_{a'a}^b W^b . \quad (\text{B.2.103})$$

We then define

$$\tilde{i}_{a;z} := \tilde{f}_{a;z}^{-1} \circ \tilde{g}_{a;z} : V \longrightarrow (W^a)' \boxtimes_{P(z)} W^a . \quad (\text{B.2.104})$$

The left creation morphism is defined by

$$\tilde{i}_a := \tilde{i}_{a;1} . \quad (\text{B.2.105})$$

We can again compute the composition of maps

$$\gamma_a := \ell_a \circ (\hat{e} \boxtimes \mathbf{1}_{W^a}) \circ \alpha_{W^a, (W^a)', W^a} \circ (\mathbf{1}_{W^a} \boxtimes \tilde{i}_a) \circ \tilde{r}_a^{-1} \in \text{Hom}_V(W^a, W^a) \cong \mathbb{C} , \quad (\text{B.2.106})$$

that appears in Diagram B.6. Similar computation shows that $\gamma_a \neq 0$. We can then define the left annihilation morphism

$$\tilde{e}_a := \frac{1}{\gamma_a} \hat{e}_a . \quad (\text{B.2.107})$$

Similarly, one can check that other coherence conditions in Diagram B.5 and Diagram B.6 is satisfied.

The twist map is defined by

$$\theta_W := e^{2\pi i L_0} . \quad (\text{B.2.108})$$

Straightforward calculation shows that θ_W satisfies the balancing axioms.

To prove the nondegeneracy in $\mathcal{C}_V$, we need the Verlinde formula and its consequences. We first briefly discuss the Verlinde conjecture and its proof following [Hua04] and then describe its application in proving the nondegeneracy of $\mathcal{C}_V$ following [Hua05].

Verlinde formula

For an irreducible V -module W^a , we define the characters

$$\chi_a(u, \tau, z) := \text{Tr}_{W^a} Y_{W^a}(e^{2\pi i z L_0} u, e^{2\pi i z}) q^{L_0 - \frac{c}{24}}, \quad q := e^{2\pi i \tau}, \quad \tau \in \mathbb{H}, \quad (\text{B.2.109})$$

where c is the central charge of the VOA V .

Theorem B.2.5. [Zhu95] *Under the (slightly stronger than) Conditions [item 1](#), [item 2](#) and¹⁴ [item 3](#), the following statements hold:*

1. *The q -series $\chi_a(u, \tau, z)$ converges absolutely in the domain $|q| < 1$.*
2. *$\chi_a(u, \tau, z)$ is independent of z and the set of maps $\{u \mapsto \chi_a(u, \tau, z)\}_{a \in \mathcal{I}_V}$ is linearly independent.*
3. *The following modular invariance property holds: there exists a homomorphism¹⁵*

$$\rho_V : \text{SL}(2, \mathbb{Z}) \longrightarrow M(|\mathcal{I}_V|, \mathbb{C}), \quad (\text{B.2.110})$$

such that

$$\chi_{a_1} \left(\left(\frac{1}{c\tau + d} \right)^{L_0} u, \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = \sum_{a_2 \in \mathcal{I}_V} \rho_V(\gamma)_{a_2}^{a_1} \chi_{a_2}(u, \tau, z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \quad (\text{B.2.111})$$

We define the S -matrix to be

$$S_{a_1}^{a_2} := \rho_V \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)_{a_1}^{a_2}. \quad (\text{B.2.112})$$

For $u = \mathbf{1}$, we see that the S -matrix acts on the space spanned by the characters

$$\chi_a(\tau) := \chi_a(\mathbf{1}, \tau, z). \quad (\text{B.2.113})$$

For $a \in \mathcal{I}_V$, let $\mathcal{N}(a) \in M(|\mathcal{I}_V|, \mathbb{C})$ be the matrix with entries

$$\mathcal{N}(a)_{a_2}^{a_1} := \mathcal{N}_{aa_1}^{a_2}. \quad (\text{B.2.114})$$

Theorem B.2.6. [Ver88, Verlinde Conjecture] *The S -matrix $S = S_{a_1}^{a_2}$ diagonalizes the fusion matrix $\mathcal{N}(a)$ for each $a \in \mathcal{I}_V$. Moreover, $S_0^a \neq 0$ and we have the following Verlinde formula*

$$\mathcal{N}_{a_1 a_2}^{a_3} = \sum_{a_4 \in \mathcal{I}_V} \frac{S_{a_1}^{a_4} S_{a_2}^{a_4} S_{a_4}^{a_3}}{S_0^{a_4}}. \quad (\text{B.2.115})$$

¹⁴Dong-Li-Mason in [DLM97] improved Zhu's results by weakening the assumptions.

¹⁵We denote the group of $n \times n$ complex matrices by $M(n, \mathbb{C})$.

We will now describe the main steps in the proof of this theorem. We start by discussing the matrix elements of braiding and fusing isomorphisms. For $a_1, a_2, a_3 \in \mathcal{I}_V$ and $p \in \mathbb{N}$, we will denote by

$$\left\{ \mathcal{Y}_{a_1 a_2; i}^{a_3; (p)} : i = 1, \dots, \mathcal{N}_{a_1 a_2}^{a_3} \right\}, \quad (\text{B.2.116})$$

various bases of $\mathcal{V}_{a_1 a_2}^{a_3}$. Then it was proved in [Hua95a, Hua00, Hua02] that for $a_1, \dots, a_6 \in \mathcal{I}_V, i = 1, \dots, \mathcal{N}_{a_1 a_5}^{a_4}, j = 1, \dots, \mathcal{N}_{a_2 a_3}^{a_5}, k = 1, \dots, \mathcal{N}_{a_6 a_3}^{a_4}$ and $l = 1, \dots, \mathcal{N}_{a_1 a_2}^{a_6}$, there exists

$$F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}, \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \in \mathbb{C} \quad (\text{B.2.117})$$

such that

$$\begin{aligned} & \langle w'_{a_4}, \mathcal{Y}_{a_1 a_5; i}^{a_4; (1)}(w_{a_1}, z_1) \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}(w_{a_2}, z_2) w_{a_3} \rangle \\ &= \sum_{a_6 \in \mathcal{A}} \sum_{k=1}^{\mathcal{N}_{a_6 a_3}^{a_4}} \sum_{l=1}^{\mathcal{N}_{a_1 a_2}^{a_6}} F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}, \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \\ & \quad \times \langle w'_{a_4}, \mathcal{Y}_{a_6 a_3; k}^{a_4; (3)}(\mathcal{Y}_{a_1 a_2; l}^{a_6; (4)}(w_{a_1}, z_1 - z_2) w_{a_2}, z_2) w_{a_3} \rangle, \end{aligned} \quad (\text{B.2.118})$$

for $|a_1| > |z_2| > |z_1 - z_2| > 0$ and $w_{a_1} \in W^{a_1}, w_{a_2} \in W^{a_2}, w_{a_3} \in W^{a_3}, w'_{a_4} \in (W^{a_4})'$. The numbers

$$F(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}, \mathcal{Y}_{a_6 a_3; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_2; k}^{a_6; (4)}) \in \mathbb{C} \quad (\text{B.2.119})$$

are matrix elements of a linear transformation

$$\bigoplus_{a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \longrightarrow \bigoplus_{a_1, a_2, a_3, a_4, a_6 \in \mathcal{A}} \mathcal{V}_{a_6 a_3}^{a_4} \otimes \mathcal{V}_{a_1 a_2}^{a_6}, \quad (\text{B.2.120})$$

called the *fusing isomorphism*. It was proved in [Hua95b, Hua00, Hua02], that for $a_1, \dots, a_6 \in \mathcal{A}, i = 1, \dots, \mathcal{N}_{a_1 a_5}^{a_4}, j = 1, \dots, \mathcal{N}_{a_2 a_3}^{a_5}, k = 1, \dots, \mathcal{N}_{a_2 a_6}^{a_4}, l = 1, \dots, \mathcal{N}_{a_1 a_3}^{a_6}$, and $r \in \mathbb{Z}$, there exist

$$B^{(r)}(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}, \mathcal{Y}_{a_2 a_6; l}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_3; k}^{a_6; (4)}) \in \mathbb{C} \quad (\text{B.2.121})$$

such that the analytic extension of the single-valued analytic function

$$\langle w'_{a_4}, \mathcal{Y}_{a_1 a_5; i}^{a_4; (1)}(w_{a_1}, z_1) \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}(w_{a_2}, z_2) w_{a_3} \rangle \quad (\text{B.2.122})$$

on the region $|z_1| > |z_2| > 0, 0 \leq \arg z_1, \arg z_2 < 2\pi$ along the path

$$t \mapsto \left(\frac{3}{2} - \frac{e^{(2r+1)\pi i t}}{2}, \frac{3}{2} + \frac{e^{(2r+1)\pi i t}}{2} \right) \quad (\text{B.2.123})$$

to the region $|z_2| > |z_1| > 0, 0 \leq \arg z_1, \arg z_2 < 2\pi$ is

$$\begin{aligned} & \sum_{a_6 \in \mathcal{A}} \sum_{k=1}^{\mathcal{N}_{a_2 a_6}^{a_4}} \sum_{l=1}^{\mathcal{N}_{a_1 a_3}^{a_6}} B^{(r)}(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}, \mathcal{Y}_{a_2 a_6; k}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_3; l}^{a_6; (4)}) \\ & \quad \times \langle w'_{a_4}, \mathcal{Y}_{a_2 a_6; k}^{a_4; (3)}(w_{a_2}, z_1) \mathcal{Y}_{a_1 a_3; l}^{a_6; (4)}(w_{a_1}, z_2) w_{a_3} \rangle. \end{aligned} \quad (\text{B.2.124})$$

The numbers

$$B^{(r)}(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}; \mathcal{Y}_{a_2 a_6; k}^{a_4; (3)} \otimes \mathcal{Y}_{a_1 a_3; l}^{a_6; (4)}) \quad (\text{B.2.125})$$

are matrix elements of a linear isomorphism

$$B^{(r)} : \bigoplus_{a_1, a_2, a_3, a_4, a_5 \in \mathcal{A}} \mathcal{V}_{a_1 a_5}^{a_4} \otimes \mathcal{V}_{a_2 a_3}^{a_5} \longrightarrow \bigoplus_{a_1, a_2, a_3, a_4, a_6 \in \mathcal{I}_V} \mathcal{V}_{a_2 a_6}^{a_4} \otimes \mathcal{V}_{a_1 a_3}^{a_6}, \quad (\text{B.2.126})$$

called the *braiding isomorphism*. We will need the matrix elements of the square of the braiding isomorphism. The matrix elements of the square of the braiding isomorphism is denoted by

$$(B^{(r)})^2(\mathcal{Y}_{a_1 a_5; i}^{a_4; (1)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_5; (2)}; \mathcal{Y}_{a_1 a_6; i}^{a_4; (3)} \otimes \mathcal{Y}_{a_2 a_3; j}^{a_6; (4)}) . \quad (\text{B.2.127})$$

Note that $(B^{(r)})^2$ give the monodromy of the correlation function (B.2.122) corresponding to $\log(z_1 - z_2) \rightarrow \log(z_1 - z_2) + 2(2r+1)\pi i$. Braiding and fusing matrices give the genus-zero data of an RCFT. The next step is to relate genus-zero data to genus-one data. To this end, we consider genus-one correlation functions of the form

$$\begin{aligned} \chi_{a_1, a_2, a_3}^{i, j}(w_a, w_{a'}; z_1, z_2; \tau) \\ = \delta_{a_2}^a \text{Tr}_{W^{a_1}} \mathcal{Y}_{a_3 a_1; k}^{a_1; (1)} \left(\mathcal{U}(e^{2\pi i z_2}) \mathcal{Y}_{a_2 a'; j}^{a_3; (2)}(w_a, z_1 - z_2) w_{a'}, e^{2\pi i z_2} \right) q^{L_0 - \frac{c}{24}}, \end{aligned} \quad (\text{B.2.128})$$

where $\mathcal{U}(x)$ is defined as

$$\mathcal{U}(x) := (e^{2\pi i x})^{L_0} e^{-L^+(A)}, \quad (\text{B.2.129})$$

where $L^+(A) = \sum_{n \geq 0} A_n L_n$ and A_n is defined by the expansion

$$\frac{1}{2\pi i} \log(1 + 2\pi i y) = \left(\exp \left(\sum_{n \geq 0} A_n y^{n+1} \frac{\partial}{\partial y} \right) \right) y. \quad (\text{B.2.130})$$

These are two point torus correlation functions constructed by geometrically modifying the intertwining operators from sphere to torus. This is the origin of the operator $\mathcal{U}(x)$. Note that

$$\chi_{a, 0, 0}^{1, 1}(\mathbf{1} \otimes u; z_1, z_2; \tau) = \chi_a(u, z_2, \tau). \quad (\text{B.2.131})$$

We denote by $\mathcal{F}_{1;2}$ the space spanned by the maps

$$w_a \otimes w_{a'} \mapsto \chi_{a_1, a_2, a_3}^{i, j}(w_a \otimes w_{a'}; z_1, z_2; \tau). \quad (\text{B.2.132})$$

We denote by $\mathcal{F}_{1;2}^0$ the subspace of $\mathcal{F}_{1;2}$ spanned by¹⁶ $\{\chi_{a_1, a_2, 0}^{1, 1}\}$ and by $\mathcal{F}_{1;2}^{\neq 0}$ the subspace of $\mathcal{F}_{1;2}$ spanned by $\{\chi_{a_1, a_2, a_3}^{i, j}\}_{a_3 \neq 0}$. We have the following generalization of Zhu's result.

¹⁶In defining $\chi_{a_1, a_2, 0}^{1, 1}$, we use the intertwining operators defined in (B.2.72) and (B.2.73).

Theorem B.2.7. [Hua04, Hua02] Under the Conditions [item 1](#), [item 2](#) and [item 3](#), the following statements hold:

1. The q -series $\chi_{a_1, a_2, a_3}^{i, j}(w_a \otimes w_{a'}; z_1, z_2; \tau)$ converges absolutely in the domain $|q| < 1$.
2. The maps $w_a \otimes w_{a'} \mapsto \chi_{a_1, a_2, a_3}^{i, j}(w_a \otimes w_{a'}; z_1, z_2; \tau)$ are linearly independent. Moreover, $\{\chi_{a_1, a_2, 0}^{1, 1}\}_{a_1 \in \mathcal{I}_V}$ forms a basis of $\mathcal{F}_{1;2}^0$.
3. We have

$$\mathcal{F}_{1;2} = \mathcal{F}_{1;2}^0 \oplus \mathcal{F}_{1;2}^{\neq 0}, \quad (\text{B.2.133})$$

and we have the projection map $\pi_0 : \mathcal{F}_{1;2} \rightarrow \mathcal{F}_{1;2}^0$.

We can now define the action of $S \in \text{SL}(2, \mathbb{Z})$ on $\mathcal{F}_{1;2}^0$ as follows:

$$\left(S(\chi_{a_1, a_2, 0}^{1, 1}) \right) (w_a \otimes w_{a'}; z_1, z_2; \tau) := \delta_{a_2}^a \chi_{a_1, a_2, e}^{1, 1} \left(\left(-\frac{1}{\tau} \right)^{L(0)} w_a, \left(-\frac{1}{\tau} \right)^{L(0)} w_{a'}; -\frac{z_1}{\tau}, -\frac{z_2}{\tau}; -\frac{1}{\tau} \right). \quad (\text{B.2.134})$$

By Theorem [B.2.7](#), we see that there exists $S_{a_3}^{a_1} \in \mathbb{C}$ such that

$$S(\chi_{a_1, a_2, 0}^{1, 1}) = \sum_{a_3 \in \mathcal{I}_V} S_{a_3}^{a_1} \chi_{a_3, a_2, 0}^{1, 1}. \quad (\text{B.2.135})$$

These S -matrix elements are the same as [\(B.2.112\)](#) as can be seen from [\(B.2.131\)](#).

Now define the maps

$$(\alpha(\chi_{a_1, a_2, e}^{1, 1}))(w_a, w_{a'}; z_1, z_2; \tau) = \pi_0(\chi_{a_1, a_2, e}^{1, 1}(w_a, w_{a'}; z_1, z_2 - 1; \tau)), \quad (\text{B.2.136})$$

$$(\beta(\chi_{a_1, a_2, e}^{1, 1}))(w_a, w_{a'}; z_1, z_2; \tau) = \pi_0(\chi_{a_1, a_2, e}^{1, 1}(w_a, w_{a'}; z_1, z_2 + \tau; \tau)). \quad (\text{B.2.137})$$

From [\(B.2.134\)](#), [\(B.2.136\)](#) and [\(B.2.137\)](#), it can easily be checked that as operators on $\mathcal{F}_{1;2}^0$, we have the operator identity¹⁷

$$S\alpha S^{-1} = \beta. \quad (\text{B.2.138})$$

We now want to write the action of α, β in terms of the braiding and fusing matrix elements. For that we need to introduce the action of the permutation group $\mathcal{S}_3$ on the space of intertwining operators: for $\mathcal{Y} \in \mathcal{V}_{a_1 a_2}^{a_3}$, define

$$\sigma_{12}(\mathcal{Y}) = e^{\pi i \Delta(\mathcal{Y})} \Omega_{-1}(\mathcal{Y}), \quad \sigma_{23}(\mathcal{Y}) = e^{\pi i h_{a_1}} A_{-1}(\Omega), \quad (\text{B.2.139})$$

¹⁷This equation was first obtained in [MS88, MS89c] by using the geometric description of modular transformation on the torus. More precisely, the S -transformation exchanges the α and β cycles on the torus and hence gives rise to [\(B.2.138\)](#) on torus correlators.

where

$$\Delta(\mathcal{Y}) = h_{a_3} - h_{a_1} - h_{a_2} , \quad (\text{B.2.140})$$

and h_a denotes the conformal dimension of the lowest weight vector of W^a . Then by direct calculation using the definition (B.1.71) and (B.1.74) of Ω_p and A_r , one can show that

$$\mathcal{S}_3 \cong \langle \sigma_{12}, \sigma_{23} \rangle . \quad (\text{B.2.141})$$

Let $\sigma_{123} = \sigma_{12}\sigma_{23}$. Then we have the following Moore-Seiberg equations [MS88, MS89c].

Theorem B.2.8. [Hua04] *The following relations hold:*

1. *We have*

$$\begin{aligned} \sum_{i=1}^{\mathcal{N}_{a_1 a_2}^{a_3}} \sum_{k=1}^{\mathcal{N}_{a_1' a_3}^{a_2}} F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a_3' a_3;1}^0; \mathcal{Y}_{a_1' a_3;k}^{a_2} \otimes \mathcal{Y}_{a_2 a_3';i}^{a_1'}) F(\mathcal{Y}_{a_1' a_3;k}^{a_2} \otimes \sigma_{123}(\mathcal{Y}_{a_2 a_3';i}^{a_1'}); \mathcal{Y}_{0 a_2;1}^{a_2} \otimes \mathcal{Y}_{a_1' a_1;1}^0) \\ = \mathcal{N}_{a_1 a_2}^{a_3} F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a_2' a_2;1}^0; \mathcal{Y}_{0 a_2;1}^{a_2} \otimes \mathcal{Y}_{a_2 a_2';1}^0) . \end{aligned} \quad (\text{B.2.142})$$

2. *The maps $\alpha, \beta : \mathcal{F}_{1;2}^0 \rightarrow \mathcal{F}_{1;2}^0$ are well defined. Moreover, if we denote the matrix elements of α, β for fixed a_2 by $\alpha_{a_1}^{a_3}(a_2), \beta_{a_1}^{a_3}(a_2)$, then we have*

$$\begin{aligned} \alpha_{a_1}^{a_3}(a_2) &= \delta_{a_1}^{a_3} (B^{(-1)})^2 (\mathcal{Y}_{a_1 0;1}^{a_1} \otimes \mathcal{Y}_{a_2' a_2;1}^0; \mathcal{Y}_{a_1 0;1}^{a_1} \otimes \mathcal{Y}_{a_2' a_2;1}^0) , \\ \beta_{a_1}^{a_3}(a_2) &= e^{-2\pi i h_{a_2}} \sum_{i=1}^{N_{a_2 a_3}^{a_1}} \sum_{j=1}^{N_{a_2' a_1}^{a_3}} F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a_1' a_1;1}^0; \mathcal{Y}_{a_3' a_1;j}^{a_2} \otimes \mathcal{Y}_{a_2 a_1';i}^{a_3'}) \\ &\quad \times F(\mathcal{Y}_{a_3' a_1;j}^{a_2} \otimes \sigma_{123}(\mathcal{Y}_{a_2 a_1';i}^{a_3'}); \mathcal{Y}_{0 a_2;1}^{a_2} \otimes \mathcal{Y}_{a_3' a_3;1}^0) . \end{aligned} \quad (\text{B.2.143})$$

Sketch of Proof of Verlinde Conjecture B.2.6. Using (B.2.138) and Theorem B.2.8 immediately imply

$$\begin{aligned} \sum_{a_4 \in \mathcal{I}_V} S_{a_1}^{a_4} (B^{(-1)})^2 (\mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a_2' a_2;1}^0; \mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a_2' a_2;1}^0) (S^{-1})_{a_4}^{a_3} \\ = \sum_{i=1}^{\mathcal{N}_{a_1 a_2}^{a_3}} \sum_{k=1}^{\mathcal{N}_{a_1' a_3}^{a_2}} F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a_3' a_3;1}^0; \mathcal{Y}_{a_1' a_3;k}^{a_2} \otimes \mathcal{Y}_{a_2 a_3';i}^{a_1'}) \\ \quad \times F(\mathcal{Y}_{a_1' a_3;k}^{a_2} \otimes \sigma_{123}(\mathcal{Y}_{a_2 a_3';i}^{a_1'}); \mathcal{Y}_{0 a_2;1}^{a_2} \otimes \mathcal{Y}_{a_1' a_1;1}^0) \\ = \mathcal{N}_{a_1 a_2}^{a_3} F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a_2' a_2;1}^0; \mathcal{Y}_{0 a_2;1}^{a_2} \otimes \mathcal{Y}_{a_2 a_2';1}^0) . \end{aligned} \quad (\text{B.2.144})$$

From (B.2.144), we see that we must have

$$F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a_2' a_2;1}^0; \mathcal{Y}_{0 a_2;1}^{a_2} \otimes \mathcal{Y}_{a_2 a_2';1}^0) \neq 0 . \quad (\text{B.2.145})$$

Indeed, if otherwise then by (B.2.144), we must have

$$(B^{(-1)})^2(\mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a'_2 a_2;1}^0) . \quad (\text{B.2.146})$$

But since the vertex operators are single valued, we know that

$$(B^{(-1)})^2(\mathcal{Y}_{00;1}^0 \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{00;1}^0 \otimes \mathcal{Y}_{a'_2 a_2;1}^0) = 1 , \quad (\text{B.2.147})$$

which is a contradiction. Thus, we are allowed to divide by $F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{0a_2;1}^{a_2} \otimes \mathcal{Y}_{a_2 a'_2;1}^0)$. We can then rewrite (B.2.144) to get

$$\sum_{a_1, a_3 \in \mathcal{I}_V} (S^{-1})_{a_4}^{a_1} \mathcal{N}_{a_1 a_2}^{a_3} S_{a_3}^{a_5} = \delta_{a_4}^{a_5} \frac{(B^{(-1)})^2(\mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a'_2 a_2;1}^0)}{F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{0a_2;1}^{a_2} \otimes \mathcal{Y}_{a_2 a'_2;1}^0)} . \quad (\text{B.2.148})$$

This proves the first part of the Verlinde conjecture that S -matrix diagonalizes the fusion matrix.

To prove the Verlinde formula (B.2.115), let us define

$$\lambda_{a_2}^{a_4} := \frac{(B^{(-1)})^2(\mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{a_4 0;1}^{a_4} \otimes \mathcal{Y}_{a'_2 a_2;1}^0)}{F(\mathcal{Y}_{a_2 0;1}^{a_2} \otimes \mathcal{Y}_{a'_2 a_2;1}^0; \mathcal{Y}_{0a_2;1}^{a_2} \otimes \mathcal{Y}_{a_2 a'_2;1}^0)} , \quad (\text{B.2.149})$$

and invert (B.2.148) to get

$$\mathcal{N}_{a_1 a_2}^{a_3} = \sum_{a_4 \in \mathcal{I}_V} S_{a_1}^{a_4} \lambda_{a_2}^{a_4} (S^{-1})_{a_4}^{a_3} . \quad (\text{B.2.150})$$

For $a_1 = 0$ and using $\mathcal{N}_{0a_3}^{a_3} = \delta_{a_3}^{a_2}$, we get from (B.2.150)

$$\lambda_{a_4}^{a_2} = \frac{S_{a_2}^{a_4}}{S_0^{a_2}} . \quad (\text{B.2.151})$$

Finally, we calculate that

$$S^2(\chi_a) = \chi_{a'} , \quad a \in \mathcal{I}_V . \quad (\text{B.2.152})$$

This means that $(S^{-1})_{a_2}^{a_4} = S_{a_2}^{a'_4}$. Substituting this and (B.2.151) into (B.2.150), we get the Verlinde formula. $\square$

Another computation shows that the S -matrix is symmetric.

Remark B.2.2. The proof of rigidity also uses a consequence of the Verlinde conjecture. Indeed, the number λ_a, γ_a from (B.2.95), (B.2.107) are given by [Hua05]

$$\lambda_a = \gamma_a = F_a := F(\mathcal{Y}_{a 0;1}^a \otimes \mathcal{Y}_{a' a;1}^0; \mathcal{Y}_{0a;1}^a \otimes \mathcal{Y}_{aa';1}^0) . \quad (\text{B.2.153})$$

The fact that $\lambda_a, \gamma_a \neq 0$ then follows from (B.2.145) which can be thought of as a consequence of the Verlinde conjecture.

Nondegeneracy of $\mathcal{C}_V$

To prove the nondegeneracy, we need to compute the composition of maps in (B.2.35). This computation has been done in detail in [Hua05]. The result is

$$(S_{\text{MTC}})_{ab} = \frac{(B^{(-1)})_{b,a}^2}{F_a F_b} , \quad (\text{B.2.154})$$

where

$$(B^{(-1)})_{b,a}^2 := (\mathcal{Y}_{b0;1}^b \otimes \mathcal{Y}_{a'a;1}^0; \mathcal{Y}_{b0;1}^b \otimes \mathcal{Y}_{a'a;1}^0) , \quad (\text{B.2.155})$$

and we have used the subscript MTC to differentiate the S -matrix defined for the MTC using the composition of maps in (B.2.35) and the S -matrix defined in (B.2.112) in the VOA. From (B.2.151) we find that

$$(S_{\text{MTC}})_{ab} = \frac{S_a^b}{S_0^a} , \quad (\text{B.2.156})$$

which is invertible by the Verlinde formula. This proves the nondegeneracy of $\mathcal{C}_V$.

B.3 The dictionary from VOA to 2d CFT

We now give the dictionary between the vertex operator algebra literature and rational conformal field theory. We will assume familiarity with the discussion of Moore-Seiberg [MS89c]. Consider an RCFT with chiral algebra $\mathcal{A}$ and antichiral algebra $\widetilde{\mathcal{A}}$.¹⁸ In the mathematical language, the chiral algebra $\mathcal{A}, \widetilde{\mathcal{A}}$ are vertex operator algebras, which we denote by $V, \widetilde{V}$. For rational CFT, $V, \widetilde{V}$ are “nice” VOAs in the sense that they satisfy the Conditions [item 1](#), [item 2](#) and [item 3](#). Thus there are finitely many isomorphism classes of irreducible $V, \widetilde{V}$ -modules. Following our earlier notation, we will label the irreducible modules by

$$\{W^a\}_{a \in \mathcal{I}_V} , \quad \{\widetilde{W}^{\tilde{a}}\}_{\tilde{a} \in \mathcal{I}_{\widetilde{V}}} . \quad (\text{B.3.1})$$

We first decompose the physical Hilbert space of the CFT into conformal families:

$$\mathcal{H}_{\text{phys}} = \bigoplus_{\alpha=0}^M \mathcal{H}_{\alpha} , \quad (\text{B.3.2})$$

¹⁸In general, the chiral and the antichiral algebra may be non-isomorphic. Most notable example are the Narain CFTs [Nar86, NSW87, NSV87] which appear in the toroidal compactification of string theory [Pol07a]. Another example is the worldsheet CFT of the heterotic string [Pol07b]. A rigorous mathematical formulation of such theories was given using the formalism of nonchiral vertex operator algebra in [SS23, SST24].

where the index α runs over the chiral primary fields of the theory and $\alpha = 0$ corresponds to the identity primary field. We can also decompose the Hilbert space into irreducible representations of the chiral algebra $\mathcal{A} \otimes \widetilde{\mathcal{A}}$:

$$\mathcal{H}_{\text{phys}} = \bigoplus_{\alpha=0}^M \mathcal{H}_{r(\alpha)} \otimes \widetilde{\mathcal{H}}_{\tilde{r}(\alpha)} , \quad (\text{B.3.3})$$

where $r, \tilde{r}$ maps a primary field to the appropriate representation of the chiral algebra. In mathematical language, the vector space $\mathcal{H}_{r(\alpha)}$ is an irreducible V -module and $\widetilde{\mathcal{H}}_{\tilde{r}(\alpha)}$ an irreducible $\widetilde{V}$ -module. Thus we can think of $r, \tilde{r}$ as maps

$$r : \{0, \dots, M\} \longrightarrow \mathcal{I}_V , \quad \tilde{r} : \{0, \dots, M\} \longrightarrow \mathcal{I}_{\widetilde{V}} , \quad (\text{B.3.4})$$

mapping 0 to 0. Depending on the RCFT, the map $r, \tilde{r}$ may or may not be surjective¹⁹. We will denote the map from $\mathcal{H}_\alpha$ to $\mathcal{H}_{r(\alpha)}, \widetilde{\mathcal{H}}_{\tilde{r}(\alpha)}$ by the same symbol:

$$r : \mathcal{H}_\alpha \longrightarrow \mathcal{H}_{r(\alpha)} , \quad \tilde{r} : \mathcal{H}_\alpha \longrightarrow \widetilde{\mathcal{H}}_{\tilde{r}(\alpha)} . \quad (\text{B.3.5})$$

Now recall that given three modules $\mathcal{H}_a, \mathcal{H}_b, \mathcal{H}_c$ of the chiral algebra $\mathcal{A}$, a chiral vertex operator $\mathcal{Y}$ of type $\left(\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{H}_a \mathcal{H}_b \end{smallmatrix} \right)$ is a linear map

$$\begin{aligned} \mathcal{Y} : \mathcal{H}_a \otimes \mathcal{H}_b &\longrightarrow \mathcal{H}_c \{z\} , \\ w_{(a)} \otimes w_{(b)} &\longmapsto \mathcal{Y}(w_{(a)}, z)w_{(b)} , \end{aligned} \quad (\text{B.3.6})$$

which satisfies certain locality properties²⁰. Chiral vertex operators in the mathematical language are intertwining operators defined in Definition B.1.4. Let us denote the vector space of intertwining operators of type $\left(\begin{smallmatrix} \mathcal{H}_c \\ \mathcal{H}_a \mathcal{H}_b \end{smallmatrix} \right)$ by $\mathcal{V}_{ab}^c$. The space of intertwining operators $\widetilde{\mathcal{V}}_{\tilde{a}\tilde{b}}^{\tilde{c}}$ for the antichiral algebra $\widetilde{\mathcal{A}}$ is defined similarly. The fusion rules $\mathcal{N}_{ab}^c, \widetilde{\mathcal{N}}_{\tilde{a}\tilde{b}}^{\tilde{c}}$ of the CFT are defined as the dimension of the spaces $\mathcal{V}_{ab}^c, \widetilde{\mathcal{V}}_{\tilde{a}\tilde{b}}^{\tilde{c}}$ respectively. Let us denote the space of all intertwining operators by $\mathcal{V}$:

$$\mathcal{V} := \bigoplus_{a,b,c \in \mathcal{I}_V} \mathcal{V}_{ab}^c , \quad \widetilde{\mathcal{V}} := \bigoplus_{\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{I}_{\widetilde{V}}} \widetilde{\mathcal{V}}_{\tilde{a}\tilde{b}}^{\tilde{c}} . \quad (\text{B.3.7})$$

Define the ‘‘OPE coefficients’’ as a linear map

$$d_{\alpha\beta}' : \mathcal{V} \otimes \widetilde{\mathcal{V}} \longrightarrow \mathbb{C} , \quad (\text{B.3.8})$$

¹⁹The *diagonal modular invariant* corresponds to the choice $\mathcal{A} \cong \widetilde{\mathcal{A}}$, i.e., $V \cong \widetilde{V}$, $M+1 = |\mathcal{I}_V|$, r is a bijective map of sets and $\tilde{r}(\alpha) = r(\alpha)'$. Here we are following the notation that a' labels the contragredient of the V -module labeled by a .

²⁰Chiral vertex operators were first introduced in [TK87] and then generalized to general setting by Moore-Seiberg in [MS89c]. They were then formalized in [FHL93b] to *intertwining operators*.

which is nonzero only on the subspace $\mathcal{V}_{r(\alpha)r(\beta)}^{r(\gamma)} \otimes \tilde{\mathcal{V}}_{\tilde{r}(\alpha)\tilde{r}(\beta)}^{\tilde{r}(\gamma)}$. This function will relate the three point function of primary fields to the matrix elements of intertwining operator. Let $\pi_\alpha : \mathcal{H} \longrightarrow \mathcal{H}_\alpha$ be the projection map and let

$$\begin{aligned} \{\mathcal{Y}_{ab;i}^c : i = 1, \dots, \mathcal{N}_{ab}^c\} , \\ \{\tilde{\mathcal{Y}}_{\tilde{a}\tilde{b};\tilde{i}}^{\tilde{c}} : \tilde{i} = 1, \dots, \tilde{\mathcal{N}}_{\tilde{a}\tilde{b}}^{\tilde{c}}\} , \end{aligned} \quad (\text{B.3.9})$$

be a basis for $\mathcal{V}_{ab}^c, \tilde{\mathcal{V}}_{\tilde{a}\tilde{b}}^{\tilde{c}}$ respectively. Then a field corresponding to a state $v \in \mathcal{H}_\alpha$ is given in terms of the intertwining operators by [MS89b]

$$\phi_v(z, \bar{z}) = \sum_{\beta, \gamma=0}^M \sum_{i=1}^{\mathcal{N}_{r(\alpha)r(\beta)}^{r(\gamma)}} \sum_{\tilde{i}=1}^{\tilde{\mathcal{N}}_{\tilde{r}(\alpha)\tilde{r}(\beta)}^{\tilde{r}(\gamma)}} d_{\alpha\beta}^\gamma \left(\mathcal{Y}_{r(\alpha)r(\beta);i}^{r(\gamma)}, \tilde{\mathcal{Y}}_{\tilde{r}(\alpha)\tilde{r}(\beta);\tilde{i}}^{\tilde{r}(\gamma)} \right) \pi_\gamma \left(\mathcal{Y}_{r(\alpha)r(\beta);i}^{r(\gamma)}(r(v), z) \otimes \tilde{\mathcal{Y}}_{\tilde{r}(\alpha)\tilde{r}(\beta);\tilde{i}}^{\tilde{r}(\gamma)}(\tilde{r}(v), \bar{z}) \right) \pi_\beta . \quad (\text{B.3.10})$$

Assuming that the fields ϕ_{v_1}, ϕ_{v_2} corresponding to states $v_1 \in \mathcal{H}_\alpha, v_2 \in \mathcal{H}_\beta$ are BPZ self-conjugate, this implies the relation between the OPE coefficients and three point functions:

$$\begin{aligned} \lim_{z_1, z_2 \rightarrow 0} \langle \phi_{v_1}(z_1, \bar{z}_1) \phi_v(z, \bar{z}) \phi_{v_2}(z_2, \bar{z}_2) \rangle \\ = \sum_{i=1}^{\mathcal{N}_{r(\alpha)r(\beta)}^{r(\gamma)}} \sum_{\tilde{i}=1}^{\tilde{\mathcal{N}}_{\tilde{r}(\alpha)\tilde{r}(\beta)}^{\tilde{r}(\gamma)}} d_{\alpha\beta}^\gamma \left(\mathcal{Y}_{r(\alpha)r(\beta);i}^{r(\gamma)}, \tilde{\mathcal{Y}}_{\tilde{r}(\alpha)\tilde{r}(\beta);\tilde{i}}^{\tilde{r}(\gamma)} \right) \left\langle r(v_1), \mathcal{Y}_{r(\alpha)r(\beta);i}^{r(\gamma)}(r(v), z) r(v_2) \right\rangle \left\langle \tilde{r}(v_1), \tilde{\mathcal{Y}}_{\tilde{r}(\alpha)\tilde{r}(\beta);\tilde{i}}^{\tilde{r}(\gamma)}(\tilde{r}(v), \bar{z}) \tilde{r}(v_2) \right\rangle . \end{aligned} \quad (\text{B.3.11})$$

Solving this equation using known 3-point functions of the RCFT, we can determine the OPE coefficient map.

Conversely, given a “nice” VOA, we can construct the diagonal RCFT (see Footnote 19) except for the data of OPE coefficient map (B.3.8). Equivalently, the chiral RCFT can be fully reconstructed from the data of VOA and a subset of its irreducible modules. Constructing other modular invariants requires further analysis because we need to glue chiral and antichiral sector in a modular invariant way. A systematic study of constructing the full RCFT from the modular tensor category $\mathcal{C}_V$ of a “nice” VOA was carried out in a series of papers [FRS02, FRS04a, FRS03, FRS04b, FFRS05]. A review of this construction is beyond the scope of this thesis.

Appendix C

Appendix for Chapter 3

C.1 Affine Lie Algebras

In this appendix, we review the essential aspects of affine Lie algebra theory needed in our discussion. We refer the reader to [DFMS97, Hum95, FS97] for more details.

C.1.1 Lie Algebra Conventions

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. An element $x \in \mathfrak{g}$ is called *nilpotent* or *ad-nilpotent* if $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by

$$\text{ad}_x(y) = [x, y] , \quad (\text{C.1.1})$$

is nilpotent. If $\text{ad}(x)$ is diagonalizable, then x is called *semisimple*. A subalgebra of $\mathfrak{g}$ consisting of semisimple elements is called a *toral subalgebra*. One can prove that a toral subalgebra of a complex Lie algebra is abelian¹. A maximal toral subalgebra $\mathfrak{t}_{\mathfrak{g}}$ of $\mathfrak{g}$ is called the *Cartan subalgebra* of $\mathfrak{g}$. Considering the complexification $\mathfrak{g}^{\mathbb{C}}$, since $\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}$ is abelian, the elements of $\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}$ can be simultaneously diagonalized. More precisely, the set of operators $\{\text{ad}_x : x \in \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}\}$ are simultaneously diagonalizable. This means that the complexification $\mathfrak{g}^{\mathbb{C}}$ can be decomposed as a direct sum

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{\alpha \in \mathfrak{t}_{\mathfrak{g}}^*} \mathfrak{g}_{\alpha}^{\mathbb{C}} , \quad (\text{C.1.2})$$

where $\mathfrak{g}_{\alpha} := \{y \in \mathfrak{g} : \text{ad}_x(y) = \alpha(x)y \quad \forall x \in \mathfrak{t}_{\mathfrak{g}}\}$. Let $\Phi := \{\alpha \in \mathfrak{t}_{\mathfrak{g}}^* : \alpha \neq 0, \mathfrak{g}_{\alpha} \neq 0\}$. Then Φ is called the *set of roots* of $\mathfrak{g}$. Note that $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{g}})$, the centralizer of $\mathfrak{t}_{\mathfrak{g}}$ in $\mathfrak{g}$. One can show that $C_{\mathfrak{g}}(\mathfrak{t}_{\mathfrak{g}}) = \mathfrak{t}_{\mathfrak{g}}$. Thus, we have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}^{\mathbb{C}} . \quad (\text{C.1.3})$$

Recall that the Killing form is defined by

$$\kappa(x, y) := \text{Tr}_{\mathfrak{g}}(\text{ad}_x \circ \text{ad}_y) . \quad (\text{C.1.4})$$

¹This result works for Lie algebra over algebraically closed fields. Thus, we need to consider complex Lie algebras.

A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form is non-degenerate. From now on, we shall assume that $\mathfrak{g}$ is semisimple. One can show that the restriction of the Killing form κ to $\mathfrak{t}_{\mathfrak{g}}$ is non-degenerate. Thus, given $\phi \in \mathfrak{t}_{\mathfrak{g}}^*$, there exists a unique $t_{\phi} \in \mathfrak{t}_{\mathfrak{g}}$ such that $\phi(x) = \kappa(t_{\phi}, x)$ for all $x \in \mathfrak{t}_{\mathfrak{g}}$. In particular, $\Phi \subset \mathfrak{t}_{\mathfrak{g}}^*$ corresponds to $\{t_{\alpha} : \alpha \in \Phi\} \subset \mathfrak{t}_{\mathfrak{g}}$.

Theorem C.1.1. *The following is true:*

1. $\kappa(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}) = 0$ if $\alpha, \beta \in \mathfrak{t}_{\mathfrak{g}}^*$ with $\alpha + \beta \neq 0$. In particular $\kappa(\mathfrak{t}_{\mathfrak{g}}, \mathfrak{g}_{\alpha}) = 0$ for all $\alpha \in \Phi$.
2. Φ spans $\mathfrak{t}_{\mathfrak{g}}^*$.
3. If $\alpha \in \Phi$ then $-\alpha \in \Phi$.
4. If $\alpha \in \Phi$, then for $x \in \mathfrak{g}_{\alpha}, y \in \mathfrak{g}_{-\alpha}$, we have $[x, y] = \kappa(x, y)t_{\alpha}$. In particular $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]$ is one-dimensional with basis t_{α} .
5. $\alpha(t_{\alpha}) = \kappa(t_{\alpha}, t_{\alpha}) \neq 0$ for $\alpha \in \Phi$.
6. If $\alpha \in \Phi$ and $x_{\alpha} \in \mathfrak{g}_{\alpha} \setminus \{0\}$, then there exists $y_{\alpha} \in \mathfrak{g}_{-\alpha}$ such that the subalgebra spanned by x_{α}, y_{α} and $z_{\alpha} := [x_{\alpha}, y_{\alpha}]$ is isomorphic to $\mathfrak{su}(2)$ under the map

$$x_{\alpha} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y_{\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad z_{\alpha} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{C.1.5})$$

Moreover

$$z_{\alpha} = \frac{2t_{\alpha}}{\kappa(t_{\alpha}, t_{\alpha})}, \quad z_{\alpha} = -z_{-\alpha}. \quad (\text{C.1.6})$$

7. $\dim(\mathfrak{g}_{\alpha}) = 1$ for $\alpha \in \Phi$.

Since Φ spans $\mathfrak{t}_{\mathfrak{g}}^*$, we can choose a basis $\{\alpha_1, \dots, \alpha_r\} \subset \Phi$ of $\mathfrak{t}_{\mathfrak{g}}^*$. For $\beta \in \Phi$, we can write

$$\beta = \sum_{i=1}^r c_i \alpha_i \quad (\text{C.1.7})$$

Then one can prove that $c_i \in \mathbb{Q}$. Let $E_{\mathbb{Q}}$ be the $\mathbb{Q}$ -span of all the roots. Then $\dim_{\mathbb{Q}} E_{\mathbb{Q}} = \dim_{\mathbb{C}} \mathfrak{t}_{\mathfrak{g}}^* = \text{rank}(\mathfrak{g})$. Introduce a bilinear form on $\mathfrak{t}_{\mathfrak{g}}^*$ as

$$\langle \alpha, \beta \rangle := \kappa(t_{\alpha}, t_{\beta}) \quad (\text{C.1.8})$$

Then one can show that

$$\langle \alpha, \beta \rangle \in \mathbb{Q} \quad \text{and} \quad \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}, \quad \text{for } \alpha, \beta \in \Phi. \quad (\text{C.1.9})$$

Moreover this form is positive-definite on $E_{\mathbb{Q}}$. Let $E := \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$. Then Φ spans E and $0 \notin \Phi$.

Moreover if $\alpha \in \Phi$ then the only multiple of α in Φ is $-\alpha$. For $\alpha, \beta \in \Phi$, we have

$$\beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Phi. \quad (\text{C.1.10})$$

A subset $\Delta \subset \Phi$ is called a *base* if Δ is a basis of E and for each root $\beta \in \Phi$, when we write

$$\beta = \sum_{\alpha \in \Delta} k_{\alpha} \alpha \quad (\text{C.1.11})$$

then $k_{\alpha} \in \mathbb{Z}$ and either all k_{α} is non-negative or non-positive. The crucial result is that there exists a base of Φ . The elements of Δ are called *simple roots*. If all $k_{\alpha} \geq 0$ ($k_{\alpha} \leq 0$) then we call β *positive* (*negative*). We write $\beta \succ 0$ ($\beta \prec 0$). The collection of positive (negative) roots is denoted by Φ^+ (Φ^-). Clearly $\Phi^- = -\Phi^+$. We put a partial order on E by defining

$$\beta \prec \alpha \Leftrightarrow \alpha - \beta \succ 0 \text{ or } \beta = \alpha . \quad (\text{C.1.12})$$

Let $r := \dim(\mathfrak{t}_{\mathfrak{g}}) = \text{rank}(\mathfrak{g})$ be the rank of $\mathfrak{g}$ and let $\{\alpha_1, \dots, \alpha_r\} = \Delta$ be a set of simple roots. Then the matrix $A_{ij} := (\alpha_i, \alpha_j)$ is called the Cartan matrix, where

$$(\alpha_i, \alpha_j) := 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} \in \mathbb{Z} . \quad (\text{C.1.13})$$

Let $\Lambda_{\mathbf{w}} \subset E$ be defined as

$$\Lambda_{\mathbf{w}} := \{ \lambda \in E : (\lambda, \alpha) \in \mathbb{Z} \quad \forall \alpha \in \Phi \} . \quad (\text{C.1.14})$$

Clearly $\Lambda_{\mathbf{w}}$ is an additive subgroup of E since (λ, α) is linear in λ . $\Lambda_{\mathbf{w}}$ equipped with the pairing $(\cdot, \cdot)$ is called the *weight lattice* of $\mathfrak{g}$ and E is called the *weight vector space*. It is clear that $\lambda \in \Lambda_{\mathbf{w}}$ if and only if $(\lambda, \alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Clearly $\Phi \subset \Lambda_{\mathbf{w}}$. Let Λ_r be the subgroup of $\Lambda_{\mathbf{w}}$ generated by Φ . We call it the *root lattice*. By definition, the lattice dual $\Lambda_r^{\vee} \cong \Lambda_{\mathbf{w}}$. It is a lattice in E since it is the $\mathbb{Z}$ -span of an $\mathbb{R}$ -basis of E . Fix a base $\Delta \subset \Phi$. Then $\lambda \in \Lambda_{\mathbf{w}}$ is called *dominant* if $(\lambda, \alpha) \geq 0$ for all $\alpha \in \Delta$ and *strongly dominant* if $(\lambda, \alpha) > 0$ for all $\alpha \in \Delta$. Let $\Lambda_{\mathbf{w}}^+ \subset \Lambda_{\mathbf{w}}$ be the set of all dominant weights. A $\lambda \in \Lambda_{\mathbf{w}}^+$ is called *minimal* if $\mu \in \Lambda_{\mathbf{w}}^+, \mu \prec \lambda$ implies $\mu = \lambda$. One can show that the set of minimal weights is in 1-1 correspondence with the coset $\Lambda_{\mathbf{w}}/\Lambda_r$. In other words, every coset of $\Lambda_{\mathbf{w}}/\Lambda_r$ contains a unique minimal weight.

The lattice

$$\Lambda_{\text{cr}} := \left\{ \alpha^{\vee} := \frac{2t_{\alpha}}{\langle \alpha, \alpha \rangle} : \alpha \in \Lambda_r \right\} = \text{Span}_{\mathbb{Z}} \{ \alpha^{\vee} : \alpha \in \Delta \} \subset \mathfrak{t}_{\mathfrak{g}} , \quad (\text{C.1.15})$$

equipped with the Killing form is called the coroot lattice. The dual

$$\Lambda_{\text{cw}} := \Lambda_{\text{cr}}^{\vee} = \{ t \in \mathfrak{t}_{\mathfrak{g}} : \kappa(t, \alpha^{\vee}) \in \mathbb{Z} \quad \forall \alpha^{\vee} \in \Lambda_{\text{cr}} \} \subset \mathfrak{t}_{\mathfrak{g}} , \quad (\text{C.1.16})$$

of the coroot lattice is called the coweight lattice. It is easy to see that the simple coroots $\{\alpha_i^{\vee}\}_{i=1}^r$ is a basis for the coroot lattice. The basis $\{\omega_i\}_{i=1}^r$ of $\Lambda_{\mathbf{w}}$ dual to the basis of simple roots, i.e., $(\omega_i, \alpha_j) = \delta_{ij}$, is called a *fundamental basis* of $\Lambda_{\mathbf{w}}$ and the basis vectors ω_i are called *fundamental*

weights. Any weight vector $\lambda \in \Lambda_w$ can then be expanded as

$$\lambda = \sum_{i=1}^r \lambda_i \omega_i . \quad (\text{C.1.17})$$

The coefficients λ_i are called the *Dynkin labels* of λ . A weight vector is called *integral* if all its Dynkin labels are integers. Then *dominant integral weights* are precisely those weights whose Dynkin labels are all non-negative integers. There is a distinguished element $\theta \in \Phi$, called the *highest root*, uniquely characterized by the fact that in its expansion

$$\theta = \sum_{i=1}^r a_i \alpha_i , \quad (\text{C.1.18})$$

the sum $\sum_{i=1}^r a_i$ is maximized. The coefficients a_i are called *marks*. The corresponding vector $t_\theta \in \mathfrak{t}_{\mathfrak{g}}$ can also be expanded as

$$t_\theta = \sum_{i=1}^r a_i^\vee \alpha_i^\vee . \quad (\text{C.1.19})$$

The coefficients $a_i^\vee$ are called *comarks*. The *dual Coxeter number* $h_{\mathfrak{g}}^\vee$ is defined as

$$h_{\mathfrak{g}}^\vee := 1 + \sum_{i=1}^r a_i^\vee . \quad (\text{C.1.20})$$

For each root $\alpha \in \Phi$, we have a map, called a *Weyl reflection*, defined by

$$\begin{aligned} \sigma_\alpha : \mathfrak{t}_{\mathfrak{g}}^* &\longrightarrow \mathfrak{t}_{\mathfrak{g}}^* \\ \beta &\longmapsto \sigma_\alpha(\beta) := \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha . \end{aligned} \quad (\text{C.1.21})$$

By (C.1.10), we see that the restriction of σ_α maps roots to roots: $\sigma_\alpha|_\Phi : \Phi \longrightarrow \Phi$. The *Weyl group* is the group generated by the Weyl reflection:

$$W := \langle \sigma_\alpha : \alpha \in \Phi \rangle . \quad (\text{C.1.22})$$

The action of the Weyl group on the weight vector space decomposes it into various chambers called *Weyl chambers*:

$$C_w := \{ \lambda \in E : \langle w(\lambda), \alpha_i \rangle \geq 0, i = 1, \dots, r \} , \quad w \in W . \quad (\text{C.1.23})$$

The chamber corresponding to the identity element, denoted by C_0 , is called the *fundamental chamber*. One can show that for any weight $\lambda \in E$, there exists a $w \in W$ such that $w(\lambda) \in C_0$.

We will often work with the Cartan-Weyl basis of $\mathfrak{g}$ defined as follows: let $t_i := t_{\alpha_i} \in \mathfrak{t}_{\mathfrak{g}}$, $i = 1, \dots, r$ be the dual of the simple roots under the Killing form. Since each fg_α is one-dimensional, for each $\alpha \in \Phi$, we can choose $\mathbf{e}_\alpha \in \mathfrak{g}_\alpha$ such that

$$[t_i, \mathbf{e}_\alpha] = \alpha(t_i) \mathbf{e}_\alpha . \quad (\text{C.1.24})$$

Then $\{t_i\}_{i=1}^r \cup \{e_\alpha\}_{\alpha \in \Phi_{\mathfrak{g}}}$ is a Cartan-Weyl basis for $\mathfrak{g}$. The explicit Lie bracket is given by

$$\begin{aligned} [t_i, t_j] &= 0, \\ [t_i, e_\alpha] &= \alpha(t_i) e_\alpha, \\ [e_\alpha, e_\beta] &= \begin{cases} \frac{2}{\|\alpha\|^2} \sum_{i=1}^r \alpha(t_i) t_i & \alpha = -\beta, \\ N_{\alpha\beta} e_{\alpha+\beta} & \alpha + \beta \in \Delta, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (\text{C.1.25})$$

where $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$ and $N_{\alpha\beta}$ is a normalization constant.

Let $\tilde{G}$ be the unique simply-connected Lie group such that $\mathfrak{g} \cong \text{Lie}(\tilde{G})$. Then for any $\alpha^\vee \in \Lambda_{\text{cr}}$, we see that $e^{-2\pi i \alpha^\vee} = 1 \in \tilde{G}$. Moreover, the map $h \mapsto e^{-2\pi i h}$ defines an isomorphism

$$Z(\tilde{G}) \cong \Lambda_{\text{cw}}^\vee / \Lambda_{\text{cr}}^\vee. \quad (\text{C.1.26})$$

We also have a canonical isomorphism

$$\begin{aligned} \Lambda_{\text{w}} / \Lambda_{\text{r}} &\xrightarrow{\cong} \widehat{Z(\tilde{G})} \\ \lambda &\longmapsto [e^{-2\pi i \alpha^\vee} \mapsto e^{-2\pi i \langle \lambda, \alpha^\vee \rangle}] , \end{aligned} \quad (\text{C.1.27})$$

where $\widehat{Z(\tilde{G})} := \text{Hom}(Z(\tilde{G}), \text{U}(1))$ is the Pontryagin dual of $Z(\tilde{G})$ and $\langle \lambda, \alpha^\vee \rangle$ is the natural pairing between $\mathfrak{t}_{\mathfrak{g}}^*$ and $\mathfrak{t}_{\mathfrak{g}}$. Using the non-canonical isomorphism between $Z(\tilde{G})$ and $\widehat{Z(\tilde{G})}$, we have

$$\Lambda_{\text{w}} / \Lambda_{\text{r}} \cong \widehat{Z(\tilde{G})} \cong Z(\tilde{G}) \cong \Lambda_{\text{cw}}^\vee / \Lambda_{\text{cr}}^\vee. \quad (\text{C.1.28})$$

Let M be a $\mathfrak{g}$ -module. Then for each $\lambda \in \mathfrak{t}_{\mathfrak{g}}^*$, we can define the weight λ subspace of M as

$$M_\lambda := \{v \in M : x \cdot v = \lambda(x)v \ \forall x \in \mathfrak{t}_{\mathfrak{g}}\}. \quad (\text{C.1.29})$$

$\lambda \in \mathfrak{t}_{\mathfrak{g}}^*$ is called a weight of M if $M_\lambda \neq 0$. Let λ, μ be two weights of M , then λ is *higher* than μ if $\lambda - \mu$ is expressible as a linear combination of positive roots with non-negative real coefficients. A weight of M is called *highest* if it is higher than every weight of M . When $\mathfrak{g}$ is a simple Lie algebra, the *theorem of the highest weight* classifies finite-dimensional irreducible $\mathfrak{g}$ -modules. The theorem states that for every dominant integral weight λ of $\mathfrak{g}$, there exists a unique (up to isomorphism) finite-dimensional irreducible $\mathfrak{g}$ -module, which we will denote by $L_{\mathfrak{g}}(\lambda)$, whose highest weight is λ .

C.1.2 Affine Lie Algebra Conventions

For simplicity of later discussions, we take $\mathfrak{g}$ to be simple, the generalization to semisimple case is straightforward. Associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$ equipped with a non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$, its *affinization* produces an infinite-dimensional

Lie algebra. We call this algebra the (*untwisted*) *affine Lie algebra* or a *Kac-Moody algebra*. To construct it, we start with

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}k, \quad (\text{C.1.30})$$

and equip it with the Lie bracket

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b \rangle \delta_{m+n,0} k, \quad (\text{C.1.31})$$

$$[a \otimes t^m, k] = 0, \quad a, b \in \mathfrak{g}, \quad m, n \in \mathbb{Z}. \quad (\text{C.1.32})$$

Since k commutes with every element, it must act as a scalar multiple of the identity on irreducible representations. We call the scalar the *level* of the affine Lie algebra and sometimes indicate it by writing $\hat{\mathfrak{g}}_k$ for the affine Lie algebra.

A Cartan-Weyl basis for $\hat{\mathfrak{g}}$ is given by $t_i \otimes t^n, e_\alpha \otimes t^n, k$ with Lie bracket

$$\begin{aligned} [t_i \otimes t^m, t_j \otimes t^n] &= mk\delta_{ij}\delta_{m+n,0}, \\ [t_i \otimes t^m, e_\alpha \otimes t^n] &= \alpha(t_i)(e_\alpha \otimes t^{m+n}), \\ [e_\alpha \otimes t^m, e_\beta \otimes t^n] &= \begin{cases} \frac{2}{\|\alpha\|^2} (\alpha(t_i)(t_i \otimes t^{m+n}) + km\delta_{m+n,0}) & \alpha = -\beta, \\ N_{\alpha\beta}(e_{\alpha+\beta} \otimes t^{m+n}) & \alpha + \beta \in \Delta_{\mathfrak{g}}, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (\text{C.1.33})$$

where $\{t_i\}_{i=1}^{\text{rank}(\mathfrak{g})} \cup \{e_\alpha\}_{\alpha \in \Phi_{\mathfrak{g}}}$ is a Cartan-Weyl basis for $\mathfrak{g}$, $\Phi_{\mathfrak{g}} \subset \mathfrak{t}_{\mathfrak{g}}^* := \text{Cartan}(\mathfrak{g})^*$ denotes a set of roots of $\mathfrak{g}$, $\Delta_{\mathfrak{g}} \subset \Phi_{\mathfrak{g}}$ is a set of simple roots of $\mathfrak{g}$, $\|\alpha\|^2 = \langle \alpha, \alpha \rangle$ and $N_{\alpha\beta}$ is a normalization constant.

There is another operator d with the following Lie bracket²

$$[d, a \otimes t^n] = n(a \otimes t^n), \quad a \in \mathfrak{g}, \quad (\text{C.1.34})$$

and all other Lie brackets vanish. The *extended affine Lie algebra* is the space

$$\tilde{\mathfrak{g}} := \hat{\mathfrak{g}} \oplus \mathbb{C}d, \quad (\text{C.1.35})$$

with the above Lie bracket. The generators $\{t_i \otimes t^0, d, k\}$ is the maximal commuting set of generators and hence span the Cartan subalgebra $\mathfrak{t}_{\tilde{\mathfrak{g}}}$ of the extended affine Lie algebra. As in the classical Lie algebra case, *affine weight vectors* $\hat{\lambda}$ are vectors whose components are the eigenvalues of a vector in an irreducible representation of $\tilde{\mathfrak{g}}$ which is a simultaneous eigenvector of all the generators of $\mathfrak{t}_{\tilde{\mathfrak{g}}}$. It is clear that an affine weight vector has the form

$$\hat{\lambda} = (\lambda; k_\lambda; n_\lambda), \quad (\text{C.1.36})$$

²In affine vertex operator algebra $d = -L(0)$.

where λ is a weight vector of $\mathfrak{g}$ and k_λ, n_λ are the eigenvalues of k, d respectively. Similarly, *affine roots* are the affine weight vectors in the adjoint representation. We also write

$$\lambda \equiv (\lambda; 0; 0) . \quad (\text{C.1.37})$$

We introduce the bilinear form

$$\langle \hat{\lambda}, \hat{\lambda}' \rangle = \langle \lambda, \lambda' \rangle + k_\lambda n_{\lambda'} + k_{\lambda'} n_\lambda . \quad (\text{C.1.38})$$

Affine roots have the form

$$\hat{\alpha} = (\alpha; 0; n) , \quad (\text{C.1.39})$$

and hence

$$\langle \hat{\alpha}, \hat{\beta} \rangle = \langle \alpha, \beta \rangle . \quad (\text{C.1.40})$$

We define *affine fundamental weights* $\{\hat{\omega}_i\}_{i=0}^{\text{rank}(\mathfrak{g})}$ by

$$\hat{\omega}_0 = (0; 1; 0) , \quad \hat{\omega}_i = (\omega_i; a_i^\vee; 0) , \quad i \neq 0 , \quad (\text{C.1.41})$$

where $\{\omega_i\}_{i=1}^{\text{rank}(\mathfrak{g})}$ are the fundamental weights and $a_i^\vee$ are the comarks of $\mathfrak{g}$. One can check that

$$\langle \hat{\omega}_0, \hat{\omega}_j \rangle = \langle \hat{\omega}_0, \hat{\omega}_0 \rangle = 0, \quad \langle \hat{\omega}_i, \hat{\omega}_j \rangle = \langle \omega_i, \omega_j \rangle = F_{ij} , \quad i, j \neq 0 , \quad (\text{C.1.42})$$

where

$$F_{ij} = (A^{-1})_{ij} \frac{\|\alpha_j\|^2}{2} , \quad (\text{C.1.43})$$

and A is the Cartan matrix of $\mathfrak{g}$. If we ignore the d -eigenvalue, then affine weight vectors can be expanded in affine fundamental weights:

$$\hat{\lambda} = \sum_{i=0}^r \lambda_i \hat{\omega}_i . \quad (\text{C.1.44})$$

We write the

$$\hat{\lambda} = [\lambda_0, \lambda_1, \dots, \lambda_r] . \quad (\text{C.1.45})$$

One can show that

$$\lambda_0 = k_\lambda - \langle \lambda, \theta \rangle . \quad (\text{C.1.46})$$

In particular, dominant integral weights at fixed level satisfy

$$\lambda_0 \in \mathbb{Z}_{\geq 0} . \quad (\text{C.1.47})$$

We now consider affine embeddings. Suppose $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. We can consider affine Lie algebras $\hat{\mathfrak{h}}, \hat{\mathfrak{g}}$ associated to $\mathfrak{h}, \mathfrak{g}$ respectively. It turns out the levels $\tilde{k}, k$ of $\hat{\mathfrak{h}}, \hat{\mathfrak{g}}$ are related via the *embedding index*:

$$\tilde{k} = x_e k , \quad (\text{C.1.48})$$

which is defined as follows. We first define a *projection matrix* $\mathcal{P}$ which maps the weight vectors of $\mathfrak{g}$ to weight vectors of $\mathfrak{h}$. To define $\mathcal{P}$, we start with the irreducible $\mathfrak{g}$ -modules $L_{\mathfrak{g}}(\omega_i)$ corresponding to the fundamental weights³ of $\mathfrak{g}$. Then, considering $L_{\mathfrak{g}}(\omega_i)$ as an $\mathfrak{h}$ -module, we decompose it as direct sum of irreducible $\mathfrak{h}$ -modules:

$$L_{\mathfrak{g}}(\omega_i) \cong \bigoplus_{\mu_i \in \Lambda_{\mathfrak{w}}^{\mathfrak{h}+}} n_{\mu_i} L_{\mathfrak{h}}(\mu_i) , \quad n_{\mu_i} \in \mathbb{Z}_{\geq 0} . \quad (\text{C.1.49})$$

Here $\Lambda_{\mathfrak{w}}^{\mathfrak{h}+}$ denotes the set of dominant integral weights of $\mathfrak{h}$. We then define

$$\begin{aligned} \mathcal{P} : \Lambda_{\mathfrak{w}}^{\mathfrak{g}} &\longrightarrow \Lambda_{\mathfrak{w}}^{\mathfrak{h}} \\ \omega_i &\longmapsto \mathcal{P}\omega_i := \sum_{\mu_i \in \Lambda_{\mathfrak{w}}^{\mathfrak{h}+}} n_{\mu_i} \mu_i , \end{aligned} \quad (\text{C.1.50})$$

by $\mathbb{Z}$ -linearly extending the action from fundamental weights to the weight lattice. The matrix for $\mathcal{P}$ can be obtained by expressing the images of ω_i as $\mathbb{Z}$ -linear combinations of fundamental weights of $\mathfrak{h}$.

Let θ, ϑ be the highest roots of $\mathfrak{g}, \mathfrak{h}$ respectively. Then the embedding index x_e of $\mathfrak{h} \subset \mathfrak{g}$ is defined by

$$x_e := \frac{\|\mathcal{P}\theta\|^2}{\|\vartheta\|^2} . \quad (\text{C.1.51})$$

Example: Induced Level Of $\mathfrak{u}(1)$ Embedded In $\mathfrak{sl}(2, \mathbb{R})_k$

In this appendix, we compute the induced level of $\mathfrak{u}(1)$ embedded in $\mathfrak{sl}(2, \mathbb{R})$ at level k . The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is 3-dimensional spanned by generators h, e, f with Lie bracket

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h . \quad (\text{C.1.52})$$

A standard 2×2 real matrix representation of this algebra is given by

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \quad (\text{C.1.53})$$

³Note that any fundamental weight ω is a dominant integral weight and hence there exists a unique (up to isomorphism) irreducible $\mathfrak{g}$ -module $L_{\mathfrak{g}}(\omega)$ with highest weight ω .

There are several $U(1)$ embedded inside $SL(2, \mathbb{R})$. The compact $U(1)$ is unique up to conjugation and is given by

$$C := \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2) \subset SL(2, \mathbb{R}) : \theta \in \mathbb{R} \right\} \cong U(1) . \quad (C.1.54)$$

The $\mathfrak{u}(1)_C$ subalgebra corresponding to this $U(1)$ is spanned by

$$J = e - f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \quad (C.1.55)$$

There are two inequivalent noncompact $U(1)$, called *null* $U(1)$ subgroups of $SL(2, \mathbb{R})$ given by

$$\begin{aligned} A &:= \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \in SL(2, \mathbb{R}) : t \in \mathbb{R} \right\} = \{e^{th} : t \in \mathbb{R}\} \cong \mathbb{R} , \\ N &:= \left\{ \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} \in SL(2, \mathbb{R}) : t \in \mathbb{R} \right\} = \{e^{te} : t \in \mathbb{R}\} \cong \mathbb{R} . \end{aligned} \quad (C.1.56)$$

To get the Killing form, we need the adjoint representation matrices. It is straightforward to check that

$$\text{ad}_h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \text{ad}_e = \begin{pmatrix} 0 & 0 & -1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{ad}_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}. \quad (C.1.57)$$

Using this, it is straightforward to check that for

$$X = x_1 h + x_2 e + x_3 f, \quad Y = y_1 h + y_2 e + y_3 f, \quad (C.1.58)$$

we have

$$\kappa(X, Y) = \text{Tr}(\text{ad}_X \circ \text{ad}_Y) = 8x_1 y_1 - 2(x_2 y_2 + x_3 y_3) . \quad (C.1.59)$$

In particular,

$$\|J\|^2 = \kappa(J, J) = 4, \quad \|h\|^2 = 8, \quad \|e\|^2 = -2 . \quad (C.1.60)$$

Now let $J^a(z)$, $a = h, e, f$ be the $\mathfrak{sl}(2, \mathbb{R})$ -WZW currents. Then

$$J^a(z)J^b(0) \sim \frac{k \kappa(a, b)}{2z^2} + \frac{if^{ab}_c J^c(0)}{z}. \quad (C.1.61)$$

The $\mathfrak{u}(1)$ current is $J(z) = J^e(z) - J^f(z)$. Then we have

$$J(z)J(0) \sim \frac{k \kappa(J, J)}{2z^2} =: \frac{k_{\mathfrak{u}(1)}}{2z^2} . \quad (C.1.62)$$

where

$$k_{\mathfrak{u}(1)_C} = k \kappa(J, J) = 4k . \quad (C.1.63)$$

Similarly

$$k_{\mathfrak{u}(1)_A} = 8k \ , \quad k_{\mathfrak{u}(1)_N} = -2k \ . \quad (\text{C.1.64})$$

C.1.3 Outer Automorphisms

The automorphisms of the simple Lie algebra $\mathfrak{g}$ induced from the symmetries of the corresponding Dynkin diagram is called an *outer automorphism*. Similarly, one defines the outer automorphisms of the affine Lie algebra $\hat{\mathfrak{g}}$ using symmetries of the Dynkin diagram of $\hat{\mathfrak{g}}$. Let $D(\mathfrak{g})$ and $D(\hat{\mathfrak{g}})$ be the symmetries of the Dynkin diagrams for $\mathfrak{g}$ and $\hat{\mathfrak{g}}$. Then clearly $D(\mathfrak{g}) \subset D(\hat{\mathfrak{g}})$. The outer automorphism group of $\hat{\mathfrak{g}}$ is defined by the quotient

$$\mathcal{O}(\hat{\mathfrak{g}}) := D(\hat{\mathfrak{g}})/D(\mathfrak{g}) \ . \quad (\text{C.1.65})$$

Table C.1 summarizes the action of outer automorphisms of affine Lie algebras on affine weight vectors.

$\mathfrak{g}$	$\mathcal{O}(\hat{\mathfrak{g}})$	Action of the $\mathcal{O}(\hat{\mathfrak{g}})$ generators
A_r	$\mathbb{Z}_{r+1}$	$a[\lambda_0, \lambda_1, \dots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_0, \dots, \lambda_{r-2}, \lambda_{r-1}]$
B_r	$\mathbb{Z}_2$	$a[\lambda_0, \lambda_1, \dots, \lambda_{r-1}, \lambda_r] = [\lambda_1, \lambda_0, \dots, \lambda_{r-1}, \lambda_r]$
C_r	$\mathbb{Z}_2$	$a[\lambda_0, \lambda_1, \dots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_{r-1}, \dots, \lambda_1, \lambda_0]$
$D_{r=2\ell}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$a[\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_r] = [\lambda_1, \lambda_0, \lambda_2, \dots, \lambda_r, \lambda_{r-1}]$ $\tilde{a}[\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_r] = [\lambda_r, \lambda_{r-1}, \lambda_{r-2}, \dots, \lambda_1, \lambda_0]$
$D_{r=2\ell+1}$	$\mathbb{Z}_4$	$a[\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_r] = [\lambda_{r-1}, \lambda_r, \lambda_{r-2}, \dots, \lambda_1, \lambda_0]$
E_6	$\mathbb{Z}_3$	$a[\lambda_0, \lambda_1, \dots, \lambda_6] = [\lambda_1, \lambda_5, \lambda_4, \lambda_3, \lambda_6, \lambda_0, \lambda_2]$
E_7	$\mathbb{Z}_2$	$a[\lambda_0, \lambda_1, \dots, \lambda_7] = [\lambda_6, \lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0, \lambda_7]$

Table C.1: Outer automorphism group action on affine weights.

One has the isomorphism

$$\mathcal{O}(\hat{\mathfrak{g}}) \cong Z(\tilde{G}) \ , \quad (\text{C.1.66})$$

where as before $\tilde{G}$ is the unique compact connected simply-connected Lie group such that $\mathfrak{g} \cong \text{Lie}(\tilde{G})$.

The explicit isomorphism is given by composing the map

$$\begin{aligned} \mathcal{O}(\hat{\mathfrak{g}}) &\longrightarrow \Lambda_w / \Lambda_r \\ A &\longmapsto A\hat{\omega}_0 \ , \end{aligned} \quad (\text{C.1.67})$$

with the isomorphisms in (C.1.28). This isomorphism also gives us the action of $Z(\tilde{G})$ on any highest weight module $L_{\mathfrak{g}}(\lambda)$:

$$b \cdot v = e^{-2\pi i \langle A(\tilde{\omega}_0), \lambda \rangle} v, \quad \forall v \in L_{\mathfrak{g}}(\lambda). \quad (\text{C.1.68})$$

Isomorphism classes of irreducible *integrable-highest weight modules* of $\hat{\mathfrak{g}}$ at level $k \in \mathbb{Z}_{>0}$ are in one-to-one correspondence with the set of *dominant integral affine weights* defined by

$$P_k^+(\mathfrak{g}) := \{\lambda \in \Lambda_w : \langle \lambda, \alpha^i \rangle \geq 0, i = 1, \dots, r, \langle \lambda, \theta \rangle \leq k\}, \quad (\text{C.1.69})$$

where $\alpha_i, i = 1, \dots, r$ are the simple roots of $\mathfrak{g}$. We will denote the irreducible integrable highest weight irreducible module by $L_{\mathfrak{g}}(k, \hat{\lambda})$. We have emphasized the dependence of the irreducible module on the level by using the affine weight $\hat{\lambda}$ rather than the weight λ . Note that $\lambda_0 = k - \langle \lambda, \theta \rangle$ as in (C.1.46). This is also important because the outer automorphism of $\hat{\mathfrak{g}}$ acts on affine weights and not on weights.

The center $Z(\tilde{G})$ acts on $L_{\mathfrak{g}}(k, \hat{\lambda})$ identical to (C.1.68):

$$b \cdot v = e^{-2\pi i \langle A(\tilde{\omega}_0), \lambda \rangle} v, \quad \forall v \in L_{\mathfrak{g}}(k, \hat{\lambda}). \quad (\text{C.1.70})$$

C.2 Selection Rules, Field Identifications And Fixed Point Resolutions

In this appendix, we would like to characterize the set of equivalence classes $\mathcal{E} / \sim$. More specifically, we want to characterize the set of isomorphism classes of irreducible modules of the coset VOA $V_{\mathfrak{g}/\mathfrak{h}}$. As we have already noted in Theorem 3.2.1 that under some conditions, all irreducible modules of $V_{\mathfrak{g}/\mathfrak{h}}$ are of the form $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$. We will see below that this is not always true.

C.2.1 Selection Rules

As noted in Section 3.3.2, while constructing the irreducible modules of the coset VOA $V_{\mathfrak{g}/\mathfrak{h}}$ using the irreducible modules of $L_{\mathfrak{g}}(\mathbf{k}, 0)$, not all modules $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ are nontrivial. Indeed, a necessary condition for $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ or equivalently the coset character $\chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau)$ to be nonzero is [DFMS97]

$$\mathcal{P}\lambda - \mu \in \mathcal{P}\Lambda_{\mathfrak{g}}^r, \quad (\text{C.2.1})$$

where $\mathcal{P}$ is a projection matrix that maps weight vectors of $\mathfrak{g}$ to weight vectors of $\mathfrak{h}$ and $\Lambda_{\mathfrak{g}}^r$ is the root lattice of $\mathfrak{g}$. Generically, (C.2.1) is the only selection rule. But there are examples in which the coset character $\chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau)$ is zero even if (C.2.1) is satisfied. Such examples are called *Maverick cosets* [DJ93b, DJ93a, PSW99]. An explanation of such selection rules based on non-abelian gauging has

been proposed in [CGS23] which we have reviewed in Section 3.2.3. So, for non-Maverick cases, we can write

$$\mathcal{E} = \{(\hat{\lambda}, \hat{\mu}) : \lambda \in P_k^+(\mathfrak{g}), \mu \in P_k^+(\mathfrak{h}), \mathcal{P}\lambda - \mu \in \mathcal{P}\Lambda_{\mathfrak{g}}^r\} . \quad (\text{C.2.2})$$

C.2.2 Field Identifications

Our next goal is to determine the equivalence relation on $\mathcal{E}$.

Field Identifications From Outer Automorphism Branching

We first explain field identifications algebraically and then we will review its physical origin from Chern-Simons theory. Consider an embedding $\iota : \mathfrak{h} \hookrightarrow \mathfrak{g}$ of semisimple Lie algebras. Let $\tilde{G}, \tilde{H}$ be the unique compact, connected and simply connected Lie groups with $\text{Lie}(\tilde{G}) = \mathfrak{g}$ and $\text{Lie}(\tilde{H}) = \mathfrak{h}$. Composing ι with exponential map, we obtain a homomorphism $\iota : H \hookrightarrow G$ as shown in (C.2.3).

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{exp}} & \tilde{G} \\ \uparrow \iota & & \uparrow \iota \\ \mathfrak{h} & \xrightarrow{\text{exp}} & \tilde{H} \end{array} \quad (\text{C.2.3})$$

Note that in general $\iota : \tilde{H} \rightarrow \tilde{G}$ is not an injective map and hence not an embedding⁴. We will *assume* that $\iota : \tilde{H} \rightarrow \tilde{G}$ is an embedding. Consider the common center of $\tilde{G}, \tilde{H}$ defined by

$$Z := Z(\tilde{G}) \cap \iota(Z(\tilde{H})) . \quad (\text{C.2.4})$$

Suppose $g \in Z \subset Z(G)$. Then there exists $\tilde{g} \in Z(H)$ such that $g = \iota(\tilde{g})$. Let $\mathbf{A}_g \in \mathcal{O}(\hat{\mathfrak{g}})$ and $\tilde{\mathbf{A}}_{\tilde{g}} \in \mathcal{O}(\hat{\mathfrak{h}})$ be the image of $g, \tilde{g}$ under the isomorphism (C.1.67). Then we say that $\mathbf{A}_g$ *branches to* $\tilde{\mathbf{A}}_{\tilde{g}}$ if $\iota(\tilde{g}) = g \in Z$. We write this as

$$\mathbf{A}_g \mapsto \tilde{\mathbf{A}}_{\tilde{g}} , \quad \text{for } \iota(\tilde{g}) = g \in Z . \quad (\text{C.2.5})$$

Given $(\mathbf{A}, \tilde{\mathbf{A}}) \in \mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathfrak{h}})$, we would like to know if $\mathbf{A} \mapsto \tilde{\mathbf{A}}$. An algebraic condition for this is derived in [DFMS97]: for all weight vectors λ of $\mathfrak{g}$,

$$\langle \mathbf{A}\hat{\omega}_0, \lambda \rangle = \langle \tilde{\mathbf{A}}\hat{\omega}_0, \mathcal{P}\lambda \rangle \bmod 1 . \quad (\text{C.2.6})$$

⁴An example of such a Lie algebra embedding is $\hat{\mathfrak{su}}(2)_8 \subset \hat{\mathfrak{su}}(3)_2$. Under this embedding, the three dimensional fundamental representation of $\mathfrak{su}(3)$ becomes the adjoint representation of $\mathfrak{su}(2)$ [DFMS97]. Clearly the center of $\text{SU}(2)$ acts trivially but the center of $\text{SU}(3)$ acts nontrivially on this representation and hence $Z(\text{SU}(2))$ maps to the identity under the induced group homomorphism.

Note that, to define the bilinear pairing, we use (C.1.37) and $\hat{\omega}_0$ on RHS is the fundamental weight for $\hat{\mathfrak{h}}$. Define

$$G_{\text{id}} := \{(\mathbf{A}, \tilde{\mathbf{A}}) \in \mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathfrak{h}}) : \mathbf{A} \mapsto \tilde{\mathbf{A}}\} . \quad (\text{C.2.7})$$

Using the isomorphism $\mathcal{O}(\hat{\mathfrak{g}}) \cong Z(\tilde{G})$, $\mathcal{O}(\hat{\mathfrak{h}}) \cong Z(\tilde{H})$ and the fact that

$$\mathbf{A} \mapsto \tilde{\mathbf{A}} \iff \iota(\tilde{g}_{\tilde{\mathbf{A}}}) = g_{\mathbf{A}} \in Z , \quad (\text{C.2.8})$$

where $g_{\mathbf{A}}$ is the image of $\mathbf{A}$ under the isomorphism $\mathcal{O}(\hat{\mathfrak{g}}) \cong Z(G)$, we conclude that⁵

$$\begin{aligned} Z &\longrightarrow G_{\text{id}} \\ g &\longmapsto (\mathbf{A}_g, \tilde{\mathbf{A}}_{\iota^{-1}(g)}) \end{aligned} \quad (\text{C.2.9})$$

is an isomorphism. Thus, G_{id} is a group, which is called the *identification group*, the reason for this name will become clear below.

One can also show that if there is an outer automorphism branching $\mathbf{A} \mapsto \tilde{\mathbf{A}}$, then [DFMS97]

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \cong L_{\mathfrak{g}/\mathfrak{h}}(\mathbf{A}\hat{\lambda}, \tilde{\mathbf{A}}\hat{\mu}) . \quad (\text{C.2.10})$$

Equivalently, one can prove that [DFMS97]

$$\chi_{(\hat{\lambda}, \hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau) = \chi_{(\mathbf{A}\hat{\lambda}, \tilde{\mathbf{A}}\hat{\mu})}^{\mathfrak{g}/\mathfrak{h}}(\tau) . \quad (\text{C.2.11})$$

It turns out that these are all the identifications among modules $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$. Thus, for $(\hat{\lambda}, \hat{\mu}) \in \mathcal{E}$, we can write the equivalence class as

$$[(\hat{\lambda}, \hat{\mu})] = \{(\mathbf{A}\hat{\lambda}, \tilde{\mathbf{A}}\hat{\mu}) : (\mathbf{A}, \tilde{\mathbf{A}}) \in G_{\text{id}}\} . \quad (\text{C.2.12})$$

It is widely believed that if there are no fixed points of the outer automorphisms in the branching $\mathbf{A} \mapsto \tilde{\mathbf{A}}$, i.e., there are *no* nonzero dominant integrable weight vectors $(\hat{\lambda}, \hat{\mu}) \in \mathcal{E}$ such that

$$\mathbf{A}\hat{\lambda} = \hat{\lambda}, \quad \tilde{\mathbf{A}}\hat{\mu} = \hat{\mu} , \quad (\text{C.2.13})$$

then all irreducible modules of $V_{\mathfrak{g}/\mathfrak{h}}$ is of the form $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ and the number of elements in each equivalence class $[(\hat{\lambda}, \hat{\mu})]$ is the same and equal to the number of outer automorphisms of $\hat{\mathfrak{g}}$ that branch into outer automorphisms of $\hat{\mathfrak{h}}$ (i.e., the order $|Z|$ of Z). Thus, under this assumption, we can write the GKO Hilbert space as

$$\mathcal{H}_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}} := \bigoplus_{\substack{(\hat{\lambda}_L, \hat{\mu}_L) \in \mathcal{E}/\sim \\ (\hat{\lambda}_R, \hat{\mu}_R) \in \mathcal{E}/\sim}} \mathcal{M}_{\hat{\lambda}_L \hat{\lambda}_R}^{\mathfrak{g}} \mathcal{M}_{\hat{\mu}_L \hat{\mu}_R}^{\mathfrak{h}} L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}_L, \hat{\mu}_L) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}_R, \hat{\mu}_R)} , \quad (\text{C.2.14})$$

⁵Note that $\iota^{-1} : Z \rightarrow Z(\tilde{H})$ is well-defined since ι is assumed to be injective.

where $\mathcal{M}_{\hat{\lambda}_L \hat{\lambda}_R}^{\mathfrak{g}}, \mathcal{M}_{\hat{\mu}_L \hat{\mu}_R}^{\mathfrak{h}}$ are modular invariants of $L_{\mathfrak{g}}(\mathbf{k}, 0)$ and $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$. In this paper, we are only interested in the diagonal modular invariants for the GKO theorem, for which

$$\mathcal{H}_{\mathfrak{g}/\mathfrak{h}}^{\text{GKO}} := \bigoplus_{(\hat{\lambda}, \hat{\mu}) \in \mathcal{E}/\sim} L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \otimes \widetilde{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} . \quad (\text{C.2.15})$$

Field Identifications From Chern-Simons Theory

Let us explain the origin of field identifications from the Chern-Simons theory point of view. We assume that G and $H \subset G$ are simple and simply connected. For a compact Lie group G , the Chern-Simons-Witten (CSW) theory is a 3d topological field theory [Wit89]. It was argued in [Wit89] that the Hilbert space of the CSW theory on a compact Riemann surface on Σ is the (finite-dimensional) space of chiral conformal blocks of the WZW model based on G on Σ and the gauge invariant observables – the Wilson loops – correspond to the primary fields in the WZW model. It was shown in [MS89d, EFR91] that the CSW action on three manifold with boundary reduces to the WZW action on the boundary, i.e., Chern-Simons theory is the holographic dual of WZW models. The natural question is to understand the holographic dual of gauged WZW models. It was argued in [MS89d] that the holographic dual of the G/H -WZW model is the Chern-Simons theory for the group⁶ $(G_k \times H_{-\tilde{k}})/Z$ where $Z = Z(G) \cap Z(H)$ is the common center of G, H embedded diagonally in $G \times H$. Here k is the level of G and $\tilde{k}$ is the induced level from the embedding $H \subset G$. The Wilson loops of the $(G_k \times H_{-\tilde{k}})/Z$ theory are

$$W_{\lambda, \mu}(C) = \text{Tr}_{\lambda} \text{P exp} \left(\oint_C A \right) \text{Tr}_{\mu} \text{P exp} \left(\oint_C B \right) , \quad (\text{C.2.16})$$

where A, B are G, H -gauge fields respectively. Naively, from this we discover that the state space is $L_{\mathfrak{g}}(k, \hat{\lambda}) \otimes L_{\mathfrak{h}}(\tilde{k}, \hat{\mu})^{\vee}$ [MS89d]. But the boundary path integral of the $G \times H$ CSW theory imposes first order constraints which restricts the state space to be the LH -invariant in $L_{\mathfrak{g}}(k, \hat{\lambda}) \otimes L_{\mathfrak{h}}(\tilde{k}, \hat{\mu})^{\vee}$ [MS89d]. These are precisely the space of LH -homomorphisms:

$$(L_{\mathfrak{g}}(k, \hat{\lambda}) \otimes L_{\mathfrak{h}}(\tilde{k}, \hat{\mu})^{\vee})^{LH} \cong \text{Hom}(L_{\mathfrak{h}}(\tilde{k}, \hat{\mu}), L_{\mathfrak{g}}(k, \hat{\lambda})) = L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) . \quad (\text{C.2.17})$$

Comparing with (3.3.40), we see that we have recovered the modules of the GKO vertex operator algebra $V_{\mathfrak{g}/\mathfrak{h}}$. We now discuss the effect of division by Z and explain how it gives rise to selection rules and field identifications. Gauging Z implies two conditions:

1. $L_{\mathfrak{g}}(k, \hat{\lambda}), L_{\mathfrak{h}}(\tilde{k}, \hat{\mu})$ must transform in the same way under Z .

⁶There are counter examples to this proposal [CGS23], they mostly appear for conformal embeddings or Maverick cosets.

2. Wilson lines related by the action of spectral flow corresponding to elements of Z must be identified.

Let us see how Condition 1 gives selection rules. A central element $g \in Z(G)$ acts on the representation $L_{\mathfrak{g}}(k, \hat{\lambda})$ as a phase given by (C.1.70)

$$g \cdot v = e^{-2\pi i \langle A_g(\hat{\omega}_0), \lambda \rangle} v, \quad \forall v \in L_{\mathfrak{g}}(k, \hat{\lambda}), \quad (\text{C.2.18})$$

where $g \mapsto A_g \in \mathcal{O}(\hat{\mathfrak{g}})$ is the group isomorphism $Z(G) \cong \mathcal{O}(\hat{\mathfrak{g}})$. Similarly, $h \in Z(H)$ acts on $L_{\mathfrak{h}}(\tilde{k}, \hat{\mu})$ as a phase

$$h \cdot v = e^{-2\pi i \langle A_h(\hat{\omega}_0), \mu \rangle} v, \quad \forall v \in L_{\mathfrak{h}}(\tilde{k}, \hat{\mu}), \quad (\text{C.2.19})$$

where $h \mapsto A_h \in \mathcal{O}(\hat{\mathfrak{h}})$ is the group isomorphism $Z(H) \cong \mathcal{O}(\hat{\mathfrak{h}})$. Now, suppose $g \in Z = Z(G) \cap Z(H)$. We will write the corresponding outer elements of $\mathcal{O}(\hat{\mathfrak{g}})$ and $\mathcal{O}(\hat{\mathfrak{h}})$ by $A, \tilde{A}$ respectively. Then we see that $g \in Z$ acts in the same way on $L_{\mathfrak{g}}(k, \hat{\lambda}), L_{\mathfrak{h}}(\tilde{k}, \hat{\mu})$ if and only if

$$\langle A\hat{\omega}_0, \lambda \rangle = \langle \tilde{A}\hat{\omega}_0, \mu \rangle \pmod{1}. \quad (\text{C.2.20})$$

Because of the isomorphism $Z \cong G_{\text{id}}$ in (C.2.9), $A \mapsto \tilde{A}$ or equivalently satisfies

$$\langle A\hat{\omega}_0, \lambda \rangle = \langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle \pmod{1}. \quad (\text{C.2.21})$$

Combining (C.2.20) and (C.2.21) we get

$$\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda - \mu \rangle = 0 \pmod{1}. \quad (\text{C.2.22})$$

A sufficient condition for (C.2.22) to hold is

$$\mathcal{P}\lambda - \mu \in \mathcal{P}\Lambda_r^{\mathfrak{g}}, \quad (\text{C.2.23})$$

which is precisely (C.2.1). Condition 2 implies that we must identify sectors related by spectral flow isomorphism. Using the isomorphism $Z \cong G_{\text{id}}$, we see that the spectral flow isomorphism gives the identification:

$$L_{\mathfrak{g}/\mathfrak{h}}(\lambda, \mu) \cong L_{\mathfrak{g}/\mathfrak{h}}(A\lambda, \tilde{A}\mu), \quad (A, \tilde{A}) \in G_{\text{id}}, \quad (\text{C.2.24})$$

which gives us field identifications. This explains field identifications from Chern-Simons theory point of view. Finally, if the action of Z has fixed points, there are new operators in the theory and one has to do *fixed point resolution* as explained in Appendix C.2.4 below.

Example: $\widehat{\mathfrak{g}} = \widehat{\mathfrak{so}}(4n)_1$, $\widehat{\mathfrak{h}} = \widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1 \oplus \widehat{\mathfrak{su}}(2)_n$

In this appendix, we give an explicit example of field identification. We consider the following embedding [BRS88, ABR88] of $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)^a \oplus \mathfrak{su}(2)^b$ in $\mathfrak{so}(4n)$. First embed $\mathfrak{so}(4n-4) \oplus \mathfrak{su}(2)^a \oplus \mathfrak{su}(2)^c$ in $\mathfrak{so}(4n)$ in the standard way. To embed $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)^d$ into $\mathfrak{so}(4n-4)$, we will give the embedding of the corresponding Lie group. For $g \in \mathrm{Sp}(n-1) = \mathrm{Sp}(2n-2, \mathbb{C}) \cap \mathrm{su}(2n-2)$ and $h \in \mathfrak{su}(2)$, denote by $\hat{h}$ the diagonal embedding of $\mathrm{SU}(2)$ in $\mathrm{Sp}(n-1)$:

$$\hat{h} := \mathrm{diag}(h, \dots, h) \in \mathrm{Sp}(n-1) , \quad (\text{C.2.25})$$

and consider the embedding

$$\begin{aligned} \mathrm{Sp}(n-1) \times \mathrm{SU}(2) &\longrightarrow \mathrm{SO}(4n-4) \\ (g, h) &\longmapsto \begin{pmatrix} \mathrm{Re}(g\hat{h}) & \mathrm{Im}(g\hat{h}) \\ -\mathrm{Im}(g\hat{h}) & \mathrm{Re}(g\hat{h}) \end{pmatrix} . \end{aligned} \quad (\text{C.2.26})$$

Finally, embed $\mathfrak{su}(2)^b$ diagonally into $\mathfrak{su}(2)^c \oplus \mathfrak{su}(2)^d$.

To figure out the factor corresponding to field identifications, we start with the weight vectors for $\mathfrak{so}(4n)$. Let $e_1, \dots, e_{2n}$ be an orthonormal basis of $\mathbb{R}^{2n}$. A choice of roots of $\mathfrak{so}(4n)$ is

$$\pm e_i \pm e_j, \quad i \neq j , \quad (\text{C.2.27})$$

and a choice of simple roots is

$$\alpha_j = e_j - e_{j+1} , \quad j = 1, \dots, 2n-1, \quad \alpha_{2n} = e_{2n-1} + e_{2n} . \quad (\text{C.2.28})$$

A choice of fundamental weights is

$$\begin{aligned} \omega_j &= e_1 + \dots + e_j , \quad j = 1, \dots, 2n-2 , \\ \omega_{2n-1} &= \frac{1}{2} (e_1 + \dots + e_{2n-1} - e_{2n}) , \\ \omega_{2n} &= \frac{1}{2} (e_1 + \dots + e_{2n-1} + e_{2n}) . \end{aligned} \quad (\text{C.2.29})$$

Some of the fundamental weights and their corresponding representation is shown in Table C.2.

Representation	Fundamental weight
Vector	$\omega_1 = e_1$
Adjoint	$\omega_2 = e_1 + e_2$
Spinor	$\omega_{2n-1} = \frac{1}{2} (e_1 + \dots + e_{2n-1} - e_{2n})$
Conjugate spinor	$\omega_{2n} = \frac{1}{2} (e_1 + \dots + e_{2n-1} + e_{2n})$

Table C.2: Some representations of $\mathfrak{so}(4n)$ and the corresponding fundamental representation

Under the embedding $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)^a \oplus \mathfrak{su}(2)^d$ described above, we want to find a projection matrix $\mathcal{P}$ such that $\mathcal{P}\lambda$ is a weight vector for $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ for every weight vector λ of $\mathfrak{so}(4n)$. Let us expand a weight vector $\mathbf{x}$ of $\mathfrak{so}(4n)$ as

$$\mathbf{x} = \sum_{i=1}^{2n} x_i e_i . \quad (\text{C.2.30})$$

Now, to project onto weights of $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)^a \oplus \mathfrak{su}(2)^b$, we first note that under the embedding $\mathfrak{so}(4n-4) \oplus \mathfrak{su}(2)^a \oplus \mathfrak{su}(2)^c \hookrightarrow \mathfrak{so}(4n)$, we can take the weight space of $\mathfrak{so}(4n-4)$ to be spanned by $e_1, \dots, e_{2n-2}$ and weight space of $\mathfrak{su}(2)^a \oplus \mathfrak{su}(2)^c$ to be spanned by e_{2n-1}, e_{2n} respectively. Now, group the $2n-2$ basis vectors $e_1, e_2, \dots, e_{2n-2}$ into pairs

$$(e_1, e_2), (e_3, e_4), \dots, (e_{2n-3}, e_{2n-2}) . \quad (\text{C.2.31})$$

For the coefficients in (C.2.30), define pair sums

$$\mu_k = x_{2k-1} + x_{2k} , \quad k = 1, \dots, n-1 . \quad (\text{C.2.32})$$

Under the embedding $\mathfrak{sp}(n-1) \oplus \mathfrak{su}(2)^d \hookrightarrow \mathfrak{so}(4n-4)$, the Dynkin labels $(b_1, \dots, b_{n-1})$ of $\mathfrak{sp}(n-1)$ are given by

$$b_i = \mu_i - \mu_{i+1} , \quad i = 1, \dots, n-2, \quad b_{n-1} = \mu_{n-1} . \quad (\text{C.2.33})$$

The roots of $\mathfrak{su}(2)^a, \mathfrak{su}(2)^c$ and $\mathfrak{su}(2)^d$ is given respectively by

$$\alpha_a = e_{2n-1} + e_{2n}, \quad \alpha_c = e_{2n-1} - e_{2n}, \quad \alpha_d = \sum_{j=1}^{n-2} [(e_{2j-1} + e_{2j}) - (e_{2j+1} + e_{2j+2})] . \quad (\text{C.2.34})$$

Then the Dynkin labels for $\mathfrak{su}(2)^a$ and $\mathfrak{su}(2)^b = \text{diag}(\mathfrak{su}(2)^c \oplus \mathfrak{su}(2)^d)$ are

$$h_a = x_{2n-1} + x_{2n}, \quad h_b = x_{2n-1} - x_{2n} + \sum_{i=1}^{2n-2} b_i . \quad (\text{C.2.35})$$

In summary, the projection map $\mathcal{P} : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n-1} \oplus \mathbb{R} \oplus \mathbb{R}$ is given by

$$\mathbf{x} \mapsto (b_1, \dots, b_{n-1}; h_a; h_b) . \quad (\text{C.2.36})$$

Here we must keep in mind that the first $n-1$ coordinates are in the basis of fundamental weights of $\mathfrak{sp}(n-1)$ and the last two coordinates are in the standard basis of $\mathbb{R} \oplus \mathbb{R}$. To emphasize this, let us choose the standard orthonormal basis $\{e_1^{(1)}, \dots, e_{n-1}^{(1)}\}$ of $\mathbb{R}^{n-1}$ and $e^{(2)}, e^{(3)}$ of the $\mathbb{R} \oplus \mathbb{R}$. Then a choice of fundamental weights for $\mathfrak{sp}(n-1)$ is

$$\omega_i^{(1)} := e_1^{(1)} + \dots + e_i^{(1)} , \quad i = 1, \dots, n-1 , \quad (\text{C.2.37})$$

while that of $\mathfrak{su}(2)^a, \mathfrak{su}(2)^b$ is

$$\omega^{(2)} := e^{(2)} , \quad \omega^{(3)} := e^{(3)} . \quad (\text{C.2.38})$$

Thus, the projection map can be written as

$$\mathbf{x} \mapsto \sum_{i=1}^{n-1} b_i \omega_i^{(1)} + h_a \omega^{(2)} + h_b \omega^{(3)} . \quad (\text{C.2.39})$$

For affine $\widehat{\mathfrak{so}}(4n)_1$, we fix the fundamental weights $\hat{\omega}_0, \hat{\omega}_1, \dots, \hat{\omega}_{2n}$ defined using the fundamental weights of $\mathfrak{so}(4n)$ as in (C.1.41). Then

$$\langle \hat{\omega}_i, \hat{\omega}_j \rangle = F_{ij} = (A^{-1})_{ij} \frac{\|\alpha_j\|^2}{2}, \quad i, j = 1, \dots, 2n , \quad (\text{C.2.40})$$

where A is the Cartan matrix of $\mathfrak{so}(4n)$ which we take to be

$$A(\mathfrak{so}(4n)) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & -1 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 2 \end{pmatrix} . \quad (\text{C.2.41})$$

We can check that

$$F(\mathfrak{so}(4n)) = \frac{1}{2} \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 & 1 \\ 2 & 4 & 4 & \cdots & 4 & 2 & 2 \\ 2 & 4 & 6 & \cdots & 6 & 3 & 3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 2 & 4 & 6 & \cdots & 2(2n-2) & 2n-2 & 2n-2 \\ 1 & 2 & 3 & \cdots & 2n-2 & n & n-1 \\ 1 & 2 & 3 & \cdots & 2n-2 & n-1 & n \end{pmatrix} . \quad (\text{C.2.42})$$

Now for each automorphism A of $\mathfrak{so}(4n)_1$, we need to check if there is an outer automorphism $\tilde{A}$ of $\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1^a \oplus \widehat{\mathfrak{su}}(2)_n^b$ satisfying

$$\langle A\hat{\omega}_0, \lambda \rangle \equiv \langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle \pmod{1}, \quad \text{for all } \lambda \in \Lambda_{\mathfrak{so}(4n)}^w , \quad (\text{C.2.43})$$

where $\hat{\omega}_0$ on the RHS is the first fundamental weight of $\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1^a \oplus \widehat{\mathfrak{su}}(2)_n^b$. We now want to calculate $\mathcal{P}\lambda$ for any $\lambda \in \Lambda_{\mathfrak{so}(4n)}^w$. In the basis $\{e_1, \dots, e_{2n}\}$ we can write the fundamental weights $\omega_1, \dots, \omega_{2n}$ of $\mathfrak{so}(4n)$ given in (C.2.29) as

$$\begin{aligned} \omega_i &= (\underbrace{1, 1, \dots, 1}_{i \text{ entries}}, 0, \dots, 0) , \quad i = 1, \dots, 2n-2 , \\ \omega_{2n-1} &= \frac{1}{2}(1, \dots, 1, -1) , \\ \omega_{2n} &= \frac{1}{2}(1, \dots, 1) . \end{aligned} \quad (\text{C.2.44})$$

Thus we have

$$\begin{aligned}
\mathcal{P}\omega_{2j} &= 2\omega_j^{(1)} + 2\omega^{(3)} , \quad j = 1, \dots, n-1 , \\
\mathcal{P}\omega_{2j+1} &= \omega_j^{(1)} + \omega_{j+1}^{(1)} + 2\omega^{(3)} , \quad j = 0, \dots, n-2 , \\
\mathcal{P}\omega_{2n-1} &= \omega_{n-1}^{(1)} + \omega^{(3)} , \\
\mathcal{P}\omega_{2n} &= \omega_{n-1}^{(1)} + \omega^{(2)} .
\end{aligned} \tag{C.2.45}$$

Take a general weight vector λ of $\mathfrak{so}(4n)$:

$$\lambda = \sum_{i=1}^{2n} \lambda_i \omega_i, \quad \lambda_i \in \mathbb{Z} . \tag{C.2.46}$$

Then

$$\begin{aligned}
\mathcal{P}\lambda &= \sum_{i=1}^{2n} \lambda_i \mathcal{P}\omega_i = \sum_{j=1}^{n-1} \lambda_{2j} 2 \left(\omega_j^{(1)} + \omega^{(3)} \right) + \sum_{j=0}^{n-2} \lambda_{2j+1} \left(\omega_j^{(1)} + \omega_{j+1}^{(1)} + 2\omega^{(3)} \right) \\
&\quad + \lambda_{2n-1} \left(\omega_{n-1}^{(1)} + \omega^{(3)} \right) + \lambda_{2n} \left(\omega_{n-1}^{(1)} + \omega^{(2)} \right) .
\end{aligned} \tag{C.2.47}$$

The outer automorphism group of $\widehat{\mathfrak{so}}(4n)_1$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ which acts on $\hat{\omega}_0$ as follows: let a and $\tilde{a}$ be the generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$, then

$$a \cdot \hat{\omega}_0 = \hat{\omega}_1, \quad \tilde{a} \cdot \hat{\omega}_0 = \hat{\omega}_{2n}, \quad (a\tilde{a}) \cdot \hat{\omega}_0 = \hat{\omega}_{2n-1} . \tag{C.2.48}$$

Let

$$\lambda = \sum_{i=1}^{2n} \lambda_i \omega_i, \quad \lambda_i \in \mathbb{Z} , \tag{C.2.49}$$

be a weight vector of $\mathfrak{so}(4n)$. Then using (C.2.48), (C.2.40) and (C.2.42), we have

$$\begin{aligned}
\langle \hat{\omega}_0, \lambda \rangle &= 0 , \\
\langle a \cdot \hat{\omega}_0, \lambda \rangle &= \langle \hat{\omega}_1, \lambda \rangle = \sum_{i=1}^{2n-2} \lambda_i + \frac{1}{2} (\lambda_{2n-1} + \lambda_{2n}) \\
&\equiv \frac{1}{2} (\lambda_{2n-1} + \lambda_{2n}) \pmod{1} , \\
\langle \tilde{a} \cdot \hat{\omega}_0, \lambda \rangle &= \langle \hat{\omega}_{2n}, \lambda \rangle = \frac{1}{2} \sum_{i=1}^{2n-2} i \lambda_i + \frac{1}{2} ((n-1)\lambda_{2n-1} + n\lambda_{2n}) \\
&\equiv \frac{1}{2} \sum_{j=0}^{n-2} \lambda_{2j+1} + \frac{1}{2} ((n-1)\lambda_{2n-1} + n\lambda_{2n}) \pmod{1} , \\
\langle a\tilde{a} \cdot \hat{\omega}_0, \lambda \rangle &= \langle \hat{\omega}_{2n-1}, \lambda \rangle = \frac{1}{2} \sum_{i=1}^{2n-2} i \lambda_i + \frac{1}{2} (n\lambda_{2n-1} + (n-1)\lambda_{2n}) \\
&\equiv \frac{1}{2} \sum_{j=0}^{n-2} \lambda_{2j+1} + \frac{1}{2} (n\lambda_{2n-1} + (n-1)\lambda_{2n}) \pmod{1} .
\end{aligned} \tag{C.2.50}$$

For $\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1^{\mathfrak{a}} \oplus \widehat{\mathfrak{su}}(2)_1^{\mathfrak{b}}$, we have

$$\hat{\omega}_0 \equiv \left(\hat{\omega}_0^{(1)}, \hat{\omega}_0^{(2)}, \hat{\omega}_0^{(3)} \right) , \tag{C.2.51}$$

where $\hat{\omega}_0^{(1)}, \hat{\omega}_0^{(2)}, \hat{\omega}_0^{(3)}$ are the first fundamental weights of $\widehat{\mathfrak{sp}}(n-1)_1$, $\widehat{\mathfrak{su}}(2)_1^a$, $\widehat{\mathfrak{su}}(2)_1^b$ respectively. We will denote the generators of the $\mathbb{Z}_2$ outer automorphism group of $\widehat{\mathfrak{sp}}(n-1)_1$, $\widehat{\mathfrak{su}}(2)_1^a$, $\widehat{\mathfrak{su}}(2)_1^b$ by $a^{(1)}, a^{(2)}, a^{(3)}$ respectively. Then for any outer automorphism $\tilde{A} = (\tilde{A}^{(1)}, \tilde{A}^{(2)}, \tilde{A}^{(3)})$ of $\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1^a \oplus \widehat{\mathfrak{su}}(2)_1^b$, using (C.2.47), we have

$$\begin{aligned}
\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle &= \sum_{j=1}^{n-1} \lambda_{2j} \langle \tilde{A}\hat{\omega}_0, 2(\omega_j^{(1)} + \omega^{(3)}) \rangle + \sum_{j=0}^{n-2} \lambda_{2j+1} \langle \tilde{A}\hat{\omega}_0, (\omega_j^{(1)} + \omega_{j+1}^{(1)} + 2\omega^{(3)}) \rangle \\
&\quad + \lambda_{2n-1} \langle \tilde{A}\hat{\omega}_0, (\omega_{n-1}^{(1)} + \omega^{(3)}) \rangle + \lambda_{2n} \langle \tilde{A}\hat{\omega}_0, (\omega_{n-1}^{(1)} + \omega^{(2)}) \rangle \\
&= 2 \sum_{j=1}^{n-1} \lambda_{2j} \langle \tilde{A}^{(1)}\hat{\omega}_0^{(1)}, \omega_j^{(1)} \rangle + \sum_{j=0}^{n-2} \lambda_{2j+1} \langle \tilde{A}^{(1)}\hat{\omega}_0^{(1)}, \omega_j^{(1)} + \omega_{j+1}^{(1)} \rangle \\
&\quad + 2 \sum_{i=1}^{2n-2} \lambda_{2j+1} \langle \tilde{A}^{(3)}\hat{\omega}_0^{(3)}, \omega^{(3)} \rangle + (\lambda_{2n-1} + \lambda_{2n}) \langle \tilde{A}^{(1)}\hat{\omega}_0^{(1)}, \omega_{n-1}^{(1)} \rangle \\
&\quad + \lambda_{2n-1} \langle \tilde{A}^{(3)}\hat{\omega}_0^{(3)}, \omega^{(3)} \rangle + \lambda_{2n} \langle \tilde{A}^{(2)}\hat{\omega}_0^{(2)}, \omega^{(2)} \rangle .
\end{aligned} \tag{C.2.52}$$

To compute the RHS of (C.2.56), we need the Cartan matrix $A(\widehat{\mathfrak{sp}}(n-1))$ and $F(\widehat{\mathfrak{sp}}(n-1))$:

$$A(\widehat{\mathfrak{sp}}(n-1)) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{pmatrix}, \quad F(\widehat{\mathfrak{sp}}(n-1)) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 2 & 3 & \cdots & n-2 & n-2 \\ 1 & 2 & 3 & \cdots & n-2 & n-1 \end{pmatrix}. \tag{C.2.53}$$

For $\widehat{\mathfrak{su}}(2)$, we have

$$A(\widehat{\mathfrak{su}}(2)) = (2), \quad F(\widehat{\mathfrak{su}}(2)) = \left(\frac{1}{2}\right). \tag{C.2.54}$$

The action of the outer automorphism group of $\widehat{\mathfrak{sp}}(n-1)_1$, $\widehat{\mathfrak{su}}(2)_1^a$, $\widehat{\mathfrak{su}}(2)_1^b$ is given by

$$a^{(1)} \cdot \hat{\omega}_0^{(1)} = \hat{\omega}_{n-1}^{(1)}, \quad a^{(i)} \cdot \hat{\omega}_0^{(i)} = \hat{\omega}_1^{(i)}, \quad i = 2, 3. \tag{C.2.55}$$

Thus we have

$$\begin{aligned}
\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle &\equiv \sum_{j=0}^{n-2} \lambda_{2j+1} \langle \tilde{A}^{(1)}\hat{\omega}_0^{(1)}, \omega_j^{(1)} + \omega_{j+1}^{(1)} \rangle + (\lambda_{2n-1} + \lambda_{2n}) \langle \tilde{A}^{(1)}\hat{\omega}_0^{(1)}, \omega_{n-1}^{(1)} \rangle \\
&\quad + \lambda_{2n-1} \langle \tilde{A}^{(3)}\hat{\omega}_0^{(3)}, \omega^{(3)} \rangle + \lambda_{2n} \langle \tilde{A}^{(2)}\hat{\omega}_0^{(2)}, \omega^{(2)} \rangle \pmod{1}.
\end{aligned} \tag{C.2.56}$$

Then using (C.1.42), we have

$$\langle \hat{\omega}_0, \lambda \rangle = \langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle = 0 \quad \text{for} \quad \tilde{A} = (\mathbf{1}, \mathbf{1}, \mathbf{1}) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{C.2.57}$$

Using (C.2.55) and (C.2.56), for $\tilde{A} = (a^{(1)}, a^{(2)}, \mathbb{1})$, we have

$$\begin{aligned}
\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle &\equiv \sum_{j=0}^{n-2} \lambda_{2j+1} \langle \hat{\omega}_{n-1}^{(1)}, \hat{\omega}_j^{(1)} + \hat{\omega}_{j+1}^{(1)} \rangle + (\lambda_{2n-1} + \lambda_{2n}) \langle \hat{\omega}_{n-1}^{(1)}, \hat{\omega}_{n-1}^{(1)} \rangle \\
&\quad + \lambda_{2n} \langle \hat{\omega}_1^{(2)}, \omega^{(2)} \rangle \bmod 1 \\
&\equiv \frac{1}{2} \sum_{j=0}^{n-1} \lambda_{2j+1} (j + j + 1) + \frac{n-1}{2} (\lambda_{2n-1} + \lambda_{2n}) + \frac{1}{2} \lambda_{2n} \\
&\equiv \frac{1}{2} \sum_{j=0}^{n-1} \lambda_{2j+1} + \frac{1}{2} ((n-1)\lambda_{2n-1} + n\lambda_{2n}) \bmod 1.
\end{aligned} \tag{C.2.58}$$

Comparing with (C.2.50), we see that

$$\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle \equiv \langle \tilde{a} \cdot \hat{\omega}_0, \lambda \rangle \bmod 1, \quad \tilde{A} = (a^{(1)}, a^{(2)}, \mathbb{1}). \tag{C.2.59}$$

Similarly, for $\tilde{A} = (a^{(1)}, \mathbb{1}, a^{(3)})$ we have

$$\begin{aligned}
\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle &\equiv \sum_{j=0}^{n-2} \lambda_{2j+1} \langle \hat{\omega}_{n-1}^{(1)}, \hat{\omega}_j^{(1)} + \hat{\omega}_{j+1}^{(1)} \rangle + \sum_{j=0}^{n-2} \lambda_{2j+1} \langle a^{(3)}\hat{\omega}_0^{(3)}, \omega^{(3)} \rangle \\
&\quad + (\lambda_{2n-1} + \lambda_{2n}) \langle a^{(1)}\hat{\omega}_0^{(1)}, \omega_{n-1}^{(1)} \rangle + \lambda_{2n-1} \langle a^{(3)}\hat{\omega}_0^{(3)}, \omega^{(3)} \rangle \bmod 1 \\
&\equiv \frac{1}{2} \sum_{j=0}^{n-1} \lambda_{2j+1} + \frac{n-1}{2} (\lambda_{2n-1} + \lambda_{2n}) + \frac{1}{2} \lambda_{2n-1} \bmod 1 \\
&\equiv \frac{1}{2} \sum_{j=0}^{n-1} \lambda_{2j+1} + \frac{1}{2} (n\lambda_{2n-1} + (n-1)\lambda_{2n}) \bmod 1 \\
&\equiv \langle (a\tilde{a}) \cdot \hat{\omega}_0, \lambda \rangle \bmod 1.
\end{aligned} \tag{C.2.60}$$

Finally, we have

$$\langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle \equiv \frac{1}{2} (\lambda_{2n-1} + \lambda_{2n}) \equiv \langle a \cdot \hat{\omega}_0, \lambda \rangle \bmod 1, \quad \tilde{A} = (\mathbb{1}, a^{(2)}, a^{(3)}). \tag{C.2.61}$$

Thus, all four outer automorphisms of $\widehat{\mathfrak{so}}(4n)_1$ branch into non-trivial outer automorphisms of $\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1 \oplus \widehat{\mathfrak{su}}(2)_n$.

C.2.3 Selection Rules And Field Identifications From Simple Currents

Selection rules and field identifications can be given a uniform description in terms of *simple currents* of $L_{\mathfrak{g}}(k, 0)$ and $L_{\mathfrak{h}}(\tilde{k}, 0)$. Let us begin by defining simple currents.

Definition C.2.1. A simple current for a rational VOA V is an irreducible module J for which the *quantum dimension*, defined as

$$d_W := \frac{S_{VJ}}{S_{VV}}, \tag{C.2.62}$$

is 1. Here $S_{WW'}$ is the modular S -matrix.

Note that V is trivially a simple current. We will call simple currents $J \not\cong V$ as nontrivial simple currents. This definition has many important consequences which we summarize below.

1. For any simple current J and an irreducible V -module W , we have

$$\sum_a \mathcal{N}_{JW}^{W^a} = 1, \quad (\text{C.2.63})$$

where W^a runs over the finite set of isomorphism classes of irreducible V -modules. This implies that $J \boxtimes W$ is an irreducible V -module for any irreducible V -module W . In particular, the contragredient module J' of J satisfies $J \boxtimes J' \cong V$.

2. For two simple currents J_1, J_2 , the tensor product $J_1 \boxtimes J_2$ is also a simple current. Thus the set of all simple currents of V form an abelian group with respect to the tensor product.

Given a simple current J and an irreducible V -module W , we can define the *monodromy charge* of W with respect to J as

$$Q_J(W) := (h_J + h_W - h_{J \boxtimes W}) \bmod 1, \quad (\text{C.2.64})$$

where h_W is the conformal dimension of the irreducible V -module W . The monodromy charge satisfies

$$Q_J(W_1 \boxtimes W_2) = (Q_J(W_1) + Q_J(W_2)) \bmod 1. \quad (\text{C.2.65})$$

One can also show that [SY90b] for any irreducible V -modules W_1, W_2 and a simple current J , the S -matrix satisfies

$$S_{J^\alpha \boxtimes W_1, W_2} = e^{2\pi i \alpha Q_J(W_2)} S_{W_1, W_2}, \quad (\text{C.2.66})$$

where $J^\alpha := J \boxtimes \cdots \boxtimes J$ denotes the α times tensor product. Simple currents of affine VOAs $L_{\mathfrak{g}}(k, 0)$ for simple Lie algebras $\mathfrak{g}$ have been classified in [SY90b, SY90a]. We quote the theorem from [Fuc91] here.

Theorem C.2.1. *Let $\mathfrak{g}$ be a simple Lie algebra and $k \in \mathbb{Z}_{\geq 1}$. Then $L_{\mathfrak{g}}(k, \lambda)$ is a nontrivial simple current of $L_{\mathfrak{g}}(k, 0)$ if and only if either*

$$\lambda = k\omega_i, \quad a_i = 1, \quad (\text{C.2.67})$$

or

$$\mathfrak{g} = \mathfrak{e}_8, \quad k = 2, \quad \lambda = \omega_7, \quad (\text{C.2.68})$$

where a_i is the mark of $\mathfrak{g}$ defined in (C.1.18).

Fundamental weights ω_i such that the corresponding mark $a_i = 1$ are called *cominimal weights*. It was also shown in [Fuc91] that for any outer automorphism $A \in \mathcal{O}(\hat{\mathfrak{g}})$, the finite part of $A\omega_0$ is a cominimal weight and in fact all cominimal weights of $\mathfrak{g}$ are of this form. This establishes a 1-1 correspondence between the simple currents of $L_{\mathfrak{g}}(k, 0)$ and $\mathcal{O}(\hat{\mathfrak{g}})$ except for the case $\hat{\mathfrak{g}}_k = (\hat{\mathfrak{e}}_8)_2$. We will restrict to cases where such a correspondence is true. We will denote the simple current corresponding to an outer automorphism A by J_A and label the set of simple currents of $L_{\mathfrak{g}}(k, 0)$ by $\mathcal{O}(\hat{\mathfrak{g}})$.

Consider now the set

$$G'_{\text{id}} := \{(J_A, J'_{\tilde{A}}) : (A, \tilde{A}) \in \mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathfrak{h}}), h_{J_A}^{\mathfrak{g}} - h_{J'_{\tilde{A}}}^{\mathfrak{h}} \in \mathbb{Z}, \chi_{(J_A, J'_{\tilde{A}})}^{\mathfrak{g}/\mathfrak{h}}(\tau) \neq 0\} . \quad (\text{C.2.69})$$

Selection rules can then be written as follows [SY90a, SY90b]:

$$\chi_{(\lambda, \mu)}(\tau) \neq 0 \iff Q_J(\lambda) = Q_{J'}(\mu) . \quad (\text{C.2.70})$$

We can now show that G'_{id} is in fact a group. Indeed, suppose $(J_1, J'_1), (J_2, J'_2) \in G'_{\text{id}}$. Then using (C.2.65), we have for all $(J, J') \in G'_{\text{id}}$

$$Q_J(J_1 \boxtimes J_2) - Q_{J'}(J'_1 \boxtimes J'_2) = Q_J(J_1) - Q_{J'}(J'_1) + Q_J(J_2) - Q_{J'}(J'_2) = 0 . \quad (\text{C.2.71})$$

This means that

$$\chi_{(J_1 \boxtimes J_2, J'_1 \boxtimes J'_2)}^{\mathfrak{g}/\mathfrak{h}}(\tau) \neq 0 . \quad (\text{C.2.72})$$

Next, we also have

$$\begin{aligned} h_{J_1 \boxtimes J_2} - h_{J'_1 \boxtimes J'_2} &= h_{J_1} + h_{J_2} - Q_{J_1}(J_2) - h_{J'_1} - h_{J'_2} + Q_{J'_1}(J'_2) \\ &= (h_{J_1} - h_{J'_1}) + (h_{J_2} - h_{J'_2}) \in \mathbb{Z} . \end{aligned} \quad (\text{C.2.73})$$

Thus G'_{id} is closed under tensor product and hence a group with identity given by $(L_{\mathfrak{g}}(k, 0), L_{\mathfrak{h}}(\tilde{k}, 0))$.

Let $J(\lambda) \in P_k^+(\mathfrak{g})$ be such that

$$J \boxtimes L_{\mathfrak{g}}(k, \lambda) \cong L_{\mathfrak{g}}(k, J(\lambda)) . \quad (\text{C.2.74})$$

Then G'_{id} acts on the set $\mathcal{E}$ as

$$(J, J') \cdot (\lambda, \mu) = (J(\lambda), J'(\mu)) . \quad (\text{C.2.75})$$

One can show that

$$L_{\mathfrak{g}/\mathfrak{h}}(\lambda, \mu) = L_{\mathfrak{g}/\mathfrak{h}}(J(\lambda), J'(\mu)) , \quad (J, J') \in G'_{\text{id}} . \quad (\text{C.2.76})$$

We now show that $G'_{\text{id}} \cong G_{\text{id}}$.

Proposition C.2.1. *The map*

$$\begin{aligned} G'_{\text{id}} &\longrightarrow G_{\text{id}} \\ (J_A, J'_A) &\longmapsto (A, \tilde{A}) . \end{aligned} \quad (\text{C.2.77})$$

is a group isomorphism.

We will need the following lemma to show that the above map is well-defined.

Lemma C.2.1. *Let $A \in \mathcal{O}(\hat{\mathfrak{g}})$ and J_A be the corresponding simple current. Then for any dominant integral weight $\lambda \in P_k^+(\mathfrak{g})$, we have*

$$Q_{J_A}(\lambda) = -\langle A\hat{\omega}_0, \lambda \rangle \bmod 1 . \quad (\text{C.2.78})$$

Proof. For any semisimple Lie algebra $\mathfrak{g}$, the fusion rules satisfy [DFMS97]

$$\mathcal{N}_{\hat{\lambda}_1 \hat{\lambda}_2}^{\hat{\lambda}_3} = \mathcal{N}_{A_1 \hat{\lambda}_1, A_2 \hat{\lambda}_2}^{A_1 A_2 \hat{\lambda}_3} , \quad \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3 \in P_k^+(\mathfrak{g}), \quad A_1, A_2 \in \mathcal{O}(\hat{\mathfrak{g}}) . \quad (\text{C.2.79})$$

This implies that

$$\mathcal{N}_{A\hat{\omega}_0, \hat{\lambda}_1}^{\hat{\lambda}_2} = \mathcal{N}_{\hat{\omega}_0, \hat{\lambda}_1}^{A^{-1}\hat{\lambda}_2} = \delta_{\hat{\lambda}_1}^{A^{-1}\hat{\lambda}_2} , \quad (\text{C.2.80})$$

where we used the fact that $L_{\mathfrak{g}}(\mathbf{k}, \hat{\omega}_0) = L_{\mathfrak{g}}(\mathbf{k}, 0)$. This implies that

$$J_A \boxtimes L_{\mathfrak{g}}(k, \lambda) = L_{\mathfrak{g}}(k, A\hat{\lambda}) . \quad (\text{C.2.81})$$

One can show that [DFMS97]

$$S_{A\hat{\lambda}_1, \hat{\lambda}_2}^{\mathfrak{g}} = e^{-2\pi i \langle A\hat{\omega}_0, \lambda_2 \rangle} S_{\hat{\lambda}_1, \hat{\lambda}_2}^{\mathfrak{g}} . \quad (\text{C.2.82})$$

Using (C.2.81) and comparing (C.2.66) with (C.2.82) we get⁷

$$Q_{J_A}(\lambda) = -\langle A\hat{\omega}_0, \lambda \rangle \bmod 1 . \quad (\text{C.2.83})$$

□

Proof of Proposition C.2.1. Consider any pair of weights $(\lambda, \mu) \in \mathcal{E}$, then selection rule implies that

$$Q_{J_A}(\lambda) = Q_{J'_A}(\mu) . \quad (\text{C.2.84})$$

Using the selection rule $\mathcal{P}\lambda - \mu \in \mathcal{P}\Lambda_r^{\mathfrak{g}}$ and (C.2.83) we get

$$\langle A\hat{\omega}_0, \lambda \rangle = \langle \tilde{A}\hat{\omega}_0, \mathcal{P}\lambda \rangle \bmod 1 . \quad (\text{C.2.85})$$

⁷Note that (C.2.83) is true for dominant integrable weights at arbitrary level k and hence we can conclude that it is true for arbitrary weight vector λ .

This implies that $A \mapsto \tilde{A}$ and hence the map (C.2.77) is well defined. To show that the map is group homomorphism, it is enough to show that $J_A \mapsto A$ is a group homomorphism. Using (C.2.79), the fact that $\mathcal{N}_{\hat{\omega}_0 \hat{\omega}_0}^{\hat{\omega}_0} = 1$ and $L_{\mathfrak{g}}(k, A\hat{\omega}_0)$ is a simple current, we find that

$$J_A \boxtimes J_{A'} \cong J_{AA'} . \quad (\text{C.2.86})$$

This proves that $J_A \mapsto A$ is a group homomorphism. Injectivity of the map is obvious. To prove surjectivity, consider the pair $(J_A, J'_{\tilde{A}})$ for $(A, \tilde{A}) \in G_{\text{id}}$. We want to show that $(J_A, J'_{\tilde{A}}) \in G'_{\text{id}}$. Since $J_A = L_{\mathfrak{g}}(k, A\hat{\omega}_0)$ and similarly for $J'_{\tilde{A}}$, it is clear that

$$\chi_{(J_A, J'_{\tilde{A}})}^{\mathfrak{g}/\mathfrak{h}}(\tau) = \chi_{(A\hat{\omega}_0, \tilde{A}\hat{\omega}_0)}^{\mathfrak{g}/\mathfrak{h}}(\tau) = \chi_{(0,0)}^{\mathfrak{g}/\mathfrak{h}}(\tau) \neq 0 . \quad (\text{C.2.87})$$

Next, for any $(\lambda, \mu) \in \mathcal{E}$, by selection rules and (C.2.81), we have

$$h_{J_A}^{\mathfrak{g}} + h_{\lambda}^{\mathfrak{g}} - h_{A\lambda}^{\mathfrak{g}} = Q_{J_A}(\lambda) = Q_{J'_{\tilde{A}}}(\mu) = h_{J'_{\tilde{A}}}^{\mathfrak{h}} + h_{\mu}^{\mathfrak{h}} - h_{\tilde{A}\mu}^{\mathfrak{h}} \bmod 1 . \quad (\text{C.2.88})$$

Since $L_{\mathfrak{g}/\mathfrak{h}}(\lambda, \mu) \cong L_{\mathfrak{g}/\mathfrak{h}}(A\lambda, \tilde{A}\mu)$, we have

$$h_{\lambda}^{\mathfrak{g}} - h_{A\lambda}^{\mathfrak{g}} = h_{\mu}^{\mathfrak{h}} - h_{\tilde{A}\mu}^{\mathfrak{h}} . \quad (\text{C.2.89})$$

Thus, we conclude that $h_{J_A}^{\mathfrak{g}} - h_{J'_{\tilde{A}}}^{\mathfrak{h}} \in \mathbb{Z}$. The proof is complete. $\square$

We will omit the prime from now on.

It turns out that for generic cases,

$$\mathcal{E}/\sim = \mathcal{E}/G_{\text{id}} . \quad (\text{C.2.90})$$

Thus, field identifications can be described by choosing one representative from each G_{id} -orbit of $\mathcal{E}$.

Selection rules and field identifications from Wilson lines

Selection rules and field identifications can be understood in a geometric way as the trivialization of Wilson lines corresponding to the elements of Z in the $G_k \times H_{-\tilde{k}}$ CSW theory. As discussed in C.2.2, Wilson lines in the $G_k \times H_{-\tilde{k}}$ CSW theory are labeled by pair of weights $(\lambda, \mu) \in P_k^+(\mathfrak{g}) \times P_{\tilde{k}}(\mathfrak{h})$. Gauging Z is equivalent to trivializing the Wilson lines corresponding to Z via the isomorphism with G_{id} in terms of simple currents as discussed above. Wilson lines can link with other Wilson lines in two ways:

1. Two Wilson lines can be parallel in which case they fuse to another Wilson line.
2. One Wilson line can loop around another Wilson line.

This is shown in Figure C.1 and Figure C.2.

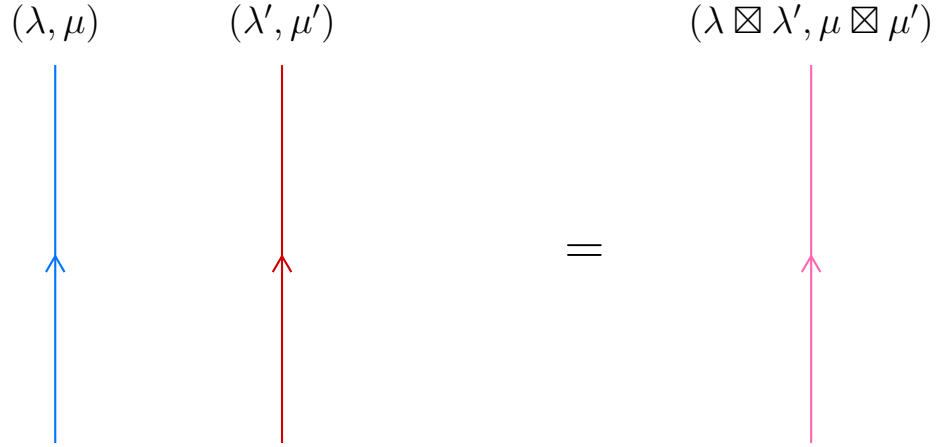


Figure C.1: Fusion of two Wilson lines.

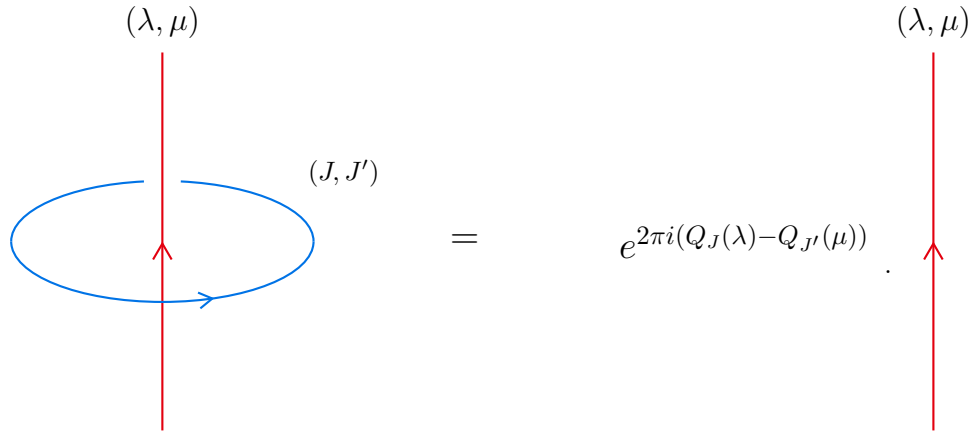


Figure C.2: Wilson loop with nontrivial linking with another Wilson line.

If $(J, J') \in G_{\text{id}} \cong Z$ loops around another Wilson line (λ, μ) , it acts by the phase $e^{2\pi i(Q_J(\lambda) - Q_{J'}(\mu))}$. It is now clear that trivializing first action leads to field identifications and trivializing second action leads to selection rules.

C.2.4 Fixed Point Resolution

When an element $(\hat{\lambda}, \hat{\mu}) \in \mathcal{E}$ is fixed by some branching outer automorphism, i.e., there exists $(\mathbf{A}, \tilde{\mathbf{A}}) \in \mathcal{O}(\hat{\mathfrak{g}}) \times \mathcal{O}(\hat{\mathfrak{h}})$ with $\mathbf{A} \mapsto \tilde{\mathbf{A}}$ such that

$$\mathbf{A}\hat{\lambda} = \hat{\lambda}, \quad \tilde{\mathbf{A}}\hat{\mu} = \hat{\mu}, \quad (\text{C.2.91})$$

then the module $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is not irreducible. Assuming that $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ is completely reducible as a $V_{\mathfrak{g}/\mathfrak{h}}$ -module, there are other irreducible modules of $V_{\mathfrak{g}/\mathfrak{h}}$, which are not obtained by decomposing $L_{\mathfrak{g}}(\mathbf{k}, 0)$ in terms of $L_{\mathfrak{h}}(\tilde{\mathbf{k}}, 0)$ -modules and these new $V_{\mathfrak{g}/\mathfrak{h}}$ -modules, that appear in the decomposition of $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ in terms of irreducible $V_{\mathfrak{g}/\mathfrak{h}}$ -modules. More precisely, let

$$G_{\text{id}}^{\hat{\lambda}, \hat{\mu}} := \{(A, \tilde{A}) \in G_{\text{id}} : A\lambda = \lambda, \tilde{A}\mu = \mu\} , \quad (\text{C.2.92})$$

be the stabilizer subgroup of $(\hat{\lambda}, \hat{\mu})$. Then the $V_{\mathfrak{g}/\mathfrak{h}}$ -module $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ splits into $|G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}|$ number of distinct irreducible modules and one needs to construct these distinct irreducible modules using different methods. This is called *fixed point resolution*. Some progress has been made in regard to solving this issue in [SY90a, FSS95a, FSS95c, SY90b]. For diagonal embedding $\mathfrak{g} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}$, this problem was solved in [FSS95b, FSS96] (see also [FSS95a]), but it is expected to work for generic coset theories, the only known counterexamples appear at level $k = 1, 2$. These are all either conformal embeddings or Maverick cosets [DJ93a, DJ93b, PSW99]. We now review the solution proposed in [FSS95b]. By spectral flow isomorphism, there is a vector space isomorphism

$$\tau_{A, \tilde{A}} : L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) \longrightarrow L_{\mathfrak{g}/\mathfrak{h}}(A\hat{\lambda}, \tilde{A}\hat{\mu}) . \quad (\text{C.2.93})$$

One can show that $\tau_{A, \tilde{A}}$ is actually a VOA automorphism with respect to the Virasoro sub-VOA of $V_{\mathfrak{g}/\mathfrak{h}}$ [FSS95b]. When $(A, \tilde{A}) \in G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}$, then $\tau_{A, \tilde{A}}$ is an automorphism. Then we can decompose

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) = \bigoplus_{\Psi \in \text{Hom}(G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}, \mathbb{C}^\times)} L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})^\Psi , \quad (\text{C.2.94})$$

where

$$L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})^\Psi = \left\{ v \in L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu}) : \tau_{A, \tilde{A}}(v) = \Psi(A, \tilde{A})v \right\} . \quad (\text{C.2.95})$$

It has been shown in [FSS95b] that the spaces $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})^\Psi$ are $V_{\mathfrak{g}/\mathfrak{h}}$ -submodules of $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})$ and can be consistently taken to be irreducible. The characters $\chi_{(\hat{\lambda}, \hat{\mu})}^{\Psi, \mathfrak{g}/\mathfrak{h}}(\tau)$ for the irreducible modules $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})^\Psi$ can be obtained by following the steps below:

1. Define the *Fourier transform* of $\chi_{(\hat{\lambda}, \hat{\mu})}^{\Psi, \mathfrak{g}/\mathfrak{h}}(\tau)$ by

$$\hat{\chi}_{(\hat{\lambda}, \hat{\mu})}^{(A, \tilde{A}), \mathfrak{g}/\mathfrak{h}}(\tau) := \sum_{\Psi \in \text{Hom}(G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}, \mathbb{C}^\times)} \Psi(A, \tilde{A}) \chi_{(\hat{\lambda}, \hat{\mu})}^{\Psi, \mathfrak{g}/\mathfrak{h}}(\tau) . \quad (\text{C.2.96})$$

Then $\hat{\chi}_{(\hat{\lambda}, \hat{\mu})}^{(A, \tilde{A}), \mathfrak{g}/\mathfrak{h}}(\tau)$ can be identified with the *twining character*

$$\hat{\chi}_{(\hat{\lambda}, \hat{\mu})}^{(A, \tilde{A}), \mathfrak{g}/\mathfrak{h}}(\tau) = \text{Tr}_{L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})} \left[\tau_{(A, \tilde{A})} q^{L_{\mathfrak{g}/\mathfrak{h}}(0) - \frac{c_{\mathfrak{g}/\mathfrak{h}}}{24}} \right] , \quad q := e^{2\pi i \tau} . \quad (\text{C.2.97})$$

2. Let $\tau_{\mathbf{A}} : L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda}) \longrightarrow L_{\mathfrak{g}}(\mathbf{k}, \mathbf{A}\hat{\lambda})$ and $\tau_{\tilde{\mathbf{A}}} : L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu}) \longrightarrow L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \tilde{\mathbf{A}}\hat{\mu})$ be the spectral flow isomorphism of $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$ -modules respectively. Then for $(\mathbf{A}, \tilde{\mathbf{A}}) \in G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}$, then these maps are automorphisms and we can consider the twined characters

$$\chi_{\hat{\lambda}}^{\mathbf{A}, \mathfrak{g}}(\tau) := \text{Tr}_{L_{\mathfrak{g}}(\mathbf{k}, \hat{\lambda})} \left[\tau_{\mathbf{A}} q^{L_{\mathfrak{g}}(0) - \frac{c_{\mathfrak{g}}}{24}} \right], \quad \chi_{\hat{\mu}}^{\tilde{\mathbf{A}}, \mathfrak{h}}(\tau) := \text{Tr}_{L_{\mathfrak{h}}(\tilde{\mathbf{k}}, \hat{\mu})} \left[\tau_{\tilde{\mathbf{A}}} q^{L_{\mathfrak{h}}(0) - \frac{c_{\mathfrak{h}}}{24}} \right]. \quad (\text{C.2.98})$$

The branching functions are then precisely $\hat{\chi}_{(\hat{\lambda}, \hat{\mu})}^{(\mathbf{A}, \tilde{\mathbf{A}}), \mathfrak{g}/\mathfrak{h}}(\tau)$:

$$\chi_{\hat{\lambda}}^{\mathbf{A}, \mathfrak{g}}(\tau) = \sum_{\mu \in P_{\hat{\mathbf{k}}}^+(\mathfrak{h})} \hat{\chi}_{(\hat{\lambda}, \hat{\mu})}^{(\mathbf{A}, \tilde{\mathbf{A}}), \mathfrak{g}/\mathfrak{h}}(\tau) \chi_{\hat{\mu}}^{\tilde{\mathbf{A}}, \mathfrak{h}}(\tau). \quad (\text{C.2.99})$$

3. The twined characters $\chi_{\hat{\lambda}}^{\mathbf{A}, \mathfrak{g}}(\tau), \chi_{\hat{\mu}}^{\tilde{\mathbf{A}}, \mathfrak{h}}(\tau)$ can be computed using the ordinary character of the orbit Lie algebras $\mathfrak{g}^{\circ}$ and $\mathfrak{h}^{\circ}$ of $\mathfrak{g}, \mathfrak{h}$ respectively:

$$\begin{aligned} \chi_{\hat{\lambda}}^{\mathbf{A}, \mathfrak{g}}(\tau) &= \chi_{\hat{\lambda}^{\circ}}^{\mathfrak{g}^{\circ}}(\tau) := \text{Tr}_{L_{\mathfrak{g}^{\circ}}(\mathbf{k}^{\circ}, \hat{\lambda}^{\circ})} q^{L_{\mathfrak{g}^{\circ}}(0) - \frac{c_{\mathfrak{g}^{\circ}}}{24}}, \\ \chi_{\hat{\mu}}^{\tilde{\mathbf{A}}, \mathfrak{h}}(\tau) &= \chi_{\hat{\mu}^{\circ}}^{\mathfrak{h}^{\circ}}(\tau) := \text{Tr}_{L_{\mathfrak{h}^{\circ}}(\tilde{\mathbf{k}}^{\circ}, \hat{\mu}^{\circ})} q^{L_{\mathfrak{h}^{\circ}}(0) - \frac{c_{\mathfrak{h}^{\circ}}}{24}}. \end{aligned} \quad (\text{C.2.100})$$

The precise construction of orbit Lie algebras and the mapping of weights of $\mathfrak{g}, \mathfrak{h}$ to weights of $\mathfrak{g}^{\circ}, \mathfrak{h}^{\circ}$ is explained in [FSS95c]. Using these twined characters, we can compute the branching functions $\chi_{\hat{\lambda}}^{\mathbf{A}, \mathfrak{g}}(\tau), \chi_{\hat{\mu}}^{\tilde{\mathbf{A}}, \mathfrak{h}}(\tau)$.

4. Finally, we compute the character $\chi_{(\hat{\lambda}, \hat{\mu})}^{\Psi, \mathfrak{g}/\mathfrak{h}}(\tau)$ of the irreducible $V_{\mathfrak{g}/\mathfrak{h}}$ -module $L_{\mathfrak{g}/\mathfrak{h}}(\hat{\lambda}, \hat{\mu})^{\Psi}$ by inverse Fourier transformation of (C.2.96) using the orthogonality of characters:

$$\sum_{(\mathbf{A}, \tilde{\mathbf{A}}) \in G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}} \Psi_1(\mathbf{A}, \tilde{\mathbf{A}}) \Psi_2(\mathbf{A}, \tilde{\mathbf{A}}) = \left| G_{\text{id}}^{\hat{\lambda}, \hat{\mu}} \right| \delta_{\Psi_1, \Psi_2}. \quad (\text{C.2.101})$$

More explicitly, we have

$$\chi_{(\hat{\lambda}, \hat{\mu})}^{\Psi, \mathfrak{g}/\mathfrak{h}}(\tau) = \frac{1}{\left| G_{\text{id}}^{\hat{\lambda}, \hat{\mu}} \right|} \sum_{(\mathbf{A}, \tilde{\mathbf{A}}) \in G_{\text{id}}^{\hat{\lambda}, \hat{\mu}}} \Psi(\mathbf{A}, \tilde{\mathbf{A}}) \hat{\chi}_{(\hat{\lambda}, \hat{\mu})}^{(\mathbf{A}, \tilde{\mathbf{A}}), \mathfrak{g}/\mathfrak{h}}(\tau). \quad (\text{C.2.102})$$

C.3 Lagrangians For Unitary Minimal Series

For each $h, c \in \mathbb{C}$, there exists a unique module of the Virasoro VOA with central charge c . We denote this module by $\text{Vir}(c, h)$ and identify $\text{Vir}(c, 0)$ with Virasoro VOA itself. For $0 < c < 1$, unitary Virasoro VOA modules have been completely classified. There is a discrete series of allowed central charges

$$c_n := 1 - \frac{6}{(n+1)(n+2)}, \quad n = 2, 3, \dots, \quad (\text{C.3.1})$$

and for each c_n , the only allowed values of h are labeled by two integers p, q :

$$h_{n,p,q} := \frac{[(n+2)p - (n+1)q]^2 - 1}{4(n+1)(n+2)}, \quad 1 \leq p \leq n, \quad 1 \leq q \leq p. \quad (\text{C.3.2})$$

The character of the module $\text{Vir}(c_n, h_{n,p,q})$ is denoted by

$$\chi_{n,p,q}(\tau) := \text{Tr}_{\text{Vir}(c_n, h_{n,p,q})} e^{2\pi i L(0)\tau} . \quad (\text{C.3.3})$$

There is also an identification of modules:

$$\chi_{n,p,q}(\tau) = \chi_{n,n+1-p,n+2-q}(\tau) . \quad (\text{C.3.4})$$

The Virasoro VOA with this irreducible modules constitute the unitary minimal models.

It was shown by Goddard-Kent-Olive [GKO86] that these modules can be obtained as modules of the coset VOA $L_{\mathfrak{g}/\mathfrak{h}}$ for

$$\begin{aligned} \hat{\mathfrak{g}} &= \hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_n , \\ \hat{\mathfrak{h}} &= \hat{\mathfrak{su}}(2)_{n+1} \cong \text{diag}(\hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_n) \hookrightarrow \hat{\mathfrak{su}}(2)_1 \oplus \hat{\mathfrak{su}}(2)_n . \end{aligned} \quad (\text{C.3.5})$$

Cappelli-Itzykson-Zuber [CIZ87] classified all modular invariants for $\hat{\mathfrak{su}}(2)_k$ WZW model and using the coset description of the discrete unitary minimal series, they classified all modular invariants that can be constructed using $\chi_{n,p,q}(\tau)$. It follows an ADE classification. The simplest one is the A_n modular invariant obtained from diagonal gluing of the left and right chiral parts:

$$Z_{A_n}(\tau) = \frac{1}{2} \sum_{p=1}^n \sum_{q=1}^{n+1} |\chi_{n,p,q}(\tau)|^2 . \quad (\text{C.3.6})$$

In [BRS88, ABR88] a different coset realization of minimal models was given using gauged free fermions. One takes

$$\begin{aligned} \hat{\mathfrak{g}} &= \hat{\mathfrak{so}}(4n)_1 , \\ \hat{h}_{\tilde{\mathbf{k}}} &= \hat{\mathfrak{p}}_{k_1} \oplus \hat{\mathfrak{su}}(2)_{k_2} \oplus \hat{\mathfrak{su}}(2)_{k_3} , \end{aligned} \quad (\text{C.3.7})$$

where the subalgebras $\hat{\mathfrak{p}}$ and levels $\tilde{k}_i$ are listed in [ABR88, Table 2] for various modular invariants in the CIZ classification.

Type	$\mathcal{M}^{\mathfrak{g}}$	$\mathcal{M}^{\mathfrak{h}}$
A_n	Diagonal: $\delta_{\lambda_L, \lambda_R}$	$\delta_{\mu_L^*, \mu_R}$
$D_{4\ell}$	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\delta_{\mu_L^*, \mu_R}$
$D_{2\ell+2}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\delta_{\mu_L^*, \mu_R}$
E_6	Diagonal: $\delta_{\lambda_L, \lambda_R}$	$\delta_{\mu_L^*, \mu_R}$
E_7	Diagonal: $\delta_{\lambda_L, \lambda_R}$	$\delta_{\mu_L^*, \mu_R}$
E_8	Diagonal: $\delta_{\lambda_L, \lambda_R}$	$\delta_{\mu_L^*, \mu_R}$

Table C.3: The modular invariants for affine VOA $L_{\mathfrak{g}}(k, 0)$ and affine sub-VOA $L_{\mathfrak{h}}(\tilde{k}, 0)$ used in the construction of modular invariant partition function of the coset VOA $L_{\mathfrak{g}/\mathfrak{h}}(0, 0)$. The ordering of characters used to write down the 4×4 matrices $\mathcal{M}^{\mathfrak{g}}$ is $\mathbf{o}, \mathbf{s}, \mathbf{v}, \mathbf{t}$.

The embedding of

$$\widehat{\mathfrak{h}}_{\tilde{k}} \subset \widehat{\mathfrak{so}}(4n)_1, \quad (\text{C.3.8})$$

can be found in [ABR88, BRS88]. The coset VOA $L_{\mathfrak{g}/\mathfrak{h}}$ in each of the cases listed in [ABR88, Table 2] is the Virasoro algebra with appropriate central charge from the CIZ classification. In particular, the central charge of the coset Virasoro VOA for A_n, D_n series is c_n defined in (C.3.1). There are four dominant integral weights of $\mathfrak{so}(4n)$ which lead to irreducible modules of $\widehat{\mathfrak{so}}(4n)_1$:

$$\lambda \in \{\mathbf{o}, \mathbf{v}, \mathbf{s}, \mathbf{c}\}, \quad (\text{C.3.9})$$

where $\mathbf{o}, \mathbf{v}, \mathbf{s}, \mathbf{c}$ corresponds to the trivial, vector, spinor and conjugate representation of $\mathfrak{so}(4n)$. The modular invariants used for $\widehat{\mathfrak{so}}(4n)_1$ WZW model and $\widehat{\mathfrak{h}}_{\tilde{k}}$ WZW model to get various modular invariants in the CIZ classification is listed in Table C.3. One can then use the formula (3.3.56) to compute the partition function of the coset model. Here we reproduce the calculation for A_n modular invariant with a discussion of field identifications. In this case,

$$\widehat{h}_{\tilde{k}} = \widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1 \oplus \widehat{\mathfrak{su}}(2)_n. \quad (\text{C.3.10})$$

In the notation of Section 3.3.1, the decomposition of the four dominant integral highest weight modules of $\widehat{\mathfrak{so}}(4n)_1$ in terms of modules of $(\widehat{\mathfrak{sp}}(n-1)_1 \oplus \widehat{\mathfrak{su}}(2)_1 \oplus \widehat{\mathfrak{su}}(2)_n) \otimes \text{Vir}(c_n, 0)$ is given by

[ABR88]

$$\begin{aligned}
L_{\mathfrak{so}(4n)}(1, \mathbf{o}) &\cong \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ even}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 0, m) \right) \otimes \text{Vir}(c_n, h_{n,j+1,m+1}) \\
&\oplus \bigoplus_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ even}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 1, m) \right) \otimes \text{Vir}(c_n, h_{n,j+1,m+1}) ,
\end{aligned} \tag{C.3.11}$$

$$\begin{aligned}
L_{\mathfrak{so}(4n)}(1, \mathbf{v}) &\cong \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ odd}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 1, m) \right) \otimes \text{Vir}(c_n, h_{n,j+1,m+1}) \\
&\oplus \bigoplus_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ odd}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 0, m) \right) \otimes \text{Vir}(c_n, h_{n,j+1,m+1}) ,
\end{aligned} \tag{C.3.12}$$

$$\begin{aligned}
L_{\mathfrak{so}(4n)}(1, \mathbf{c}) &\cong \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ odd}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 1, m) \right) \otimes \text{Vir}(c_n, h_{n,n-j,m+1}) \\
&\oplus \bigoplus_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ odd}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 0, m) \right) \otimes \text{Vir}(c_n, h_{n,n-j,m+1}) ,
\end{aligned} \tag{C.3.13}$$

$$\begin{aligned}
L_{\mathfrak{so}(4n)}(1, \mathbf{s}) &\cong \bigoplus_{\substack{j=0 \\ j \text{ even}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ even}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 0, m) \right) \otimes \text{Vir}(c_n, h_{n,n-j,m+1}) \\
&\oplus \bigoplus_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \bigoplus_{\substack{m=0 \\ m \text{ even}}}^n L_{\mathfrak{h}} \left(\tilde{\mathbf{k}}, (\Lambda_j, 1, m) \right) \otimes \text{Vir}(c_n, h_{n,n-j,m+1}) ,
\end{aligned} \tag{C.3.14}$$

where Λ_j is the j -th fundamental weight of $\mathfrak{sp}(n-1)$, and m denotes the spin- $\frac{m}{2}$ representation of $\mathfrak{su}(2)$. Using this and the modular invariants in Table C.3, we can write coset partition function for A_n and D_n modular invariants. It is also easy to see that the outer automorphisms of $\widehat{\mathfrak{so}}(4n)_1$ do not fix any of the fundamental weights in Table C.2, and hence we do not need to worry about fixed

point resolution. Thus, the GKO coset partition function must be divided by 4. We have

$$\begin{aligned}
Z_{A_n}(\tau) &= \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ even}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^n |\chi_{n,j+1,m+1}(\tau)|^2 + \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^n |\chi_{n,j+1,m+1}(\tau)|^2 \\
&+ \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ even}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ odd}}}^n |\chi_{n,j+1,m+1}(\tau)|^2 + \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ odd}}}^n |\chi_{n,j+1,m+1}(\tau)|^2 \\
&+ \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ even}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^n |\chi_{n,n-j,m+1}(\tau)|^2 + \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ even}}}^n |\chi_{n,n-j,m+1}(\tau)|^2 \\
&+ \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ even}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ odd}}}^n |\chi_{n,n-j,m+1}(\tau)|^2 + \frac{1}{4} \sum_{\substack{j=0 \\ j \text{ odd}}}^{n-1} \sum_{\substack{m=0 \\ m \text{ odd}}}^n |\chi_{n,n-j,m+1}(\tau)|^2, \\
&= \frac{1}{2} \sum_{p=1}^n \sum_{q=1}^{n+1} |\chi_{n,p,q}(\tau)|^2,
\end{aligned} \tag{C.3.15}$$

where we used the identity (C.3.4) and renamed variables $p = j + 1, q = m + 1$ in the last step. Similarly, for the D_n -modular invariant we again reproduce the partition function from Table using the coset partition function formula (3.3.56) and modular invariants from Table C.3.

One can repeat the exercise for the modular invariants for E_n from Table C.3 and reproduce the partition function for E_n minimal models in the CIZ classification.

C.4 Some Categorical Notions

In this appendix, we briefly review some definitions required in Section 3.2, we refer the reader to [Tur94, BK01, ?] for more details.

Recall that a modular tensor category $\mathcal{C}$ is equipped with the following structures:

1. Tensor bifunctor $\boxtimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, with *unit object* $\mathbf{1}_{\mathcal{C}}$, *associator isomorphism*

$$\alpha_{W_1, W_2, W_3} : W_1 \boxtimes (W_2 \boxtimes W_3) \longrightarrow (W_1 \boxtimes W_2) \boxtimes W_3, \tag{C.4.1}$$

left and right unit isomorphism

$$\ell_W : \mathbf{1}_{\mathcal{C}} \boxtimes W \longrightarrow W, \quad r_W : W \boxtimes \mathbf{1}_{\mathcal{C}} \longrightarrow W. \tag{C.4.2}$$

Along with coherence conditions on these isomorphisms, the category $\mathcal{C}$ is called *tensor* or *monoidal* category.

2. A braiding isomorphism $c_{W_1, W_2} : W_1 \boxtimes W_2 \longrightarrow W_2 \boxtimes W_1$ satisfying compatibility with the tensor bifunctor. This structure makes $\mathcal{C}$ a *braided tensor category*.

3. Rigidity axioms: for every object W , there is a (*left*) *dual object* $W^\vee$ along with morphisms

$$e_W : W^\vee \boxtimes W \longrightarrow \mathbf{1}_\mathcal{C} , \quad i_W : \mathbf{1}_\mathcal{C} \longrightarrow W \boxtimes W^\vee , \quad (\text{C.4.3})$$

called *annihilation* and *creation morphisms* compatible with the tensor and braiding structure. Similarly, *right duals*, denoted by ${}^\vee W$ and corresponding annihilation, creation morphisms

$$\tilde{e}_W : W \boxtimes {}^\vee W \longrightarrow \mathbf{1}_\mathcal{C} , \quad \tilde{i}_W : \mathbf{1}_\mathcal{C} \longrightarrow {}^\vee W \boxtimes W , \quad (\text{C.4.4})$$

satisfying compatibility with tensor and braiding and tensor structure must exist. A braided tensor category satisfying rigidity axioms is called a *rigid braided tensor category*.

4. Linearity and Semisimplicity: The space of morphisms $\text{Hom}_\mathcal{C}(W_1, W_2)$ are vector spaces⁸ and $\mathcal{C}$ is endowed with the structure of direct sum and that every object is isomorphic to the direct sum of a finite number of *simple objects*⁹.
5. Finiteness: $\mathcal{C}$ must have finitely many isomorphism classes of simple objects and the unit object $\mathbf{1}_\mathcal{C}$ must be one of them. Moreover $\text{Hom}_\mathcal{C}(\mathbf{1}_\mathcal{C}, \mathbf{1}_\mathcal{C}) = \text{End}_\mathcal{C}(\mathbf{1}_\mathcal{C})$ must be one-dimensional ($\mathbb{C}$ for us). These structures define a *rigid braided fusion category*.
6. Ribbon structure: For every object W , there is an isomorphism $\theta_W : W \longrightarrow W$, called the *twist map*, satisfying the balancing axioms, i.e., compatible with the braiding isomorphism and the rigid structure. This is the additional data for a *ribbon category*. In a ribbon category, the left and right duals coincide.
7. Nondegeneracy: Let $\{W_a\}_{a \in \mathcal{A}}$ be the isomorphism classes of simple objects with $W_0 = \mathbf{1}_\mathcal{C}$. Then the *S-matrix*¹⁰ defined by

$$S_{ab} := \left[\ell_{\mathbf{1}_\mathcal{C}} \circ \left(e_{W_a} \otimes e_{W_b^\vee} \right) \circ \left(\mathbf{1}_{W_a^\vee} \otimes c_{W_a W_b} \otimes \mathbf{1}_{W_b^\vee} \right) \right. \\ \left. \circ \left(\mathbf{1}_{W_a^\vee} \otimes c_{W_a W_b} \otimes \mathbf{1}_{W_b^\vee} \right) \circ \left(i_{W_a^\vee} \otimes i_{W_a} \right) \circ \ell_{\mathbf{1}_\mathcal{C}}^{-1} \right]_{ab} , \quad (\text{C.4.5})$$

where $\mathbf{1}_{W_a} : W_a \longrightarrow W_a$ is the identity morphism, is invertible. This condition on a rigid braided ribbon fusion category defines a modular tensor category (MTC). The number of isomorphism classes of simple objects is called the rank of the MTC.

⁸We will work with complex vector spaces.

⁹Linearity involves other technical conditions which allow us to define *subobjects* of an object. Simple objects are those objects whose only subobjects are the *zero object* and itself. We will not go into the details, the reader is referred to [Tur94, BK01] for details.

¹⁰The map $S_{a,b} \in \text{Hom}_\mathcal{C}(\mathbf{1}_\mathcal{C}, \mathbf{1}_\mathcal{C}) \cong \mathbb{C}$ and hence is a complex number.

Definition C.4.1. An object A in a braided fusion category $\mathcal{C}$ is called an *algebra object* if there exists morphisms $\mu_A \in \text{Hom}_{\mathcal{C}}(A \boxtimes A, A)$ and $\eta_A \in \text{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A)$ such that

$$\begin{aligned} \mu_A \circ (\mu_A \boxtimes \mathbf{1}_A) \circ \alpha_{A,A,A} &= \mu_A \circ (\mathbf{1}_A \boxtimes \mu_A) , \\ \mu_A \circ (\eta_A \boxtimes \mathbf{1}_A) \circ \ell_A^{-1} &= \mathbf{1}_A = \mu_A \circ (\mathbf{1}_A \boxtimes \eta_A) \circ r_A^{-1} , \end{aligned} \quad (\text{C.4.6})$$

and $\text{Hom}(\mathbf{1}_{\mathcal{C}}, A)$ is one-dimensional¹¹. A is called *commutative* if in addition

$$\mu_A = \mu_A \circ c_{A,A} . \quad (\text{C.4.7})$$

Definition C.4.2. Let A be a commutative algebra object in $\mathcal{C}$. A *left A -module* is a tuple (W, μ_W) with W an object in $\mathcal{C}$ and μ_W a morphism $\mu_W \in \text{Hom}_{\mathcal{C}}(A \boxtimes W, W)$ such that

$$\mu_W \circ (\mu_A \boxtimes \mathbf{1}_W) \circ \alpha_{A,A,W} = \mu_W \circ (\mathbf{1}_A \boxtimes \mu_W) . \quad (\text{C.4.8})$$

We will call left A -modules as simply A -modules. An A -module (W, μ_W) is called *local* if

$$\mu_W \circ c_{W,A} \circ c_{A,W} = \mu_W . \quad (\text{C.4.9})$$

An A -module homomorphism between A -modules (W_1, μ_{W_1}) and (W_2, μ_{W_2}) is a morphism $f \in \text{Hom}_{\mathcal{C}}(W_1, W_2)$ satisfying

$$f \circ \mu_{W_1} = \mu_{W_2} \circ (\mathbf{1}_A \boxtimes f) . \quad (\text{C.4.10})$$

We will denote the space of A -module homomorphism by $\text{Hom}_A(W_1, W_2)$. An A -module (W, μ_W) is called *simple* if $\text{Hom}_A(W, W) \cong \mathbb{C}\mathbf{1}_W$.

The set of all A -modules in the category $\mathcal{C}$ forms a category with morphisms given by A -module homomorphisms. We will denote the A -module category in $\mathcal{C}$ by $\mathcal{C}_A$ and the local A -module category by $\mathcal{C}_A^0$.

Theorem C.4.1. *Let $\mathcal{C}$ be a braided tensor category and A a commutative algebra object in $\mathcal{C}$. Then the module category $\mathcal{C}_A$ is a fusion category and the local A -module category $\mathcal{C}_A^0$ is a braided fusion category. Moreover, if $\mathcal{C}$ is an MTC and A is simple¹² an A -module with $\theta_A = \mathbf{1}_A$, then $\mathcal{C}_A^0$ is also an MTC.*

¹¹Many references do not include this condition in the definition of an algebra object. Rather, this condition defines a *haploid algebra object*. In this paper, we will always assume this condition in the definition of an algebra object.

¹²In [FFRS03b], an algebra object is defined to be simple if the space of *bimodule* homomorphisms from A to A is spanned by the identity morphism. But it is noted in [FFRS03b, Remark 2.28] that for commutative algebra objects, the space of bimodule homomorphisms are same as the space of module homomorphism and our definition of simplicity is enough.

Definition C.4.3. A *coalgebra object* in a braided fusion category $\mathcal{C}$ is a tuple $(A, \Delta_A, \varepsilon_A)$ where A is an object, $\Delta_A \in \text{Hom}_{\mathcal{C}}(A, A \boxtimes A)$, $\varepsilon_A \in \text{Hom}_{\mathcal{C}}(A, \mathbf{1}_{\mathcal{C}})$ which satisfy the following:

$$\begin{aligned} (\mathbf{1}_A \boxtimes \Delta_A) \circ \Delta_A &= (\Delta_A \boxtimes \mathbf{1}_A) \circ \Delta_A, \\ r_A \circ (\mathbf{1}_A \boxtimes \varepsilon_A) \circ \Delta_A &= \mathbf{1}_A = \ell_A \circ (\varepsilon_A \boxtimes \mathbf{1}_A) \circ \Delta_A, \end{aligned} \quad (\text{C.4.11})$$

and $\text{Hom}_{\mathcal{C}}(\mathbf{1}_{\mathcal{C}}, A)$ is one-dimensional. A is called *cocommutative* if in addition

$$c_{A,A} \circ \Delta_A = \Delta_A. \quad (\text{C.4.12})$$

Definition C.4.4. A *Frobenius algebra object* in a braided fusion category $\mathcal{C}$ is a quintuple $(A, \mu_A, \eta_A, \Delta_A, \varepsilon_A)$ such that (A, μ_A, η_A) is a regular algebra object and $(A, \Delta_A, \varepsilon_A)$ is a regular coalgebra object, such that the following *Frobenius condition* is satisfied:

$$(\mathbf{1}_A \otimes \mu_A) \circ (\Delta_A \otimes \mathbf{1}_A) = \Delta_A \circ \mu_A = (\mu_A \otimes \mathbf{1}_A) \circ (\mathbf{1}_A \otimes \Delta_A). \quad (\text{C.4.13})$$

A is called *commutative* if (A, μ_A, η_A) is also commutative. A Frobenius algebra is called *special* if

$$\varepsilon_A \circ \eta_A = \beta_{\mathbf{1}_{\mathcal{C}}} \mathbf{1}_{\mathbf{1}_{\mathcal{C}}}, \quad \mu_A \circ \Delta_A = \beta_A \mathbf{1}_A, \quad (\text{C.4.14})$$

for non-zero complex numbers $\beta_{\mathbf{1}_{\mathcal{C}}}, \beta_A$.

Definition C.4.5. An algebra object A together with a morphism $\varepsilon_A \in \text{Hom}_{\mathcal{C}}(A, \mathbf{1}_{\mathcal{C}})$ in a braided fusion ribbon category $\mathcal{C}$ is called *symmetric* if the two morphisms defined by

$$\begin{aligned} \Phi_1 &:= [(\varepsilon_A \circ \mu_A) \boxtimes \mathbf{1}_{A^\vee}] \circ \alpha_{A,A,A^\vee} \circ (\mathbf{1}_A \boxtimes i_A) \circ r_A^{-1}, \\ \Phi_2 &:= [\mathbf{1}_{\vee A} \boxtimes (\varepsilon_A \circ \mu_A)] \circ \alpha_{\vee A,A,A}^{-1} \circ (\tilde{i}_A \boxtimes \mathbf{1}_A) \circ \ell_A^{-1}, \end{aligned} \quad (\text{C.4.15})$$

are equal¹³ in $\text{Hom}_{\mathcal{C}}(A, A^\vee)$.

Given two categories $\mathcal{C}_1, \mathcal{C}_2$, one can define the *Deligne product* $\mathcal{C}_1 \boxtimes_{\mathbf{D}} \mathcal{C}_2$ whose objects are pairs written as $X \boxtimes_{\mathbf{D}} Y$ where X, Y are objects of $\mathcal{C}_1, \mathcal{C}_2$ respectively and morphisms are

$$\text{Hom}_{\mathcal{C}_1 \boxtimes_{\mathbf{D}} \mathcal{C}_2}(X_1 \boxtimes_{\mathbf{D}} Y_1, X_2 \boxtimes_{\mathbf{D}} Y_2) := \text{Hom}_{\mathcal{C}_1}(X_1, X_2) \otimes_{\mathbb{C}} \text{Hom}_{\mathcal{C}_2}(Y_1, Y_2). \quad (\text{C.4.16})$$

We will denote morphisms in $\mathcal{C}_1 \boxtimes_{\mathbf{D}} \mathcal{C}_2$ by $f \boxtimes_{\mathbf{D}} g$. Various structures on $\mathcal{C}_1, \mathcal{C}_2$ can be extended to $\mathcal{C}_1 \boxtimes_{\mathbf{D}} \mathcal{C}_2$ in the obvious way. For example, if $\mathcal{C}_1, \mathcal{C}_2$ are MTCs then so is $\mathcal{C}_1 \boxtimes_{\mathbf{D}} \mathcal{C}_2$, see [FFRS03b, Proposition 6.3] for details.

We are now ready to state a simplified version of the main theorem of [FFRS03b].

¹³Note that in a ribbon category ${}^\vee A = A^\vee$ and hence the equality of the maps makes sense.

Theorem C.4.2. *Let $\mathcal{C}_1, \mathcal{C}_2$ be two modular tensor categories¹⁴ and A be a special symmetric commutative Frobenius algebra object in $\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2$. Let $\mathcal{C}_3$ be another modular tensor category equivalent to the category of local A -modules $(\mathcal{C}_1 \boxtimes_{\mathcal{D}} \mathcal{C}_2)_A^0$. If the only subobject of A of the form $U \boxtimes_{\mathcal{D}} \mathbf{1}_{\mathcal{C}_2}$ is $\mathbf{1}_{\mathcal{C}_1} \boxtimes_{\mathcal{D}} \mathbf{1}_{\mathcal{C}_2}$, then there exists a special symmetric commutative Frobenius algebra object B in $\mathcal{C}_3 \boxtimes_{\mathcal{D}} \mathcal{C}_2^{\text{opp}}$ such that*

$$\mathcal{C}_1 \cong (\mathcal{C}_3 \boxtimes_{\mathcal{D}} \mathcal{C}_2^{\text{opp}})_B^0, \quad (\text{C.4.17})$$

where $\mathcal{C}_2^{\text{opp}}$ is the opposite category of $\mathcal{C}_2$.

The explicit description of B is also given in [FFRS03b], we will not give its detailed description but write it explicitly for the setting in which we will apply this theorem in Section 3.2.3.

¹⁴The general statement of this theorem in [FFRS03b] requires the categories to be only rigid braided ribbon categories. In that case the Deligne product is replaced by *Karoubian product*. Since modular tensor categories are automatically Karoubian, the Karoubian product coincides with the Deligne product. Since any abelian semisimple category is Karoubian, we can further relax the requirement of $\mathcal{C}_1$ being an MTC to an abelian semisimple rigid braided ribbon category.

Appendix D

Appendix for Chapter 4

D.1 Supersymmetry conventions

We consider the Euclidean flat space $\mathbb{R}^4$ with real coordinates x^μ . The Lorentz group is $\text{SO}(4)$. To define spinors, we consider its double cover

$$\text{Spin}(4) \cong \text{SU}(2)_- \times \text{SU}(2)_+ . \quad (\text{D.1.1})$$

Left and right handed Weyl spinors $\psi_\alpha, \tilde{\psi}^{\dot{\alpha}}$ respectively are defined as the $(\mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2})$ representation of $\text{Spin}(4)$ where $(\ell, \mathbf{m})$ indicates the dimension of the representation under the two $\text{SU}(2)$ factors in $\text{Spin}(4)$. The indices on the Weyl spinors is raised and lowered using the North-West South-West (NWSW) convention¹

$$\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = -\epsilon^{\alpha\beta} = -\epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon_{\alpha\beta}\epsilon^{\beta\delta} = \delta_\alpha^\delta, \quad \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\delta}} = \delta_{\dot{\alpha}}^{\dot{\delta}}, \quad (\text{D.1.2})$$

as follows

$$\begin{aligned} \psi^\alpha &= \epsilon^{\alpha\beta}\psi_\beta, & \psi_\alpha &= \epsilon_{\alpha\beta}\psi^\beta \\ \psi^{\dot{\alpha}} &= \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}, & \psi_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}} \end{aligned} \quad (\text{D.1.3})$$

Spinor contractions are given by

$$\begin{aligned} \psi\chi &= \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha = \chi^\alpha\psi_\alpha = \chi\psi \\ \tilde{\psi}\tilde{\chi} &= \tilde{\psi}_{\dot{\alpha}}\tilde{\chi}^{\dot{\alpha}} = -\tilde{\psi}^{\dot{\alpha}}\tilde{\chi}_{\dot{\alpha}} = \tilde{\chi}_{\dot{\alpha}}\tilde{\psi}^{\dot{\alpha}} = \tilde{\chi}\tilde{\psi}. \end{aligned} \quad (\text{D.1.4})$$

The following *exchange identity* holds:

$$A_{\dot{\alpha}}B_{\dot{\beta}} = A_{\dot{\beta}}B_{\dot{\alpha}} - \epsilon_{\dot{\alpha}\dot{\beta}}A_{\dot{\gamma}}B^{\dot{\gamma}}. \quad (\text{D.1.5})$$

The Euclidean Pauli 4-vector is given by

$$\sigma_{\alpha\dot{\beta}}^\mu = (\sigma^i, i\mathbb{1})_{\alpha\dot{\beta}}, \quad (\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} = (-\sigma^i, i\mathbb{1})^{\dot{\alpha}\beta}, \quad (\text{D.1.6})$$

¹The Appendices of [CMRS23] follow the North-West South-East convention.

so that

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \quad (\text{D.1.7})$$

We define the matrices

$$(\sigma^{\mu\nu})^\alpha{}_\beta = \frac{1}{4} (\sigma^\mu \tilde{\sigma}^\nu - \sigma^\nu \tilde{\sigma}^\mu)^\alpha{}_\beta, \quad (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{4} (\tilde{\sigma}^\mu \sigma^\nu - \tilde{\sigma}^\nu \sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}}. \quad (\text{D.1.8})$$

These matrices are symmetric

$$\begin{aligned} (\sigma_{\mu\nu})_{\alpha\beta} &= (\sigma_{\mu\nu})_{\beta\alpha}, \\ (\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} &= (\tilde{\sigma}_{\mu\nu})^{\dot{\beta}\dot{\alpha}}, \end{aligned} \quad (\text{D.1.9})$$

and generate rotations under $\text{SU}(2)_-$, and $\text{SU}(2)_+$ on ψ and $\tilde{\chi}$ respectively. They also satisfy

$$\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \sigma_{\rho\lambda} = -\sigma^{\mu\nu}, \quad \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} \tilde{\sigma}_{\rho\lambda} = \tilde{\sigma}^{\mu\nu}. \quad (\text{D.1.10})$$

Thus $\sigma^{\mu\nu}$ and $\tilde{\sigma}^{\mu\nu}$ are generators of Lorentz transformation on the space of anti-selfdual (ASD) and selfdual (SD) rank 2-tensors. Some identities useful in calculations involving sigma matrices are

$$(\sigma^\mu)_{\alpha\dot{\alpha}} = -(\tilde{\sigma}^\mu)_{\dot{\alpha}\alpha}, \quad (\text{D.1.11})$$

$$(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} = -(\sigma^\mu)^{\alpha\dot{\alpha}}, \quad (\text{D.1.12})$$

$$(\tilde{\sigma}_\mu)^{\dot{\beta}}_{\alpha} = (\sigma_\mu)_{\alpha}^{\dot{\beta}} \quad (\text{D.1.13})$$

$$(\tilde{\sigma}_\mu)^{\dot{\alpha}\beta}(\sigma_\nu)_{\beta\dot{\alpha}} = -2\delta_{\mu\nu}, \quad (\text{D.1.14})$$

$$(\tilde{\sigma}^\mu)^{\dot{\alpha}\beta}(\sigma_\mu)_{\gamma\dot{\beta}} = -2\delta_\gamma^\beta \delta_{\dot{\alpha}}^{\dot{\gamma}}, \quad (\text{D.1.15})$$

$$(\tilde{\sigma}_\mu)^{\dot{\alpha}\beta}(\sigma_\nu)_{\beta\dot{\gamma}} = -\delta_{\mu\nu} \delta_{\dot{\gamma}}^{\dot{\alpha}} + 2(\tilde{\sigma}_{\mu\nu})_{\dot{\gamma}}^{\dot{\alpha}}, \quad (\text{D.1.16})$$

$$(\sigma_\mu)_{\alpha\dot{\beta}}(\tilde{\sigma}_\nu)^{\dot{\beta}\gamma} = -\delta_{\mu\nu} \delta_\alpha^\gamma + 2(\sigma_{\mu\nu})_\alpha^\gamma \quad (\text{D.1.17})$$

$$\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = -2\delta^{\mu\nu}, \quad (\text{D.1.18})$$

$$\tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu = -2\delta^{\mu\nu}, \quad (\text{D.1.19})$$

$$(\tilde{\sigma}_{\rho\sigma})_{\dot{\beta}}^{\dot{\alpha}} (\tilde{\sigma}_\mu)^{\dot{\beta}\alpha} = \epsilon_{\rho\sigma\mu\nu} (\tilde{\sigma}^\nu)^{\dot{\alpha}\alpha} + \delta_{\rho\mu} (\tilde{\sigma}_\sigma)^{\dot{\alpha}\alpha} - \delta_{\sigma\mu} (\tilde{\sigma}_\rho)^{\dot{\alpha}\alpha} \quad (\text{D.1.20})$$

$$(\sigma_{\rho\sigma})_{\alpha}^{\beta} (\sigma_\mu)_{\beta\dot{\alpha}} = -\epsilon_{\rho\sigma\mu\nu} (\sigma^\nu)_{\alpha\dot{\alpha}} + \delta_{\rho\mu} (\sigma_\sigma)_{\alpha\dot{\alpha}} - \delta_{\sigma\mu} (\sigma_\rho)_{\alpha\dot{\alpha}} \quad (\text{D.1.21})$$

$$(\sigma_\mu)_{\alpha\dot{\beta}} (\tilde{\sigma}_{\rho\sigma})_{\dot{\gamma}}^{\beta} = -\frac{1}{2} \epsilon_{\rho\sigma\mu\nu} (\sigma^\nu)_{\alpha\dot{\gamma}} - \frac{1}{2} \delta_{\mu\rho} (\sigma_\sigma)_{\alpha\dot{\gamma}} + \frac{1}{2} \delta_{\mu\sigma} (\sigma_\rho)_{\alpha\dot{\gamma}}, \quad (\text{D.1.22})$$

$$(\tilde{\sigma}_\mu)^{\dot{\alpha}\beta} (\sigma_{\rho\sigma})_{\beta}^{\gamma} = \frac{1}{2} \epsilon_{\rho\sigma\mu\nu} (\tilde{\sigma}^\nu)^{\dot{\alpha}\gamma} - \frac{1}{2} \delta_{\mu\rho} (\tilde{\sigma}_\sigma)^{\dot{\alpha}\gamma} + \frac{1}{2} \delta_{\mu\sigma} (\tilde{\sigma}_\rho)^{\dot{\alpha}\gamma} \quad (\text{D.1.23})$$

$$(\sigma_{\mu\nu})_{\alpha}^{\beta} (\sigma_{\rho\sigma})_{\beta}^{\alpha} = \epsilon_{\mu\nu\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}, \quad (\text{D.1.24})$$

$$(\tilde{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} (\tilde{\sigma}_{\rho\sigma})_{\dot{\alpha}}^{\dot{\beta}} = -\epsilon_{\mu\nu\rho\sigma} - \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} \quad (\text{D.1.25})$$

$$(\tilde{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})_{\dot{\delta}}^{\dot{\gamma}} = \delta_{\dot{\delta}}^{\dot{\gamma}} \delta_{\dot{\beta}}^{\dot{\alpha}} - 2\delta_{\dot{\delta}}^{\dot{\alpha}} \delta_{\dot{\beta}}^{\dot{\gamma}}, \quad (\text{D.1.26})$$

$$(\sigma_{\mu\nu})_{\alpha}^{\beta} (\sigma^{\mu\nu})_{\gamma}^{\delta} = \delta_{\alpha}^{\delta} \delta_{\gamma}^{\beta} - 2\delta_{\alpha}^{\beta} \delta_{\gamma}^{\delta}, \quad (\text{D.1.27})$$

$$(\tilde{\sigma}_{\mu\nu})_{\dot{\beta}}^{\dot{\alpha}} (\tilde{\sigma}_{\rho\sigma})_{\dot{\gamma}}^{\beta} = -\frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \delta_{\dot{\gamma}}^{\dot{\alpha}} + \frac{1}{2} \delta_{\mu\rho} (\tilde{\sigma}_{\nu\sigma})_{\dot{\gamma}}^{\dot{\alpha}} - \frac{1}{2} \delta_{\mu\sigma} (\tilde{\sigma}_{\nu\rho})_{\dot{\gamma}}^{\dot{\alpha}} + \frac{1}{2} \delta_{\nu\sigma} (\tilde{\sigma}_{\mu\rho})_{\dot{\gamma}}^{\dot{\alpha}} \quad (\text{D.1.28})$$

$$- \frac{1}{2} \delta_{\nu\rho} (\tilde{\sigma}_{\mu\sigma})_{\dot{\gamma}}^{\dot{\alpha}} - \frac{1}{4} \delta_{\mu\rho} \delta_{\nu\sigma} \delta_{\dot{\gamma}}^{\dot{\alpha}} + \frac{1}{4} \delta_{\mu\sigma} \delta_{\nu\rho} \delta_{\dot{\gamma}}^{\dot{\alpha}}, \quad (\text{D.1.29})$$

$$(\sigma_{\mu\nu})_{\alpha}^{\beta} (\sigma_{\rho\sigma})_{\beta}^{\gamma} = \frac{1}{4} \epsilon_{\mu\nu\rho\sigma} \delta_{\alpha}^{\gamma} + \frac{1}{2} \delta_{\mu\rho} (\sigma_{\nu\sigma})_{\alpha}^{\gamma} - \frac{1}{2} \delta_{\mu\sigma} (\sigma_{\nu\rho})_{\alpha}^{\gamma} + \frac{1}{2} \delta_{\nu\sigma} (\sigma_{\mu\rho})_{\alpha}^{\gamma} \quad (\text{D.1.30})$$

$$- \frac{1}{2} \delta_{\nu\rho} (\sigma_{\mu\sigma})_{\alpha}^{\gamma} - \frac{1}{4} \delta_{\mu\rho} \delta_{\nu\sigma} \delta_{\alpha}^{\gamma} + \frac{1}{4} \delta_{\mu\sigma} \delta_{\nu\rho} \delta_{\alpha}^{\gamma}. \quad (\text{D.1.31})$$

A vector can be written as a bispinor by using the sigma matrices as follows:

$$A_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^{\mu} A_{\mu}, \quad A_{\dot{\alpha}\alpha} = \tilde{\sigma}_{\dot{\alpha}\alpha}^{\mu} A_{\mu}, \quad (\text{D.1.32})$$

and the inverse is given by

$$A_{\mu} = \frac{1}{2} \sigma_{\mu}^{\alpha\dot{\alpha}} A_{\alpha\dot{\alpha}}, \quad A_{\mu} = \frac{1}{2} \tilde{\sigma}_{\mu}^{\dot{\alpha}\alpha} A_{\dot{\alpha}\alpha}. \quad (\text{D.1.33})$$

Thus

$$A_{\mu} B^{\mu} = \frac{1}{2} A_{\alpha\dot{\alpha}} B^{\alpha\dot{\alpha}}. \quad (\text{D.1.34})$$

The antisymmetric part of a rank 2-tensor is decomposed into its SD and ASD part as

$$A_{[\mu\nu]} = A_{[\mu\nu]}^+ + A_{[\mu\nu]}^- , \quad (\text{D.1.35})$$

where

$$A_{[\mu\nu]}^\pm = \frac{1}{2}A_{[\mu\nu]} \pm \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}A^{\rho\sigma} . \quad (\text{D.1.36})$$

Using $\sigma^{\mu\nu}$ and $\tilde{\sigma}^{\mu\nu}$ we can project onto the ASD and SD part of a rank 2 tensor in the spinor notation:

$$A_{\alpha\beta} = A_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}, \quad A_{\dot{\alpha}\dot{\beta}} = A_{\mu\nu}(\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} , \quad (\text{D.1.37})$$

and the inverse is given by

$$A_{\mu\nu} = \frac{1}{2}A_{\alpha\beta}(\sigma_{\mu\nu})^{\alpha\beta}, \quad A_{\mu\nu} = \frac{1}{2}A_{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} , \quad (\text{D.1.38})$$

In the bispinor indices, we get

$$A_{[\alpha\dot{\alpha},\beta\dot{\beta}]} = A_{\dot{\alpha}\dot{\beta}}\epsilon_{\alpha\beta} + A_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} , \quad (\text{D.1.39})$$

so that

$$A_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}\epsilon^{\alpha\beta}A_{[\alpha\dot{\alpha},\beta\dot{\beta}]}, \quad A_{\alpha\beta} = -\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}A_{[\alpha\dot{\alpha},\beta\dot{\beta}]} . \quad (\text{D.1.40})$$

On curved Riemannian 4-manifolds, we introduce local orthonormal frames or veirbein $\{e_\mu^a\}$ so that

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \delta_{ab}e_\mu^a e_\nu^b dx^\mu dx^\nu . \quad (\text{D.1.41})$$

Then we have Pauli matrices $\sigma^a, \tilde{\sigma}^a$ and the generators $\sigma^{ab}, \tilde{\sigma}^{ab}$ of left and right-handed Lorentz transformations given by the same formula as (D.1.6), (D.1.8). The veilbein in spinor indices is given by

$$e_{\alpha\dot{\alpha}}^\mu = (\sigma^a)_{\alpha\dot{\alpha}}e_\mu^a, \quad e_{\dot{\alpha}\alpha}^\mu = (\tilde{\sigma}^a)_{\dot{\alpha}\alpha}e_\mu^a . \quad (\text{D.1.42})$$

The spin connection is given by

$$\omega_\mu^{ab} = e_\nu^a \Gamma_{\sigma\mu}^\nu e^{\sigma b} + e_\nu^b \partial_\mu e^{\nu a} = e_\nu^a \Gamma_{\sigma\mu}^\nu e^{\sigma b} - e^{\nu b} \partial_\mu e_\nu^a , \quad (\text{D.1.43})$$

where

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) , \quad (\text{D.1.44})$$

The Riemman curvature tensor is given by

$$R_{\mu\nu\dot{a}\dot{b}} = \partial_\mu \omega_{\nu\dot{a}\dot{b}} - \partial_\nu \omega_{\mu\dot{a}\dot{b}} - \omega_{\mu\dot{a}\dot{c}}\omega_{\nu\dot{c}\dot{b}} + \omega_{\nu\dot{a}\dot{c}}\omega_{\mu\dot{c}\dot{b}} . \quad (\text{D.1.45})$$

In spinor indices, the Riemann tensor can be decomposed into irreducibles as

$$\begin{aligned} R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}} &= \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}}C_{\alpha\beta\gamma\delta} + \epsilon_{\alpha\beta}\epsilon_{\dot{\gamma}\dot{\delta}}R_{\dot{\alpha}\dot{\beta}\gamma\delta} + \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta}R_{\alpha\beta\dot{\gamma}\dot{\delta}} \\ &+ \frac{R}{24} \left(\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \left(\epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\beta}\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\delta}} \right) + \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\dot{\gamma}\dot{\delta}} (\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\beta\gamma}\epsilon_{\alpha\delta}) \right) , \end{aligned} \quad (\text{D.1.46})$$

where $C_{\alpha\beta\gamma\delta}$ and $C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ are Weyl tensors, $R_{\alpha\beta\dot{\gamma}\dot{\delta}}$ is the Ricci tensor and R is the Ricci scalar. From the symmetries of the Riemann tensor we get

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}, \quad C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = C_{\dot{\gamma}\dot{\delta}\dot{\alpha}\dot{\beta}}, \quad R_{\alpha\beta\dot{\gamma}\dot{\delta}} = R_{\dot{\gamma}\dot{\delta}\alpha\beta}. \quad (\text{D.1.47})$$

Define

$$R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} := \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} R_{\alpha\dot{\alpha}\beta\dot{\beta}\gamma\dot{\gamma}\delta\dot{\delta}}. \quad (\text{D.1.48})$$

Then from the expansion of Riemann tensor we have

$$R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 4C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} - \frac{R}{6} \left(\epsilon_{\dot{\alpha}\dot{\gamma}} \epsilon_{\dot{\beta}\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\gamma}} \epsilon_{\dot{\alpha}\dot{\delta}} \right) \quad (\text{D.1.49})$$

Thus

$$R_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \frac{R}{2} \epsilon_{\dot{\alpha}\dot{\delta}} \epsilon_{\dot{\beta}\dot{\gamma}}. \quad (\text{D.1.50})$$

D.2 Transfer Of Structure Group

In this appendix, we recall the notion of transfer of structure group. For simplicity, we will work in the category of smooth manifolds (where the objects are smooth manifolds and morphisms are smooth maps). Let G be a Lie group and $\mathbb{X}$ a smooth manifold. Let $\mathcal{C}(G, \mathbb{X})$ be the groupoid of principal G -bundles over $\mathbb{X}$. Let G_1, G_2 be two Lie groups. Given a homomorphism $\phi : G_1 \rightarrow G_2$, we have a functor $\phi_* : \mathcal{C}(G_1, \mathbb{X}) \rightarrow \mathcal{C}(G_2, \mathbb{X})$ that maps a principal G_1 -bundle $P_1 \xrightarrow{\pi_1} \mathbb{X}$ to $P_1 \times_{\phi} G_2 \xrightarrow{\pi'_1} \mathbb{X}$ where

$$P_1 \times_{\phi} G_2 := (P_1 \times G_2) / \sim_{\phi}, \quad (\text{D.2.1})$$

with equivalence relation is defined by

$$(p_1 \cdot h, g) \sim_{\phi} (p_1, \phi(h)g), \quad (p_1, g) \in P_1 \times G_2, h \in G_1. \quad (\text{D.2.2})$$

The projection $\pi'_1 : P_1 \times_{\phi} G_2 \rightarrow \mathbb{X}$ is defined as $\pi'_1([(p_1, g)]) = \pi_1(p_1)$ and the right G_2 action is given by

$$[(p_1, g_1)] \cdot g_2 = [(p_1, g_1 \cdot g_2)]. \quad (\text{D.2.3})$$

Let $P_1 \xrightarrow{\pi_1} \mathbb{X}$ be a principal G_1 -bundle. The *transfer of structure group* of $P_1 \xrightarrow{\pi_1} \mathbb{X}$ from G_1 to G_2 using the homomorphism $\phi : G_1 \rightarrow G_2$ is, by definition, the image $\phi_*(P_1 \xrightarrow{\pi_1} \mathbb{X})$ under the above functor. One can show that in terms of patches and transition functions, this means that if the local trivialization of $P_1 \xrightarrow{\pi_1} \mathbb{X}$ is given by $\{U_p, t_{pq}\}$, then the local trivialization of $\phi_*(P_1 \xrightarrow{\pi_1} \mathbb{X})$ is given by $\{U_p, \tilde{t}_{pq}\}$ with $\tilde{t}_{pq} = \phi \circ t_{pq}$.

A related notion is a *reduction of structure group* defined as follows: let $P_2 \xrightarrow{\pi_2} \mathbb{X}$ be a principal G_2 -bundle. A reduction of structure group of $P_2 \xrightarrow{\pi_2} \mathbb{X}$ from G_2 to G_1 using the homomorphism

$\phi : G_1 \longrightarrow G_2$ is a principal G_1 -bundle $P_1 \xrightarrow{\pi_1} \mathbb{X}$ together with an isomorphism of $\phi_*(P_1 \xrightarrow{\pi_1} \mathbb{X})$ with $P_2 \xrightarrow{\pi_2} \mathbb{X}$. One can show that in terms of patches and transition functions, this means that there is an atlas and local trivialization of $P_2 \xrightarrow{\pi_2} \mathbb{X}$ given by $\{U_p, \tilde{t}_{pq}\}$ such that there exists a G_1 bundle $P_1 \rightarrow \mathbb{X}$ with local trivializations $\{U_p, t_{pq}\}$ such that $\tilde{t}_{pq} = \phi \circ t_{pq}$.

It is worth noting that there is never an obstruction to transferring the structure group of a principal bundle, but there are, in general, nontrivial obstructions to the existence of a reduction of structure group. Here are some examples of reductions of structure groups:

Fix a positive integer $n \in \mathbb{Z}_{>0}$. Given an n -manifold $\mathbb{X}$ and a structure group G , a G -structure [Che66] on $\mathbb{X}$ is a principal G -subbundle of the frame bundle $F\mathbb{X}$. Cases where the G -structure corresponds to standard tensor fields are familiar. For example, an $O(n)$ structure defines a Riemannian metric on $\mathbb{X}$ and can be viewed as a principal $O(n)$ subbundle of the $GL(n, \mathbb{R})$ frame bundle, or equivalently, a reduction of structure group from $GL(n, \mathbb{R})$ to $O(n)$ via the natural inclusion. A further reduction to $SO(n)$ structure defines an orientation² on $\mathbb{X}$, and a reduction to $Spin(n)$ via the standard homomorphism $Spin(n) \rightarrow O(n)$ defines a spin structure. Indeed, some of these reductions might not exist – obstructions are quantified by certain characteristic classes. The aforementioned examples are of tangential structures, i.e., structure group reductions for the frame bundle. However, the idea of the reduction of structure group applies more generally to principal bundles.

The notion of transfer of structure group can be extended to include the data of a connection. Let $\mathcal{C}^{\text{conn}}(G, \mathbb{X})$ be the groupoid of principal G -bundles with connection. If we view a connection as a rule for the coherent lifting of paths then a connection ∇_1 on $P_1 \xrightarrow{\pi_1} \mathbb{X}$ lifts a path $\gamma : [0, 1] \rightarrow \mathbb{X}$ to a path $\tilde{\gamma} : [0, 1] \rightarrow P_1$, uniquely specified once we are given the initial point $\tilde{\gamma}(0)$ in the fiber above $\gamma(0)$. The image connection $\phi_*(\nabla_1)$ is simply the connection on $P_1 \times_{\phi} G_2 \xrightarrow{\pi'_1} \mathbb{X}$ given by the rule to lift the path $\gamma(t)$ to $[(\tilde{\gamma}(t), 1)]$ given the initial point $[(\tilde{\gamma}(0), 1)]$.

In terms of local patches and transition functions $\{U_p, t_{pq}\}$ for P_1 we have locally-defined $\text{Lie}(G_1)$ -valued one-forms A_p , then $\phi_*(P_1 \xrightarrow{\pi_1} \mathbb{X})$ has local trivializations $\{U_p, \phi \circ t_{pq}\}$ and the pushed-forward connection $\phi_*(\nabla_1)$ is represented by the locally-defined $\text{Lie}(G_2)$ -valued one-forms $\phi_*(A_p)$ where $\phi_* : \text{Lie}(G_1) \rightarrow \text{Lie}(G_2)$ is the induced morphism of Lie algebras.

²An orientation *without* a choice of Riemannian metric corresponds to a reduction from $GL(n, \mathbb{R})$ to $GL^+(n, \mathbb{R})$, the subgroup of matrices of positive determinant. A volume form corresponds to a reduction from $GL(n, \mathbb{R})$ to $SL(n, \mathbb{R})$, the subgroup of matrices of determinant $+1$.

D.3 Generalized Spin^c Structure

In this section, we elaborate on the notion of generalized spin^c structures on smooth, oriented 4-manifolds and give some examples. Our conventions and notation for Čech cohomology are from Section 4.4. Define the generalized spin^c group

$$\text{Spin}_C^d(4) = \frac{\text{Spin}(4) \times \text{U}(1)^d}{C}, \quad (\text{D.3.1})$$

where C is a finite subgroup of the center:

$$C \subset Z(\text{Spin}(4) \times \text{U}(1)^d) \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \text{U}(1)^d, \quad (\text{D.3.2})$$

such that

$$p_1(C) = \langle (-1, -1) \rangle \subset Z(\text{SU}(2) \times \text{SU}(2)) = Z(\text{Spin}(4)). \quad (\text{D.3.3})$$

This ensures that

$$p_1(\text{Spin}_C^d(4)) \cong \text{Spin}(4)/p_1(C) \cong \text{SO}(4). \quad (\text{D.3.4})$$

We define a principal $\text{Spin}_C^d(4)$ -bundle $P \rightarrow \mathbb{X}$ to be a generalized spin^c structure if

$$p_{1,*}(P) \cong \text{O}(\text{Fr}(T\mathbb{X})). \quad (\text{D.3.5})$$

A connection ∇ on a principal $\text{Spin}_C^d(4)$ -bundle P is called a generalized spin^c connection if $p_{1,*}(P)(\nabla) \cong \nabla^{\text{LC}}$. A generalized spin^c structure is thus a particular case of a G -structure on a 4-manifold with $G = \text{Spin}_C^d(4)$.³ Below, we present some examples to illustrate that a generalized spin^c structure is in general different from the standard spin^c structure. The latter exists on all oriented four-manifolds.⁴

Example D.3.1. Consider the subgroup

$$C = \langle (-1, -1, -1, \dots, -1) \rangle = \text{diag}(\mathbb{Z}_2^{d+2}) \subset \mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{U}(1)^d. \quad (\text{D.3.6})$$

³Reference [BI23] discusses “ $\text{Spin}_G(4)$ -structures,” which can be given a similar interpretation. It is possible that these are relevant to special topological twistings with nongeneric masses. See also references [AM20, LJ23] for other generalizations of spin^c such as $\text{Spin}^h(n) := (\text{Spin}(n) \times \text{Sp}(1))/\mathbb{Z}_2$ and $\text{Spin}^k(n) := (\text{Spin}(n) \times \text{Spin}(k))/\mathbb{Z}_2$ in n -dimensions (and any positive integer $k \in \mathbb{Z}_{>0}$). Reference [AI80] is an interesting early work on generalizations of spin structures, motivated by quantum gravity. [AI80] considers a reduction of structure group of the tangent bundle to $\text{Spin}_G(4) := (\text{Spin}(4) \times G)/\mathbb{Z}_2$ where G is a Lie group with center $\mathbb{Z}_2$. See [WWW18] for a recent discussion of “ $\text{Spin} - \text{SU}(2)$ -structures,” which correspond to $G = \text{SU}(2)$.

⁴This follows from the important fact that on such manifolds, the second Stiefel-Whitney class of the tangent bundle, i.e., $w_2(T\mathbb{X}) \in H^2(\mathbb{X}; \mathbb{Z}_2)$, admits an integral lift, due to the existence of a characteristic element [HH58]. This is equivalent to the vanishing of $W_3 = \text{Bock}(w_2(T\mathbb{X}))$ where $\text{Bock} : H^2(\mathbb{X}; \mathbb{Z}_2) \rightarrow H^3(\mathbb{X}; \mathbb{Z})$ is the Bockstein map at degree 2, associated with the short exact sequence $0 \rightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}_2 \rightarrow 0$. See [TV94, GS99, Sco05] for details.

We first show that for this choice of $\mathbb{C}$, a $\text{Spin}^d_{\mathbb{C}}(4)$ -structure exists on any orientable 4-manifold, and is equivalent to a choice of d independent spin^c structures.

We start with transition functions

$$t_{pq}^{\text{SO}(4)} : U_{pq} \longrightarrow \text{SO}(4) , \quad (\text{D.3.7})$$

for $\text{O}(\mathbb{X})$, and choose local lifts

$$(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-) : U_{pq} \longrightarrow \text{SU}(2) \times \text{SU}(2) \cong \text{Spin}(4) , \quad (\text{D.3.8})$$

such that the diagram

$$\begin{array}{ccc} U_{pq} & \xrightarrow{(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-)} & \text{Spin}(4) \\ & \searrow t_{pq}^{\text{SO}(4)} & \swarrow \pi \\ & \text{SO}(4) & \end{array} \quad (\text{D.3.9})$$

commutes, where $\pi : \text{Spin}(4) \rightarrow \text{SO}(4)$ is the projection map. The $\mathbb{Z}_2$ -valued cochain

$$\zeta_{pqr}^{\pm} := \tilde{t}_{pq}^{\pm} \tilde{t}_{qr}^{\pm} \tilde{t}_{rp}^{\pm} , \quad (\text{D.3.10})$$

is a Čech representative of the Stiefel-Whitney class of $T\mathbb{X}$:

$$[\zeta_{pqr}^{\pm}] := w_2(T\mathbb{X}) \in H^2(\mathbb{X}; \mathbb{Z}_2) . \quad (\text{D.3.11})$$

Choose d integral lifts of $w_2(T\mathbb{X})$, i.e., d integer classes $\{c_1(u) \in H^2(\mathbb{X}; \mathbb{Z})\}_{u=1}^d$ such that

$$c_1(u) \equiv w_2(T\mathbb{X}) \bmod 2 . \quad (\text{D.3.12})$$

(Such integral lifts always exist – see footnote 4.) There exist complex line bundles $\{\mathcal{L}_u\}_{u=1}^d$ with first Chern classes $\{c_1(u)\}_{u=1}^d$. Endowing each $\mathcal{L}_u$ with a Hermitian metric on its fibers, we can regard $\mathcal{L}_u$ as a $\text{U}(1)$ -bundle. Let us choose transition functions $(h_u)_{pq} : U_{pq} \rightarrow \text{U}(1)$ on $\mathcal{L}_u$. We further choose local lifts $(\tilde{h}_u)_{pq} : U_{pq} \rightarrow \text{U}(1)$ such that the diagram

$$\begin{array}{ccc} U_{pq} & \xrightarrow{(\tilde{h}_u)_{pq}} & \text{U}(1) \\ & \searrow (h_u)_{pq} & \swarrow P_2 \\ & \text{U}(1) & \end{array} \quad (\text{D.3.13})$$

commutes, where $P_2 : \text{U}(1) \rightarrow \text{U}(1)$ is the squaring map, that acts as $z \mapsto z^2$ for $z \in \text{U}(1)$. Therefore,

$$(\eta_u)_{pqr} := (\tilde{h}_u)_{pq} (\tilde{h}_u)_{qr} (\tilde{h}_u)_{rp} : U_{pqr} \longrightarrow \mathbb{Z}/2\mathbb{Z} \subset \text{U}(1) , \quad u = 1, \dots, d , \quad (\text{D.3.14})$$

represents the $\bmod 2$ reduction of $c_1(u)$:

$$[(\eta_u)_{pqr}] = r_2(c_1(u)) \in H^2(\mathbb{X}; \mathbb{Z}/2\mathbb{Z}) , \quad u = 1, \dots, d , \quad (\text{D.3.15})$$

or equivalently, the obstruction to finding a line bundle $\mathcal{W}_u$ such that $\mathcal{W}_u^{\otimes 2} \cong \mathcal{L}_u$. Note that by definition, this mod 2 reduction is simply the second Stiefel-Whitney class $w_2(T\mathbb{X})$ due to (D.3.12). So the representatives $\zeta_{pqr}^\pm$ and $(\eta_u)_{pqr}$ are cohomologous in $\mathbb{Z}/2\mathbb{Z}$. Therefore, there exist coboundaries $(\varepsilon_u)_{pqr} : U_{pqr} \rightarrow \mathbb{Z}_2$ such that

$$\zeta_{pqr}^\pm = (\varepsilon_u)_{pqr} (\eta_u)_{pqr} . \quad (\text{D.3.16})$$

Let $(\varepsilon_u)_{pq} : U_{pq} \rightarrow \text{U}(1)$ denote the trivializing maps of these coboundaries, that is,

$$(\varepsilon_u)_{pqr} = (\varepsilon_u)_{pq} (\varepsilon_u)_{qr} (\varepsilon_u)_{rp} . \quad (\text{D.3.17})$$

We define transition functions

$$\tilde{t}_{pq} : U_{pq} \longrightarrow \text{Spin}(4) \times \text{U}(1)^d , \quad (\text{D.3.18})$$

via

$$\tilde{t}_{pq} := \left(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, (\varepsilon_1)_{pq} (\tilde{h}_1)_{pq}, \dots, (\varepsilon_d)_{pq} (\tilde{h}_d)_{pq} \right) . \quad (\text{D.3.19})$$

Then (D.3.16) implies that

$$\tilde{t}_{pq} \tilde{t}_{qr} \tilde{t}_{rp} = \left(\zeta_{pqr}^+, \zeta_{pqr}^-, (\varepsilon_1)_{pqr} (\eta_1)_{pqr}, \dots, (\varepsilon_d)_{pqr} (\eta_d)_{pqr} \right) \in \mathbb{C} . \quad (\text{D.3.20})$$

Thus the transition functions

$$t_{pq} := [\tilde{t}_{pq}] : U_{pq} \longrightarrow \text{Spin}_\mathbb{C}^d(4) , \quad (\text{D.3.21})$$

(where $[\cdot]$ indicates an equivalence class due to the quotient by $\mathbb{C}$), satisfy the cocycle condition

$$t_{pq}(x) t_{qr}(x) t_{rp}(x) = \mathbb{1}_{\text{Spin}_\mathbb{C}^d(4)} . \quad (\text{D.3.22})$$

Therefore, using these transition functions, we can construct a $\text{Spin}_\mathbb{C}^d(4)$ -bundle $P \rightarrow \mathbb{X}$. Finally, because of (D.3.4), the bundle P satisfies

$$(p_1)_*(P) \cong \text{O}(\mathbb{X}) . \quad (\text{D.3.23})$$

The defining data of a $\text{Spin}_\mathbb{C}^d(4)$ structure is a choice of d integral lifts of $w_2(T\mathbb{X})$. This is the same data needed to choose d spin^c -structures. The groups $\text{Spin}_\mathbb{C}^d(4)$ and $(\text{Spin}^c(4))^d$ are not isomorphic: they have distinct third homotopy groups:

$$\begin{aligned} \pi_3((\text{Spin}^c(4))^d) &\cong (\mathbb{Z} \oplus \mathbb{Z})^{\otimes d} , \\ \pi_3(\text{Spin}_\mathbb{C}^d(4)) &\cong \mathbb{Z} \oplus \mathbb{Z} . \end{aligned} \quad (\text{D.3.24})$$

The groupoids of principal bundles based on these two structure groups are not isomorphic. However, once the latter bundles are subject to constraint (D.3.5), with a similar constraint for each of the factors in $(\text{Spin}^c(4))^d$ there will be a 1-1 correspondence. Let us elaborate on this.

An element of the set of d spin^c -structures is a collection of equivalence classes of pairs of maps $\{[\tilde{t}_{pq}^i, h_{pq}^i]\}_{i=1}^d$ where $\tilde{t}_{pq}^i : U_{pq} \rightarrow \text{Spin}(4)$ and $h_{pq}^i : U_{pq} \rightarrow \text{U}(1)$. For a fixed i , both elements of this pair fail by a common $\mathbb{Z}_2$ -valued cochain. However, since the Čech 2-cocycle constructed out of any choice⁵ of $\tilde{t}_{pq}^i$ is a representative of the second Stiefel-Whitney class of $T\mathbb{X}$, the only choice one has is in picking the transition functions h_{pq}^i . With this Čech data, we can define a map

$$\begin{aligned} \text{Set of } d \text{ spin}^c \text{ structures} &\longrightarrow \text{Set of } \text{Spin}_{\mathbb{C}}^d(4) \text{ structures} \\ \{[\tilde{t}_{pq}^i, h_{pq}^i]\}_{i=1}^d &\longmapsto [\tilde{t}_{pq}^i, h_{pq}^1, \dots, h_{pq}^d], \end{aligned} \quad (\text{D.3.25})$$

where we have picked – without loss of generality – one of the transition functions to $\text{Spin}(4)$, namely the i^{th} one. We can also define another map

$$\begin{aligned} \text{Set of } \text{Spin}_{\mathbb{C}}^d(4) \text{ structures} &\longrightarrow \text{Set of } d \text{ spin}^c \text{ structures} \\ [\tilde{t}_{pq}^i, h_{pq}^1, \dots, h_{pq}^d] &\longmapsto \{[\tilde{t}_{pq}^i, h_{pq}^j]\}_{j=1}^d, \end{aligned} \quad (\text{D.3.26})$$

The composition of the maps (D.3.25) and (D.3.26) clearly defines a one-to-one correspondence of the set of d spin^c structures to itself. This example illustrates that it is not sufficient to say that the groups are not isomorphic to claim that this structure is different from a choice of d spin^c structures.

Example D.3.2. Consider a finite subgroup $\mathbb{C}$ of the following subgroup of the center of $\text{Spin}(4) \times \text{U}(1)^d$:

$$\mathbb{C} \subset \left\{ (\zeta, \zeta, \mu_1, \dots, \mu_d) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{U}(1)^d \mid \zeta \prod_{k=1}^d \mu_k = 1 \right\}. \quad (\text{D.3.27})$$

Let us begin with transition functions

$$t_{pq}^{\text{SO}(4)} : U_{pq} \longrightarrow \text{SO}(4), \quad (\text{D.3.28})$$

for $\text{O}(\mathbb{X})$, and choose local lifts

$$(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-) : U_{pq} \longrightarrow \text{SU}(2) \times \text{SU}(2) \cong \text{Spin}(4), \quad (\text{D.3.29})$$

such that the diagram

$$\begin{array}{ccc} U_{pq} & \xrightarrow{(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-)} & \text{Spin}(4) \\ & \searrow t_{pq}^{\text{SO}(4)} & \swarrow \pi \\ & \text{SO}(4) & \end{array} \quad (\text{D.3.30})$$

commutes, where $\pi : \text{Spin}(4) \rightarrow \text{SO}(4)$ is the projection map. The $\mathbb{Z}_2$ -valued cochain

$$\zeta_{pqr}^{\pm} := \tilde{t}_{pq}^{\pm} \tilde{t}_{qr}^{\pm} \tilde{t}_{rp}^{\pm}, \quad (\text{D.3.31})$$

⁵The transition functions are determined up to possible bundle automorphisms (a.k.a. gauge transformations or frame rotations).

is a Čech representative of the Stiefel-Whitney class of $T\mathbb{X}$:

$$[\zeta_{pqr}^{\pm}] := w_2(T\mathbb{X}) \in H^2(\mathbb{X}; \mathbb{Z}_2) . \quad (\text{D.3.32})$$

Note that $p_{u+1}(\mathbb{C})$ (for any $u = 1, \dots, d$) is a cyclic group, since all finite subgroups of $\text{U}(1)$ are cyclic. Therefore, we define d integers $\{n_u\}_{u=1}^d$ by

$$p_{u+1}(\mathbb{C}) \cong \mathbb{Z}/n_u\mathbb{Z} \subset \text{U}(1) , \quad u = 1, \dots, d , \quad (\text{D.3.33})$$

where $p_{u+1}(\mathbb{C})$ denotes the projection onto the $(u+1)^{\text{th}}$ factor of $\mathbb{C}$.

For each $\text{U}(1)$ factor in $\text{Spin}(4) \times \text{U}(1)^d$, we introduce a complex line bundle $\mathcal{L}_u$ with first Chern class $c_1(u) \in H^2(\mathbb{X}; \mathbb{Z})$. At this point, this is a completely arbitrary choice, but we will momentarily present a cohomological constraint for the existence of a $\text{Spin}_{\mathbb{C}}^d(4)$ -structure which will involve these Chern classes as well as the second Stiefel-Whitney class of $T\mathbb{X}$.

Let us endow all complex line bundles with Hermitian inner products on their fibers so that they become $\text{U}(1)$ bundles. Let $(h_u)_{pq} : U_{pq} \rightarrow \text{U}(1)$ be transition functions on the bundle $\mathcal{L}_u$. Also choose local lifts $(\tilde{h}_u)_{pq} : U_{pq} \rightarrow \text{U}(1)$ such that the diagram

$$\begin{array}{ccc} U_{pq} & \xrightarrow{(\tilde{h}_u)_{pq}} & \text{U}(1) \\ & \searrow (h_u)_{pq} \quad \swarrow P_{n_u} & \\ & \text{U}(1) & \end{array} \quad (\text{D.3.34})$$

commutes, where $P_{n_u} : \text{U}(1) \rightarrow \text{U}(1)$ is the n_u^{th} power map, that acts as $z \mapsto z^{n_u}$ for $z \in \text{U}(1)$. Then the cochain defined by

$$(\eta_u)_{pqr} := (\tilde{h}_u)_{pq}(\tilde{h}_u)_{qr}(\tilde{h}_u)_{rp} : U_{pqr} \longrightarrow \mathbb{Z}/n_u\mathbb{Z} \subset \text{U}(1) , \quad u = 1, \dots, d . \quad (\text{D.3.35})$$

represents the $\text{mod } n_u$ reduction of $c_1(u)$:

$$[(\eta_u)_{pqr}] = r_{n_u}(c_1(u)) \in H^2(\mathbb{X}; \mathbb{Z}/n_u\mathbb{Z}) , \quad u = 1, \dots, d . \quad (\text{D.3.36})$$

For any fixed u , the class $[(\eta_u)_{pqr}]$ (or equivalently, the $\mathbb{Z}_{n_u}$ -valued cochain $(\eta_u)_{pqr}$) measures the obstruction to the existence of a globally-defined line bundle $\mathcal{W}_u$ such that $\mathcal{W}_u^{\otimes n_u} \cong \mathcal{L}_u$.

At this point, we have $\mathbb{Z}_2$ -valued cochains $\zeta_{pqr}^{\pm}$, and the d cochains of the form $(\eta_u)_{pqr}$ for $u = 1, \dots, d$, where, for a fixed u , $(\eta_u)_{pqr}$ is $\mathbb{Z}_{n_u}$ -valued.⁶

Let us define

$$N := \text{lcm}(2, n_1, \dots, n_d) . \quad (\text{D.3.37})$$

⁶It is instructive to contrast this with the Čech arguments that establish the existence of a standard spin^c structure. There, $d = 1$, $u = 1$, and $n_u = 2$. One has two $\mathbb{Z}_2$ -valued cochains, and due to the result of footnote 4, they represent the *same* $\mathbb{Z}_2$ cohomology class, and must consequently differ by a coboundary.

Therefore, $N = 2p$ for some $p \in \mathbb{Z}_{\geq 1}$. Hence, the maps

$$\rho_s : \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} , \quad (\text{D.3.38})$$

$$\iota_u : \mathbb{Z}/n_u\mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} , \quad u = 1, \dots, d , \quad (\text{D.3.39})$$

are well-defined. They induce the following maps in cohomology

$$\rho_s : H^\bullet(X; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^\bullet(\mathbb{X}; \mathbb{Z}/N\mathbb{Z}) , \quad (\text{D.3.40})$$

$$\iota_u : H^\bullet(X; \mathbb{Z}/n_u\mathbb{Z}) \longrightarrow H^\bullet(\mathbb{X}; \mathbb{Z}/N\mathbb{Z}) , \quad u = 1, \dots, d . \quad (\text{D.3.41})$$

which, in general, are neither injective nor surjective.⁷

We define transition functions

$$\tilde{t}_{pq} : U_{pq} \longrightarrow \text{Spin}(4) \times \text{U}(1)^d , \quad (\text{D.3.42})$$

by

$$\tilde{t}_{pq} := \left(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, (\tilde{h}_1)_{pq}, \dots, (\tilde{h}_d)_{pq} \right) . \quad (\text{D.3.43})$$

With this definition,

$$\tilde{t}_{pq} \tilde{t}_{qr} \tilde{t}_{rp} = \left(\zeta_{pqr}^+, \zeta_{pqr}^-, (\eta_1)_{pqr}, \dots, (\eta_d)_{pqr} \right) . \quad (\text{D.3.44})$$

It follows from the definition (D.3.27) that if

$$\zeta_{pqr}^\pm \cdot \prod_{u=1}^d (\eta_u)_{pqr} = 1 , \quad (\text{D.3.45})$$

the transition functions $\tilde{t}_{pq}$ satisfy

$$\tilde{t}_{pq} \tilde{t}_{qr} \tilde{t}_{rp} \in \mathbb{C} , \quad (\text{D.3.46})$$

as desired. In particular, the transition functions

$$t_{pq} := [\tilde{t}_{pq}] : U_{pq} \longrightarrow \text{Spin}_{\mathbb{C}}^d(4) , \quad (\text{D.3.47})$$

(where $[\cdot]$ indicates an equivalence class due to the quotient by $\mathbb{C}$), satisfy the cocycle condition

$$t_{pq}(x) t_{qr}(x) t_{rp}(x) = \mathbb{1}_{\text{Spin}_{\mathbb{C}}^d(4)} . \quad (\text{D.3.48})$$

Then, following the same line of reasoning as in Section 4.4, (D.3.45) implies the following cohomological condition:

$$\boxed{\rho_s(w_2(\mathbb{X})) + \sum_{u=1}^d \iota_u \circ r_{n_u}(c_1(u)) = 0 \in H^2(\mathbb{X}; \mathbb{Z}/N\mathbb{Z}) .} \quad (\text{D.3.49})$$

⁷For example, the induced map ρ_s on cohomology has a kernel isomorphic to the p -torsion submodule of $H^\bullet(\mathbb{X}; \mathbb{Z}/2\mathbb{Z})$, whereas the induced map ι_u has a kernel isomorphic to the (N/n_u) -torsion submodule of $H^\bullet(\mathbb{X}; \mathbb{Z}/n_u\mathbb{Z})$.

We now argue that this cohomological condition enables us to construct explicitly a $\text{Spin}_{\mathbb{C}}^d(4)$ cocycle. For this, we assume that a collection of line bundles $\mathcal{L}_u$ exists with its first Chern classes chosen to satisfy (D.3.49).

The cohomological condition (D.3.49) implies that the product of the Čech cochains corresponding to the individual terms represents a coboundary:

$$\rho_s(\zeta_{pqr}^{\pm}) \cdot \left[\prod_{u=1}^d \iota_u((\eta_u)_{pqr}) \right] = \varepsilon_{pqr} : U_{pqr} \rightarrow \mathbb{Z}/N\mathbb{Z} \subset \text{U}(1) . \quad (\text{D.3.50})$$

Since ε_{pqr} is a coboundary, there exist trivializing maps

$$\varepsilon_{pq} : U_{pq} \rightarrow \mathbb{Z}/N\mathbb{Z} \subset \text{U}(1) , \quad (\text{D.3.51})$$

such that

$$\varepsilon_{pqr} = \varepsilon_{pq} \varepsilon_{qr} \varepsilon_{rp} . \quad (\text{D.3.52})$$

Therefore,

$$\rho_s(\zeta_{pqr}^{\pm}) \cdot \left[\prod_{u=1}^d \iota_u((\eta_u)_{pqr}) \right] = \varepsilon_{pq} \varepsilon_{qr} \varepsilon_{rp} . \quad (\text{D.3.53})$$

Now, we can define the modified transition functions⁸

$$(\tilde{h}_u)'_{pq} := (\tilde{h}_u)_{pq} (\varepsilon_{pq})^{-1/d} , \quad (\text{D.3.54})$$

as well as maps $\tilde{s}_{pq} : U_{pq} \rightarrow \text{Spin}(4) \times \text{U}(1)^d$ via

$$\begin{aligned} \tilde{s}_{pq} &:= \left(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, (\tilde{h}_1)'_{pq}, \dots, (\tilde{h}_d)'_{pq} \right) \\ &= \left(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, (\tilde{h}_1)_{pq} \varepsilon_{pq}^{-1/d}, \dots, (\tilde{h}_d)_{pq} \varepsilon_{pq}^{-1/d} \right) . \end{aligned} \quad (\text{D.3.55})$$

Note that

$$\tilde{s}_{pq} \tilde{s}_{qr} \tilde{s}_{rp} = (\zeta_{pqr}^+, \zeta_{pqr}^-, (\eta_1)_{pqr} \varepsilon_{pqr}^{-1/d}, \dots, (\eta_d)_{pqr} \varepsilon_{pqr}^{-1/d}) . \quad (\text{D.3.56})$$

It follows directly from (D.3.53) that

$$\tilde{s}_{pq} \tilde{s}_{qr} \tilde{s}_{rp} \in \mathbb{C} . \quad (\text{D.3.57})$$

⁸By the d^{th} root of a generic phase ε_{pq} , we mean the following: we choose a solution y to the equation $y^d = \varepsilon_{pq}$ and use it in (D.3.54)–(D.3.56). The conclusion (D.3.57) remains valid because the d^{th} power of any solution by definition equals ε_{pq} . So (D.3.57) is valid and independent of the branch chosen to define the d^{th} root. We could have avoided taking the d^{th} root – at the expense of an undemocratic treatment of the $\text{U}(1)$ bundles – by arbitrarily picking one of the d $\text{U}(1)$ bundles – say the u_0^{th} one – and defining its modified transition functions as $(\tilde{h}_{u_0})'_{pq} := (\tilde{h}_{u_0})_{pq} \cdot (\varepsilon_{pq})^{-1}$, whereas $(\tilde{h}_u)'_{pq} := (\tilde{h}_u)_{pq}$ for $u = 1, \dots, d, u \neq u_0$.

Therefore, the transition functions,

$$s_{pq} := [\tilde{s}_{pq}] : U_{pq} \longrightarrow (\mathrm{Spin}(4) \times \mathrm{U}(1)^d)/C , \quad (\text{D.3.58})$$

satisfy the cocycle condition

$$s_{pq}s_{qr}s_{rp} = \mathbb{1}_{\mathrm{Spin}_C^d(4)} . \quad (\text{D.3.59})$$

Moreover, because of (D.3.30) the $\mathrm{Spin}_C^d(4)$ -bundle $P \rightarrow \mathbb{X}$ with transition functions t_{pq} satisfies

$$(p_1)_*(P) \cong \mathrm{O}Fr(T\mathbb{X}) . \quad (\text{D.3.60})$$

Thus we have constructed a $\mathrm{Spin}_C^d(4)$ -structure on $\mathbb{X}$. For $d = 1$, this is precisely the standard $\mathrm{Spin}^c(4)$ -structure.

It follows from the above argument that the set of $\mathrm{Spin}_C^d(4)$ -structures on $\mathbb{X}$ is a torsor for the subgroup of elements

$$(c(1), \dots, c(d)) \in H^2(\mathbb{X}; \mathbb{Z})^d . \quad (\text{D.3.61})$$

that satisfy:

$$\sum_{u=1}^d \iota_u \circ r_{n_u}(c(u)) = 0 \in H^2(\mathbb{X}; \mathbb{Z}/N\mathbb{Z}) . \quad (\text{D.3.62})$$

Example D.3.3. Fix a positive even integer $n \in \mathbb{Z}_{>0}$ and consider the finite group

$$C_n = \left\{ (\zeta, \zeta, \mu) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{U}(1) \mid \mu^{n/2}\zeta = 1 \right\} . \quad (\text{D.3.63})$$

Then C_n is generated by

$$(-1, -1, g) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathrm{U}(1) , \quad (\text{D.3.64})$$

where $g \in \mathrm{U}(1)$ is any element of order n . Moreover, since n is even,

$$C_n \cong \mathbb{Z}/n\mathbb{Z} . \quad (\text{D.3.65})$$

We denote the corresponding generalized spin^c group by

$$\mathrm{Spin}_{C_n}^1(4) = (\mathrm{Spin}(4) \times \mathrm{U}(1))/C_n . \quad (\text{D.3.66})$$

As in Example D.3.2, we introduce transition functions $t_{pq}^{\mathrm{SO}(4)}$ for $\mathrm{O}Fr(\mathbb{X})$ and choose local lifts $(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-) : U_{pq} \rightarrow \mathrm{Spin}(4)$ which satisfy a commutative diagram of the form (D.3.30). Therefore, as before, the $\mathbb{Z}_2$ -valued cochain

$$\zeta_{pqr}^\pm := \tilde{t}_{pqr}^\pm \tilde{t}_{pqr}^\pm \tilde{t}_{pqr}^\pm , \quad (\text{D.3.67})$$

is a Čech representative of the second Stiefel-Whitney class of $T\mathbb{X}$:

$$[\zeta_{pqr}^{\pm}] := w_2(T\mathbb{X}) \in H^2(\mathbb{X}; \mathbb{Z}_2) . \quad (\text{D.3.68})$$

Let us introduce a complex line bundle $\mathcal{L}$ with a first Chern class $c_1(\mathcal{L})$ which is at the moment an arbitrary integral class, but will be further constrained by a cohomological condition later. Endowing $\mathcal{L}$ with a Hermitian metric on its fibers, we can regard $\mathcal{L}$ as $\text{U}(1)$ -bundle. Let $\{h_{pq}\}$ denote transition functions on this bundle. As in (D.3.34), we choose local lifts $\tilde{h}_{pq} : U_{pq} \rightarrow \text{U}(1)$ such that the diagram

$$\begin{array}{ccc} U_{pq} & \xrightarrow{\tilde{h}_{pq}} & \text{U}(1) \\ & \searrow h_{pq} & \swarrow P_n \\ & \text{U}(1) & \end{array} \quad (\text{D.3.69})$$

commutes, where P_n is the n^{th} power map, which acts as $z \mapsto z^n$ for $z \in \text{U}(1)$. Note that the global existence of the lifts $\tilde{h}_{pq}$ is equivalent to the existence of a complex line bundle $\mathcal{W}$ such that $\mathcal{W}^{\otimes n} = \mathcal{L}$. Existence of $\mathcal{W}$ is equivalent to $c_1(\mathcal{L})$ being n times an integral class $y \in H^2(\mathbb{X}; \mathbb{Z})$ (which is then the first Chern class of $\mathcal{W}$, i.e., $y = c_1(\mathcal{W})$).

The cochain

$$\eta_{pqr} := \tilde{h}_{pq} \tilde{h}_{qr} \tilde{h}_{rp} , \quad (\text{D.3.70})$$

represents the mod n reduction of $c_1(\mathcal{L})$:

$$[\eta_{pqr}] := r_n(c_1(\mathcal{L})) , \quad (\text{D.3.71})$$

where r_n is reduction modulo n , and measures the obstruction to the existence of a line bundle $\mathcal{W}$.

We define transition functions

$$\tilde{t}_{pq} := \left(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, (\tilde{h})_{pq} \right) . \quad (\text{D.3.72})$$

With this definition,

$$\tilde{t}_{pq} \tilde{t}_{qr} \tilde{t}_{rp} = (\zeta_{pqr}^+, \zeta_{pqr}^-, \eta_{pqr}) . \quad (\text{D.3.73})$$

It follows from the definition (D.3.63) that if

$$\zeta_{pqr}^{\pm} \cdot (\eta_{pqr})^{n/2} = 1 , \quad (\text{D.3.74})$$

the transition functions $\tilde{t}_{pq}$ satisfy

$$\tilde{t}_{pq} \tilde{t}_{qr} \tilde{t}_{rp} \in \mathbb{C}_n , \quad (\text{D.3.75})$$

as desired. If there exist functions

$$\varepsilon_{pq} : U_{pq} \rightarrow \mathbb{Z}/n\mathbb{Z} , \quad (\text{D.3.76})$$

such that

$$\iota(\zeta_{pqr}^{\pm}) \cdot (\eta_{pqr})^{n/2} = \varepsilon_{pq}\varepsilon_{qr}\varepsilon_{rp} \equiv \varepsilon_{pqr} , \quad (\text{D.3.77})$$

this is equivalent to the following cohomological condition

$$\boxed{\iota(w_2(\mathbb{X})) + \frac{n}{2}r_n(c_1(\mathcal{L})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}/n\mathbb{Z}) ,} \quad (\text{D.3.78})$$

which should be viewed as a constraint on $c_1(\mathcal{L})$.

We now show that the condition (D.3.77) (or equivalently (D.3.78)) enables us to satisfy the cocycle condition for $\text{Spin}_{\mathbb{C}_n}^1(4)$. We simply define the modified transition functions⁹

$$\tilde{h}'_{pq} := \tilde{h}_{pq}(\varepsilon_{pq})^{-2/n} , \quad (\text{D.3.79})$$

and maps $\tilde{s}_{pq} : U_{pq} \rightarrow \text{Spin}(4) \times \text{U}(1)$ via

$$\tilde{s}_{pq} := (\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{h}'_{pq}) = (\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{h}_{pq}\varepsilon_{pq}^{-2/n}) . \quad (\text{D.3.80})$$

Note that

$$\tilde{s}_{pq}\tilde{s}_{qr}\tilde{s}_{rp} = (\zeta_{pqr}^+, \zeta_{pqr}^-, \eta_{pqr}\varepsilon_{pqr}^{-2/n}) . \quad (\text{D.3.81})$$

It follows directly from (D.3.77) that

$$\tilde{s}_{pq}\tilde{s}_{qr}\tilde{s}_{rp} \in \mathbb{C}_n . \quad (\text{D.3.82})$$

Therefore, the transition functions,

$$s_{pq} := [\tilde{s}_{pq}] : U_{pq} \longrightarrow (\text{Spin}(4) \times \text{U}(1))/\mathbb{C}_n , \quad (\text{D.3.83})$$

satisfy the cocycle condition

$$s_{pq}s_{qr}s_{rp} = \mathbb{1}_{\text{Spin}_{\mathbb{C}_n}^d(4)} . \quad (\text{D.3.84})$$

A few remarks are in order. First of all, if $n = 2$, the map ι reduces to the identity map, and the condition (D.3.78) reduces to the condition for a standard spin^c structure, namely $w_2(\mathbb{X}) = c_1(\mathcal{L}) \bmod 2$. In this case, one recovers the familiar fact that the evenness of $c_1(\mathcal{L})$ is equivalent to the manifold being a spin manifold.

For $n > 2$, if $c_1(\mathcal{L})$ is n times an integral class, i.e., if there exists an integral class $y \in H^2(\mathbb{X}; \mathbb{Z})$ such that $c_1(\mathcal{L}) = n \cdot y$, then the cohomological condition (D.3.78) is equivalent to $\iota(w_2(\mathbb{X})) = 0$. In this case, one possibility is that $\mathbb{X}$ be spin, i.e., $w_2(\mathbb{X}) = 0$. The other possibility is that $w_2(\mathbb{X})$ be in the kernel of the map $\iota : H^2(\mathbb{X}; \mathbb{Z}_2) \rightarrow H^2(\mathbb{X}; \mathbb{Z}/n\mathbb{Z})$. This possibility merits further study.

⁹We choose a solution y to $y^{n/2} = \varepsilon_{pq}$, use it in (D.3.79)–(D.3.81). The conclusion (D.3.82) remains true independent of the choice of branch for the $(n/2)^{th}$ root because by definition, the $(n/2)^{th}$ power of *any* solution is the same. The reasoning is similar to what was described in footnote 8 for Example D.3.1, except in this example, we do not have the option to be undemocratic with only one $\text{U}(1)$ bundle at our disposal.

D.4 Derivation Of The Twisted Action (4.7.8)

In this appendix, we present the detailed calculation leading to the twisted action (4.7.8). We will use various facts about supermultiplets of 4d $\mathcal{N} = 1, 2$ supersymmetry, for which we refer the reader to [WB92, LMn05, GGRS83]. We will use the same conventions as in [CMRS23, Appendix A]. Our definition of the selfdual tensor $\chi_{\mu\nu}$ is

$$\begin{aligned}\chi_{\mu\nu} &= \frac{1}{2}(\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\chi_{\dot{\alpha}\dot{\beta}}, \\ \chi_{\dot{\alpha}\dot{\beta}} &= (\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}}\chi^{\mu\nu}.\end{aligned}\tag{D.4.1}$$

We start by listing some formulas which will be useful in the computation below.

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta,\tag{D.4.2}$$

$$\theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta\tag{D.4.3}$$

$$\tilde{\theta}^{\dot{\alpha}}\tilde{\theta}^{\dot{\beta}} = -\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\tilde{\theta}\tilde{\theta},\tag{D.4.4}$$

$$\tilde{\theta}_{\dot{\alpha}}\tilde{\theta}_{\dot{\beta}} = \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\tilde{\theta}\tilde{\theta}.\tag{D.4.5}$$

D.4.1 Lagrangian for 4d $\mathcal{N} = 2$ supermultiplets

Let us start with the 4d $\mathcal{N} = 2$ supersymmetry algebra in Euclidean signature:

$$\{\mathcal{Q}_\alpha^I, \tilde{\mathcal{Q}}_{\dot{\beta}J}\} = 2\sigma_{\alpha\dot{\beta}}^\mu\delta_J^I,\tag{D.4.6}$$

$$\{\mathcal{Q}_\alpha^I, \mathcal{Q}_\beta^J\} = 2\epsilon_{\alpha\beta}\epsilon^{IJ}\bar{Z}\tag{D.4.7}$$

$$\{\tilde{\mathcal{Q}}_{\dot{\alpha}I}, \tilde{\mathcal{Q}}_{\dot{\beta}J}\} = 2\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{IJ}Z,\tag{D.4.8}$$

where I, J are $\text{SU}(2)_R$ indices. Upper $\text{SU}(2)_R$ index is in the $\mathbf{2}$ of $\text{SU}(2)$ and lower $\text{SU}(2)_R$ index is in the $\bar{\mathbf{2}}$ of $\text{SU}(2)$ and the indices are raised and lowered as follows:

$$\varphi_I = \epsilon_{IJ}\varphi^J, \quad \varphi^I = \epsilon^{IJ}\varphi_J\tag{D.4.9}$$

and ϵ_{IJ} is the antisymmetric matrix with $\epsilon_{12} = \epsilon^{21} = 1$. The supersymmetry transformation is generated by

$$\delta \equiv i\xi_I\mathcal{Q}^I + i\tilde{\xi}^I\tilde{\mathcal{Q}}_I = i\xi^\alpha_I\mathcal{Q}_\alpha^I + i\tilde{\xi}_{\dot{\alpha}}^I\tilde{\mathcal{Q}}^{\dot{\alpha}}_I\tag{D.4.10}$$

where $\xi_I, \tilde{\xi}^I$ are (Grassmann even) parameters of supersymmetry transformation and transform in the $\bar{\mathbf{2}}$ and $\mathbf{2}$ of $\text{SU}(2)_R$.

Vector Multiplet

In Wess-Zumino gauge, the $\mathcal{N} = 2$ vector multiplet is a collection of the following fields

$$V_{\mathcal{N}=2} = (A_\mu, \phi, \tilde{\phi}, \psi^I, \tilde{\psi}_I, D_{IJ}) , \quad (\text{D.4.11})$$

all in the adjoint representation of the gauge group. In terms of $\mathcal{N} = 1$ superfields, $V_{\mathcal{N}=2}$ is the combination of the $\mathcal{N} = 1$ vector multiplet $V_{\mathcal{N}=1}$ and the $\mathcal{N} = 1$ chiral multiplets $\Phi_{\mathcal{N}=1}, \tilde{\Phi}_{\mathcal{N}=1}$:

$$V_{\mathcal{N}=1} = -\theta\sigma^\mu\tilde{\theta}A_\mu + i\theta\theta\tilde{\theta}\tilde{\psi}_2 - i\tilde{\theta}\tilde{\theta}\theta\psi^2 + \frac{1}{2}\tilde{\theta}\tilde{\theta}\theta\theta D, \quad (\text{D.4.12})$$

$$\Phi_{\mathcal{N}=1} = \phi + i\theta\sigma^\mu\tilde{\theta}\partial_\mu\phi - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\phi + \sqrt{2}\theta\psi^1 - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi^1\sigma^\mu\tilde{\theta} + \theta\theta F, \quad (\text{D.4.13})$$

$$\tilde{\Phi}_{\mathcal{N}=1} = \tilde{\phi} - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{\phi} - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{\phi} + \sqrt{2}\tilde{\theta}\tilde{\psi}^1 + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\psi}^1\theta + \tilde{\theta}\tilde{\theta}\tilde{F}. \quad (\text{D.4.14})$$

The superpotential for the theory is given by

$$W_\alpha = -\frac{1}{4}\tilde{D}\tilde{D}e^{-V_{\mathcal{N}=1}}D_\alpha e^{V_{\mathcal{N}=1}}, \quad \tilde{W}_{\dot{\alpha}} = -\frac{1}{4}DDe^{V_{\mathcal{N}=1}}\tilde{D}_{\dot{\alpha}}e^{-V_{\mathcal{N}=1}} \quad (\text{D.4.15})$$

where the supercovariant derivatives are

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}\partial_\mu, \quad \tilde{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\tilde{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu. \quad (\text{D.4.16})$$

We have

$$e^{-V_{\mathcal{N}=1}}D_\alpha e^{V_{\mathcal{N}=1}} = D_\alpha V_{\mathcal{N}=1} - \frac{1}{2}[V_{\mathcal{N}=1}, D_\alpha V_{\mathcal{N}=1}]. \quad (\text{D.4.17})$$

Then

$$D_\alpha V_{\mathcal{N}=1} = D_\alpha \left(-\theta\sigma^\mu\tilde{\theta}A_\mu + i\theta\theta\tilde{\theta}\tilde{\psi}_2 - i\tilde{\theta}\tilde{\theta}\theta\psi^2 + \frac{1}{2}\tilde{\theta}\tilde{\theta}\theta\theta D \right) \quad (\text{D.4.18})$$

$$= \left(\frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}\partial_\mu \right) \left(-\theta\sigma^\nu\tilde{\theta}A_\nu + i\theta\theta\tilde{\theta}\tilde{\psi}_2 - i\tilde{\theta}\tilde{\theta}\theta\psi^2 + \frac{1}{2}\tilde{\theta}\tilde{\theta}\theta\theta D \right) \quad (\text{D.4.19})$$

$$= -\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}A_\mu + 2i\theta_\alpha\tilde{\theta}\tilde{\psi}_2^{\dot{\alpha}} - i\tilde{\theta}\tilde{\theta}\psi_\alpha^2 + \tilde{\theta}\tilde{\theta}\theta_\alpha D \\ + i\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}} \left(-\theta\sigma^\nu\tilde{\theta}\partial_\mu A_\nu + i\theta\theta\tilde{\theta}\partial_\mu\tilde{\psi}_2 - i\tilde{\theta}\tilde{\theta}\partial_\mu\psi^2 + \frac{1}{2}\tilde{\theta}\tilde{\theta}\theta\partial_\mu D \right), \quad (\text{D.4.20})$$

where we used

$$\frac{\partial}{\partial\theta^\alpha}\theta\theta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta\theta_\beta = \frac{\partial}{\partial\theta^\alpha}\epsilon_{\beta\gamma}\theta^\beta\theta^\gamma = \epsilon_{\beta\gamma}\delta_\alpha^\beta\theta^\gamma - \epsilon_{\beta\gamma}\theta^\beta\delta_\alpha^\gamma \quad (\text{D.4.21})$$

$$= \theta_\alpha + \theta_\alpha \quad (\text{D.4.22})$$

$$= 2\theta_\alpha. \quad (\text{D.4.23})$$

Since the highest power of θ and $\tilde{\theta}$ that is nonvanishing is 2, we can ignore several terms. We have

$$D_\alpha V_{\mathcal{N}=1} = -\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}A_\mu + 2i\theta_\alpha\tilde{\theta}\tilde{\psi}_2^{\dot{\alpha}} - i\tilde{\theta}\tilde{\theta}\psi_\alpha^2 + \tilde{\theta}\tilde{\theta}\theta_\alpha D - i\sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}\theta\sigma^\nu\tilde{\theta}\partial_\mu A_\nu \\ - \sigma^\mu_{\alpha\dot{\alpha}}\tilde{\theta}^{\dot{\alpha}}\theta\theta\tilde{\theta}\partial_\mu\tilde{\psi}_2. \quad (\text{D.4.24})$$

Thus

$$\begin{aligned}
e^{-V_{\mathcal{N}=1}} D_\alpha e^{V_{\mathcal{N}=1}} &= D_\alpha V_{\mathcal{N}=1} - \frac{1}{2} [V_{\mathcal{N}=1}, D_\alpha V_{\mathcal{N}=1}] \\
&= -\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} A_\mu + 2i\theta_\alpha \tilde{\theta}_{\dot{\alpha}} \tilde{\psi}_2^{\dot{\alpha}} - i\tilde{\theta}\tilde{\theta}\psi_\alpha^2 + \tilde{\theta}\tilde{\theta}\theta_\alpha D - i\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \sigma^\nu \tilde{\theta} \partial_\mu A_\nu \\
&\quad - \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \tilde{\theta} \partial_\mu \tilde{\psi}_2 - \frac{1}{2} \left[-\theta \sigma^\mu \tilde{\theta} A_\mu + i\theta \tilde{\theta} \tilde{\psi}_2 - i\tilde{\theta}\tilde{\theta}\psi^2 + \frac{1}{2} \tilde{\theta}\tilde{\theta}\theta D, \right. \\
&\quad - \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} A_\mu + 2i\theta_\alpha \tilde{\theta}_{\dot{\alpha}} \tilde{\psi}_2^{\dot{\alpha}} - i\tilde{\theta}\tilde{\theta}\psi_\alpha^2 + \tilde{\theta}\tilde{\theta}\theta_\alpha D \\
&\quad \left. - i\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \sigma^\nu \tilde{\theta} \partial_\mu A_\nu - \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \tilde{\theta} \partial_\mu \tilde{\psi}_2 \right] \\
&= -\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} A_\mu + 2i\theta_\alpha \tilde{\theta}_{\dot{\alpha}} \tilde{\psi}_2^{\dot{\alpha}} - i\tilde{\theta}\tilde{\theta}\psi_\alpha^2 + \tilde{\theta}\tilde{\theta}\theta_\alpha D - i\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \sigma^\nu \tilde{\theta} \partial_\mu A_\nu \\
&\quad - \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \tilde{\theta} \partial_\mu \tilde{\psi}_2 - \frac{1}{2} \left\{ -\theta \sigma^\mu \tilde{\theta} \left(-\sigma_{\alpha\dot{\alpha}}^\nu \tilde{\theta}^{\dot{\alpha}} [A_\mu, A_\nu] + 2i\theta_\alpha \tilde{\theta}_{\dot{\alpha}} [A_\mu, \tilde{\psi}_2^{\dot{\alpha}}] \right) \right. \\
&\quad \left. - i\theta \tilde{\theta} \tilde{\theta} \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} [\tilde{\psi}_2, A_\mu] \right\} \\
&= -\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} A_\mu + 2i\theta_\alpha \tilde{\theta}_{\dot{\alpha}} \tilde{\psi}_2^{\dot{\alpha}} - i\tilde{\theta}\tilde{\theta}\psi_\alpha^2 + \tilde{\theta}\tilde{\theta}\theta_\alpha D \\
&\quad - i\sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \sigma^\nu \tilde{\theta} \partial_\mu A_\nu - \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} \theta \tilde{\theta} \partial_\mu \tilde{\psi}_2 - \frac{1}{2} \theta \sigma^\mu \tilde{\theta} \sigma_{\alpha\dot{\alpha}}^\nu \tilde{\theta}^{\dot{\alpha}} [A_\mu, A_\nu] \\
&\quad + i\theta \sigma^\mu \tilde{\theta} \theta_\alpha \tilde{\theta}_{\dot{\alpha}} [A_\mu, \tilde{\psi}_2^{\dot{\alpha}}] + \frac{1}{2} \theta \tilde{\theta} \tilde{\theta} [\tilde{\psi}_2, A_\mu] \sigma_{\alpha\dot{\alpha}}^\mu \tilde{\theta}^{\dot{\alpha}} .
\end{aligned} \tag{D.4.25}$$

Next, using

$$\begin{aligned}
\frac{\partial}{\partial \tilde{\theta}^{\dot{\beta}}} \tilde{\theta} \tilde{\theta} &= \frac{\partial}{\partial \tilde{\theta}^{\dot{\beta}}} \epsilon^{\dot{\alpha}\dot{\gamma}} \tilde{\theta}_{\dot{\alpha}} \tilde{\theta}_{\dot{\gamma}} = \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon_{\dot{\alpha}\dot{\beta}} \tilde{\theta}_{\dot{\gamma}} - \epsilon^{\dot{\alpha}\dot{\gamma}} \epsilon_{\dot{\gamma}\dot{\beta}} \tilde{\theta}_{\dot{\alpha}} \\
&= -\delta_{\dot{\beta}}^{\dot{\gamma}} \tilde{\theta}_{\dot{\gamma}} - \delta_{\dot{\beta}}^{\dot{\alpha}} \tilde{\theta}_{\dot{\alpha}} \\
&= -2\tilde{\theta}_{\dot{\beta}} ,
\end{aligned} \tag{D.4.26}$$

and

$$\frac{\partial}{\partial \tilde{\theta}^{\dot{\beta}}} \theta_\alpha = \epsilon_{\dot{\alpha}\dot{\gamma}} \frac{\partial}{\partial \tilde{\theta}^{\dot{\beta}}} \theta^{\dot{\gamma}} = \epsilon_{\dot{\alpha}\dot{\beta}} \tag{D.4.27}$$

we get

$$\begin{aligned}
\tilde{D}_{\dot{\beta}} e^{-V_{\mathcal{N}=1}} D_{\alpha} e^{V_{\mathcal{N}=1}} &= \left(\frac{\partial}{\partial \tilde{\theta}^{\dot{\beta}}} + i\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \partial_{\rho} \right) \left(\sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} A_{\mu} - 2i\theta_{\alpha} \tilde{\theta}_{\dot{\alpha}} \tilde{\psi}_2^{\dot{\alpha}} + i\tilde{\theta}\tilde{\theta}\psi_2^{\alpha} - \tilde{\theta}\tilde{\theta}\theta_{\alpha} D \right. \\
&\quad + i\sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \sigma^{\nu} \tilde{\theta} \partial_{\mu} A_{\nu} + \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \partial_{\mu} \tilde{\psi}_2 \\
&\quad + \frac{1}{2} \theta \sigma^{\mu} \tilde{\theta} \sigma_{\alpha\dot{\alpha}}^{\nu} \tilde{\theta}^{\dot{\alpha}} [A_{\mu}, A_{\nu}] - i\theta \sigma^{\mu} \tilde{\theta} \theta_{\alpha} \tilde{\theta}_{\dot{\alpha}} [A_{\mu}, \tilde{\psi}_2^{\dot{\alpha}}] \\
&\quad \left. - \frac{i}{2} \theta \tilde{\theta} [\tilde{\psi}_2, A_{\mu}] \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \right) \\
&= \sigma_{\alpha\dot{\beta}}^{\mu} A_{\mu} + 2i\epsilon_{\alpha\dot{\beta}} \theta_{\alpha} \tilde{\psi}_2^{\dot{\alpha}} - 2i\tilde{\theta}_{\dot{\beta}} \psi_2^{\alpha} + 2\tilde{\theta}_{\dot{\beta}} \theta_{\alpha} D + i\sigma_{\alpha\dot{\beta}}^{\mu} \theta \sigma^{\nu} \tilde{\theta} \partial_{\mu} A_{\nu} + i\sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \sigma^{\nu} \epsilon_{\dot{\gamma}\dot{\beta}} \partial_{\mu} A_{\nu} \\
&\quad + \sigma_{\alpha\dot{\beta}}^{\mu} \theta \partial_{\mu} \tilde{\psi}_2 - \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \epsilon_{\dot{\gamma}\dot{\beta}} \partial_{\mu} \tilde{\psi}_2^{\dot{\gamma}} - \frac{1}{2} \theta \sigma^{\mu} \epsilon_{\dot{\gamma}\dot{\beta}} \sigma_{\alpha\dot{\alpha}}^{\nu} \tilde{\theta}^{\dot{\alpha}} [A_{\mu}, A_{\nu}] + \frac{1}{2} \theta \sigma^{\mu} \tilde{\theta} \sigma_{\alpha\dot{\beta}}^{\nu} [A_{\mu}, A_{\nu}] \\
&\quad + i\theta \sigma^{\mu} \epsilon_{\dot{\gamma}\dot{\beta}} \theta_{\alpha} \tilde{\theta}_{\dot{\alpha}} [A_{\mu}, \tilde{\psi}_2^{\dot{\alpha}}] + i\theta \sigma^{\mu} \tilde{\theta} \theta_{\alpha} \epsilon_{\dot{\gamma}\dot{\beta}} [A_{\mu}, \tilde{\psi}_2^{\dot{\gamma}}] - \frac{i}{2} \theta \theta \epsilon_{\dot{\alpha}\dot{\beta}} [\tilde{\psi}_2, A_{\mu}] \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \\
&\quad - \frac{i}{2} \theta \tilde{\theta} [\tilde{\psi}_2, A_{\mu}] \sigma_{\alpha\dot{\beta}}^{\mu} + i\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \partial_{\rho} A_{\mu} + 2\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \theta_{\alpha} \tilde{\theta}_{\dot{\alpha}} \partial_{\rho} \tilde{\psi}_2^{\dot{\alpha}} - \theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \tilde{\theta} \partial_{\rho} \psi_2^{\alpha} \\
&\quad - i\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \tilde{\theta} \theta_{\alpha} \partial_{\rho} D - \theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \sigma^{\nu} \tilde{\theta} \partial_{\rho} \partial_{\mu} A_{\nu} \\
&\quad + \frac{i}{2} \theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \theta \sigma^{\mu} \tilde{\theta} \sigma_{\alpha\dot{\alpha}}^{\nu} \tilde{\theta}^{\dot{\alpha}} ([\partial_{\rho} A_{\mu}, A_{\nu}] + [A_{\mu}, \partial_{\rho} A_{\nu}]) .
\end{aligned} \tag{D.4.28}$$

Next

$$\begin{aligned}
\epsilon^{\dot{\gamma}\dot{\beta}} \tilde{D}_{\dot{\gamma}} \tilde{D}_{\dot{\beta}} e^{-V_{\mathcal{N}=1}} D_{\alpha} e^{V_{\mathcal{N}=1}} &= -\epsilon^{\dot{\gamma}\dot{\beta}} \left(\frac{\partial}{\partial \tilde{\theta}^{\dot{\gamma}}} + i\theta^{\gamma} \sigma_{\gamma\dot{\gamma}}^{\rho} \partial_{\rho} \right) \left[\sigma_{\alpha\dot{\beta}}^{\mu} A_{\mu} + 2i\epsilon_{\alpha\dot{\beta}} \theta_{\alpha} \tilde{\psi}_2^{\dot{\alpha}} - 2i\tilde{\theta}_{\dot{\beta}} \psi_2^{\alpha} \right. \\
&\quad + 2\tilde{\theta}_{\dot{\beta}} \theta_{\alpha} D + i\sigma_{\alpha\dot{\beta}}^{\mu} \theta \sigma^{\nu} \tilde{\theta} \partial_{\mu} A_{\nu} + i\sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \sigma^{\nu} \epsilon_{\dot{\gamma}'\dot{\beta}} \partial_{\mu} A_{\nu} + \sigma_{\alpha\dot{\beta}}^{\mu} \theta \partial_{\mu} \tilde{\psi}_2 - \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \epsilon_{\dot{\gamma}'\dot{\beta}} \partial_{\mu} \tilde{\psi}_2^{\dot{\gamma}'} \\
&\quad - \frac{1}{2} \theta \sigma^{\mu} \epsilon_{\dot{\gamma}'\dot{\beta}} \sigma_{\alpha\dot{\alpha}}^{\nu} \tilde{\theta}^{\dot{\alpha}} [A_{\mu}, A_{\nu}] + \frac{1}{2} \theta \sigma^{\mu} \tilde{\theta} \sigma_{\alpha\dot{\beta}}^{\nu} [A_{\mu}, A_{\nu}] + i\theta \sigma^{\mu} \epsilon_{\dot{\gamma}'\dot{\beta}} \theta_{\alpha} \tilde{\theta}_{\dot{\alpha}} [A_{\mu}, \tilde{\psi}_2^{\dot{\alpha}}] \\
&\quad + i\theta \sigma^{\mu} \tilde{\theta} \theta_{\alpha} \epsilon_{\dot{\gamma}\dot{\beta}} [A_{\mu}, \tilde{\psi}_2^{\dot{\gamma}}] - \frac{i}{2} \theta \theta \epsilon_{\dot{\alpha}\dot{\beta}} [\tilde{\psi}_2, A_{\mu}] \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} - \frac{i}{2} \theta \tilde{\theta} [\tilde{\psi}_2, A_{\mu}] \sigma_{\alpha\dot{\beta}}^{\mu} \\
&\quad + i\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \partial_{\rho} A_{\mu} + 2\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \theta_{\alpha} \tilde{\theta}_{\dot{\alpha}} \partial_{\rho} \tilde{\psi}_2^{\dot{\alpha}} - \theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \tilde{\theta} \partial_{\rho} \psi_2^{\alpha} - i\theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \tilde{\theta} \theta_{\alpha} \partial_{\rho} D \\
&\quad \left. - \theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \sigma_{\alpha\dot{\alpha}}^{\mu} \tilde{\theta}^{\dot{\alpha}} \theta \sigma^{\nu} \tilde{\theta} \partial_{\rho} \partial_{\mu} A_{\nu} + \frac{i}{2} \theta^{\beta} \sigma_{\beta\dot{\beta}}^{\rho} \theta \sigma^{\mu} \tilde{\theta} \sigma_{\alpha\dot{\alpha}}^{\nu} \tilde{\theta}^{\dot{\alpha}} ([\partial_{\rho} A_{\mu}, A_{\nu}] + [A_{\mu}, \partial_{\rho} A_{\nu}]) \right] .
\end{aligned} \tag{D.4.29}$$

The Super Yang-Mills action is given by

$$\mathcal{L}_{\text{SYM}} = \frac{1}{4\pi} \text{Im} \left(\tau \text{Tr}_R \int d^2\theta W^{\alpha} W_{\alpha} \right) , \tag{D.4.30}$$

where

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2} , \tag{D.4.31}$$

is the complexified coupling and the trace is taken in some representation of the Lie algebra. We take the generators to be antihermitian and normalise the trace as

$$\text{Tr}(T_R^a T_R^b) = C(R) \delta^{ab}, \tag{D.4.32}$$

where $C(R)$ is the index of the representation. In explicit computations we will take the trace in adjoint representation in which case $C(\text{adj}) = h^\vee$ is the dual Coxeter number. So to compute the Lagrangian, we only need terms in the superpotential which has no $\tilde{\theta}$ dependence. This simplifies the calculation a little bit. We get

$$\begin{aligned} \epsilon^{\dot{\gamma}\dot{\beta}} \tilde{D}_{\dot{\gamma}} \tilde{D}_{\dot{\beta}} e^{-V_{N=1}} D_\alpha e^{V_{N=1}} = & -\epsilon^{\dot{\gamma}\dot{\beta}} \left[-2i\epsilon_{\dot{\beta}\dot{\gamma}} \psi_2^\alpha + 2\epsilon_{\dot{\beta}\dot{\gamma}} \theta_\alpha D - i\sigma_{\alpha\dot{\beta}}^\mu \theta \sigma^\nu \epsilon_{\dot{\gamma}} \partial_\mu A_\nu \right. \\ & + \sigma_{\alpha\dot{\beta}}^\mu \theta \theta \epsilon_{\dot{\gamma}} \partial_\mu \tilde{\psi}_2^\alpha - \sigma_{\alpha\dot{\gamma}}^\mu \theta \theta \epsilon_{\dot{\beta}} \partial_\mu \tilde{\psi}_2^\alpha + \frac{1}{2} \theta \sigma^\mu \epsilon_{\dot{\gamma}\dot{\beta}} \sigma_{\alpha\dot{\gamma}}^\nu [A_\mu, A_\nu] - \frac{1}{2} \theta^\beta \sigma_{\beta\dot{\gamma}}^\mu \sigma_{\alpha\dot{\beta}}^\nu [A_\mu, A_\nu] \\ & + i\theta^\gamma \sigma_{\gamma\dot{\beta}}^\mu \theta_\alpha \epsilon_{\dot{\gamma}} [A_\mu, \tilde{\psi}_2^\alpha] - i\theta^\beta \sigma_{\beta\dot{\gamma}}^\mu \theta_\alpha \epsilon_{\dot{\beta}} [A_\mu, \tilde{\psi}_2^\alpha] + \frac{i}{2} \theta \theta \epsilon_{\dot{\alpha}\dot{\beta}} [\tilde{\psi}_2^\alpha, A_\mu] \sigma_{\alpha\dot{\gamma}}^\mu \\ & - \frac{i}{2} \theta \theta \epsilon_{\dot{\alpha}\dot{\gamma}} [\tilde{\psi}_2^\alpha, A_\mu] \sigma_{\alpha\dot{\beta}}^\mu - i\theta^\beta \sigma_{\beta\dot{\beta}}^\rho \sigma_{\alpha\dot{\gamma}}^\mu \partial_\rho A_\mu + 2\theta^\beta \sigma_{\beta\dot{\beta}}^\rho \theta_\alpha \epsilon_{\dot{\gamma}} \partial_\rho \tilde{\psi}_2^\alpha \\ & \left. + i\sigma_{\alpha\dot{\gamma}}^\mu \theta^\beta \sigma_{\beta\dot{\beta}}^\rho \partial_\rho A_\mu + i\theta^\gamma \sigma_{\gamma\dot{\gamma}}^\rho \sigma_{\alpha\dot{\beta}}^\mu \partial_\rho A_\mu + 2\theta^\gamma \sigma_{\gamma\dot{\gamma}}^\rho \theta_\alpha \epsilon_{\dot{\beta}} \partial_\rho \tilde{\psi}_2^\alpha \right] \end{aligned} \quad (\text{D.4.33})$$

This gives the superpotential to be

$$\begin{aligned} W_\alpha = & -\frac{1}{4} \tilde{D} \tilde{D} e^{-V_{N=1}} D_\alpha e^{V_{N=1}} = \frac{1}{4} [-4i\psi_2^\alpha + 4\theta_\alpha D - i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\mu A_\nu \\ & + \sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \tilde{\psi}_2^\beta \theta \theta + \sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \tilde{\psi}_2^\beta \theta \theta - \frac{1}{2} (\sigma^\nu \tilde{\sigma}^\mu \theta)_\alpha [A_\mu, A_\nu] - \frac{1}{2} (\sigma^\nu \tilde{\sigma}^\mu \theta)_\alpha [A_\mu, A_\nu] \\ & + i\theta^\gamma \sigma_{\gamma\dot{\beta}}^\mu \theta_\alpha [A_\mu, \tilde{\psi}_2^\beta] + i\theta^\beta \sigma_{\beta\dot{\gamma}}^\mu \theta_\alpha [A_\mu, \tilde{\psi}_2^\gamma] - \frac{i}{2} [\tilde{\psi}_2^\gamma, A_\mu] \sigma_{\alpha\dot{\gamma}}^\mu \theta \theta - \frac{i}{2} [\tilde{\psi}_2^\gamma, A_\mu] \sigma_{\alpha\dot{\gamma}}^\mu \theta \theta \\ & + i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\nu A_\mu + 2\theta^\beta \sigma_{\beta\dot{\beta}}^\mu \theta_\alpha \partial_\mu \tilde{\psi}_2^\beta + i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\nu A_\mu + 2\theta^\gamma \sigma_{\gamma\dot{\gamma}}^\rho \theta_\alpha \partial_\rho \tilde{\psi}_2^\gamma] \\ & = \frac{1}{4} (-4i\psi_2^\alpha + 4\theta_\alpha D + i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\mu A_\nu - i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\mu A_\nu - i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\nu A_\mu \\ & + (\sigma^\nu \tilde{\sigma}^\mu \theta)_\alpha [A_\mu, A_\nu] + 2\sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \tilde{\psi}_2^\beta \theta \theta + 2i\theta^\beta \sigma_{\beta\dot{\beta}}^\mu \theta_\alpha [A_\mu, \tilde{\psi}_2^\beta] + 4\theta^\beta \sigma_{\beta\dot{\beta}}^\mu \theta_\alpha \partial_\mu \tilde{\psi}_2^\beta \\ & - i\sigma_{\alpha\dot{\gamma}}^\mu [\tilde{\psi}_2^\gamma, A_\mu] \theta \theta - i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha \partial_\mu A_\nu) , \end{aligned} \quad (\text{D.4.34})$$

where we used

$$\begin{aligned} (\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha &= \sigma_{\alpha\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \theta_\gamma = -\sigma_{\alpha\dot{\beta}}^\mu (\sigma^\nu)^{\gamma\dot{\beta}} \theta_\gamma \quad \left((\sigma^\mu)^{\gamma\dot{\beta}} = -(\tilde{\sigma}^\mu)^{\dot{\beta}\gamma} \right) \\ &= -\epsilon^{\gamma\delta} \epsilon^{\dot{\beta}\dot{\gamma}} \sigma_{\alpha\dot{\beta}}^\mu \sigma_{\delta\dot{\gamma}}^\nu \theta_\gamma \\ &= -\epsilon^{\dot{\beta}\dot{\gamma}} \sigma_{\alpha\dot{\beta}}^\mu \sigma_{\delta\dot{\gamma}}^\nu \theta^\delta \\ &= \epsilon^{\dot{\gamma}\dot{\beta}} \sigma_{\alpha\dot{\beta}}^\mu \sigma_{\delta\dot{\gamma}}^\nu \theta^\delta . \end{aligned} \quad (\text{D.4.35})$$

Note that

$$\begin{aligned} \theta^\alpha \theta^\beta &= -\frac{1}{2} \epsilon^{\alpha\beta} \theta \theta \implies \theta^\beta \sigma_{\beta\dot{\beta}}^\mu \theta_\alpha = \theta^\beta \sigma_{\beta\dot{\beta}}^\mu \epsilon_{\alpha\dot{\gamma}} \theta^\gamma \\ &= -\frac{1}{2} \sigma_{\beta\dot{\beta}}^\mu \epsilon_{\alpha\dot{\gamma}} \epsilon^{\beta\gamma} \theta \theta \\ &= \frac{1}{2} \sigma_{\alpha\dot{\beta}}^\mu \theta \theta . \end{aligned} \quad (\text{D.4.36})$$

We get

$$W_\alpha = \frac{1}{4} \left(-4i\psi_2^\alpha + 4\theta_\alpha D - i(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha (2(\partial_\mu A_\nu - \partial_\nu A_\mu) - i[A_\mu, A_\nu]) + 2\sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \tilde{\psi}_2^{\dot{\beta}} \theta\theta \right. \\ \left. + i\sigma_{\alpha\dot{\beta}}^\mu [A_\mu, \tilde{\psi}_2^{\dot{\beta}}] \theta\theta + 2\sigma_{\alpha\dot{\beta}}^\mu \partial_\mu \tilde{\psi}_2^{\dot{\beta}} \theta\theta + i\sigma_{\alpha\dot{\beta}}^\mu [A_\mu, \tilde{\psi}_2^{\dot{\beta}}] \theta\theta \right) \quad (\text{D.4.37})$$

To make the field strength gauge covariant, we scale $V_{\mathcal{N}=1} \longrightarrow 2V_{\mathcal{N}=1}$ and $W_\alpha \longrightarrow \frac{1}{2}W_\alpha$. We then get

$$W_\alpha = -i\psi_2^\alpha + \theta_\alpha D - \frac{i}{2}(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha F_{\mu\nu} + (\sigma^\mu D_\mu \tilde{\psi}_2)_\alpha \theta\theta, \quad (\text{D.4.38})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \quad (\text{D.4.39}) \\ D_\mu \tilde{\psi}_2 = \partial_\mu \tilde{\psi}_2 + i[A_\mu, \tilde{\psi}_2].$$

We can now switch to antihermitian generators for Lie algebra and write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \quad (\text{D.4.40}) \\ D_\mu \tilde{\psi}_2 = \partial_\mu \tilde{\psi}_2 + [A_\mu, \tilde{\psi}_2].$$

We now compute $W^\alpha W_\alpha$. We have

$$W^\alpha W_\alpha = \epsilon^{\alpha\beta} \left(-i\psi_\beta^2 + \theta_\beta D - \frac{i}{2}(\sigma^\mu \tilde{\sigma}^\nu \theta)_\beta F_{\mu\nu} + (\sigma^\mu D_\mu \tilde{\psi}_2)_\beta \theta\theta \right) \\ \cdot \left(-i\psi_\alpha^2 + \theta_\alpha D - \frac{i}{2}(\sigma^\mu \tilde{\sigma}^\nu \theta)_\alpha F_{\mu\nu} + (\sigma^\mu D_\mu \tilde{\psi}_2)_\alpha \theta\theta \right) \\ = -\psi^2 \psi^2 - iD(\theta\psi^2) - \frac{1}{2}F_{\mu\nu}(\sigma^\mu \tilde{\sigma}^\nu \theta\psi^2) - i\sigma^\mu D_\mu \tilde{\psi}_2 \psi^2 \theta\theta - i(\theta\psi^2)D + \theta\theta D^2 \\ - \frac{i}{2}(\sigma^\mu \tilde{\sigma}^\nu \theta)F_{\mu\nu}D - \frac{1}{2}(\psi^2 \sigma^\mu \tilde{\sigma}^\nu \theta)F_{\mu\nu} - \frac{i}{2}(\theta\sigma^\mu \tilde{\sigma}^\nu \theta)DF_{\mu\nu} \\ - \frac{1}{4}(\sigma^\mu \tilde{\sigma}^\nu \theta)(\sigma^\rho \tilde{\sigma}^\lambda \theta)F_{\mu\nu}F_{\rho\lambda} - i(\psi^2 \sigma^\mu D_\mu \tilde{\psi}_2)\theta\theta. \quad (\text{D.4.41})$$

Now we have the following identities

$$\theta_\alpha \theta_\beta = \epsilon_{\alpha\gamma} \epsilon_{\beta\delta} \theta^\delta \theta^\gamma = -\frac{1}{2}\epsilon_{\alpha\gamma} \epsilon^{\gamma\delta} \epsilon_{\beta\delta} \theta\theta = -\frac{1}{2}\epsilon_{\beta\alpha} \theta\theta = \frac{1}{2}\epsilon_{\alpha\beta} \theta\theta. \quad (\text{D.4.42})$$

$$(\sigma^\mu \tilde{\sigma}^\nu \theta)\theta = \epsilon^{\alpha\beta} (\sigma^\mu \tilde{\sigma}^\nu \theta)_\beta \theta_\alpha = \epsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \theta_\gamma \theta_\alpha \\ = \epsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \epsilon_{\gamma\delta} \theta^\delta \epsilon_{\alpha\xi} \theta^\xi \\ = -\frac{1}{2}\epsilon^{\alpha\beta} \epsilon_{\gamma\delta} \epsilon_{\alpha\xi} \epsilon^{\delta\xi} \sigma_{\beta\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} (\theta\theta) \\ = -\frac{1}{2}\epsilon^{\alpha\beta} \epsilon_{\alpha\gamma} \sigma_{\beta\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} (\theta\theta) \\ = \frac{1}{2}(\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \sigma_{\gamma\dot{\beta}}^\mu \theta\theta \\ = -\delta^{\mu\nu} \theta\theta. \quad (\text{D.4.43})$$

$$\begin{aligned}
\theta \sigma^\mu \tilde{\sigma}^\nu \theta &= \theta^\alpha \sigma_{\alpha\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \theta_\gamma = -\frac{1}{2} \epsilon_{\gamma\delta} \epsilon^{\alpha\delta} \sigma_{\alpha\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \theta \theta \\
&= \frac{1}{2} (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \sigma_{\gamma\dot{\beta}}^\mu \theta \theta \\
&= -\delta^{\mu\nu} \theta \theta .
\end{aligned} \tag{D.4.44}$$

$$\begin{aligned}
(\sigma^\mu \tilde{\sigma}^\nu \theta)(\sigma^\rho \tilde{\sigma}^\lambda \theta) &= \epsilon^{\beta\alpha} \sigma_{\alpha\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \theta_\gamma \sigma_{\beta\dot{\gamma}}^\rho (\tilde{\sigma}^\lambda)^{\dot{\gamma}\delta} \theta_\delta \\
&= \frac{1}{2} \epsilon^{\beta\alpha} \epsilon_{\gamma\delta} \sigma_{\alpha\dot{\beta}}^\mu (\tilde{\sigma}^\nu)^{\dot{\beta}\gamma} \sigma_{\beta\dot{\gamma}}^\rho (\tilde{\sigma}^\lambda)^{\dot{\gamma}\delta} \theta \theta \\
&= \frac{1}{2} \epsilon^{\beta\alpha} \epsilon_{\gamma\delta} (-\delta^{\mu\nu} \delta_\gamma^\alpha + 2(\sigma^{\mu\nu})_\gamma^\alpha) \left(-\delta^{\rho\lambda} \delta_\delta^\beta + 2(\sigma^{\rho\lambda})_\delta^\beta \right) \theta \theta \\
&= \frac{1}{2} \left(\delta^{\mu\nu} \delta^{\rho\lambda} - \delta^{\mu\nu} (\sigma^{\rho\lambda})_\beta^\beta \epsilon^{\beta\alpha} \epsilon_{\gamma\alpha} - \delta^{\rho\lambda} (\sigma^{\mu\nu})_\alpha^\alpha \epsilon^{\beta\alpha} \epsilon_{\gamma\delta} \right. \\
&\quad \left. + 2(\sigma^{\mu\nu})_\alpha^\gamma (\sigma^{\rho\lambda})_\beta^\delta \epsilon^{\beta\alpha} \epsilon_{\gamma\delta} \right) \theta \theta .
\end{aligned} \tag{D.4.45}$$

When contracting with $F_{\mu\nu} F_{\rho\lambda}$, only last term gives nonzero result. So we simplify it using

$$\begin{aligned}
(\sigma^{\mu\nu})_\alpha^\gamma (\sigma^{\rho\lambda})_\beta^\delta \epsilon^{\beta\alpha} \epsilon_{\gamma\delta} &= (\sigma^{\mu\nu})_\alpha^\gamma (\sigma^{\rho\lambda})_\gamma^\alpha \\
&= \frac{1}{2} (\epsilon^{\mu\nu\rho\lambda} - \delta^{\mu\rho} \delta^{\nu\lambda} + \delta^{\mu\lambda} \delta^{\nu\rho}) .
\end{aligned} \tag{D.4.46}$$

Thus we get

$$\begin{aligned}
W^\alpha W_\alpha|_{\theta\theta} &= D^2 - \frac{1}{4} (\epsilon^{\mu\nu\rho\lambda} - \delta^{\mu\rho} \delta^{\nu\lambda} + \delta^{\mu\lambda} \delta^{\nu\rho}) F_{\mu\nu} F_{\rho\lambda} - 2i \left(\psi^2 \sigma^\mu D_\mu \tilde{\psi}_2 \right) \\
&= D^2 - \frac{1}{4} (-F_{\mu\nu} F^{\mu\nu} + F_{\mu\nu} F^{\nu\mu}) - 2i \left(\psi^2 \sigma^\mu D_\mu \tilde{\psi}_2 \right) - \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \\
&= D^2 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i \left(\psi^2 \sigma^\mu D_\mu \tilde{\psi}_2 \right) - \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} .
\end{aligned} \tag{D.4.47}$$

We thus get

$$\begin{aligned}
\mathcal{L}_{V_{\mathcal{N}=1}} &= \frac{1}{8\pi} \text{Tr} [\text{Im} (\tau W^\alpha W_\alpha|_{\theta\theta})] \\
&= \frac{1}{8\pi} \text{Tr} \left[\text{Im} \left\{ \left(i \frac{4\pi}{g^2} + \frac{\Theta}{2\pi} \right) \left(D^2 + \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2i (\psi^2 \sigma^\mu D_\mu \tilde{\psi}_2) + \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \right) \right\} \right] \\
&\stackrel{!}{=} \frac{1}{g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^2 - i \psi^2 \sigma^\mu D_\mu \tilde{\psi}_2 + \frac{\Theta g^2}{64\pi^2} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu} F_{\rho\lambda} \right] ,
\end{aligned} \tag{D.4.48}$$

where $\stackrel{!}{=}$ indicates that after taking the imaginary part, the theta term does not survive but we add it by hand with an extra factor of i owing to Wick rotation to get Euclidean action.

The Lagrangian for $\mathcal{N} = 1$ chiral multiplet in the adjoint representation of the gauge group is given by

$$\mathcal{L}_{\Phi\tilde{\Phi}} = \int d^2\theta d^2\tilde{\theta} \tilde{\Phi}_{\mathcal{N}=1} e^{2\rho_{\text{adj}}(V_{\mathcal{N}=1})} \Phi_{\mathcal{N}=1} \tag{D.4.49}$$

where ρ_{adj} is the adjoint representation of the Lie algebra. We will omit it from now on. We begin by computing the exponential. We have

$$e^{2V_{\mathcal{N}=1}} = 1 + 2V_{\mathcal{N}=1} + 2V_{\mathcal{N}=1}^2 . \tag{D.4.50}$$

Next, we compute

$$\begin{aligned}
V_{\mathcal{N}=1}^2 &= \left(-\theta\sigma^\mu\tilde{\theta}A_\mu + i\theta\theta\tilde{\theta}\tilde{\psi}_2 - i\tilde{\theta}\theta\theta\psi^2 + \frac{1}{2}\theta\theta\tilde{\theta}D \right) \left(-\theta\sigma^\nu\tilde{\theta}A_\nu + i\theta\theta\tilde{\theta}\tilde{\psi}_2 - i\tilde{\theta}\theta\theta\psi^2 + \frac{1}{2}\theta\theta\tilde{\theta}D \right) \\
&= (\theta\sigma^\mu\tilde{\theta})(\theta\sigma^\nu\tilde{\theta})A_\mu A_\nu \\
&= -\frac{1}{2}A_\mu A^\mu\theta\theta\tilde{\theta}\tilde{\theta} ,
\end{aligned} \tag{D.4.51}$$

where we used:

$$\begin{aligned}
(\theta\sigma^\mu\tilde{\theta})(\theta\sigma^\nu\tilde{\theta}) &= \theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\tilde{\theta}^{\dot{\alpha}}\theta^\beta\sigma_{\beta\dot{\beta}}^\nu\tilde{\theta}^{\dot{\beta}} \\
&= -\frac{1}{4}\sigma_{\alpha\dot{\alpha}}^\mu\sigma_{\beta\dot{\beta}}^\nu\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\theta\theta\tilde{\theta}\tilde{\theta} \\
&= \frac{1}{4}(\sigma^\mu)^{\dot{\beta}\beta}(\sigma^\nu)_{\beta\dot{\beta}}\theta\theta\tilde{\theta}\tilde{\theta} \\
&= -\frac{1}{4}(\tilde{\sigma}^\mu)^{\dot{\beta}\beta}(\sigma^\nu)_{\beta\dot{\beta}}\theta\theta\tilde{\theta}\tilde{\theta} \\
&= \frac{1}{2}\delta^{\mu\nu}\theta\theta\tilde{\theta}\tilde{\theta} .
\end{aligned} \tag{D.4.52}$$

Next

$$\begin{aligned}
\tilde{\Phi}_{\mathcal{N}=1}\Phi_{\mathcal{N}=1}\Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} &= \left(\tilde{\phi} - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{\phi} - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\Box\tilde{\phi} + \sqrt{2}\theta\tilde{\psi}_1 + \frac{i}{\sqrt{2}}\tilde{\theta}\theta\sigma^\mu\partial_\mu\tilde{\psi}_1 + \tilde{\theta}\tilde{\theta}\tilde{F} \right) \\
&\quad \cdot \left(\phi + i\theta\sigma^\nu\tilde{\theta}\partial_\nu\phi - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\Box\phi + \sqrt{2}\theta\psi^1 - \frac{i}{\sqrt{2}}\theta\theta(\partial_\nu\psi^1\sigma^\nu\tilde{\theta}) + \theta\theta F \right)\Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} \\
&= -\frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\tilde{\phi}\Box\phi + (\theta\sigma^\mu\tilde{\theta})(\theta\sigma^\nu\tilde{\theta})\partial_\mu\tilde{\phi}\partial_\nu\phi - \frac{1}{4}(\theta\theta\tilde{\theta}\tilde{\theta})\Box\tilde{\phi}\phi \\
&\quad - i\tilde{\theta}\tilde{\psi}_1\theta\theta\partial_\nu(\psi^1\sigma^\nu\tilde{\theta}) + \frac{i}{\sqrt{2}}\tilde{\theta}\theta\sigma^\mu\partial_\mu\tilde{\psi}_1\sqrt{2}\theta\psi^1 + \tilde{\theta}\tilde{\theta}\tilde{F}\theta\theta F \\
&= -\frac{1}{4}(\tilde{\phi}\Box\phi + \Box\tilde{\phi}\phi) + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\phi - \frac{i}{2}\tilde{\psi}_1^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\nu\partial_\nu\psi^{1\alpha} \\
&\quad + \frac{i}{2}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\tilde{\psi}_1^{\dot{\alpha}}\psi^{1\alpha} + \tilde{F}F ,
\end{aligned} \tag{D.4.53}$$

where we used

$$\begin{aligned}
\tilde{\psi}_1\partial_\nu\psi^1\sigma^\nu\tilde{\theta} &= \tilde{\psi}_{1\dot{\alpha}}\partial_\nu\psi^{1\alpha}\sigma_{\alpha\dot{\beta}}^\nu\tilde{\theta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\beta}\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\gamma}}\tilde{\psi}_1^{\dot{\gamma}}\partial_\nu\psi^{1\alpha}\sigma_{\alpha\dot{\beta}}^\nu\tilde{\theta}^{\dot{\beta}} \\
&= \frac{1}{2}\tilde{\psi}_1^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\nu\partial_\nu\psi^{1\alpha}\tilde{\theta}^{\dot{\beta}} ,
\end{aligned} \tag{D.4.54}$$

and

$$\begin{aligned}
\theta\sigma^\mu\partial_\mu\tilde{\psi}_1\theta\psi^1 &= \theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\tilde{\psi}_1^{\dot{\alpha}}\theta^\beta\psi_\beta^1 = \frac{1}{2}\epsilon^{\alpha\beta}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\tilde{\psi}_1^{\dot{\alpha}}\psi_\beta^1\theta\theta \\
&= \frac{1}{2}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\tilde{\psi}_1^{\dot{\alpha}}\psi^{1\alpha}\theta\theta .
\end{aligned} \tag{D.4.55}$$

For the next term, we will only write the $\theta\theta\tilde{\theta}\tilde{\theta}$ term

$$\begin{aligned}
\tilde{\Phi}_{\mathcal{N}=1} V_{\mathcal{N}=1} \Phi_{\mathcal{N}=1} \Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} &= \left(\tilde{\phi} - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{\phi} - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{\phi} + \sqrt{2}\tilde{\theta}\tilde{\psi}_1 + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\psi}_1 + \tilde{\theta}\tilde{\theta}\tilde{F} \right) \\
&\quad \cdot \left(-\theta\sigma^\mu\tilde{\theta}A_\mu + i\theta\theta\tilde{\theta}\tilde{\psi}_2 - i\tilde{\theta}\tilde{\theta}\theta\psi^2 + \frac{1}{2}\theta\theta\tilde{\theta}\tilde{\theta}D \right) \\
&\quad \cdot \left(\phi + i\theta\sigma^\nu\tilde{\theta}\partial_\nu\phi - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\phi + \sqrt{2}\theta\psi^1 - \frac{i}{\sqrt{2}}\theta\theta(\partial_\nu\psi^1\sigma^\nu\tilde{\theta}) + \theta\theta F \right) \Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} \\
&= \left(-\tilde{\phi}(\theta\sigma^\mu\tilde{\theta})A_\mu i(\theta\sigma^\nu\tilde{\theta})\partial_\nu\phi - i\tilde{\phi}\tilde{\theta}\tilde{\theta}\theta\psi^2\sqrt{2}\theta\psi^1 + \frac{1}{2}\theta\theta\tilde{\theta}\tilde{\theta}\tilde{\phi}\phi D \right. \\
&\quad \left. - i(\theta\sigma^\mu\tilde{\theta})\partial_\mu\tilde{\phi} \left(-\theta\sigma^\nu\tilde{\theta}A_\nu \right) \phi + \sqrt{2}\tilde{\theta}\tilde{\psi}_1 \left(-\theta\sigma^\mu\tilde{\theta}A_\mu \right) \sqrt{2}\theta\psi^1 + \sqrt{2}\tilde{\theta}\tilde{\psi}_1 i\theta\theta\tilde{\theta}\tilde{\psi}_2\phi \right) \Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} \\
&= \left(\frac{i}{2}\tilde{\phi}A_\mu\partial^\mu\phi\theta\theta\tilde{\theta}\tilde{\theta} - i\sqrt{2}\tilde{\phi}\tilde{\theta}\tilde{\theta}\theta^\alpha\psi_\alpha^2\theta^\beta\psi_\beta^1 + \frac{1}{2}\tilde{\phi}D\phi\theta\theta\tilde{\theta}\tilde{\theta} - \frac{i}{2}\partial_\mu\tilde{\phi}A^\mu\phi\theta\theta\tilde{\theta}\tilde{\theta} \right. \\
&\quad \left. - 2\tilde{\theta}_\alpha\tilde{\psi}_1^{\dot{\alpha}}\theta^\alpha\sigma_{\alpha\beta}^\mu\tilde{\theta}^{\dot{\beta}}A_\mu\theta^\gamma\psi_\gamma^1 + \sqrt{2}i\tilde{\theta}_\alpha\tilde{\psi}_1^{\dot{\alpha}}\theta\theta\tilde{\theta}^{\dot{\beta}}\tilde{\psi}_{2\dot{\beta}}\phi \right) \Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} \\
&= \frac{i}{2}\tilde{\phi}A_\mu\partial^\mu\phi - \frac{i\sqrt{2}}{2}\tilde{\phi}\epsilon^{\alpha\beta}\psi_\alpha^2\psi_\beta^1 + \frac{1}{2}\tilde{\phi}D\phi - \frac{i}{2}\partial_\mu\tilde{\phi}A^\mu\phi \\
&\quad \left[-2\epsilon^{\dot{\beta}\dot{\alpha}}\tilde{\psi}_{1\dot{\alpha}}\sigma_{\alpha\beta}^\mu A_\mu\psi_\gamma^1 \left(\frac{1}{2}\epsilon^{\alpha\gamma}\theta\theta \right) \left(-\frac{1}{2}\epsilon_{\epsilon\lambda}\tilde{\theta}\tilde{\theta} \right) - \sqrt{2}i\tilde{\psi}_1^{\dot{\alpha}}\tilde{\psi}_{2\dot{\alpha}}\phi \left(\frac{1}{2}\epsilon_{\alpha\beta}\tilde{\theta}\tilde{\theta}\theta\theta \right) \right] \Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} \\
&= \frac{i}{2}\tilde{\phi}A_\mu\partial^\mu\phi - \frac{i\sqrt{2}}{2}\tilde{\phi}\epsilon^{\alpha\beta}\psi_\alpha^2\psi_\beta^1 + \frac{1}{2}\tilde{\phi}D\phi - \frac{i}{2}\partial_\mu\tilde{\phi}A^\mu\phi - \frac{1}{2}\tilde{\psi}_1^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu A_\mu\psi^{1\alpha} + \frac{1}{\sqrt{2}}i\tilde{\psi}_1\tilde{\psi}_2\phi .
\end{aligned} \tag{D.4.56}$$

Finally

$$\tilde{\Phi}_{\mathcal{N}=1} V_{\mathcal{N}=1}^2 \Phi_{\mathcal{N}=1} \Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} = -\frac{1}{2}\tilde{\phi}A_\mu A^\mu\phi . \tag{D.4.57}$$

Thus the full Lagrangian is

$$\begin{aligned}
\mathcal{L}_{\Phi\tilde{\Phi}} &= \text{Tr} \left[-\frac{1}{4}(\tilde{\phi}\square\phi + \square\tilde{\phi}\phi) + \frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\phi - \frac{i}{2}\tilde{\psi}_1^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\nu\partial_\nu\psi^{1\alpha} + \frac{i}{2}\sigma_{\alpha\dot{\alpha}}^\mu\partial_\mu\tilde{\psi}_1^{\dot{\alpha}}\psi^{1\alpha} \right. \\
&\quad \left. + \tilde{F}F + 2 \left(\frac{i}{2}\tilde{\phi}A_\mu\partial^\mu\phi - \frac{i\sqrt{2}}{2}\tilde{\phi}\epsilon^{\alpha\beta}\psi_\alpha^2\psi_\beta^1 + \frac{1}{2}\tilde{\phi}D\phi - \frac{i}{2}\partial_\mu\tilde{\phi}A^\mu\phi \right. \right. \\
&\quad \left. \left. - \frac{1}{2}\tilde{\psi}_1^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu A_\mu\psi^{1\alpha} + \frac{1}{\sqrt{2}}i\tilde{\psi}_1\tilde{\psi}_2\phi - \frac{1}{2}\tilde{\phi}A_\mu A^\mu\phi \right) \right] \\
&= \text{Tr} \left[\partial_\mu\tilde{\phi}\partial^\mu\phi + i\tilde{\phi}A_\mu\partial^\mu\phi - i\partial_\mu\tilde{\phi}A^\mu\phi - \tilde{\phi}A_\mu A^\mu\phi + \tilde{\phi}D\phi + \tilde{F}F + \sqrt{2}i\tilde{\phi}\psi^2\psi^1 \right. \\
&\quad \left. + \sqrt{2}i\tilde{\psi}_1\tilde{\psi}_2\phi + \frac{i}{2}\partial_\mu\tilde{\psi}_1^{\dot{\alpha}}\sigma_{\alpha\dot{\alpha}}^\mu\psi^{1\alpha} + \frac{i}{2}\partial_\mu\tilde{\psi}_1\sigma^\mu\psi^1 - \tilde{\psi}_1^{\dot{\beta}}\sigma_{\alpha\dot{\beta}}^\mu A_\mu\psi^{1\alpha} \right] \\
&= \text{Tr} \left[D_\mu\tilde{\phi}D^\mu\phi + D \cdot [\phi, \tilde{\phi}] + \tilde{F}F - i\tilde{\psi}^1\sigma^\mu D_\mu\psi_1 - \frac{i}{\sqrt{2}}\psi^1[\tilde{\phi}, \psi^2] \right. \\
&\quad \left. + \frac{i}{\sqrt{2}}\tilde{\psi}_1[\tilde{\psi}_2, \phi] \right] ,
\end{aligned} \tag{D.4.58}$$

where we dropped some total derivatives owing to fact that we will work on closed 4-manifolds. We also used the fact that $V_{\mathcal{N}=1}$ acts on $\Phi, \tilde{\Phi}$ in the adjoint representation so that

$$\begin{aligned}
\text{Tr}(\tilde{\phi} \cdot D \cdot \phi) &= \text{Tr}(\tilde{\phi}[D, \phi]) = \text{Tr}(\tilde{\phi}D\phi - \tilde{\phi}\phi D) \\
&= \text{Tr}(\phi\tilde{\phi}D - \tilde{\phi}\phi D) = \text{Tr}(D \cdot [\phi, \tilde{\phi}]) .
\end{aligned} \tag{D.4.59}$$

Also

$$\begin{aligned}
\text{Tr} \left(\tilde{\phi} \psi^2 \cdot \psi^1 \right) &= \text{Tr} \left(\tilde{\phi} [\psi^2, \psi^1] \right) \\
&= \text{Tr} \left(\tilde{\phi}^c T^c \psi^{2a} \psi^{1b} [T^a, T^b] \right) \\
&= \text{Tr} \left(\tilde{\phi}^c \psi^{1b} \psi^{2a} (T^c T^a T^b - T^c T^b T^a) \right) \\
&= \text{Tr} \left(\psi^1 (\tilde{\phi}^c \psi^{2a} T^b T^c T^a - \tilde{\phi}^c \psi^{2a} T^a T^b T^c) \right) \\
&= \text{Tr} \left(\psi^1 [\tilde{\phi}, \psi^2] \right) .
\end{aligned} \tag{D.4.60}$$

Similarly

$$\begin{aligned}
\text{Tr} \left(\tilde{\phi} \psi^2 \cdot \psi^1 \right) &= \text{Tr} \left(\tilde{\phi}^c \psi^{1b} \psi^{2a} (T^c T^a T^b - T^c T^b T^a) \right) \\
&= \text{Tr} \left(\psi^2 \tilde{\phi}^c \psi^{1b} (T^b T^c - T^c T^b) \right) \\
&= -\text{Tr} \left(\psi^2 [\tilde{\phi}, \psi^1] \right)
\end{aligned} \tag{D.4.61}$$

Also

$$\begin{aligned}
\text{Tr} \left(\tilde{\psi}_1 [\tilde{\psi}_2, \phi] \right) &= \text{Tr} \left(\tilde{\psi}_1^a \tilde{\psi}_2^b \phi^c T^a (T^b T^c - T^c T^b) \right) \\
&= \text{Tr} \left(\tilde{\psi}_2 \tilde{\psi}_1^a \phi^c (T^c T^a - T^a T^c) \right) \\
&= \text{Tr} \left(\tilde{\psi}_2 [\phi, \tilde{\psi}_1] \right) .
\end{aligned} \tag{D.4.62}$$

Rescaling $\Phi_{\mathcal{N}=1} \longrightarrow \frac{1}{g} \Phi_{\mathcal{N}=1}$, the full Lagrangian can be written as

$$\begin{aligned}
\mathcal{L}_{\text{SYM}} &= \mathcal{L}_{V_{\mathcal{N}=1}} + \mathcal{L}_{\Phi \tilde{\Phi}} \\
&= \frac{1}{g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi - i \psi^1 \sigma^\mu D_\mu \tilde{\psi}_1 - i \psi^2 \sigma^\mu D_\mu \tilde{\psi}_2 + \tilde{F} F + \frac{1}{2} D^2 \right. \\
&\quad \left. + D[\phi, \tilde{\phi}] + \frac{i}{\sqrt{2}} \psi^1 [\tilde{\phi}, \psi^2] - \frac{i}{\sqrt{2}} \psi^2 [\tilde{\phi}, \psi^1] + \frac{i}{\sqrt{2}} \tilde{\psi}_1 [\tilde{\psi}_2, \phi] - \frac{i}{\sqrt{2}} \tilde{\psi}_2 [\tilde{\psi}_1, \phi] \right] \\
&\quad + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\lambda} \text{Tr} (F_{\mu\nu} F_{\rho\lambda}) \\
&= \frac{1}{g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi - i \psi^I \sigma^\mu D_\mu \tilde{\psi}_I - i \psi^2 \sigma^\mu D_\mu \tilde{\psi}_2 + \tilde{F} F + \frac{1}{2} D^2 \right. \\
&\quad \left. + D[\phi, \tilde{\phi}] - \frac{i}{\sqrt{2}} \psi_I [\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}} \tilde{\psi}_I [\phi, \tilde{\psi}^I] \right] + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\lambda} \text{Tr} (F_{\mu\nu} F_{\rho\lambda})
\end{aligned} \tag{D.4.63}$$

If we now define

$$D_{IJ} := \begin{pmatrix} \sqrt{2} \tilde{F} & iD + i[\phi, \tilde{\phi}] \\ iD + i[\phi, \tilde{\phi}] & \sqrt{2} F \end{pmatrix} , \tag{D.4.64}$$

then using

$$\begin{aligned}
D^{11} &= \epsilon^{1I} \epsilon^{1J} D_{IJ} = D_{22}, \quad D^{22} = \epsilon^{2I} \epsilon^{2J} D_{IJ} = D_{11} \\
D^{12} &= \epsilon^{1I} \epsilon^{2J} D_{IJ} = -D_{21} ,
\end{aligned} \tag{D.4.65}$$

we get

$$\begin{aligned}
\text{Tr} (D_{IJ} D^{IJ}) &= \text{Tr} (D_{11} D^{11} + D_{22} D^{22} + 2D_{12} D^{12}) \\
&= \text{Tr} \left(2\tilde{F}F + 2F\tilde{F} + 2 \left(D + [\phi, \tilde{\phi}] \right)^2 \right) \\
&= 4 \text{Tr} (\tilde{F}F) + 2 \text{Tr} D^2 + 2 [\phi, \tilde{\phi}]^2 + 2 \text{Tr} (D [\phi, \tilde{\phi}]) .
\end{aligned} \tag{D.4.66}$$

The Lagrangian can then be written as

$$\begin{aligned}
\mathcal{L}_{\text{SYM}} &= \frac{1}{g^2} \text{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi + i \tilde{\psi}_I \tilde{\sigma}^\mu D_\mu \psi^I + \frac{1}{4} D_{IJ} D^{IJ} - \frac{1}{2} [\phi, \tilde{\phi}]^2 \right. \\
&\quad \left. - \frac{i}{\sqrt{2}} \psi_I [\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}} \tilde{\psi}^I [\phi, \tilde{\psi}_I] \right] + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\lambda} \text{Tr} (F_{\mu\nu} F_{\rho\lambda}) ,
\end{aligned} \tag{D.4.67}$$

where we used

$$\begin{aligned}
\text{Tr} \left(\psi^{1\alpha} \sigma_{\alpha\beta}^\mu D_\mu \tilde{\psi}_1^{\dot{\beta}} \right) &= \text{Tr} \left(\psi^{1\alpha} \sigma_{\alpha\beta}^\mu \left(\partial_\mu \tilde{\psi}_1^{\dot{\beta}} + [A_\mu, \tilde{\psi}_1^{\dot{\beta}}] \right) \right) \\
&= \text{Tr} \left(-\tilde{\psi}_1^{\dot{\beta}} \tilde{\sigma}_{\beta\alpha}^\mu \partial_\mu \psi^{1\alpha} + \psi^{1\alpha} \sigma_{\alpha\beta}^\mu A_\mu \tilde{\psi}_1^{\dot{\beta}} - \psi^{1\alpha} \sigma_{\alpha\beta}^\mu \tilde{\psi}_1^{\dot{\beta}} A_\mu \right) \\
&= \text{Tr} \left(-\tilde{\psi}_1^{\dot{\beta}} \tilde{\sigma}_{\beta\alpha}^\mu \partial_\mu \psi^{1\alpha} + \tilde{\psi}_1^{\dot{\beta}} \tilde{\sigma}_{\beta\alpha}^\mu A_\mu \psi^{1\alpha} - \tilde{\psi}_1^{\dot{\beta}} \tilde{\sigma}_{\beta\alpha}^\mu \psi^{1\alpha} A_\mu \right) \\
&= -\text{Tr} \left(\tilde{\psi}_1^{\dot{\beta}} \tilde{\sigma}_{\beta\alpha}^\mu D_\mu \psi^{1\alpha} \right) .
\end{aligned} \tag{D.4.68}$$

Hypermultiplet

We now calculate the Lagrangian for a single flavor of hypermultiplet, generalization to arbitrary flavor is straightforward. The Lagrangian in terms of superfields is given by

$$\begin{aligned}
\mathcal{L}_{\text{Hyper}} &= \int d^2\theta d^2\tilde{\theta} \left[\tilde{Q} e^{2V_{\mathcal{N}=1}} Q + Q' e^{2\bar{V}_{\mathcal{N}=1}} \tilde{Q}' \right] \\
&\quad + \int d^2\theta \left[\sqrt{2} Q' \Phi_{\mathcal{N}=1} Q + \bar{m} Q' Q \right] \\
&\quad + \int d^2\tilde{\theta} \left[\sqrt{2} \tilde{Q} \tilde{\Phi}_{\mathcal{N}=1} \tilde{Q}' + m \tilde{Q} \tilde{Q}' \right]
\end{aligned} \tag{D.4.69}$$

where

$$\begin{aligned}
Q &= q^1 + i\theta\sigma^\mu\tilde{\theta}\partial_\mu q^1 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square q^1 + \sqrt{2}\theta\lambda - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\lambda\sigma^\mu\tilde{\theta} + \theta\theta F^1 \\
Q' &= -i \left(\tilde{q}_1 + i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{q}_1 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{q}_1 + \sqrt{2}\theta\chi - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\chi\sigma^\mu\tilde{\theta} + \theta\theta\tilde{F}_2 \right) \\
\tilde{Q} &= \tilde{q}_2 - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{q}_2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{q}_2 + \sqrt{2}\tilde{\theta}\tilde{\lambda} + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\lambda} + \tilde{\theta}\tilde{\theta}\tilde{F}_1 \\
\tilde{Q}' &= +i \left(q^2 - i\theta\sigma^\mu\tilde{\theta}\partial_\mu q^2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square q^2 + \sqrt{2}\tilde{\theta}\tilde{\chi} + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\chi} + \tilde{\theta}\tilde{\theta}F^2 \right)
\end{aligned} \tag{D.4.70}$$

The $\mathcal{N} = 1$ chiral superfields $Q, \tilde{Q}$ transform in some representation R of the gauge group and $Q', \tilde{Q}'$ transform in $\bar{R}$ of the gauge group. The first two terms can be obtained using the calculations

above. We have

$$\begin{aligned}\tilde{Q}e^{2V_{\mathcal{N}=1}}Q\Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} &= D_\mu\tilde{q}_2D^\mu q^1 + \tilde{q}_2Dq^1 + \tilde{F}_1F^1 - i\lambda\sigma^\mu D_\mu\tilde{\lambda} + i\sqrt{2}\tilde{q}_2\psi^\lambda\lambda + i\sqrt{2}\tilde{\lambda}\tilde{\psi}_2q^1, \\ Q'e^{-2\bar{V}_{\mathcal{N}=1}}\tilde{Q}'\Big|_{\theta\theta\tilde{\theta}\tilde{\theta}} &= D_\mu\tilde{q}_1D^\mu q^2 - \tilde{q}_1Dq^2 + \tilde{F}_2F^2 - i\chi\sigma^\mu D_\mu\tilde{\chi} - i\sqrt{2}\tilde{q}_1\tilde{\psi}_2\tilde{\chi} - i\sqrt{2}\chi\psi^2q^2.\end{aligned}\tag{D.4.71}$$

Next we calculate,

$$\begin{aligned}Q'\Phi_{\mathcal{N}=1}Q\Big|_{\theta\theta} &= -i\left(\tilde{q}_1 + i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{q}_1 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{q}_1 + \sqrt{2}\theta\chi - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\chi\sigma^\mu\tilde{\theta} + \theta\theta\tilde{F}_2\right) \\ &\quad \cdot \left(\phi + i\theta\sigma^\mu\tilde{\theta}\partial_\mu\phi - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\phi + \sqrt{2}\theta\psi^1 - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\psi^1\sigma^\mu\tilde{\theta} + \theta\theta F\right) \\ &\quad \cdot \left(q^1 + i\theta\sigma^\mu\tilde{\theta}\partial_\mu q^1 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square q^1 + \sqrt{2}\theta\lambda - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\lambda\sigma^\mu\tilde{\theta} + \theta\theta F^1\right)\Big|_{\theta\theta} \\ &= \left(-i\tilde{q}_1\sqrt{2}\theta\psi^1\sqrt{2}\theta\lambda - i\tilde{q}_1\theta\theta Fq^1 - i\tilde{q}_1\phi\theta\theta F^1 - i\sqrt{2}\theta\chi\sqrt{2}\theta\psi^1q^1 - i\sqrt{2}\theta\chi\phi\sqrt{2}\theta\lambda \right. \\ &\quad \left. - i\theta\theta\tilde{F}_2\phi q^1\right)\Big|_{\theta\theta} \\ &= -\left(2i\tilde{q}_1\theta^\alpha\psi_\alpha^1\theta^\beta\lambda_\beta + 2i\theta^\alpha\chi_\alpha\theta^\beta\psi_\beta^1q^1 + 2i\theta^\alpha\chi_\alpha\phi\theta^\beta\lambda_\beta\right)\Big|_{\theta\theta} \\ &\quad - i\tilde{q}_1Fq^1 - i\tilde{q}_1\phi F^1 - i\tilde{F}_2\phi q^1 \\ &= -i\tilde{q}_1\psi_\alpha^1\lambda_\beta\epsilon^{\alpha\beta} - i\chi_\alpha\psi_\beta^1\epsilon^{\alpha\beta}q^1 - i\chi_\alpha\phi\lambda_\beta\epsilon^{\alpha\beta} - i\tilde{q}_1Fq^1 - i\tilde{q}_1\phi F^1 - i\tilde{F}_2\phi q^1 \\ &= i\tilde{q}_1\psi^1\lambda + i\chi\psi^1q^1 + i\chi\phi\lambda - i\tilde{q}_1Fq^1 - i\tilde{q}_1\phi F^1 - i\tilde{F}_2\phi q^1\end{aligned}\tag{D.4.72}$$

Similarly

$$\begin{aligned}\tilde{Q}\tilde{\Phi}_{\mathcal{N}=1}\tilde{Q}'\Big|_{\tilde{\theta}\tilde{\theta}} &= +i\left(\tilde{q}_2 - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{q}_2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{q}_2 + \sqrt{2}\tilde{\theta}\tilde{\lambda} + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\lambda} + \tilde{\theta}\tilde{\theta}\tilde{F}_1\right) \\ &\quad \cdot \left(\tilde{\phi} - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{\phi} - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{\phi} + \sqrt{2}\tilde{\theta}\tilde{\psi}_1 + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\psi}_1 + \tilde{\theta}\tilde{\theta}\tilde{F}\right) \\ &\quad \cdot \left(q^2 - i\theta\sigma^\mu\tilde{\theta}\partial_\mu q^2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square q^2 + \sqrt{2}\tilde{\theta}\tilde{\chi} + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\chi} + \tilde{\theta}\tilde{\theta}\tilde{F}^2\right)\Big|_{\tilde{\theta}\tilde{\theta}} \\ &= \left(i\sqrt{2}\tilde{q}_2\tilde{\theta}\tilde{\psi}_1\sqrt{2}\tilde{\theta}\tilde{\chi} + \sqrt{2}i\tilde{\theta}\tilde{\lambda}\sqrt{2}\tilde{\theta}\tilde{\psi}_1q^2 + \sqrt{2}i\tilde{\theta}\tilde{\lambda}\tilde{\phi}\sqrt{2}\tilde{\theta}\tilde{\chi}\right)\Big|_{\tilde{\theta}\tilde{\theta}} \\ &\quad + i\tilde{q}_2\tilde{\phi}\tilde{F}^2 + i\tilde{q}_2\tilde{F}q^2 + i\tilde{F}_1\tilde{\phi}q^2 \\ &= 2i\tilde{q}_2\tilde{\theta}_{\dot{\alpha}}\tilde{\psi}_1^{\dot{\alpha}}\tilde{\theta}_{\dot{\beta}}\tilde{\chi}^{\dot{\beta}} + 2i\tilde{\theta}_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}\tilde{\theta}_{\dot{\beta}}\tilde{\psi}_1^{\dot{\beta}}q^2 + 2i\tilde{\theta}_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}\tilde{\phi}\tilde{\theta}_{\dot{\beta}}\tilde{\chi}^{\dot{\beta}}\Big|_{\tilde{\theta}\tilde{\theta}} \\ &\quad + i\tilde{q}_2\tilde{\phi}\tilde{F}^2 + i\tilde{q}_2\tilde{F}q^2 + i\tilde{F}_1\tilde{\phi}q^2 \\ &= -i\tilde{q}_2\tilde{\psi}_1^{\dot{\alpha}}\tilde{\chi}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}} - i\tilde{\lambda}^{\dot{\alpha}}\tilde{\psi}_1^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}q^2 - i\tilde{\lambda}^{\dot{\alpha}}\tilde{\phi}\tilde{\chi}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}} + i\tilde{q}_2\tilde{\phi}\tilde{F}^2 + i\tilde{q}_2\tilde{F}q^2 + i\tilde{F}_1\tilde{\phi}q^2 \\ &= i\tilde{q}_2\tilde{\psi}_1\tilde{\chi} + i\tilde{\lambda}\tilde{\psi}_1q^2 + i\tilde{\lambda}\tilde{\phi}\tilde{\chi} + i\tilde{q}_2\tilde{\phi}\tilde{F}^2 + i\tilde{q}_2\tilde{F}q^2 + i\tilde{F}_1\tilde{\phi}q^2.\end{aligned}\tag{D.4.73}$$

Next

$$\begin{aligned}
Q'Q|_{\theta\theta} &= -i \left(\tilde{q}_1 + i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{q}_1 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{q}_1 + \sqrt{2}\theta\chi - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\chi\sigma^\mu\tilde{\theta} + \theta\theta\tilde{F}_2 \right) \\
&\quad \cdot \left(q^2 + i\theta\sigma^\mu\tilde{\theta}\partial_\mu q^2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square q^2 + \sqrt{2}\theta\lambda - \frac{i}{\sqrt{2}}\theta\theta\partial_\mu\lambda\sigma^\mu\tilde{\theta} + \theta\theta F^1 \right) \Big|_{\theta\theta} \\
&= - \left(i\tilde{q}_1 F^1 + i\tilde{F}_2 q^2 + 2i\theta^\alpha\chi_\alpha\theta^\beta\lambda_\beta \right) \Big|_{\theta\theta} \\
&= -i\tilde{q}_1 F^1 - i\tilde{F}_2 q^2 - i\epsilon^{\alpha\beta}\chi_\alpha\lambda_\beta \\
&= -i\tilde{q}_1 F^1 - i\tilde{F}_2 q^2 + i\chi\lambda,
\end{aligned} \tag{D.4.74}$$

and

$$\begin{aligned}
\tilde{Q}\tilde{Q}'|_{\bar{\theta}\bar{\theta}} &= +i \left(\tilde{q}_2 - i\theta\sigma^\mu\tilde{\theta}\partial_\mu\tilde{q}_2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square\tilde{q}_2 + \sqrt{2}\tilde{\theta}\tilde{\lambda} + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\lambda} + \tilde{\theta}\tilde{\theta}\tilde{F}_1 \right) \\
&\quad \cdot \left(q^2 - i\theta\sigma^\mu\tilde{\theta}\partial_\mu q^2 - \frac{1}{4}\theta\theta\tilde{\theta}\tilde{\theta}\square q^2 + \sqrt{2}\tilde{\theta}\tilde{\chi} + \frac{i}{\sqrt{2}}\tilde{\theta}\tilde{\theta}\sigma^\mu\partial_\mu\tilde{\chi} + \tilde{\theta}\tilde{\theta}F^2 \right) \Big|_{\bar{\theta}\bar{\theta}} \\
&= i\tilde{q}_2 F^2 + i\tilde{F}_1 q^2 + 2i\tilde{\theta}_\alpha\tilde{\lambda}^{\dot{\alpha}}\tilde{\theta}_\beta\tilde{\chi}^{\dot{\beta}} \Big|_{\bar{\theta}\bar{\theta}} \\
&= i\tilde{q}_2 F^2 + i\tilde{F}_1 q^2 - i\tilde{\lambda}^{\dot{\alpha}}\tilde{\chi}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}} \\
&= i\tilde{q}_2 F^2 + i\tilde{F}_1 q^2 + i\tilde{\lambda}\tilde{\chi}.
\end{aligned} \tag{D.4.75}$$

Thus the full $\mathcal{N} = 2$ Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=2} &= \frac{1}{g^2} \text{Tr} \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\tilde{\phi}D^\mu\phi + i\tilde{\psi}_I\tilde{\sigma}^\mu D_\mu\psi^I + \tilde{F}F + \frac{1}{2}D^2 + D[\phi, \tilde{\phi}] \right. \\
&\quad \left. - \frac{i}{\sqrt{2}}\psi_I[\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}}\tilde{\psi}^I[\phi, \tilde{\psi}_I] \right] + \frac{\Theta i}{64\pi^2}\epsilon^{\mu\nu\rho\lambda}\text{Tr}(F_{\mu\nu}F_{\rho\lambda}) \\
&\quad + D_\mu\tilde{q}_2D^\mu q^1 + \tilde{q}_2Dq^1 + \tilde{F}_1F^1 - i\lambda\sigma^\mu D_\mu\tilde{\lambda} + i\sqrt{2}\tilde{q}_2\psi^2\lambda + i\sqrt{2}\tilde{\lambda}\tilde{\psi}_2q^1 \\
&\quad + D_\mu\tilde{q}_1D^\mu q^2 - \tilde{q}_1Dq^2 + \tilde{F}_2F^2 - i\chi\sigma^\mu D_\mu\tilde{\chi} + i\sqrt{2}\tilde{q}_1\tilde{\psi}_2\tilde{\chi} + i\sqrt{2}\chi\psi^2q^2 \\
&\quad + \sqrt{2}i\tilde{q}_1\psi^1\lambda + \sqrt{2}i\chi\psi^1q^1 + \sqrt{2}i\chi\phi\lambda - \sqrt{2}i\tilde{q}_1Fq^1 - \sqrt{2}i\tilde{q}_1\phi F^1 - \sqrt{2}i\tilde{F}_2\phi q^1 \\
&\quad + \sqrt{2}i\tilde{q}_2\tilde{\psi}_1\tilde{\chi} + \sqrt{2}i\tilde{\lambda}\tilde{\psi}_1q^2 + \sqrt{2}i\tilde{\lambda}\tilde{\phi}\tilde{\chi} + \sqrt{2}i\tilde{q}_2\tilde{\phi}F^2 + \sqrt{2}i\tilde{q}_2\tilde{F}q^2 + \sqrt{2}i\tilde{F}_1\tilde{\phi}q^2 \\
&\quad - \bar{m}i\tilde{q}_1F^1 - \bar{m}i\tilde{F}_2q^2 + \bar{m}i\chi\lambda + m i\tilde{q}_2F^2 + m i\tilde{F}_1q^2 + m i\tilde{\lambda}\tilde{\chi}.
\end{aligned} \tag{D.4.76}$$

In this action, the $\text{SU}(2)_R$ symmetry is not manifest. So we redefine auxiliary fields to make it manifest. We redefine

$$\begin{aligned}
\hat{F}^1 &= F^1 - mq^1 - i\sqrt{2}\tilde{\phi}q^1, \\
\hat{\tilde{F}}_1 &= \tilde{F}_1 - \bar{m}\tilde{q}_1 + i\sqrt{2}\tilde{q}_1\phi, \\
\hat{F}^2 &= -F^2 + \bar{m}q^2 - i\sqrt{2}\phi q^2, \\
\hat{\tilde{F}}_2 &= -\tilde{F}_2 + m\tilde{q}_2 + i\sqrt{2}\tilde{q}_2\tilde{\phi}, \\
\hat{D}^a &= D^a + [\phi, \tilde{\phi}]^a - \tilde{q}_1T_R^a q^1 + \tilde{q}_2T_R^a q^2, \\
\hat{F}^a &= F^a + i\sqrt{2}\tilde{q}_2T_R^a q^1, \\
\hat{\tilde{F}}^a &= \tilde{F}^a - i\sqrt{2}\tilde{q}_1T_R^a q^2.
\end{aligned} \tag{D.4.77}$$

Also, as before we define

$$\widehat{D}_{IJ} = \begin{pmatrix} \sqrt{2} \widehat{\widetilde{F}} & i\widehat{D} \\ i\widehat{D} & \sqrt{2} \widehat{F} \end{pmatrix}. \quad (\text{D.4.78})$$

Then, taking trace in some representation with normalization $\text{Tr}(T^a T^b) = \delta^{ab}$, we obtain

$$\begin{aligned} \text{Tr}(\widehat{D}_{IJ} \widehat{D}^{IJ}) &= \text{Tr}(\widehat{D}_{11} \widehat{D}^{11} + \widehat{D}_{22} \widehat{D}^{22} + 2\widehat{D}_{12} \widehat{D}^{12}) \\ &= \text{Tr}(\widehat{D}_{11} \widehat{D}_{22} + \widehat{D}_{22} \widehat{D}_{11} - 2\widehat{D}_{12} \widehat{D}_{21}) \\ &= 4 \left(\widetilde{F}^a - i\sqrt{2} \widetilde{q}_1 T_R^a q^2 \right) \left(F^a + i\sqrt{2} \widetilde{q}_2 T_R^a q^1 \right) \\ &\quad + 2 \left(D^a + [\phi, \widetilde{\phi}]^a - \widetilde{q}_1 T_R^a q^1 + \widetilde{q}_2 T_R^a q^2 \right) \left(D^a + [\phi, \widetilde{\phi}]^a - \widetilde{q}_1 T_R^a q^1 + \widetilde{q}_2 T_R^a q^2 \right) \\ &= 4 \widetilde{F}^a F^a + 4\sqrt{2}i \widetilde{q}_2 \widetilde{F}^a q^1 - 4\sqrt{2}i \widetilde{q}_1 F^a q^2 + 8 \widetilde{q}_1 T_R^a q^2 \widetilde{q}_2 T_R^a q^1 \\ &\quad + 2 \left(D^a D^a + ([\phi, \widetilde{\phi}]^a)^2 + \widetilde{q}_1 T_R^a q^1 \widetilde{q}_1 T_R^a q^1 + \widetilde{q}_2 T_R^a q^2 \widetilde{q}_2 T_R^a q^2 + 2D^a [\phi, \widetilde{\phi}]^a \right. \\ &\quad \left. - 2\widetilde{q}_1 D^a q^1 + 2\widetilde{q}_2 D^a q^2 - 2\widetilde{q}_1 [\phi, \widetilde{\phi}]^a q^1 + 2\widetilde{q}_2 [\phi, \widetilde{\phi}]^a q^2 \right. \\ &\quad \left. - 2\widetilde{q}_1 T_R^a q^1 \widetilde{q}_2 T_R^a q^2 \right). \end{aligned} \quad (\text{D.4.79})$$

Thus we have

$$\begin{aligned} \frac{1}{4} \text{Tr}(\widehat{D}_{IJ} \widehat{D}^{IJ}) &= \text{Tr}(\widetilde{F}F) + \frac{1}{2} \text{Tr}(D^2) + \frac{1}{2} \text{Tr}([\phi, \widetilde{\phi}]^2) + \text{Tr}(D[\phi, \widetilde{\phi}]) \\ &\quad + \sqrt{2}i \widetilde{q}_2 \widetilde{F}^a q^1 - \sqrt{2}i \widetilde{q}_1 F^a q^2 + 2 \widetilde{q}_1 T_R^a q^2 \widetilde{q}_2 T_R^a q^1 \\ &\quad + \frac{1}{2} (\widetilde{q}_1 T_R^a q^1 \widetilde{q}_1 T_R^a q^1 + \widetilde{q}_2 T_R^a q^2 \widetilde{q}_2 T_R^a q^2) - \widetilde{q}_1 D^a q^1 + \widetilde{q}_2 D^a q^2 \\ &\quad - \widetilde{q}_1 [\phi, \widetilde{\phi}]^a q^1 + \widetilde{q}_2 [\phi, \widetilde{\phi}]^a q^2 - \widetilde{q}_1 T_R^a q^1 \widetilde{q}_2 T_R^a q^2. \end{aligned} \quad (\text{D.4.80})$$

So in the Lagrangian we substitute

$$\begin{aligned} \text{Tr}(\widetilde{F}F) &+ \frac{1}{2} \text{Tr}(D^2) + \text{Tr}(D[\phi, \widetilde{\phi}]) + \widetilde{q}_2 D q^2 - \widetilde{q}_1 D q^1 - \sqrt{2}i \widetilde{q}_1 F q^2 + \sqrt{2}i \widetilde{q}_2 \widetilde{F} q^1 \\ &= \frac{1}{4} \widehat{D}_{IJ} \widehat{D}^{IJ} - \frac{1}{2} \text{Tr}([\phi, \widetilde{\phi}]^2) - 2 \widetilde{q}_1 T_R^a q^2 \widetilde{q}_2 T_R^a q^1 - \frac{1}{2} (\widetilde{q}_1 T_R^a q^1 \widetilde{q}_1 T_R^a q^1 + \widetilde{q}_2 T_R^a q^2 \widetilde{q}_2 T_R^a q^2) \\ &\quad + \widetilde{q}_1 [\phi, \widetilde{\phi}] q^1 - \widetilde{q}_2 [\phi, \widetilde{\phi}] q^2 + \widetilde{q}_1 T_R^a q^1 \widetilde{q}_2 T_R^a q^2. \end{aligned} \quad (\text{D.4.81})$$

We then get

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=2} = & \frac{1}{g^2} \text{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi + i \tilde{\psi}_I \tilde{\sigma}^\mu D_\mu \psi^I + \frac{1}{4} \hat{\tilde{D}}_{IJ} \hat{D}^{IJ} - \frac{1}{2} [\phi, \tilde{\phi}]^2 \right. \\
& - \frac{i}{\sqrt{2}} \psi_I [\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}} \tilde{\psi}^I [\phi, \tilde{\psi}_I] \left. \right] + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\lambda} \text{Tr}(F_{\mu\nu} F_{\rho\lambda}) \\
& + D_\mu \tilde{q}_2 D^\mu q^2 + \tilde{F}_1 F^1 - i \lambda \sigma^\mu D_\mu \tilde{\lambda} + i \sqrt{2} \tilde{q}_2 \psi^2 \lambda + i \sqrt{2} \tilde{\lambda} \tilde{\psi}_2 q^2 + D_\mu \tilde{q}_1 D^\mu q^1 \\
& + \tilde{F}_2 F^2 - i \chi \sigma^\mu D_\mu \tilde{\chi} - i \sqrt{2} \tilde{q}_1 \tilde{\psi}_2 \tilde{\chi} - i \sqrt{2} \chi \psi^2 q^1 + \sqrt{2} i \tilde{q}_1 \psi^1 \lambda + \sqrt{2} i \chi \psi^1 q^2 \\
& + \sqrt{2} i \chi \phi \lambda - \sqrt{2} i \tilde{q}_1 \phi F^1 - \sqrt{2} i \tilde{F}_2 \phi q^2 - \sqrt{2} i \tilde{q}_2 \tilde{\psi}_1 \tilde{\chi} + \sqrt{2} i \tilde{\lambda} \tilde{\psi}_1 q^1 + \sqrt{2} i \tilde{\lambda} \tilde{\phi} \tilde{\chi} \\
& + \sqrt{2} i \tilde{q}_2 \tilde{\phi} F^2 + \sqrt{2} i \tilde{F}_1 \tilde{\phi} q^1 - \bar{m} i \tilde{q}_1 F^1 - \bar{m} i \tilde{F}_2 q^2 + \bar{m} i \chi \lambda + m i \tilde{q}_2 F^2 \\
& + m i \tilde{F}_1 q^1 + m i \tilde{\lambda} \tilde{\chi} - 2 \tilde{q}_1 T_R^a q^2 \tilde{q}_2 T_R^a q^1 + \tilde{q}_1 [\phi, \tilde{\phi}] q^1 - \tilde{q}_2 [\phi, \tilde{\phi}] q^2 \\
& + \tilde{q}_1 T_R^a q^1 \tilde{q}_2 T_R^a q^2 - \frac{1}{2} (\tilde{q}_1 T_R^a q^1 \tilde{q}_1 T_R^a q^1 + \tilde{q}_2 T_R^a q^2 \tilde{q}_2 T_R^a q^2) .
\end{aligned} \tag{D.4.82}$$

Next we have

$$\begin{aligned}
\hat{\tilde{F}}_1 \hat{F}^1 = & \left(\tilde{F}_1 - i \bar{m} \tilde{q}_1 - i \sqrt{2} \tilde{q}_1 \phi \right) \left(F^1 + i m q^1 + i \sqrt{2} \tilde{\phi} q^1 \right) \\
= & \tilde{F}_1 F^1 + i m \tilde{F}_1 q^1 + \sqrt{2} i \tilde{F}_1 \tilde{\phi} q^1 - i \bar{m} \tilde{q}_1 F^1 + m \bar{m} \tilde{q}_1 q^1 + \sqrt{2} m \tilde{q}_1 \tilde{\phi} q^1 \\
& - i \sqrt{2} \tilde{q}_1 \phi F^1 + \sqrt{2} m \tilde{q}_1 \phi q^1 + 2 \tilde{q}_1 \phi \tilde{\phi} q^1 ,
\end{aligned} \tag{D.4.83}$$

and

$$\begin{aligned}
\hat{\tilde{F}}_2 \hat{F}^2 = & \left(-\tilde{F}_2 - i m \tilde{q}_2 - i \sqrt{2} \tilde{q}_2 \tilde{\phi} \right) \left(-F^2 + i \bar{m} q^2 + i \sqrt{2} \phi q^2 \right) \\
= & \tilde{F}_2 F^2 - i \bar{m} \tilde{F}_2 q^2 - \sqrt{2} i \tilde{F}_2 \phi q^2 + i m \tilde{q}_2 F^2 + m \bar{m} \tilde{q}_2 q^2 + \sqrt{2} m \tilde{q}_2 \phi q^2 \\
& + \sqrt{2} i \tilde{q}_2 \tilde{\phi} F^2 + \sqrt{2} m \tilde{q}_2 \tilde{\phi} q^2 + 2 \tilde{q}_2 \tilde{\phi} \phi q^2 .
\end{aligned} \tag{D.4.84}$$

In the Lagrangian we now substitute

$$\begin{aligned}
& \tilde{F}_1 F^1 + i \bar{m} \tilde{F}_1 q^1 + \sqrt{2} i \tilde{F}_1 \tilde{\phi} q^1 - \bar{m} i \tilde{q}_1 F^1 - i \sqrt{2} \tilde{q}_1 \tilde{\phi} F^1 + \tilde{F}_2 F^2 - i \bar{m} \tilde{F}_2 q^2 - \sqrt{2} i \tilde{F}_2 \tilde{\phi} q^2 + i \bar{m} \tilde{q}_2 F^2 + \sqrt{2} i \tilde{q}_2 \tilde{\phi} F^2 \\
& = \hat{\tilde{F}}_1 \hat{F}^1 - m \bar{m} \tilde{q}_1 q^1 - \sqrt{2} m \tilde{q}_1 \tilde{\phi} q^1 - \sqrt{2} m \tilde{q}_1 \tilde{\phi} q^1 - 2 \tilde{q}_1 \tilde{\phi} \phi q^1 \\
& - m \bar{m} \tilde{q}_2 q^2 + \sqrt{2} m \tilde{q}_2 \tilde{\phi} q^2 - \sqrt{2} m \tilde{q}_2 \tilde{\phi} q^2 - 2 \tilde{q}_2 \tilde{\phi} \phi q^2 \\
& = \hat{\tilde{F}}_1 \hat{F}^1 - |m|^2 \tilde{q}_I q^I - \sqrt{2} m \tilde{q}_I \tilde{\phi} q^I - \sqrt{2} m \tilde{q}_I \tilde{\phi} q^I - 2 \tilde{q}_1 \tilde{\phi} \phi q^1 - 2 \tilde{q}_2 \tilde{\phi} \phi q^2 .
\end{aligned} \tag{D.4.85}$$

We get

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=2} = & \frac{1}{g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \tilde{\phi} + i \tilde{\psi}_I \bar{\sigma}^\mu D_\mu \psi^I + \frac{1}{4} \hat{D}_{IJ} \hat{D}^{IJ} - \frac{1}{2} [\phi, \tilde{\phi}]^2 - \frac{im}{\sqrt{2}} \psi_I [\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}} \tilde{\psi}_I [\phi, \tilde{\psi}^I] \right] \\
& + \frac{\Theta i \epsilon^{\mu\nu\rho\sigma}}{64\pi^2} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
& + D_\mu \tilde{q}_2 D^\mu q^2 - i \lambda \sigma^\mu D_\mu \tilde{\lambda} + i \sqrt{2} \tilde{q}_2 \psi^2 \lambda + i \sqrt{2} \tilde{\lambda} \tilde{\psi}_2 q^2 + D_\mu \tilde{q}_1 D^\mu q^1 + \hat{F}_I \hat{F}^I \\
& - i \chi \sigma^\mu D_\mu \tilde{\chi} - i \sqrt{2} \tilde{q}_1 \tilde{\psi}_2 \tilde{\chi} - i \sqrt{2} \chi \psi^2 q^1 + \sqrt{2} i \tilde{q}_1 \psi^1 \lambda + \sqrt{2} i \chi \psi^1 q^2 \\
& + \sqrt{2} i \chi \phi \lambda + \sqrt{2} i \tilde{q}_2 \tilde{\psi}_1 \tilde{\chi} + \sqrt{2} i \tilde{\lambda} \tilde{\psi}_1 q^1 + \sqrt{2} i \tilde{\lambda} \phi \tilde{\chi} + \bar{m} i \chi \lambda + m i \tilde{\lambda} \tilde{\chi} \\
& - 2 \tilde{q}_1 T_R^a q^1 \tilde{q}_2 T_R^a q^1 + \tilde{q}_1 [\phi, \tilde{\phi}] q^1 - \tilde{q}_2 [\phi, \tilde{\phi}] q^2 + \tilde{q}_1 T_R^a q^1 \tilde{q}_2 T_R^a q^2 \\
& - \frac{1}{2} (\tilde{q}_1 T_R^a q^1 \tilde{q}_1 T_R^a q^1 + \tilde{q}_2 T_R^a q^2 \tilde{q}_2 T_R^a q^2) - |m|^2 \tilde{q}_I q^I - \sqrt{2} m \tilde{q}_I \tilde{\phi} q^I \\
& - \sqrt{2} m \tilde{q}_I \tilde{\phi} q^I - 2 \tilde{q}_1 \tilde{\phi} \tilde{\phi} q^1 - 2 \tilde{q}_2 \tilde{\phi} \tilde{\phi} q^2 .
\end{aligned} \tag{D.4.86}$$

We can then use the fact that

$$\tilde{q}_1 [\phi, \tilde{\phi}] q^1 - 2 \tilde{q}_1 \tilde{\phi} \tilde{\phi} q^1 = \tilde{q}_1 \phi \tilde{\phi} q^1 - \tilde{q}_1 \tilde{\phi} \phi q^1 - 2 \tilde{q}_1 \tilde{\phi} \tilde{\phi} q^1 = -\tilde{q}_1 \{\phi, \tilde{\phi}\} q^1 , \tag{D.4.87}$$

and similarly

$$-\tilde{q}_2 [\phi, \tilde{\phi}] q^2 - 2 \tilde{q}_2 \tilde{\phi} \tilde{\phi} q^2 = -\tilde{q}_2 \phi \tilde{\phi} q^2 + \tilde{q}_2 \tilde{\phi} \phi q^2 - 2 \tilde{q}_2 \tilde{\phi} \tilde{\phi} q^2 = -\tilde{q}_2 \{\phi, \tilde{\phi}\} q^2 , \tag{D.4.88}$$

to write the Lagrangian as

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=2} = & \frac{1}{g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi + i \tilde{\psi}_I \bar{\sigma}^\mu D_\mu \psi^I + \frac{1}{4} \hat{D}_{IJ} \hat{D}^{IJ} - \frac{1}{2} [\phi, \tilde{\phi}]^2 \right. \\
& \left. - \frac{i}{\sqrt{2}} \psi_I [\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}} \tilde{\psi}_I [\phi, \tilde{\psi}^I] \right] + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
& + D_\mu \tilde{q}_2 D^\mu q^2 + D_\mu \tilde{q}_1 D^\mu q^1 - |m|^2 \tilde{q}_I q^I - \sqrt{2} m \tilde{q}_I \phi q^I - \sqrt{2} m \tilde{q}_I \tilde{\phi} q^I \\
& - i \lambda \sigma^\mu D_\mu \tilde{\lambda} + i \sqrt{2} \tilde{q}_2 \psi^2 \lambda + \sqrt{2} i \tilde{q}_1 \psi^1 \lambda + i \sqrt{2} \tilde{\lambda} \tilde{\psi}_2 q^2 + \sqrt{2} i \tilde{\lambda} \tilde{\psi}_1 q^1 \\
& - i \sqrt{2} \tilde{q}_1 \tilde{\psi}_2 \tilde{\chi} + \sqrt{2} i \tilde{q}_2 \tilde{\psi}_1 \tilde{\chi} - \sqrt{2} i \chi \psi^1 q^2 + i \sqrt{2} \chi \psi^2 q^1 - i \chi \sigma^\mu D_\mu \tilde{\chi} \\
& + \hat{F}_I \hat{F}^I + \sqrt{2} i \chi \phi \lambda + \sqrt{2} i \tilde{\lambda} \tilde{\phi} \tilde{\chi} + \bar{m} i \chi \lambda + m i \tilde{\lambda} \tilde{\chi} + \tilde{q}_1 T_R^a q^1 \tilde{q}_2 T_R^a q^2 \\
& - 2 \tilde{q}_1 T_R^a q^2 \tilde{q}_2 T_R^a q^1 - \tilde{q}_I \{\phi, \tilde{\phi}\} q^I - \frac{1}{2} (\tilde{q}_1 T_R^a q^1 \tilde{q}_1 T_R^a q^1 + \tilde{q}_2 T_R^a q^2 \tilde{q}_2 T_R^a q^2) \\
= & \frac{1}{g^2} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi + i \tilde{\psi}_I \bar{\sigma}^\mu D_\mu \psi^I + \frac{1}{4} \hat{D}_{IJ} \hat{D}^{IJ} - \frac{1}{2} [\phi, \tilde{\phi}]^2 \right. \\
& \left. - \frac{i}{\sqrt{2}} \psi_I [\tilde{\phi}, \psi^I] + \frac{i}{\sqrt{2}} \tilde{\psi}_I [\phi, \tilde{\psi}^I] \right] + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
& + D_\mu \tilde{q}_I D^\mu q^I - |m|^2 \tilde{q}_I q^I - \sqrt{2} m \tilde{q}_I \phi q^I - \sqrt{2} m \tilde{q}_I \tilde{\phi} q^I - i \lambda \sigma^\mu D_\mu \tilde{\lambda} + i \sqrt{2} \tilde{q}_I \psi^I \lambda \\
& + \sqrt{2} i \tilde{\lambda} \tilde{\psi}_I q^I + \sqrt{2} i \tilde{q}_I \tilde{\psi}^I \tilde{\chi} + \sqrt{2} i \chi \psi_I q^I - i \chi \sigma^\mu D_\mu \tilde{\chi} + \hat{F}_I \hat{F}^I + \sqrt{2} i \chi \phi \lambda \\
& + \sqrt{2} i \tilde{\lambda} \tilde{\phi} \tilde{\chi} + \bar{m} i \chi \lambda + m i \tilde{\lambda} \tilde{\chi} + \tilde{q}^I (T_R^a q^J) \tilde{q}_I (T_R^a q_J) - \tilde{q}_I \{\phi, \tilde{\phi}\} q^I ,
\end{aligned} \tag{D.4.89}$$

where we used

$$\begin{aligned}
-i\sqrt{2}\tilde{q}_1\tilde{\psi}_2\tilde{\chi} + \sqrt{2}i\tilde{q}_2\tilde{\psi}_1\tilde{\chi} &= -i\sqrt{2}\tilde{q}_1\epsilon_{2I}\tilde{\psi}^I\tilde{\chi} + \sqrt{2}i\tilde{q}_2\epsilon_{1I}\tilde{\psi}^I\tilde{\chi} \\
&= \sqrt{2}i\tilde{q}_1\tilde{\psi}^1\tilde{\chi} + \sqrt{2}i\tilde{q}_2\tilde{\psi}^2\tilde{\chi} \\
&= \sqrt{2}i\tilde{q}_I\tilde{\psi}^I\tilde{\chi} ,
\end{aligned} \tag{D.4.90}$$

$$\begin{aligned}
-\sqrt{2}i\chi\psi^1q^2 + i\sqrt{2}\chi\psi^2q^1 &= -\sqrt{2}i\chi\epsilon^{1I}\psi_Iq^2 + i\sqrt{2}\chi\epsilon^{2I}\psi_Iq^1 \\
&= \sqrt{2}i\chi\psi_2q^2 + \sqrt{2}i\chi\psi_1q^1 \\
&= \sqrt{2}i\chi\psi_Iq^I ,
\end{aligned} \tag{D.4.91}$$

and finally

$$\begin{aligned}
&\tilde{q}_1T_R^aq^1\tilde{q}_2T_R^aq^2 - 2\tilde{q}_1T_R^aq^2\tilde{q}_2T_R^aq^1 - \frac{1}{2}(\tilde{q}_1T_R^aq^1\tilde{q}_1T_R^aq^1 + \tilde{q}_2T_R^aq^2\tilde{q}_2T_R^aq^2) \\
&= \tilde{q}^2T_R^aq^1\tilde{q}_2T_R^aq_1 + 2\tilde{q}^2T_R^aq^2\tilde{q}_1T_R^aq_2 - \frac{1}{2}(-\tilde{q}^2T_R^aq^1\tilde{q}_1T_R^aq_2 - \tilde{q}^1T_R^aq^2\tilde{q}_2T_R^aq_1) \\
&= \frac{1}{4}(\tilde{q}^2T_R^aq^1 + \tilde{q}^1T_R^aq^2)\tilde{q}_2T_R^aq_1 + \frac{1}{4}(\tilde{q}^1T_R^aq^2 + \tilde{q}^2T_R^aq^1)\tilde{q}_1T_R^aq_2 \\
&\quad + \frac{1}{4}(\tilde{q}^2T_R^aq^1 + \tilde{q}^1T_R^aq^2)\tilde{q}_2T_R^aq_1 + \frac{1}{4}(\tilde{q}^1T_R^aq^2 + \tilde{q}^2T_R^aq^1)\tilde{q}_1T_R^aq_2 \\
&\quad + \tilde{q}_1T_R^aq_1\tilde{q}^1T_R^aq^1 + \tilde{q}^2T_R^aq^2\tilde{q}_2T_R^aq_2 \\
&= \tilde{q}^{(I}T_R^aq^{J)}\tilde{q}_{(I}T_R^aq_{J)} .
\end{aligned} \tag{D.4.92}$$

Rescaling

$$Q \rightarrow \frac{1}{g}Q , \quad \tilde{Q} \rightarrow \frac{1}{g}\tilde{Q} , \quad Q' \rightarrow \frac{1}{g}Q' , \quad \tilde{Q}' \rightarrow \frac{1}{g}\tilde{Q}' , \tag{D.4.93}$$

we obtain

$$\begin{aligned}
\mathcal{L}_{\mathcal{N}=2} &= \frac{1}{g^2} \text{Tr} \left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\tilde{\phi}D^\mu\phi + i\tilde{\psi}_I\bar{\sigma}^\mu D_\mu\psi^I + \frac{1}{4}D_{IJ}D^{IJ} - \frac{1}{2}[\phi,\tilde{\phi}]^2 \right. \\
&\quad \left. - \frac{i}{\sqrt{2}}\psi_I[\tilde{\phi},\psi^I] + \frac{i}{\sqrt{2}}\tilde{\psi}_I[\phi,\tilde{\psi}^I] \right] + \frac{\Theta i}{64\pi^2}\epsilon^{\mu\nu\rho\sigma}\text{Tr}(F_{\mu\nu}F_{\rho\sigma}) \\
&\quad + \frac{1}{g^2} \left[D_\mu\tilde{q}_ID^\mu q^I - |m|^2\tilde{q}_Iq^I - \sqrt{2}m\tilde{q}_I\phi q^I - \sqrt{2}\bar{m}\tilde{q}_I\tilde{\phi}q^I - i\lambda\sigma^\mu D_\mu\tilde{\lambda} \right. \\
&\quad + \sqrt{2}i\tilde{\lambda}\tilde{\psi}_Iq^I + \sqrt{2}i\tilde{q}_I\tilde{\psi}^I\tilde{\chi} + \sqrt{2}i\chi\psi_Iq^I - i\chi\sigma^\mu D_\mu\tilde{\chi} \\
&\quad + \tilde{F}_IF^I + i\sqrt{2}\tilde{q}_I\psi^I\lambda + \sqrt{2}i\chi\phi\lambda + \sqrt{2}i\tilde{\lambda}\tilde{\phi}\tilde{\chi} + \bar{m}i\chi\lambda + mi\tilde{\lambda}\tilde{\chi} \\
&\quad \left. + \tilde{q}^{(I}T_R^aq^{J)}\tilde{q}_{(I}T_R^aq_{J)} - \tilde{q}_I\{\phi,\tilde{\phi}\}q^I \right] ,
\end{aligned} \tag{D.4.94}$$

where we omitted the hat from auxiliary fields.

D.4.2 Twisted action on spin manifolds

The twisted action can be obtained from the untwisted action (D.4.94) by minimally coupling to gravity and making the following change of variable¹⁰:

$$\begin{aligned}
A_\mu(\mathbf{2}, \mathbf{2}, \mathbf{1})^0 &\rightarrow A_\mu(\mathbf{2}, \mathbf{2})^0, \\
\psi_{\alpha A}(\mathbf{2}, \mathbf{1}, \mathbf{2})^1 &\rightarrow \psi_{\alpha\dot{\beta}}(\mathbf{2}, \mathbf{2})^1, \\
\tilde{\psi}_{\dot{\alpha} A}(\mathbf{1}, \mathbf{2}, \mathbf{2})^{-1} &\rightarrow \eta(\mathbf{1}, \mathbf{1})^{-1}, \quad \chi_{(\dot{\alpha}\dot{\beta})}(\mathbf{1}, \mathbf{3})^{-1}, \\
\phi(\mathbf{1}, \mathbf{1}, \mathbf{1})^{-2} &\rightarrow \phi(\mathbf{1}, \mathbf{1})^{-2}, \\
\tilde{\phi}(\mathbf{1}, \mathbf{1}, \mathbf{1})^2 &\rightarrow \tilde{\phi}(\mathbf{1}, \mathbf{1})^2, \\
D_{IJ}(\mathbf{1}, \mathbf{1}, \mathbf{3})^0 &\rightarrow D_{\dot{\alpha}\dot{\beta}}(\mathbf{1}, \mathbf{3})^0,
\end{aligned} \tag{D.4.95}$$

On the right hand side the representation is with respect to $\Gamma' = \text{SU}(2)_- \times \text{SU}'(2) \times \text{U}(1)_R$. Note that $\psi_{\alpha I}$, which was a spinor in the untwisted theory, is a vector $\psi_{\alpha\dot{\beta}}$ (written as a bispinor, that is $\psi_\mu \equiv (\tilde{\sigma}_\mu)^{\alpha\dot{\beta}} \psi_{\alpha\dot{\beta}}$ is a vector), the right handed spinor $\tilde{\psi}_{\dot{\alpha} I}$ turns into $\tilde{\psi}_{\dot{\alpha}\dot{\beta}}$ which decomposes as the antisymmetric piece η , which is a scalar, and the symmetric piece $\tilde{\psi}_{(\dot{\alpha}\dot{\beta})}$:

$$\eta \equiv \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\psi}_{\dot{\alpha}\dot{\beta}} = \tilde{\psi}_{[\dot{1}\dot{2}]} = -\tilde{\psi}_{[\dot{2}\dot{1}]}, \quad \chi_{(\dot{\alpha}\dot{\beta})} \equiv \frac{1}{2} (\tilde{\psi}_{\dot{\alpha}\dot{\beta}} + \tilde{\psi}_{\dot{\beta}\dot{\alpha}}) \tag{D.4.96}$$

$\tilde{\psi}_{(\dot{\alpha}\dot{\beta})}$ corresponds to a self-dual 2-form

$$\chi_{\mu\nu} \equiv \frac{1}{2} (\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}\dot{\beta}} \chi_{\dot{\alpha}\dot{\beta}}. \tag{D.4.97}$$

Note that all fields after twisting are integral spin and hence the twisted vector multiplet makes sense on an arbitrary Riemannian manifold. Twisting of the hypermultiplet gives the following set of fields:

$$\begin{aligned}
q_I(\mathbf{1}, \mathbf{1}, \mathbf{2}) &\rightarrow M_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}), \quad \tilde{q}_I(\mathbf{1}, \mathbf{1}, \mathbf{2}) \rightarrow \tilde{M}_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}), \\
\lambda_\alpha(\mathbf{2}, \mathbf{1}, \mathbf{1}) &\rightarrow \lambda_\alpha(\mathbf{2}, \mathbf{1}), \quad \tilde{\lambda}_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}, \mathbf{1}) \rightarrow \tilde{\lambda}_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}) \\
\tilde{\chi}_\alpha(\mathbf{2}, \mathbf{1}, \mathbf{1}) &\rightarrow \tilde{\mu}_\alpha(\mathbf{2}, \mathbf{1}), \quad \chi_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}, \mathbf{1}) \rightarrow \mu_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}) \\
F_I(\mathbf{1}, \mathbf{1}, \mathbf{2}) &\rightarrow F_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}), \quad \tilde{F}^I(\mathbf{1}, \mathbf{1}, \mathbf{2}) \rightarrow \tilde{F}_{\dot{\alpha}}(\mathbf{1}, \mathbf{2}).
\end{aligned} \tag{D.4.98}$$

The minimal coupling to gravity is done by writing the $\mathcal{N} = 2$ conformal supergravity action from [dWLVP85, KR88, CMRS23] and setting the variation of gravitino to zero. The resulting twisted

¹⁰The notation $(\mathbf{2}, \mathbf{2}, \mathbf{1})^0$ means that A_μ transforms in the fundamental two dimensional representation of $\text{SU}(2)_+, \text{SU}(2)_-$, as a scalar under $\text{SU}(2)_R$ and has charge 0 under $(\text{U}(1)_R)$. Here $\text{Spin}(4) \cong \text{SU}(2)_- \times \text{SU}(2)_+$.

action can be written as

$$\begin{aligned}
S_{\mathcal{N}=2}^{\text{twisted}} = & \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi - 4i \chi_{\mu\nu} D^\mu \psi^\nu - 2i\eta D_\mu \psi^\mu + \frac{1}{2} D_{\mu\nu} D^{\mu\nu} \right. \\
& \left. - \frac{1}{2} [\phi, \tilde{\phi}]^2 + \sqrt{2}i \tilde{\phi} [\psi_\mu, \psi^\mu] + \sqrt{2}i \chi_{\mu\nu} [\phi, \chi^{\mu\nu}] - \sqrt{2}i \phi [\eta, \eta] \right] \\
& + \frac{i\Theta}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
& + \int d^4x \sqrt{g} \left[D_\mu \tilde{M}_{\dot{\alpha}} D^\mu M^{\dot{\alpha}} - \frac{1}{16} R \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} - m^2 \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} - \sqrt{2}im \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} \right. \\
& + i\sqrt{2}m \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} - \sqrt{2}i \tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} - \sqrt{2}i\eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} + \sqrt{2}i \tilde{M}^{\dot{\alpha}} \tilde{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \tilde{\mu}^{\dot{\beta}} \chi_{\mu\nu} \\
& + \sqrt{2}i\eta \tilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} - i\lambda \sigma^\mu D_\mu \tilde{\lambda} - \sqrt{2}i \tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} \lambda_\alpha \psi_\mu + \sqrt{2}i \mu^\alpha \sigma^\mu_{\alpha\dot{\beta}} M^{\dot{\beta}} \psi_\mu - i\mu \sigma^\mu D_\mu \tilde{\mu} \\
& \left. + \tilde{F}_{\dot{\alpha}} F^{\dot{\alpha}} + \sqrt{2}i \mu \phi \lambda + \sqrt{2}i \tilde{\phi} \tilde{\mu} + m\mu\lambda - m\tilde{\lambda}\tilde{\mu} - \tilde{M}_{\dot{\alpha}} \{\phi, \tilde{\phi}\} M^{\dot{\alpha}} + \tilde{M}^{\dot{\alpha}} (T_R^a M^{\dot{\beta}}) \tilde{M}_{\dot{\alpha}} (T_R^a M_{\dot{\beta}}) \right] \quad (\text{D.4.99})
\end{aligned}$$

where now D_μ is the metric + gauge-covariant derivative. More explicitly we have

$$D_\mu \tilde{M}^{\dot{\alpha}} = \partial_\mu \tilde{M}^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})^{\dot{\alpha}}_{\dot{\beta}} \tilde{M}^{\dot{\beta}} + A_\mu^a T_R^a \tilde{M}^{\dot{\alpha}}, \quad (\text{D.4.100})$$

and similarly on other fields and we assumed that the mass parameter is purely imaginary. Here ω_μ^{ab} is the spin connection and $\tilde{\sigma}_{ab}$ are the Lorentz generators on the space of self-dual 2-forms and R is the Ricci scalar of the metric $g_{\mu\nu}$ on the 4-manifold $\mathbb{X}$ (see Appendix D.1 for details on our conventions). The appearance of the term proportional to R can be explained most naturally in the formalism based on coupling the untwisted matter theory to untwisted background conformal supergravity [dWLVP85, KR88, CMRS23]. In going from (D.4.94) to (D.4.113), we used the following calculations:

1.

$$\begin{aligned}
i\tilde{\psi}_I \tilde{\sigma}^\mu D_\mu \psi^I &= -i\tilde{\psi}^I \tilde{\sigma}^\mu D_\mu \psi_I \\
&\longrightarrow -i \left(\chi^{\dot{\alpha}\dot{\beta}} + \tilde{\psi}^{[\dot{\alpha}\dot{\beta}]} \right) \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^\mu D_\mu \psi_{\nu}^{\dot{\beta}} \\
&= -i \chi^{\dot{\alpha}\dot{\beta}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^\mu D_\mu \psi_{\nu}^{\dot{\beta}} - 2i\eta D_\mu \psi^\mu \\
&= -i (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} \chi_{\mu\nu} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^\rho D_\rho \psi_{\nu}^{\dot{\beta}} - 2i\eta D_\mu \psi^\mu \\
&= -i \chi_{\mu\nu} \left(-\frac{1}{2} \epsilon^{\mu\nu\rho\lambda} (\tilde{\sigma}_\lambda)^{\dot{\beta}}_{\dot{\beta}} - \frac{1}{2} \delta^{\mu\rho} (\tilde{\sigma}^\nu)^{\dot{\beta}}_{\dot{\beta}} + \frac{1}{2} \delta^{\nu\rho} (\tilde{\sigma}^\mu)^{\dot{\beta}}_{\dot{\beta}} \right) D_\rho \psi_{\nu}^{\dot{\beta}} \\
&\quad - 2i\eta D_\mu \psi^\mu \\
&= -2i \chi^{\rho\lambda} D_\rho \psi_\lambda - i \chi_{\mu\nu} \delta^{\mu\rho} D_\rho \psi^\nu + i \chi_{\mu\nu} \delta^{\nu\rho} D_\rho \psi^\mu - i\eta D_\mu \psi^\mu \\
&= -4i \chi_{\mu\nu} D^\mu \psi^\nu - 2i\eta D_\mu \psi^\mu \quad (\text{D.4.101})
\end{aligned}$$

where we used $\psi_\mu = \frac{1}{2}(\tilde{\sigma}_\mu)^{\dot{\alpha}\beta}\psi_{\beta\dot{\alpha}}$ and

$$\begin{aligned} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\beta}\tilde{\sigma}_{\dot{\alpha}\beta}^\rho &= -(\tilde{\sigma}^{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}}(\tilde{\sigma}^\rho)^{\dot{\alpha}}_{\dot{\beta}} \\ &= -\frac{1}{2}\epsilon^{\mu\nu\rho\lambda}(\tilde{\sigma}_\lambda)^{\dot{\beta}}_{\dot{\alpha}} - \frac{1}{2}\delta^{\mu\rho}(\tilde{\sigma}^\nu)^{\dot{\beta}}_{\dot{\alpha}} + \frac{1}{2}\delta^{\nu\rho}(\tilde{\sigma}^\mu)^{\dot{\beta}}_{\dot{\alpha}} \end{aligned} \quad (\text{D.4.102})$$

and the fact that $\chi_{\mu\nu}$ is a selfdual 2-form.

2.

$$\begin{aligned} D_{IJ}D^{IJ} &\longrightarrow D_{\dot{\alpha}\dot{\beta}}D^{\dot{\alpha}\dot{\beta}} \\ &= (\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}D_{\mu\nu}(\tilde{\sigma}^{\rho\sigma})^{\dot{\alpha}\dot{\beta}}D_{\rho\sigma} \\ &= -(\tilde{\sigma}^{\mu\nu})^{\dot{\beta}}_{\dot{\alpha}}(\tilde{\sigma}^{\rho\sigma})^{\dot{\alpha}}_{\dot{\beta}}D_{\mu\nu}D_{\rho\sigma} \\ &= D_{\mu\nu}D_{\rho\sigma}\left(\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} + \frac{1}{2}\delta^{\mu\rho}\delta^{\nu\sigma} - \frac{1}{2}\delta^{\mu\sigma}\delta^{\nu\rho}\right) \\ &= \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}D_{\mu\nu}D_{\rho\sigma} + \frac{1}{2}D_{\mu\nu}D^{\mu\nu} - \frac{1}{2}D_{\mu\nu}D^{\nu\mu} \\ &= D_{\mu\nu}D^{\mu\nu} + D_{\mu\nu}D^{\mu\nu} \\ &= 2D_{\mu\nu}D^{\mu\nu} \end{aligned} \quad (\text{D.4.103})$$

where we used the fact that $D_{\mu\nu}$ is selfdual.

3.

$$\begin{aligned} \psi_I [\tilde{\phi}, \psi^I] &\longrightarrow -\psi_{\dot{\alpha}\dot{\beta}} [\tilde{\phi}, \psi^{\dot{\beta}\dot{\alpha}}] \\ &= -(\sigma^\mu)_{\dot{\alpha}\dot{\beta}}\psi_\mu [\tilde{\phi}, (\tilde{\sigma}^\nu)^{\dot{\beta}\dot{\alpha}}\psi_\nu] \\ &= -(\tilde{\sigma}^\nu)^{\dot{\beta}\dot{\alpha}}(\sigma^\mu)_{\dot{\alpha}\dot{\beta}}\psi_\mu [\tilde{\phi}, \psi_\nu] \\ &= 2\delta^{\mu\nu}\psi_\mu [\tilde{\phi}, \psi_\nu] \\ &= 2\psi_\mu [\tilde{\phi}, \psi^\mu] . \end{aligned} \quad (\text{D.4.104})$$

Moreover inside trace we can write

$$\begin{aligned} \text{Tr}(\psi_\mu [\tilde{\phi}, \psi^\mu]) &= \text{Tr}(\psi_\mu^a \tilde{\phi}^b \psi^{\mu c} (T^a T^b T^c - T^a T^c T^b)) \\ &= \text{Tr}(\tilde{\phi}^b \psi_\mu^a \psi^{\mu c} [T^c, T^a]) \\ &= -\text{Tr}(\tilde{\phi} [\psi_\mu, \psi^\mu]) . \end{aligned} \quad (\text{D.4.105})$$

4.

$$\begin{aligned} \tilde{\psi}_I [\phi, \tilde{\psi}^I] &\longrightarrow \tilde{\psi}_{\dot{\alpha}\dot{\beta}} [\phi, \tilde{\psi}^{\dot{\alpha}\dot{\beta}}] \\ &= (\chi_{\dot{\alpha}\dot{\beta}} + \tilde{\psi}_{[\dot{\alpha}\dot{\beta}]}) [\phi, \chi^{\dot{\alpha}\dot{\beta}} + \tilde{\psi}^{[\dot{\alpha}\dot{\beta}]}] \\ &= \chi_{\dot{\alpha}\dot{\beta}} [\phi, \chi^{\dot{\alpha}\dot{\beta}}] + \chi_{\dot{\alpha}\dot{\beta}} [\phi, \tilde{\psi}^{[\dot{\alpha}\dot{\beta}]}] + \tilde{\psi}_{[\dot{\alpha}\dot{\beta}]} [\phi, \chi^{\dot{\alpha}\dot{\beta}}] + \tilde{\psi}_{[\dot{\alpha}\dot{\beta}]} [\phi, \tilde{\psi}^{[\dot{\alpha}\dot{\beta}]}] \\ &= 2\chi_{\mu\nu} [\phi, \chi^{\mu\nu}] + 0 + 0 + 2\eta [\phi, \eta] \\ &= 2\chi_{\mu\nu} [\phi, \chi^{\mu\nu}] - 2\phi [\eta, \eta] . \end{aligned} \quad (\text{D.4.106})$$

5.

$$\begin{aligned}
\sqrt{2i}\tilde{\lambda}\tilde{\psi}_I q^I &= -\sqrt{2i}\tilde{\lambda}\tilde{\psi}^I q_I \\
&\longrightarrow -\sqrt{2i}\tilde{\lambda}_{\dot{\alpha}} \left(\chi^{\dot{\alpha}\dot{\beta}} + \tilde{\psi}^{[\dot{\alpha}\dot{\beta}]} \right) M_{\dot{\beta}} \\
&= -\sqrt{2i}\tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} + \sqrt{2i}\eta \left(\tilde{\lambda}_1 M_{\dot{2}} - \tilde{\lambda}_2 M_{\dot{1}} \right) \\
&= -\sqrt{2i}\tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} - \sqrt{2i}\eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}}
\end{aligned} \tag{D.4.107}$$

where we used the fact that

$$\eta = \tilde{\psi}_{[\dot{1}\dot{2}]} = -\tilde{\psi}_{[\dot{2}\dot{1}]} = \tilde{\psi}^{[\dot{1}\dot{2}]} = -\tilde{\psi}^{[\dot{2}\dot{1}]} . \tag{D.4.108}$$

6.

$$\begin{aligned}
\sqrt{2i}\tilde{q}_I \tilde{\psi}^I \tilde{\chi} &= -\sqrt{2i}\tilde{q}^I \tilde{\psi}_I \tilde{\chi} \\
&\longrightarrow -\sqrt{2i}\tilde{M}^{\dot{\alpha}} \left(\chi_{\dot{\alpha}\dot{\beta}} + \tilde{\psi}_{[\dot{\alpha}\dot{\beta}]} \right) \tilde{\mu}^{\dot{\beta}} \\
&= \sqrt{2i}\tilde{M}^{\dot{\alpha}} \tilde{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \tilde{\mu}^{\dot{\beta}} \chi_{\mu\nu} - \sqrt{2i}\eta \left(\tilde{M}^{\dot{1}} \tilde{\mu}^{\dot{2}} - \tilde{M}^{\dot{2}} \tilde{\mu}^{\dot{1}} \right) \\
&= \sqrt{2i}\tilde{M}^{\dot{\alpha}} \tilde{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \tilde{\mu}^{\dot{\beta}} \chi_{\mu\nu} + \sqrt{2i}\eta \tilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} .
\end{aligned} \tag{D.4.109}$$

7.

$$\begin{aligned}
i\sqrt{2}\tilde{q}_I \psi^I \lambda &\longrightarrow \sqrt{2i}\tilde{M}_{\dot{\alpha}} \psi^{\dot{\alpha}\alpha} \lambda_{\alpha} \\
&= -\sqrt{2i}\tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^{\mu})^{\dot{\alpha}\alpha} \lambda_{\alpha} \psi_{\mu} .
\end{aligned} \tag{D.4.110}$$

8.

$$\begin{aligned}
\sqrt{2i}\chi\psi_I q^I &\longrightarrow \sqrt{2i}\chi^{\alpha} \psi_{\alpha\dot{\beta}} M^{\dot{\beta}} \\
&= \sqrt{2i}\chi^{\alpha} \sigma^{\mu}_{\alpha\dot{\beta}} M^{\dot{\beta}} \psi_{\mu} .
\end{aligned} \tag{D.4.111}$$

The next step is to perform the following simplification

$$\int d^4x \sqrt{g} \left(D_{\mu} \tilde{M}_{\dot{\alpha}} D^{\mu} M^{\dot{\alpha}} - \frac{1}{16} R \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} \right) = \int d^4x \sqrt{g} \left(D^{\alpha\dot{\alpha}} \tilde{M}_{\dot{\alpha}} D_{\alpha\dot{\beta}} M^{\dot{\beta}} - \tilde{M}_{\dot{\alpha}} (F^+)^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \right) . \tag{D.4.112}$$

This will give us the following twisted action

$$\begin{aligned}
\mathcal{S}_{\mathcal{N}=2}^{\text{twisted}} &= \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_{\mu} \tilde{\phi} D^{\mu} \phi - 4i\chi_{\mu\nu} D^{\mu} \psi^{\nu} - 2i\eta D_{\mu} \psi^{\mu} + \frac{1}{2} D_{\mu\nu} D^{\mu\nu} \right. \\
&\quad \left. - \frac{1}{2} [\phi, \tilde{\phi}]^2 + \sqrt{2i}\tilde{\phi} [\psi_{\mu}, \psi^{\mu}] + \sqrt{2i}\chi_{\mu\nu} [\phi, \chi^{\mu\nu}] - \sqrt{2i}\phi [\eta, \eta] \right] \\
&\quad + \frac{i\Theta}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
&\quad + \int d^4x \sqrt{g} \left[D^{\alpha\dot{\alpha}} \tilde{M}_{\dot{\alpha}} D_{\alpha\dot{\beta}} M^{\dot{\beta}} - \tilde{M}_{\dot{\alpha}} (F^+)^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} - m^2 \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} - \sqrt{2im} \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} \right. \\
&\quad + i\sqrt{2m} \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} - \sqrt{2i}\tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} - \sqrt{2i}\eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} + \sqrt{2i}\tilde{M}^{\dot{\alpha}} \tilde{\sigma}^{\mu\nu}_{\dot{\alpha}\dot{\beta}} \tilde{\mu}^{\dot{\beta}} \chi_{\mu\nu} \\
&\quad + \sqrt{2i}\eta \tilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} - i\lambda\sigma^{\mu} D_{\mu} \tilde{\lambda} - \sqrt{2i}\tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^{\mu})^{\dot{\alpha}\alpha} \lambda_{\alpha} \psi_{\mu} + \sqrt{2i}\mu^{\alpha} \sigma^{\mu}_{\alpha\dot{\beta}} M^{\dot{\beta}} \psi_{\mu} - i\mu\sigma^{\mu} D_{\mu} \tilde{\mu} \\
&\quad \left. + \tilde{F}_{\dot{\alpha}} F^{\dot{\alpha}} + \sqrt{2i}\mu\phi\lambda + \sqrt{2i}\tilde{\lambda}\tilde{\phi}\tilde{\mu} + m\mu\lambda - m\tilde{\lambda}\tilde{\mu} - \tilde{M}_{\dot{\alpha}} \{ \phi, \tilde{\phi} \} M^{\dot{\alpha}} + \tilde{M}^{(\dot{\alpha}} T_R^a M^{\dot{\beta})} \tilde{M}_{(\dot{\alpha}} T_R^a M_{\dot{\beta})} \right] , \tag{D.4.113}
\end{aligned}$$

where

$$\begin{aligned}
D_\mu \tilde{\phi} &= \partial_\mu \tilde{\phi} + [A_\mu, \tilde{\phi}] , \\
D_\mu \phi &= \partial_\mu \phi + [A_\mu, \phi] , \\
D_\mu \psi_\nu &= \partial_\mu \psi_\nu - \Gamma_{\mu\nu}^\rho \psi_\rho + [A_\mu, \psi_\nu] , \\
D_\mu M^{\dot{\alpha}} &= \partial_\mu M^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} M^{\dot{\beta}} + A_\mu^a T_R^a M^{\dot{\alpha}} , \\
D_\mu \widetilde{M}^{\dot{\alpha}} &= \partial_\mu \widetilde{M}^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} \widetilde{M}^{\dot{\beta}} + A_\mu^a T_R^a \widetilde{M}^{\dot{\alpha}} , \\
D_\mu \tilde{\chi}^{\dot{\alpha}} &= \partial_\mu \tilde{\chi}^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} \tilde{\chi}^{\dot{\beta}} + A_\mu^a T_R^a \tilde{\chi}^{\dot{\alpha}} , \\
D_\mu \tilde{\mu}^{\dot{\alpha}} &= \partial_\mu \tilde{\mu}^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} \tilde{\mu}^{\dot{\beta}} + A_\mu^a T_R^a \tilde{\mu}^{\dot{\alpha}} .
\end{aligned} \tag{D.4.114}$$

Since the proof of (D.4.112) is rather tedious, we present it in a separate subsection.

D.4.3 Proof of (D.4.112)

We start with

$$D_\mu \widetilde{M}_{\dot{\alpha}} D^\mu M^{\dot{\alpha}} = \frac{1}{2} D_{\alpha\dot{\alpha}} \widetilde{M}_{\dot{\beta}} D^{\alpha\dot{\alpha}} M^{\dot{\beta}} \tag{D.4.115}$$

Let

$$\mathcal{D}_\mu = \nabla_\mu - \frac{1}{2} \omega_\mu^{ab} \tilde{\sigma}_{ab} . \tag{D.4.116}$$

We first show that we can perform partial integration with $\mathcal{D}_\mu$. On a closed Riemannian 4-manifold we have

$$\begin{aligned}
\int d^4x \sqrt{g} A_\alpha^\mu \mathcal{D}_\mu B^{\dot{\alpha}} &= \int d^4x \sqrt{g} A_\alpha^\mu \left(\partial_\mu B^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} B^{\dot{\beta}} \right) \\
&= \int d^4x \left(\partial_\mu (\sqrt{g} A_\alpha^\mu B^{\dot{\alpha}}) - \partial_\mu (\sqrt{g} A_\alpha^\mu) B^{\dot{\alpha}} - \frac{\sqrt{g}}{2} A_\alpha^\mu \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} B^{\dot{\beta}} \right) \\
&= - \int d^4x \sqrt{g} \left(B^{\dot{\alpha}} \partial_\mu A_\alpha^\mu + \Gamma_{\mu\rho}^\rho A_\alpha^\mu B^{\dot{\alpha}} + \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}}^{\dot{\alpha}} A_\alpha^\mu B^{\dot{\beta}} \right) \\
&= - \int d^4x \sqrt{g} \left((\nabla_\mu A_\alpha^\mu) B^{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\beta}\dot{\alpha}} A^{\mu\dot{\alpha}} B^{\dot{\beta}} \right) \\
&= - \int d^4x \sqrt{g} (\mathcal{D}_\mu A_\alpha^\mu) B^{\dot{\alpha}} ,
\end{aligned} \tag{D.4.117}$$

where we used the fact that

$$\begin{aligned}
\mathcal{D}_\mu A_{\dot{\alpha}} &= \partial_\mu A_{\dot{\alpha}} - \frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\gamma}}^{\dot{\beta}} A^{\dot{\gamma}} \\
&= \partial_\mu A_{\dot{\alpha}} - \frac{1}{2} \omega_\mu^{ab} (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\gamma}} A^{\dot{\gamma}} .
\end{aligned} \tag{D.4.118}$$

We also need the integration by parts of the form

$$\int d^4x \sqrt{g} e_{\alpha\dot{\alpha}}^\mu e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}_\nu M_{\dot{\beta}} = - \int d^4x \sqrt{g} e_{\alpha\dot{\alpha}}^\mu e^{\nu\alpha\dot{\beta}} \mathcal{D}_\mu \widetilde{M}^{\dot{\alpha}} \mathcal{D}_\nu M_{\dot{\beta}} \tag{D.4.119}$$

To prove this note that

$$\begin{aligned}
\partial_\mu \left(\sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \right) &= \sqrt{g} \left(-\Gamma^\mu_{\mu\rho} e^\rho_{\alpha\dot{\alpha}} + \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\sigma_{\underline{a}\underline{b}})_{\alpha\beta} e^{\mu\beta}_{\dot{\alpha}} + \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})_{\dot{\alpha}\dot{\gamma}} e^{\mu\dot{\gamma}}_{\alpha} \right) e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \\
&+ \sqrt{g} e^\mu_{\alpha\dot{\alpha}} \left(-\Gamma^\nu_{\mu\rho} e^{\rho\alpha\dot{\beta}} + \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\sigma_{\underline{a}\underline{b}})^\alpha_\beta e^{\nu\beta\dot{\beta}} + \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})^\dot{\beta}_{\dot{\gamma}} e^{\nu\alpha\dot{\gamma}} \right) \widetilde{M}^{\dot{\alpha}} \\
&+ \Gamma^\rho_{\mu\rho} \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} + \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\dot{\beta}} \partial_\mu \widetilde{M}^{\dot{\alpha}} \\
&= -\frac{1}{2} \sqrt{g} e^\mu_{\alpha\dot{\alpha}} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})^\dot{\gamma}_{\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} - \sqrt{g} \Gamma^\nu_{\mu\rho} e^\mu_{\alpha\dot{\alpha}} e^{\rho\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \\
&+ \frac{1}{2} \sqrt{g} e^\mu_{\alpha\dot{\alpha}} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})^\dot{\beta}_{\dot{\gamma}} e^{\nu\alpha\dot{\gamma}} \widetilde{M}^{\dot{\alpha}} + \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \partial_\mu \widetilde{M}^{\dot{\alpha}} \\
&= \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \mathcal{D}_\mu \widetilde{M}^{\dot{\alpha}} + \frac{1}{2} \sqrt{g} e^\mu_{\alpha\dot{\alpha}} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})^\dot{\beta}_{\dot{\gamma}} e^{\nu\alpha\dot{\gamma}} \widetilde{M}^{\dot{\alpha}} \\
&- \sqrt{g} \Gamma^\nu_{\mu\rho} e^\mu_{\alpha\dot{\alpha}} e^{\rho\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} ,
\end{aligned} \tag{D.4.120}$$

where we used the fact that

$$\mathcal{D}_\mu e^\nu_{\alpha\dot{\alpha}} = 0 . \tag{D.4.121}$$

Using this we have

$$\begin{aligned}
&\int d^4x \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}_\nu M_{\dot{\beta}} \\
&= \int d^4x \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \left(\partial_\mu \left(\mathcal{D}_\nu M_{\dot{\beta}} \right) - \Gamma^\rho_{\mu\nu} \mathcal{D}_\rho M_{\dot{\beta}} - \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})_{\dot{\beta}\dot{\gamma}} \mathcal{D}_\nu M^{\dot{\gamma}} \right) \\
&= \int d^4x \left[\partial_\mu \left(\sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \mathcal{D}_\nu M_{\dot{\beta}} \right) - \partial_\mu \left(\sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \right) \mathcal{D}_\nu M_{\dot{\beta}} \right. \\
&\quad \left. - \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\gamma\alpha\dot{\beta}} \left(\Gamma^\rho_{\mu\nu} \mathcal{D}_\rho M_{\dot{\beta}} + \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})_{\dot{\beta}\dot{\gamma}} \mathcal{D}_\nu M^{\dot{\gamma}} \right) \right] \\
&= - \int d^4x \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \left(\mathcal{D}_\mu \widetilde{M}^{\dot{\alpha}} \right) \mathcal{D}_\nu M_{\dot{\beta}} - \frac{1}{2} \int d^4x \sqrt{g} e^\mu_{\alpha\dot{\alpha}} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})^\dot{\beta}_{\dot{\gamma}} e^{\nu\alpha\dot{\gamma}} \widetilde{M}^{\dot{\alpha}} \mathcal{D}_\nu M_{\dot{\beta}} \\
&+ \int d^4x \sqrt{g} \Gamma^\nu_{\mu\rho} e^\mu_{\alpha\dot{\alpha}} e^{\rho\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} \mathcal{D}_\nu M_{\dot{\beta}} - \int d^4x \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \left(\Gamma^\rho_{\mu\nu} \mathcal{D}_\rho M_{\dot{\beta}} + \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})_{\dot{\beta}\dot{\gamma}} \mathcal{D}_\nu M^{\dot{\gamma}} \right) \\
&= - \int d^4x \sqrt{g} e^\mu_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \left(\mathcal{D}_\mu \widetilde{M}^{\dot{\alpha}} \right) \mathcal{D}_\nu M_{\dot{\beta}} .
\end{aligned} \tag{D.4.122}$$

It is easy to check that $\mathcal{D}_\mu$ satisfies the following Leibniz rule:

$$\mathcal{D}_\mu \left(A_\nu M^{\dot{\alpha}} \right) = (\nabla_\mu A_\nu) M^{\dot{\alpha}} + A_\nu \mathcal{D}_\mu M^{\dot{\alpha}} . \tag{D.4.123}$$

Indeed

$$\begin{aligned}
\mathcal{D}_\mu \left(A_\nu M^{\dot{\alpha}} \right) &= \partial_\mu \left(A_\nu M^{\dot{\alpha}} \right) - \Gamma^\rho_{\mu\nu} A_\rho M^{\dot{\alpha}} - \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})_{\dot{\beta}}^{\dot{\alpha}} A_\nu M^{\dot{\beta}} \\
&= (\nabla_\mu A_\nu) M^{\dot{\alpha}} + A_\nu \left(\partial_\mu M^{\dot{\alpha}} - \frac{1}{2} \omega^\mu_{\underline{a}\underline{b}} (\widetilde{\sigma}_{\underline{a}\underline{b}})_{\dot{\beta}}^{\dot{\alpha}} M^{\dot{\beta}} \right) \\
&= (\nabla_\mu A_\nu) M^{\dot{\alpha}} + A_\nu \mathcal{D}_\mu M^{\dot{\alpha}} .
\end{aligned} \tag{D.4.124}$$

We also have that

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] M^{\dot{\beta}} = -\frac{1}{2} R_{\mu\nu\dot{a}\dot{b}} (\widetilde{\sigma}^{\dot{a}\dot{b}})_{\dot{\alpha}}^{\dot{\beta}} M^{\dot{\alpha}} . \tag{D.4.125}$$

Now we have

$$\begin{aligned}
\int d^4x \sqrt{g} D_\mu \widetilde{M}_{\dot{\alpha}} D^\mu M^{\dot{\alpha}} &= \int d^4x \sqrt{g} \left(\mathcal{D}_\mu \widetilde{M}_{\dot{\alpha}} + A_\mu \widetilde{M}_{\dot{\alpha}} \right) \left(\mathcal{D}^\mu M^{\dot{\alpha}} + A^\mu M^{\dot{\alpha}} \right) \\
&= \int d^4x \sqrt{g} \left(\mathcal{D}_\mu \widetilde{M}_{\dot{\alpha}} \mathcal{D}^\mu M^{\dot{\alpha}} + (\mathcal{D}_\mu \widetilde{M}_{\dot{\alpha}}) A^\mu M^{\dot{\alpha}} + A_\mu \widetilde{M}_{\dot{\alpha}} \mathcal{D}^\mu M^{\dot{\alpha}} \right. \\
&\quad \left. + A_\mu \widetilde{M}_{\dot{\alpha}} A^\mu M^{\dot{\alpha}} \right) \\
&= \int d^4x \sqrt{g} \left(-\widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}^\mu M^{\dot{\alpha}} - \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \left(A^\mu M^{\dot{\alpha}} \right) + A_\mu \widetilde{M}_{\dot{\alpha}} \mathcal{D}^\mu M^{\dot{\alpha}} \right. \\
&\quad \left. + \widetilde{M}_{\dot{\alpha}} A_\mu A^\mu M^{\dot{\alpha}} \right) \\
&= \int d^4x \sqrt{g} \left(-\widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}^\mu M^{\dot{\alpha}} - \widetilde{M}_{\dot{\alpha}} (\nabla_\mu A^\mu) M^{\dot{\alpha}} + \widetilde{M}_{\dot{\alpha}} A_\mu A^\mu M^{\dot{\alpha}} \right) .
\end{aligned} \tag{D.4.126}$$

Using the exchange identity (D.1.5) we have

$$\begin{aligned}
\widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}^\mu M^{\dot{\alpha}} &= \frac{1}{2} \widetilde{M}_{\dot{\beta}} \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha\dot{\alpha}} M^{\dot{\beta}} \\
&= \frac{1}{2} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} \mathcal{D}^{\alpha\dot{\alpha}} M^{\dot{\beta}} - \frac{1}{2} \epsilon_{\dot{\beta}\dot{\alpha}} \widetilde{M}_{\dot{\gamma}} \mathcal{D}_{\dot{\alpha}} \mathcal{D}^{\alpha\dot{\alpha}} M^{\dot{\beta}} \\
&= \frac{1}{2} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} \mathcal{D}^{\alpha\dot{\alpha}} M^{\dot{\beta}} + \frac{1}{2} \widetilde{M}_{\dot{\alpha}} \mathcal{D}^{\alpha\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} M^{\dot{\beta}} .
\end{aligned} \tag{D.4.127}$$

We now write

$$\begin{aligned}
\widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} \mathcal{D}^{\alpha\dot{\alpha}} M^{\dot{\beta}} &= e^\mu_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}_\nu M^{\dot{\beta}} , \\
\widetilde{M}_{\dot{\alpha}} \mathcal{D}^{\alpha\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} M^{\dot{\beta}} &= e^{\mu\alpha\dot{\alpha}} e^\nu_{\alpha\dot{\beta}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}_\nu M^{\dot{\beta}} .
\end{aligned} \tag{D.4.128}$$

Commuting $\mathcal{D}_\mu$ past $\mathcal{D}_\nu$ using (D.4.125) we get

$$\begin{aligned}
&\int d^4x \sqrt{g} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}^\mu M^{\dot{\alpha}} \\
&= \frac{1}{2} \int d^4x \sqrt{g} \left[e^\mu_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}_\nu M^{\dot{\beta}} + e^{\mu\alpha\dot{\alpha}} e^\nu_{\alpha\dot{\beta}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu \mathcal{D}_\nu M^{\dot{\beta}} \right] \\
&= \frac{1}{2} \int d^4x \sqrt{g} \left[e^\mu_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} \left(\mathcal{D}_\nu \mathcal{D}_\mu M^{\dot{\beta}} - \frac{1}{2} R_{\mu\nu\dot{a}\dot{b}} (\widetilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right) + e^{\nu\alpha\dot{\alpha}} e^\mu_{\alpha\dot{\beta}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\nu \mathcal{D}_\mu M^{\dot{\beta}} \right] \\
&= -\frac{1}{2} \int d^4x \sqrt{g} \left[2e^\mu_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} \mathcal{D}_\nu \widetilde{M}_{\dot{\alpha}} \mathcal{D}_\mu M^{\dot{\beta}} + \frac{1}{2} \widetilde{M}_{\dot{\alpha}} e^\mu_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\widetilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right] \\
&= - \int d^4x \sqrt{g} \left[\mathcal{D}^{\alpha\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} M^{\dot{\beta}} + \frac{1}{4} \widetilde{M}_{\dot{\alpha}} e^\mu_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\widetilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right] .
\end{aligned} \tag{D.4.129}$$

Next we have

$$\begin{aligned}
\widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A^{\mu}) M_{\dot{\alpha}} &= \frac{1}{2} \widetilde{M}^{\dot{\alpha}} (\nabla_{\alpha\dot{\beta}} A^{\alpha\dot{\beta}}) M_{\dot{\alpha}} \\
&= \frac{1}{2} \widetilde{M}^{\dot{\alpha}} (\nabla_{\alpha\dot{\alpha}} A^{\alpha\dot{\beta}}) M_{\dot{\beta}} - \frac{1}{2} \epsilon_{\dot{\beta}\dot{\alpha}} \widetilde{M}^{\dot{\alpha}} (\nabla_{\alpha\dot{\gamma}} A^{\alpha\dot{\beta}}) M^{\dot{\gamma}} \\
&= \frac{1}{2} \widetilde{M}^{\dot{\alpha}} (\nabla_{\alpha\dot{\alpha}} A^{\alpha\dot{\beta}}) M_{\dot{\beta}} - \frac{1}{2} \widetilde{M}_{\dot{\beta}} (\nabla_{\alpha\dot{\gamma}} A^{\alpha\dot{\beta}}) M^{\dot{\gamma}} \\
&= \frac{1}{2} e_{\alpha\dot{\alpha}}^{\mu} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A_{\nu}) M_{\dot{\beta}} - \frac{1}{2} e_{\alpha\dot{\gamma}}^{\mu} e^{\nu\alpha\dot{\beta}} \widetilde{M}_{\dot{\beta}} (\nabla_{\mu} A_{\nu}) M^{\dot{\gamma}} \\
&= \frac{1}{2} e_{\alpha\dot{\alpha}}^{\mu} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A_{\nu}) M_{\dot{\beta}} + \frac{1}{2} e^{\mu\alpha\dot{\gamma}} e_{\alpha\dot{\beta}}^{\nu} \widetilde{M}^{\dot{\beta}} (\nabla_{\mu} A_{\nu}) M_{\dot{\gamma}} \\
&= \frac{1}{2} e_{\alpha\dot{\alpha}}^{\mu} e^{\nu\alpha\dot{\beta}} \widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A_{\nu} + \nabla_{\nu} A_{\mu}) M_{\dot{\beta}} .
\end{aligned} \tag{D.4.130}$$

Similarly

$$\begin{aligned}
\widetilde{M}_{\dot{\alpha}} A_{\mu} A^{\mu} M^{\dot{\alpha}} &= \frac{1}{2} \widetilde{M}_{\dot{\alpha}} A_{\alpha\dot{\beta}} A^{\alpha\dot{\beta}} M^{\dot{\alpha}} \\
&= \frac{1}{2} \widetilde{M}_{\dot{\beta}} A_{\alpha\dot{\alpha}} A^{\alpha\dot{\beta}} M^{\dot{\alpha}} - \frac{1}{2} \epsilon_{\dot{\alpha}\dot{\beta}} \widetilde{M}_{\dot{\gamma}} A_{\alpha}^{\dot{\gamma}} A^{\alpha\dot{\beta}} M^{\dot{\alpha}} \\
&= \frac{1}{2} \widetilde{M}_{\dot{\beta}} A_{\alpha\dot{\alpha}} A^{\alpha\dot{\beta}} M^{\dot{\alpha}} - \frac{1}{2} \widetilde{M}_{\dot{\gamma}} A^{\alpha\dot{\gamma}} A_{\alpha\dot{\beta}} M^{\dot{\beta}} \\
&= \frac{1}{2} e_{\alpha\dot{\alpha}}^{\mu} e^{\nu\alpha\dot{\beta}} \widetilde{M}_{\dot{\beta}} A_{\mu} A_{\nu} M^{\dot{\alpha}} - \frac{1}{2} e^{\mu\alpha\dot{\gamma}} e_{\alpha\dot{\beta}}^{\nu} \widetilde{M}_{\dot{\gamma}} A_{\mu} A_{\nu} M^{\dot{\beta}} \\
&= \frac{1}{2} e_{\alpha\dot{\alpha}}^{\mu} e^{\nu\alpha\dot{\beta}} \widetilde{M}_{\dot{\beta}} (A_{\mu} A_{\nu} + A_{\nu} A_{\mu}) M^{\dot{\alpha}} .
\end{aligned} \tag{D.4.131}$$

Next

$$\begin{aligned}
\int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} D_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} &= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} \left(D_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} + (D_{\mu} \widetilde{M}_{\dot{\alpha}}) A_{\nu} M^{\dot{\beta}} \right. \\
&\quad \left. + A_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} + A_{\mu} \widetilde{M}_{\dot{\alpha}} A_{\nu} M^{\dot{\beta}} \right) .
\end{aligned} \tag{D.4.132}$$

We now claim that

$$\int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} \left((D_{\mu} \widetilde{M}_{\dot{\alpha}}) A_{\nu} M^{\dot{\beta}} + A_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} \right) = - \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} \widetilde{M}_{\dot{\alpha}} (\nabla_{\mu} A_{\nu}) M^{\dot{\beta}} . \tag{D.4.133}$$

To prove this we use integration by parts. We have

$$\begin{aligned}
&\int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} \left[(D_{\mu} \widetilde{M}_{\dot{\alpha}}) A_{\nu} M^{\dot{\beta}} + A_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} \right] \\
&= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} \left[-\widetilde{M}_{\dot{\alpha}} (D_{\mu} A_{\nu} M^{\dot{\beta}}) + A_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} \right] \\
&= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} \left[-\widetilde{M}_{\dot{\alpha}} (\nabla_{\mu} A_{\nu}) M^{\dot{\beta}} - A_{\nu} \widetilde{M}_{\dot{\alpha}} D_{\mu} M^{\dot{\beta}} + A_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} \right] .
\end{aligned} \tag{D.4.134}$$

Now using the exchange identity we get

$$\begin{aligned}
e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} A_{\nu} \widetilde{M}_{\dot{\alpha}} D_{\mu} M^{\dot{\beta}} &= -e^{\mu\alpha\dot{\alpha}} e_{\alpha}^{\nu\dot{\beta}} A_{\nu} \widetilde{M}_{\dot{\alpha}} D_{\mu} M_{\dot{\beta}} \\
&= -e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\alpha}} A_{\nu} \widetilde{M}_{\dot{\beta}} D_{\mu} M_{\dot{\alpha}} + \epsilon_{\dot{\alpha}\dot{\beta}} e^{\mu\alpha\dot{\alpha}} e_{\alpha}^{\nu\dot{\beta}} A_{\nu} \widetilde{M}_{\dot{\gamma}} D_{\mu} M^{\dot{\gamma}} \\
&= -e_{\alpha\dot{\alpha}}^{\mu} e^{\nu\alpha\dot{\beta}} A_{\nu} \widetilde{M}_{\dot{\beta}} D_{\mu} M^{\dot{\alpha}} \\
&= -e^{\mu\alpha\dot{\alpha}} e_{\alpha\dot{\beta}}^{\nu} A_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} ,
\end{aligned} \tag{D.4.135}$$

where we used the fact that $e^{\mu\alpha\dot{\alpha}}e^{\nu\dot{\beta}}_{\alpha} = e^{\mu\alpha\dot{\beta}}e^{\nu\dot{\alpha}}_{\alpha}$. The identity now follows. Thus we have

$$\begin{aligned} \int d^4x \sqrt{g} \mathcal{D}^{\alpha\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} M^{\dot{\beta}} &= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} \left(\widetilde{M}_{\dot{\alpha}} (\nabla_{\mu} A_{\nu}) M^{\dot{\beta}} - \widetilde{M}_{\dot{\alpha}} A_{\mu} A_{\nu} M^{\dot{\beta}} \right) \\ &+ \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} D_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} . \end{aligned} \quad (\text{D.4.136})$$

Using this relation we get

$$\begin{aligned} &\int d^4x \sqrt{g} D_{\mu} \widetilde{M}_{\dot{\alpha}} D^{\mu} M^{\dot{\alpha}} \\ &= \int d^4x \sqrt{g} \left(-\widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\mu} \mathcal{D}^{\mu} M^{\dot{\alpha}} - \widetilde{M}_{\dot{\alpha}} (\nabla_{\mu} A^{\mu}) M^{\dot{\alpha}} + \widetilde{M}_{\dot{\alpha}} A_{\mu} A^{\mu} M^{\dot{\alpha}} \right) \\ &= \int d^4x \sqrt{g} \left[\mathcal{D}^{\alpha\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} \mathcal{D}_{\alpha\dot{\beta}} M^{\dot{\beta}} + \frac{1}{4} \widetilde{M}_{\dot{\alpha}} e^{\mu}_{\alpha\dot{\beta}} e^{\nu\alpha\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right] \\ &- \frac{1}{2} \int d^4x \sqrt{g} e^{\mu}_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \left[\widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A_{\nu} + \nabla_{\nu} A_{\mu}) M_{\dot{\beta}} - \widetilde{M}_{\dot{\beta}} (A_{\mu} A_{\nu} + A_{\nu} A_{\mu}) M^{\dot{\alpha}} \right] \\ &= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} \left(D_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} + \frac{1}{4} \widetilde{M}_{\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right) \\ &+ \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} \left(\widetilde{M}_{\dot{\alpha}} (\nabla_{\mu} A_{\nu}) M^{\dot{\beta}} - \widetilde{M}_{\dot{\alpha}} A_{\mu} A_{\nu} M^{\dot{\beta}} \right) \\ &+ \frac{1}{2} \int d^4x \sqrt{g} e^{\mu}_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \left[\widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A_{\nu} + \nabla_{\nu} A_{\mu}) M_{\dot{\beta}} + \widetilde{M}_{\dot{\beta}} (A_{\mu} A_{\nu} + A_{\nu} A_{\mu}) M^{\dot{\alpha}} \right] \\ &= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} \left(D_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} + \frac{1}{4} \widetilde{M}_{\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right) \\ &- \frac{1}{2} \int d^4x \sqrt{g} e^{\mu}_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \left[\widetilde{M}^{\dot{\alpha}} (\nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu}) M_{\dot{\beta}} + \widetilde{M}_{\dot{\beta}} (A_{\mu} A_{\nu} - A_{\nu} A_{\mu}) M^{\dot{\alpha}} \right] \\ &= \int d^4x \sqrt{g} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} \left(D_{\mu} \widetilde{M}_{\dot{\alpha}} D_{\nu} M^{\dot{\beta}} + \frac{1}{4} \widetilde{M}_{\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \right) \\ &- \frac{1}{2} \int d^4x \sqrt{g} e^{\mu}_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}_{\dot{\beta}} F_{\mu\nu} M^{\dot{\alpha}} . \end{aligned} \quad (\text{D.4.137})$$

Now

$$\begin{aligned} \frac{1}{2} e^{\mu}_{\alpha\dot{\alpha}} e^{\nu\alpha\dot{\beta}} \widetilde{M}_{\dot{\alpha}} F_{\mu\nu} M^{\dot{\beta}} &= \frac{1}{2} e^{\mu}_a e^{\nu}_b (\sigma^a)_{\alpha\dot{\alpha}} (\sigma^b)^{\alpha\dot{\beta}} \widetilde{M}_{\dot{\alpha}} F_{\mu\nu} M^{\dot{\beta}} \\ &= -\frac{1}{4} e^{\mu}_a e^{\nu}_b (\tilde{\sigma}^a \sigma^b - \tilde{\sigma}^b \sigma^a)^{\dot{\beta}}_{\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} F_{\mu\nu} M^{\dot{\beta}} \\ &= -e^{\mu}_a e^{\nu}_b (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} F_{\mu\nu} M^{\dot{\beta}} \\ &= \widetilde{M}^{\dot{\alpha}} F_{\dot{\alpha}\dot{\beta}}^+ M^{\dot{\beta}} . \end{aligned} \quad (\text{D.4.138})$$

Finally

$$\begin{aligned} \frac{1}{4} e^{\mu\alpha\dot{\alpha}} e^{\nu\dot{\beta}}_{\alpha\dot{\beta}} \widetilde{M}_{\dot{\alpha}} R_{\mu\nu\dot{a}\dot{b}} (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} &= \frac{1}{4} \widetilde{M}_{\dot{\alpha}} R^{\alpha\dot{\alpha}}_{\alpha\dot{\beta}} (\tilde{\sigma}^{\dot{a}\dot{b}})^{\dot{\beta}}_{\dot{\gamma}} M^{\dot{\gamma}} \\ &= \frac{1}{4} \widetilde{M}_{\dot{\alpha}} \epsilon^{\beta\alpha} \epsilon^{\dot{\alpha}\dot{\eta}} \epsilon^{\dot{\beta}\dot{\delta}} R_{\alpha\dot{\eta}\beta\dot{\delta}} M^{\dot{\gamma}} \\ &= \frac{1}{8} \widetilde{M}_{\dot{\alpha}} \epsilon^{\beta\alpha} \epsilon^{\dot{\alpha}\dot{\eta}} \epsilon^{\dot{\beta}\dot{\delta}} \epsilon^{\gamma\delta} R_{\alpha\dot{\eta}\beta\dot{\delta}\gamma\dot{\gamma}} M^{\dot{\gamma}} \\ &= \frac{1}{16} R \epsilon^{\dot{\alpha}\dot{\eta}} \epsilon_{\dot{\eta}\dot{\gamma}} \widetilde{M}_{\dot{\alpha}} M^{\dot{\gamma}} \\ &= \frac{R}{16} \widetilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} . \end{aligned} \quad (\text{D.4.139})$$

Plugging these expressions in (D.4.137) we obtain (D.4.112).

D.4.4 $\mathcal{Q}$ -transformations, and closure of commutations and $\mathcal{Q}$ -exactness

Recall that topological twisting, as explained in Section 4.6, gives us a conserved scalar supercharge.

To construct this scalar supercharge, we first note that after twisting, the supercharges change as

$$\mathcal{Q}_{\alpha I} \rightarrow \mathcal{Q}_{\alpha\dot{\beta}}, \quad \tilde{\mathcal{Q}}_{\dot{\alpha}I} \rightarrow \mathcal{Q}_{\dot{\alpha}\dot{\beta}}. \quad (\text{D.4.140})$$

The scalar supercharge is then given by

$$\mathcal{Q} := \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\mathcal{Q}}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} (\tilde{\mathcal{Q}}_{\dot{1}\dot{2}} - \tilde{\mathcal{Q}}_{\dot{2}\dot{1}}). \quad (\text{D.4.141})$$

From the supersymmetry transformation of untwisted fields, we can find out the transformation of the twisted fields under the scalar supercharge. From the supersymmetry algebra, we see that

$$\begin{aligned} \mathcal{Q}^2 &= \frac{1}{2} \{\mathcal{Q}, \mathcal{Q}\} = \frac{1}{8} \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\gamma}\dot{\delta}} \left\{ \tilde{\mathcal{Q}}_{\dot{\alpha}\dot{\beta}}, \tilde{\mathcal{Q}}_{\dot{\gamma}\dot{\delta}} \right\} \\ &= \frac{1}{4} \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\gamma}\dot{\delta}} \epsilon_{\dot{\alpha}\dot{\gamma}} \epsilon_{\dot{\beta}\dot{\delta}} Z \\ &= -\frac{1}{4} \delta_{\dot{\delta}}^{\dot{\alpha}} \delta_{\dot{\alpha}}^{\dot{\delta}} Z \\ &= -\frac{1}{2} Z. \end{aligned} \quad (\text{D.4.142})$$

Thus on Z -invariant operators of the theory, we have a $\mathcal{Q}$ -cohomology. We need to check that this is true in field space. The $\mathcal{Q}$ -transformations of the twisted fields can be determined from the supersymmetry transformations of the untwisted fields. Since, we are working with gauge theory, the most we can expect is that

$$\mathcal{Q}^2 = -\frac{1}{2} \delta_m^{\text{central-charge}} + \delta_{\phi}^{\text{gauge}}, \quad (\text{D.4.143})$$

where $\delta_m^{\text{central-charge}}$ is the $U(1)$ -transformation by mass parameter m and $\delta_{\phi}^{\text{gauge}}$ is the gauge transformation by parameter ϕ .

It turns out that the off-shell formulation with $F, \tilde{F}$ auxiliary fields does not satisfy (D.4.142). So we need a different off-shell formulation of the action. Such an off-shell formulation was proposed

in [KR88]. Following [LMn05], introduce two new spinorial auxiliary fields $h_\alpha, \tilde{h}_\alpha$ as follows:

$$\begin{aligned}
\mathcal{S}_{N=2}^{\text{twisted}} = & \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi - 4i\chi_{\mu\nu} D^\mu \psi^\nu - 2i\eta D_\mu \psi^\mu + \frac{1}{2} D_{\mu\nu} D^{\mu\nu} \right. \\
& \left. - \frac{1}{2} [\phi, \tilde{\phi}]^2 + \sqrt{2}i\tilde{\phi} [\psi_\mu, \psi^\mu] + \sqrt{2}i\chi_{\mu\nu} [\phi, \chi^{\mu\nu}] - \sqrt{2}i\phi [\eta, \eta] \right] \\
& + \frac{i\Theta}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
& + \int d^4x \sqrt{g} \left[\tilde{h}^\alpha \sigma_{\alpha\dot{\alpha}}^\mu D_\mu M^{\dot{\alpha}} + D_\mu \tilde{M}_{\dot{\alpha}} \tilde{\sigma}_{\alpha\dot{\alpha}}^\mu h^\alpha + \tilde{h}h - m^2 \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} - \sqrt{2}im \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} \right. \\
& + i\sqrt{2}m \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} - \tilde{M}_{(\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} - \sqrt{2}i\tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} - \sqrt{2}i\eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} + \sqrt{2}i\tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} \tilde{\mu}^{\dot{\beta}} \chi_{\mu\nu} \\
& + \sqrt{2}i\eta \tilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} - i\lambda \sigma^\mu D_\mu \tilde{\lambda} - \sqrt{2}i\tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} \lambda_\alpha \psi_\mu + \sqrt{2}i\mu^\alpha \sigma_{\alpha\dot{\beta}}^\mu M^{\dot{\beta}} \psi_\mu - i\mu \sigma^\mu D_\mu \tilde{\mu} \\
& \left. + \sqrt{2}i\mu \phi \lambda + \sqrt{2}i\tilde{\lambda} \tilde{\phi} \tilde{\mu} + m\mu\lambda - m\tilde{\lambda}\tilde{\mu} - \tilde{M}_{\dot{\alpha}} \{\phi, \tilde{\phi}\} M^{\dot{\alpha}} + \tilde{M}^{(\dot{\alpha}} T_R^a M^{\dot{\beta})} \tilde{M}_{(\dot{\alpha}} T_R^a M_{\dot{\beta})} \right] .
\end{aligned} \tag{D.4.144}$$

The equation of motion of $\tilde{h}^\alpha, h_\alpha$ is

$$\begin{aligned}
h_\alpha &= \sigma_{\alpha\dot{\alpha}}^\mu D_\mu M^{\dot{\alpha}} , \\
\tilde{h}_\alpha &= -D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\alpha\dot{\alpha}}^\mu .
\end{aligned} \tag{D.4.145}$$

Plugging in the action, we obtain

$$\begin{aligned}
& \tilde{h}^\alpha \sigma_{\alpha\dot{\alpha}}^\mu D_\mu M^{\dot{\alpha}} + D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\alpha\dot{\alpha}}^\mu h^\alpha + \tilde{h}^\alpha h_\alpha \\
&= \tilde{h}^\alpha \sigma_{\alpha\dot{\beta}}^\mu D_\mu M^{\dot{\beta}} - \tilde{h}^\alpha \sigma_{\alpha\dot{\beta}}^\mu D_\mu M^{\dot{\beta}} - D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\alpha\dot{\alpha}}^\mu \epsilon^{\alpha\beta} \sigma_{\beta\dot{\beta}}^\nu D_\nu M^{\dot{\beta}} \\
&= -D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\alpha\dot{\alpha}}^\mu (\sigma^\nu)^\alpha_{\dot{\beta}} D_\nu M^{\dot{\beta}} \\
&= -D_\mu \tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} \sigma_{\alpha\dot{\beta}}^\nu D_\nu M^{\dot{\beta}} \\
&= D_\mu \tilde{M}^{\dot{\alpha}} \sigma_{\alpha\dot{\alpha}}^\mu (\sigma^\nu)^\alpha_{\dot{\beta}} D_\nu M^{\dot{\beta}} \\
&= D_{\alpha\dot{\alpha}} \tilde{M}^{\dot{\alpha}} D^\alpha_{\dot{\beta}} M^{\dot{\beta}} \\
&= D^{\alpha\dot{\alpha}} \tilde{M}_{\dot{\alpha}} D_{\alpha\dot{\beta}} M^{\dot{\beta}}
\end{aligned} \tag{D.4.146}$$

Thus we get back the old action with $F_{\dot{\alpha}}, \tilde{F}_{\dot{\alpha}}$ integrated out. To write down the $\mathcal{Q}$ -transformation, we use the following convention: $\mathcal{Q}$ acts on fields as

$$\delta_\epsilon \Phi = \epsilon \mathcal{Q} \Phi \tag{D.4.147}$$

while on operators

$$[\mathcal{Q}, \Phi] = \mathcal{Q} \cdot \Phi . \tag{D.4.148}$$

Thus we can get the $\mathcal{Q}$ -commutators from infinitesimal variation of fields by stripping off the Grassmann odd parameter ϵ . Before proceeding we rescale fields to get rid of factors of i and $\sqrt{2}$. We

make the following rescalings:

$$\begin{aligned}
\phi &\rightarrow \phi' = -\sqrt{2i}\phi, & \tilde{\phi} &\rightarrow \tilde{\phi}' = \frac{1}{\sqrt{2i}}\tilde{\phi}, & \psi_\mu &\rightarrow \psi'_\mu = \sqrt{2}\psi_\mu, \\
\chi_{\mu\nu} &\rightarrow \chi'_{\mu\nu} = i\sqrt{2}\chi_{\mu\nu}, & \eta &\rightarrow \eta' = i\sqrt{2}\eta, & D_{\mu\nu}^a &\rightarrow D_{\mu\nu}'^a = -iD_{\mu\nu}^a + \widetilde{M}^{(\dot{\alpha}}T_R^aM^{\dot{\beta})} \\
\mu &\rightarrow \mu' = i\mu, & \lambda &\rightarrow \lambda' = i\lambda, & \tilde{\lambda} &\rightarrow \tilde{\lambda}' = -\tilde{\lambda}, & \tilde{\mu} &\rightarrow \tilde{\mu}' = -\tilde{\mu}, & \tilde{h}^\alpha &\rightarrow \tilde{h}'^\alpha = -\tilde{h}^\alpha.
\end{aligned} \tag{D.4.149}$$

The action then takes the following form (after dropping the primes):

$$\begin{aligned}
\mathcal{S}_{\mathcal{N}=2}^{\text{twisted}} &= \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_{\mu\nu} D^{\mu\nu} - 2\chi_{\mu\nu} D^\mu \psi^\nu + \frac{1}{2} \chi_{\mu\nu} [\phi, \chi^{\mu\nu}] - \eta D_\mu \psi^\mu \right. \\
&\quad \left. + \tilde{\phi} D_\mu D^\mu \phi - \frac{1}{2} \phi [\eta, \eta] - \frac{1}{2} [\phi, \tilde{\phi}]^2 - \tilde{\phi} [\psi_\mu, \psi^\mu] \right] \\
&\quad + \frac{i\Theta}{64\pi^2} \int d^4x \epsilon^{\mu\nu\rho\sigma} \text{Tr} (F_{\mu\nu} F_{\rho\sigma}) \\
&\quad + \int d^4x \sqrt{g} \left[-\tilde{h}^\alpha \sigma_{\alpha\dot{\alpha}}^\mu D_\mu M^{\dot{\alpha}} + D_\mu \widetilde{M}_{\dot{\alpha}} \tilde{\sigma}_{\alpha\dot{\alpha}}^\mu h^\alpha - \tilde{h} h - m^2 \widetilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} + m \widetilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} \right. \\
&\quad - 2m \widetilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} + \tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} + \eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \widetilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} \tilde{\mu}^{\dot{\beta}} \chi_{\mu\nu} \\
&\quad - \eta \widetilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} + \lambda \sigma^\mu D_\mu \tilde{\lambda} - \widetilde{M}_{\dot{\alpha}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} \lambda_\alpha \psi_\mu + \mu^\alpha \sigma_{\alpha\dot{\beta}}^\mu \cdot M^{\dot{\beta}} \psi_\mu + \mu \sigma^\mu D_\mu \tilde{\mu} \\
&\quad + \mu \phi \lambda - 2\tilde{\lambda} \tilde{\phi} \tilde{\mu} - m \mu \lambda - m \tilde{\lambda} \tilde{\mu} \\
&\quad \left. + \widetilde{M}_{\dot{\alpha}} \{\phi, \tilde{\phi}\} M^{\dot{\alpha}} - \widetilde{M}_{(\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} + \widetilde{M}_{(\dot{\alpha}} D^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} \right]. \tag{D.4.150}
\end{aligned}$$

In this formulation, the central charge is realized as a global U(1) symmetry which acts with parameter $2m$. Vector multiplet fields are singlets under the central charge. A consistent assignment of charges for hypermultiplet fields is given in Table 4.6. This assignment is same as assigning charge $(-1, 1)$ to $\mathcal{N} = 1$ chiral multiplets Q and $\tilde{Q}'$ and charge $(1, -1)$ to Q' and $\tilde{Q}$.

The $\mathcal{Q}$ -action on fields is given in (D.4.151), we reproduce them here for ease of reference.

$$\begin{aligned}
[\mathcal{Q}, \phi] &= 0, & [\mathcal{Q}, \tilde{\phi}] &= \eta, & \{\mathcal{Q}, \eta\} &= [\phi, \tilde{\phi}], \\
[\mathcal{Q}, A_\mu] &= \psi_\mu, & \{\mathcal{Q}, \psi_\mu\} &= -D_\mu \phi, \\
\{\mathcal{Q}, \chi_{\mu\nu}\} &= F_{\mu\nu}^+ - D_{\mu\nu}, & [\mathcal{Q}, D_{\mu\nu}] &= 2(D_{[\mu} \psi_{\nu]})^+ - [\phi, \chi_{\mu\nu}], \\
[\mathcal{Q}, M_{\dot{\alpha}}] &= \tilde{\mu}_{\dot{\alpha}}, & [\mathcal{Q}, \widetilde{M}_{\dot{\alpha}}] &= \tilde{\lambda}_{\dot{\alpha}}, \\
\{\mathcal{Q}, \tilde{\mu}_{\dot{\alpha}}\} &= m M_{\dot{\alpha}} - \phi M_{\dot{\alpha}}, & \{\mathcal{Q}, \tilde{\lambda}_{\dot{\alpha}}\} &= -m \widetilde{M}_{\dot{\alpha}} + \widetilde{M}_{\dot{\alpha}} \phi, \\
\{\mathcal{Q}, \lambda_\alpha\} &= h_\alpha, & \{\mathcal{Q}, \mu_\alpha\} &= \tilde{h}_\alpha, \\
[\mathcal{Q}, h_\alpha] &= m \lambda_\alpha - \phi \lambda_\alpha, & [\mathcal{Q}, \tilde{h}_\alpha] &= -m \mu_\alpha + \mu_\alpha \phi.
\end{aligned} \tag{D.4.151}$$

On the vector multiplet fields, we have

$$\begin{aligned}
\{\mathcal{Q}, [\mathcal{Q}, \phi]\} &= 0, \\
\{\mathcal{Q}, [\mathcal{Q}, \tilde{\phi}]\} &= \{\mathcal{Q}, \eta\} = [\phi, \tilde{\phi}], \\
[\mathcal{Q}, \{\mathcal{Q}, \eta\}] &= [\mathcal{Q}, [\phi, \tilde{\phi}]] = -[\tilde{\phi}, [\mathcal{Q}, \phi]] - [\phi, [\tilde{\phi}, \mathcal{Q}]] = [\phi, \eta], \\
\{\mathcal{Q}, [\mathcal{Q}, A_\mu]\} &= \{\mathcal{Q}, \psi_\mu\} = -D_\mu \phi, \\
[\mathcal{Q}, \{\mathcal{Q}, \psi_\mu\}] &= -[\mathcal{Q}, D_\mu \phi] = -[\mathcal{Q}, \partial_\mu \phi + [A_\mu, \phi]] = [\phi, [\mathcal{Q}, A_\mu]] \\
&= [\phi, \psi_\mu] .
\end{aligned} \tag{D.4.152}$$

Next, we have

$$\begin{aligned}
[\mathcal{Q}, \{\mathcal{Q}, \chi_{\mu\nu}\}] &= [\mathcal{Q}, F_{\mu\nu}^+] - [\mathcal{Q}, D_{\mu\nu}] \\
&= 2(D_{[\mu} \psi_{\nu]})^+ - 2(D_{[\mu} \psi_{\nu]})^+ - [\chi_{\mu\nu}, \phi] \\
&= [\phi, \chi_{\mu\nu}]
\end{aligned} \tag{D.4.153}$$

where we used

$$\begin{aligned}
[\mathcal{Q}, F_{\mu\nu}] &= [\mathcal{Q}, \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]] \\
&= \partial_\mu \psi_\nu - \partial_\nu \psi_\mu - [A_\nu, [\mathcal{Q}, A_\mu]] - [A_\mu, [A_\nu, \mathcal{Q}]] \\
&= \partial_\mu \psi_\nu + [A_\mu, \psi_\nu] - \partial_\nu \psi_\mu + [A_\nu, \psi_\mu] \\
&= D_\mu \psi_\nu - D_\nu \psi_\mu = 2(D_{[\mu} \psi_{\nu]}) .
\end{aligned} \tag{D.4.154}$$

Finally, we have

$$\begin{aligned}
\{\mathcal{Q}, [\mathcal{Q}, D_{\mu\nu}]\} &= 2(\{\mathcal{Q}, D_{[\mu} \psi_{\nu]}\})^+ + \{\mathcal{Q}, [\chi_{\mu\nu}, \phi]\} \\
&= 2(\{\mathcal{Q}, \partial_{[\mu} \psi_{\nu]}\} + \{\mathcal{Q}, [A_\mu, \psi_\nu]\})^{+a} - f_{bca} T^a \chi_{\mu\nu}^b [\mathcal{Q}, \phi^c] + f_{bca} T^a \{\mathcal{Q}, \chi_{\mu\nu}^b\} \phi^c \\
&= -2(\partial_{[\mu} D_{\nu]} \phi)^+ + [A_\mu, \{\mathcal{Q}, \psi_\nu\}]^+ - \{\psi_\nu, [A_\mu, \mathcal{Q}]\}^+ + f_{bca} T^a (F_{\mu\nu}^{+b} - D_{\mu\nu}^b) \phi^c \\
&= -2(D_{[\mu} D_{\nu]} \phi)^+ + \{\psi_{[\nu}, \psi_{\mu]}\}^+ + [F_{\mu\nu}^+, \phi] + [\phi, D_{\mu\nu}] \\
&= [\phi, D_{\mu\nu}]
\end{aligned} \tag{D.4.155}$$

where we used $[D_\mu, D_\nu] = F_{\mu\nu}$. Next, on hypermultiplet, we have

$$\begin{aligned}
\{\mathcal{Q}, [\mathcal{Q}, M_{\dot{\alpha}}]\} &= \{\mathcal{Q}, \tilde{\mu}_{\dot{\alpha}}\} = -\phi M_{\dot{\alpha}} + m M_{\dot{\alpha}} \\
\{\mathcal{Q}, [\mathcal{Q}, \tilde{M}_{\dot{\alpha}}]\} &= \{\mathcal{Q}, \tilde{\lambda}_{\dot{\alpha}}\} = \tilde{M}_{\dot{\alpha}} \phi - m \tilde{M}_{\dot{\alpha}} \\
[\mathcal{Q}, \{\mathcal{Q}, \tilde{\mu}_{\dot{\alpha}}\}] &= (m - \phi) [\mathcal{Q}, M_{\dot{\alpha}}] = -\phi \tilde{\mu}_{\dot{\alpha}} + m \tilde{\mu}_{\dot{\alpha}} \\
[\mathcal{Q}, \{\mathcal{Q}, \tilde{\lambda}_{\dot{\alpha}}\}] &= [\mathcal{Q}, \tilde{M}_{\dot{\alpha}}] (-m + \phi) = \tilde{\lambda}_{\dot{\alpha}} \phi - m \tilde{\lambda}_{\dot{\alpha}} \\
[\mathcal{Q}, \{\mathcal{Q}, \lambda_\alpha\}] &= [\mathcal{Q}, h_\alpha] = -\phi \lambda_\alpha + m \lambda_\alpha \\
[\mathcal{Q}, \{\mathcal{Q}, \mu_\alpha\}] &= [\mathcal{Q}, \tilde{h}_\alpha] = \mu_\alpha \phi - m \lambda_\alpha .
\end{aligned} \tag{D.4.156}$$

This shows that on the fields, (D.4.143) is true.

$\mathcal{Q}$ -exactness of the action

We now show that the action is $\mathcal{Q}$ -exact up to topological term. We claim that

$$S_{\mathcal{N}=2}^{\text{twisted}} = \{\mathcal{Q}, V\} + \frac{i\tau}{16\pi} \int \text{Tr} (F \wedge F) \quad (\text{D.4.157})$$

where

$$\begin{aligned} V = & \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left(\frac{1}{2} \chi^{\mu\nu} (F_{\mu\nu}^+ + D_{\mu\nu}) - \frac{1}{2} \eta [\phi, \tilde{\phi}] + \psi^\mu D_\mu \tilde{\phi} \right) \\ & + \int d^4x \sqrt{g} \left(D_\mu \widetilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha + D_\mu M^{\dot{\alpha}} \sigma_{\dot{\alpha}\alpha}^\mu \mu^\alpha + \frac{1}{2} m \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} + \frac{1}{2} \mu_\alpha h^\alpha + \frac{1}{2} m \widetilde{M}^{\dot{\alpha}} \tilde{\mu}_{\dot{\alpha}} \right. \\ & \left. - \frac{1}{2} \tilde{h}_\alpha \lambda^\alpha + \frac{1}{2} \left(\widetilde{M}^{\dot{\alpha}} \tilde{\phi} \tilde{\mu}_{\dot{\alpha}} + \tilde{\lambda}^{\dot{\alpha}} \tilde{\phi} M_{\dot{\alpha}} \right) - \frac{1}{2} \widetilde{M}^{\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta}} \right), \end{aligned} \quad (\text{D.4.158})$$

and

$$\tau = \frac{4\pi i}{g^2} + \frac{\Theta}{2\pi}, \quad (\text{D.4.159})$$

is the complexified coupling. Let us check term by term. We will repeatedly use the identities:

$$\begin{aligned} [A, BC] &= [A, B]C + B[A, C], \\ \{A, BC\} &= \{A, B\}C - B\{A, C\}, \\ \{A, BC\} &= [A, B]C + B\{A, C\}. \end{aligned} \quad (\text{D.4.160})$$

We have

$$\begin{aligned} \{\mathcal{Q}, \chi_{\mu\nu} F^{+\mu\nu}\} &= -\chi_{\mu\nu} [\mathcal{Q}, F^{+\mu\nu}] + \{\mathcal{Q}, \chi_{\mu\nu}\} F^{+\mu\nu} \\ &= -2\chi_{\mu\nu} (d_A \psi)^{+\mu\nu} + (F_{\mu\nu}^+ - D_{\mu\nu}) F^{+\mu\nu} \end{aligned} \quad (\text{D.4.161})$$

$$\begin{aligned} \{\mathcal{Q}, \chi_{\mu\nu} D^{\mu\nu}\} &= -\chi_{\mu\nu} [\mathcal{Q}, D^{\mu\nu}] + \{\mathcal{Q}, \chi_{\mu\nu}\} D^{\mu\nu} \\ &= -2\chi_{\mu\nu} (d_A \psi)^{\mu\nu} - \chi_{\mu\nu} [\chi^{\mu\nu}, \phi] + F_{\mu\nu}^+ D^{\mu\nu} - D_{\mu\nu} D^{\mu\nu} \\ &= -2\chi_{\mu\nu} D^\mu \psi^\nu - \chi_{\mu\nu} [\chi^{\mu\nu}, \phi] + F_{\mu\nu}^+ D^{\mu\nu} - D_{\mu\nu} D^{\mu\nu} \\ &= -2\chi_{\mu\nu} D^\mu \psi^\nu - \chi_{\mu\nu} [\chi^{\mu\nu}, \phi] + F_{\mu\nu}^+ D^{\mu\nu} - D_{\mu\nu} D^{\mu\nu}. \end{aligned} \quad (\text{D.4.162})$$

$$\{\mathcal{Q}, \eta[\phi, \tilde{\phi}]\} = \{\mathcal{Q}, \eta\}[\phi, \tilde{\phi}] - \eta[\phi, [\tilde{\phi}, \mathcal{Q}]] = [\phi, \tilde{\phi}]^2 + \eta[\phi, \eta]. \quad (\text{D.4.163})$$

$$\begin{aligned} \{\mathcal{Q}, \psi^\mu D_\mu \tilde{\phi}\} &= \{\mathcal{Q}, \psi^\mu\} D_\mu \tilde{\phi} - \psi^\mu [\mathcal{Q}, D_\mu \tilde{\phi}] \\ &= -D^\mu \phi D_\mu \tilde{\phi} - \psi^\mu \left(\partial_\mu [\mathcal{Q}, \tilde{\phi}] + [\mathcal{Q}, [A_\mu, \tilde{\phi}]] \right) \\ &= -D^\mu \phi D_\mu \tilde{\phi} - \psi^\mu \left(\partial_\mu \eta - [\tilde{\phi}, [\mathcal{Q}, A_\mu]] - [A_\mu, [\tilde{\phi}, \mathcal{Q}]] \right) \\ &= -D^\mu \phi D_\mu \tilde{\phi} - \psi^\mu \partial_\mu \eta + \psi^\mu [\tilde{\phi}, \psi_\mu] - \psi^\mu [A_\mu, \eta] \\ &= -D^\mu \phi D_\mu \tilde{\phi} - \psi^\mu D_\mu \eta + \psi^\mu [\tilde{\phi}, \psi_\mu] \end{aligned} \quad (\text{D.4.164})$$

$$\begin{aligned}
\{\mathcal{Q}, D_\mu \widetilde{M}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha\} &= [\mathcal{Q}, \partial_\mu \widetilde{M}^{\dot{\alpha}} + A_\mu \widetilde{M}^{\dot{\alpha}}] \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha + D_\mu \widetilde{M}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu \{\mathcal{Q}, \lambda^\alpha\} \\
&= \left(\partial_\mu \widetilde{\lambda}^{\dot{\alpha}} - [\mathcal{Q}, A_\mu] \cdot \widetilde{M}^{\dot{\alpha}} - A_\mu \cdot [\widetilde{M}^{\dot{\alpha}}, \mathcal{Q}] \right) \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha + D_\mu \widetilde{M}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu h^\alpha \\
&= D_\mu \widetilde{\lambda}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha - \psi_\mu \widetilde{M}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha + D_\mu \widetilde{M}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu h^\alpha \\
&= D_\mu \widetilde{M}^{\dot{\alpha}} \widetilde{\sigma}_{\dot{\alpha}\alpha}^\mu h^\alpha + \lambda^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \widetilde{M}^{\dot{\alpha}} \psi_\mu + \lambda^\alpha \sigma_{\alpha\dot{\alpha}}^\mu D_\mu \widetilde{\lambda}^{\dot{\alpha}}
\end{aligned} \tag{D.4.165}$$

$$\begin{aligned}
\{\mathcal{Q}, D_\mu M^{\dot{\alpha}} \sigma_{\dot{\alpha}\alpha}^\mu \mu^\alpha\} &= [\mathcal{Q}, \partial_\mu M^{\dot{\alpha}} + A_\mu M^{\dot{\alpha}}] \sigma_{\dot{\alpha}\alpha}^\mu \mu^\alpha + D_\mu M^{\dot{\alpha}} \sigma_{\dot{\alpha}\alpha}^\mu \{\mathcal{Q}, \mu^\alpha\} \\
&= D_\mu \widetilde{\mu}^{\dot{\alpha}} \sigma_{\dot{\alpha}\alpha}^\mu \mu^\alpha + \psi_\mu M^{\dot{\alpha}} \sigma_{\dot{\alpha}\alpha}^\mu \mu^\alpha + D_\mu M^{\dot{\alpha}} \sigma_{\dot{\alpha}\alpha}^\mu \widetilde{h}^{\dot{\alpha}} \\
&= \mu \sigma^\mu D_\mu \widetilde{\mu} + \mu \sigma^\mu \psi_\mu M - \widetilde{h} \sigma^\mu D_\mu M
\end{aligned} \tag{D.4.166}$$

$$\begin{aligned}
\{\mathcal{Q}, M_{\dot{\alpha}} \widetilde{\lambda}^{\dot{\alpha}}\} &= [\mathcal{Q}, M_{\dot{\alpha}}] \widetilde{\lambda}^{\dot{\alpha}} + \{\mathcal{Q}, \widetilde{\lambda}^{\dot{\alpha}}\} M_{\dot{\alpha}} \\
&= \widetilde{\lambda}^{\dot{\alpha}} \widetilde{\mu}_{\dot{\alpha}} + (-m \widetilde{M}^{\dot{\alpha}} + \widetilde{M}^{\dot{\alpha}} \phi) M_{\dot{\alpha}} \\
&= -m \widetilde{M}^{\dot{\alpha}} M_{\dot{\alpha}} + \widetilde{M}_{\dot{\alpha}} \phi \cdot M_{\dot{\alpha}} + \widetilde{\lambda} \widetilde{\mu}
\end{aligned} \tag{D.4.167}$$

$$\begin{aligned}
\{\mathcal{Q}, \mu_\alpha h^\alpha\} &= \{\mathcal{Q}, \mu_\alpha\} h^\alpha - \mu_\alpha [\mathcal{Q}, h^\alpha] \\
&= \widetilde{h}_\alpha h^\alpha - m \mu_\alpha \lambda^\alpha + \mu_\alpha \phi \cdot \lambda^\alpha \\
&= \widetilde{h}_\alpha h^\alpha + m \mu \lambda - \mu \phi \cdot \lambda
\end{aligned} \tag{D.4.168}$$

$$\begin{aligned}
\{\mathcal{Q}, \widetilde{\mu}^{\dot{\alpha}} \widetilde{M}_{\dot{\alpha}}\} &= \widetilde{\mu}^{\dot{\alpha}} [\mathcal{Q}, \widetilde{M}_{\dot{\alpha}}] + \{\mathcal{Q}, \widetilde{\mu}^{\dot{\alpha}}\} \widetilde{M}_{\dot{\alpha}} \\
&= \widetilde{\mu}^{\dot{\alpha}} \widetilde{\lambda}_{\dot{\alpha}} + m M^{\dot{\alpha}} \widetilde{M}_{\dot{\alpha}} - M^{\dot{\alpha}} \phi \widetilde{M}_{\dot{\alpha}}
\end{aligned} \tag{D.4.169}$$

$$\begin{aligned}
\{\mathcal{Q}, \lambda_\alpha \widetilde{h}^\alpha\} &= -\lambda_\alpha [\mathcal{Q}, \widetilde{h}^\alpha] + \{\mathcal{Q}, \lambda_\alpha\} \widetilde{h}^\alpha = m \lambda_\alpha \mu^\alpha - \lambda_\alpha \phi \mu^\alpha + h_\alpha \widetilde{h}^\alpha \\
&= -m \lambda \mu - \lambda_\alpha \phi \mu^\alpha + h_\alpha \widetilde{h}^\alpha
\end{aligned} \tag{D.4.170}$$

$$\begin{aligned}
\{\mathcal{Q}, \widetilde{\lambda}_{\dot{\alpha}} \widetilde{\phi} M^{\dot{\alpha}}\} &= -\widetilde{\lambda}_{\dot{\alpha}} [\mathcal{Q}, \widetilde{\phi} M^{\dot{\alpha}}] + \{\mathcal{Q}, \widetilde{\lambda}_{\dot{\alpha}}\} \widetilde{\phi} M^{\dot{\alpha}} \\
&= -\widetilde{\lambda}_{\dot{\alpha}} \left([\mathcal{Q}, \widetilde{\phi}] M^{\dot{\alpha}} + \widetilde{\phi} [\mathcal{Q}, M^{\dot{\alpha}}] \right) - m \widetilde{M}_{\dot{\alpha}} \widetilde{\phi} M^{\dot{\alpha}} + \widetilde{M}_{\dot{\alpha}} \phi \widetilde{\phi} M^{\dot{\alpha}} \\
&= \eta \widetilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \widetilde{\lambda}_{\dot{\alpha}} \widetilde{\phi} \widetilde{\mu}^{\dot{\alpha}} - m \widetilde{M}_{\dot{\alpha}} \widetilde{\phi} M^{\dot{\alpha}} + \widetilde{M}_{\dot{\alpha}} \phi \widetilde{\phi} M^{\dot{\alpha}}
\end{aligned} \tag{D.4.171}$$

$$\begin{aligned}
\{\mathcal{Q}, \widetilde{M}^{\dot{\alpha}} \widetilde{\phi} \widetilde{\mu}_{\dot{\alpha}}\} &= [\mathcal{Q}, \widetilde{M}^{\dot{\alpha}} \widetilde{\phi}] \widetilde{\mu}_{\dot{\alpha}} + \widetilde{M}^{\dot{\alpha}} \phi \{\mathcal{Q}, \widetilde{\mu}_{\dot{\alpha}}\} \\
&= \left([\mathcal{Q}, \widetilde{M}^{\dot{\alpha}}] \widetilde{\phi} + \widetilde{M}^{\dot{\alpha}} [\mathcal{Q}, \widetilde{\phi}] \right) \widetilde{\mu}_{\dot{\alpha}} + m \widetilde{M}^{\dot{\alpha}} \widetilde{\phi} M_\alpha - \widetilde{M}^{\dot{\alpha}} \phi \widetilde{\phi} M_\alpha \\
&= \widetilde{\lambda}^{\dot{\alpha}} \widetilde{\phi} \widetilde{\mu}_{\dot{\alpha}} + \widetilde{M}^{\dot{\alpha}} \eta \widetilde{\mu}_{\dot{\alpha}} + m \widetilde{M}^{\dot{\alpha}} \widetilde{\phi} M_{\dot{\alpha}} - \widetilde{M}^{\dot{\alpha}} \phi \widetilde{\phi} M_{\dot{\alpha}}
\end{aligned} \tag{D.4.172}$$

$$\begin{aligned}
\{\mathcal{Q}, \widetilde{M}^{(\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}}^a M^{\dot{\beta}})\} &= \{\mathcal{Q}, \chi_{\dot{\alpha}\dot{\beta}}^a\} \widetilde{M}^{(\dot{\alpha}} T_R^a M^{\dot{\beta}}) - \chi_{\dot{\alpha}\dot{\beta}}^a [\mathcal{Q}, \widetilde{M}^{(\dot{\alpha}} T_R^a M^{\dot{\beta}})] \\
&= \left(F_{\dot{\alpha}\dot{\beta}}^{+a} - D_{\dot{\alpha}\dot{\beta}}^a \right) \widetilde{M}^{(\dot{\alpha}} T_R^a M^{\dot{\beta}}) - \chi_{\dot{\alpha}\dot{\beta}}^a [\mathcal{Q}, \widetilde{M}^{(\dot{\alpha}} T_R^a M^{\dot{\beta}}) - \chi_{\dot{\alpha}\dot{\beta}}^a \widetilde{M}^{(\dot{\alpha}} T_R^a [\mathcal{Q}, M^{\dot{\beta}}]] \\
&= \widetilde{M}^{(\dot{\alpha}} F_{\dot{\alpha}\dot{\beta}}^+ M^{\dot{\beta}}) - \widetilde{M}^{(\dot{\alpha}} D_{\dot{\alpha}\dot{\beta}}^a M^{\dot{\beta}}) - \widetilde{\lambda}^{(\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}}^a M^{\dot{\beta}}) - \widetilde{M}^{(\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}}^a \widetilde{\chi}^{\dot{\beta}}) .
\end{aligned} \tag{D.4.173}$$

Combining all terms and using

$$\begin{aligned} \text{Tr}(F^+ \wedge \star F^+) &= \text{Tr} \left(\frac{1}{2} F \wedge \star F + \frac{1}{2} F \wedge F \right) \\ \Rightarrow \frac{1}{2} \text{Tr}(F_{\mu\nu}^+ F^{+\mu\nu}) &= \frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu} F_{\rho\sigma}) , \end{aligned} \quad (\text{D.4.174})$$

we get

$$\begin{aligned} \{\mathcal{Q}, V\} &= \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left(\frac{1}{2} \{\mathcal{Q}, \chi^{\mu\nu} (F_{\mu\nu}^+ + D_{\mu\nu})\} - \frac{1}{2} \{\mathcal{Q}, \eta[\phi, \tilde{\phi}]\} + \{\mathcal{Q}, \psi^\mu D_\mu \tilde{\phi}\} \right) \\ &\quad + \int d^4x \sqrt{g} \left(\{\mathcal{Q}, D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \lambda^\alpha\} + \{\mathcal{Q}, D_\mu M^\alpha \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu \mu^{\dot{\alpha}}\} + \frac{1}{2} \{\mathcal{Q}, m \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}}\} - \frac{1}{2} \{\mathcal{Q}, m \tilde{M}^{\dot{\alpha}} \tilde{\mu}_{\dot{\alpha}}\} \right. \\ &\quad \left. + \frac{1}{2} \{\mathcal{Q}, \mu_\alpha h^\alpha\} - \frac{1}{2} \{\mathcal{Q}, \tilde{h}_{\dot{\alpha}} \lambda^{\dot{\alpha}}\} + \frac{1}{2} \{\mathcal{Q}, (\tilde{M}^{\dot{\alpha}} \tilde{\phi} \tilde{\mu}_{\dot{\alpha}} + \tilde{\lambda}^{\dot{\alpha}} \tilde{\phi} M_{\dot{\alpha}})\} - \frac{1}{2} \{\mathcal{Q}, \tilde{M}^{(\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta})}\} \right) \\ &= \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{2} (-2\chi_{\mu\nu} D^\mu \psi^\nu + F_{\mu\nu}^+ F^{+\mu\nu} - D_{\mu\nu} F^{+\mu\nu}) \right. \\ &\quad \left. + \frac{1}{2} (-2\chi_{\mu\nu} D^\mu \psi^\nu - \chi_{\mu\nu} [\chi^{\mu\nu}, \phi] + F_{\mu\nu}^+ D^{\mu\nu} - D_{\mu\nu} D^{\mu\nu} \right. \\ &\quad \left. + (\tilde{M}_{(\dot{\alpha}} \chi^{\dot{\alpha}\dot{\beta}} \tilde{\mu}_{\dot{\beta}}) + \tilde{\lambda}_{(\dot{\alpha}} \chi^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}})) \right] - \frac{1}{2} ([\phi, \tilde{\phi}]^2 + \eta[\phi, \eta]) + (-D^\mu \phi D_\mu \tilde{\phi} - \psi^\mu D_\mu \eta + \psi^\mu [\tilde{\phi}, \psi_\mu]) \\ &\quad + \frac{1}{8g^2} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\lambda} \text{Tr}(F_{\mu\nu} F_{\rho\lambda}) + \frac{1}{g^2} \int d^4x \sqrt{g} \left[(D_\mu \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu h^\alpha + \lambda^\alpha \sigma_{\dot{\alpha}\alpha}^\mu \tilde{M}^{\dot{\alpha}} \psi_\mu + \lambda^\alpha \sigma_{\dot{\alpha}\alpha}^\mu D_\mu \tilde{\lambda}^{\dot{\alpha}}) \right. \\ &\quad \left. - \frac{1}{2} (-\tilde{\lambda}_{\dot{\alpha}} \chi^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} - \tilde{M}_{\dot{\alpha}} \chi^{\dot{\alpha}\dot{\beta}} \tilde{\mu}_{\dot{\beta}} + \tilde{M}_{\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} - \tilde{M}_{\dot{\alpha}} D^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}) \right. \\ &\quad \left. + (\mu \sigma^\mu D_\mu \tilde{\mu} + \mu \sigma^\mu M \psi_\mu - \tilde{h} \sigma^\mu D_\mu M) \right] + \frac{m}{2} (-m \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} + \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} + \tilde{\lambda} \tilde{\mu}) \\ &\quad - \frac{1}{2} (\tilde{h}_\alpha h^\alpha + m \mu \lambda - \mu \phi \lambda) - \frac{m}{2} (\tilde{\mu} \tilde{\lambda} + m \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} - \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}}) \\ &\quad + \frac{1}{2} (-m \mu \lambda - \mu_\alpha \phi \lambda^\alpha + h_\alpha \tilde{h}^\alpha) + (-m \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} + \tilde{M}_{\dot{\alpha}} \phi \tilde{\phi} M^{\dot{\alpha}} + \eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \tilde{\lambda}_{\dot{\alpha}} \tilde{\phi} \tilde{\mu}^{\dot{\alpha}}) \\ &\quad + (\tilde{\lambda}^{\dot{\alpha}} \tilde{\phi} \tilde{\mu}_{\dot{\alpha}} + \tilde{M}^{\dot{\alpha}} \eta \tilde{\mu}_{\dot{\alpha}} - m \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} + \tilde{M}_{\dot{\alpha}} \phi \tilde{\phi} M^{\dot{\alpha}}) \\ &= \frac{1}{g^2} \int d^4x \sqrt{g} \text{Tr} \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \tilde{\phi} D^\mu \phi - 4i \chi_{\mu\nu} D^\mu \psi^\nu - 2i \eta D_\mu \psi^\mu + \frac{1}{2} D_{\mu\nu} D^{\mu\nu} \right. \\ &\quad \left. - \frac{1}{2} [\phi, \tilde{\phi}]^2 + \sqrt{2} i \tilde{\phi} [\psi_\mu, \psi^\mu] + \sqrt{2} i \chi_{\mu\nu} [\phi, \chi^{\mu\nu}] - \sqrt{2} i \phi [\eta, \eta] \right] \\ &\quad + \frac{1}{8g^2} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\lambda} \text{Tr}(F_{\mu\nu} F_{\rho\lambda}) \\ &\quad + \int d^4x \sqrt{g} \left[-\tilde{h}^\alpha \sigma_{\dot{\alpha}\alpha}^\mu D_\mu M^{\dot{\alpha}} + D_\mu \tilde{M}_{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^\mu h^\alpha - \tilde{h} h - m^2 \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} + m \tilde{M}_{\dot{\alpha}} \phi M^{\dot{\alpha}} \right. \\ &\quad \left. - 2m \tilde{M}_{\dot{\alpha}} \tilde{\phi} M^{\dot{\alpha}} + \tilde{\lambda}_{\dot{\alpha}} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \chi_{\mu\nu} + \eta \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} \tilde{\mu}_{\dot{\beta}} \chi_{\mu\nu} \right. \\ &\quad \left. - \eta \tilde{M}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} + \lambda \sigma^\mu D_\mu \tilde{\lambda} - \tilde{M}_{\dot{\alpha}} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} \lambda_\alpha \psi_\mu + \mu^\alpha \sigma_{\dot{\alpha}\alpha}^\mu \tilde{M}^{\dot{\alpha}} \psi_\mu + \mu \sigma^\mu D_\mu \tilde{\mu} \right. \\ &\quad \left. + \mu \phi \lambda - 2\tilde{\lambda} \tilde{\phi} \tilde{\mu} - m \mu \lambda - m \tilde{\lambda} \tilde{\mu} \right. \\ &\quad \left. + \tilde{M}_{\dot{\alpha}} \{\phi, \tilde{\phi}\} M^{\dot{\alpha}} - \tilde{M}_{(\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} + \tilde{M}_{(\dot{\alpha}} D^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta})} \right] . \end{aligned} \quad (\text{D.4.175})$$

Thus, we obtain

$$\begin{aligned}
S_{\mathcal{N}=2}^{\text{twisted}} &= \{\mathcal{Q}, V\} - \frac{1}{8g^2} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\lambda} \text{Tr}(F_{\mu\nu} F_{\rho\lambda}) + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu} F_{\rho\sigma}) \\
&= \{\mathcal{Q}, V\} + \frac{i}{32\pi} \left(\frac{4\pi}{g^2} i + \frac{\Theta}{2\pi} \right) \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu} F_{\rho\sigma}) \\
&= \{\mathcal{Q}, V\} + \frac{i\tau}{16\pi} \int \text{Tr}(F \wedge F)
\end{aligned} \tag{D.4.176}$$

It is straightforward to include multiple flavors of hypermultiplets. We now check that the twisted action is $\mathcal{Q}$ -closed. Indeed,

$$\begin{aligned}
\{\mathcal{Q}, S_{\mathcal{N}=2}^{\text{twisted}}\} &= \{\mathcal{Q}, \{\mathcal{Q}, V\}\} - \{\mathcal{Q}, \text{top. term}\} \\
&= -\frac{1}{2} Z \cdot V + \text{total derivative}
\end{aligned} \tag{D.4.177}$$

It is easy to see that $Z \cdot V = 0$ from the charges given in Table 4.6.

D.4.5 Twisted action on nonspin manifolds: inclusion of background fields

To write the twisted action with background fields such that the (untwisted) theory makes sense on nonspin manifolds (see Remark 4.2.3), we first write the untwisted action with *massless* hypermultiplets for the gauge group $G \times \mathbb{T}^f$ and then “freeze” the background $\mathbb{T}^f$ vector multiplet in a special way consistent with the cohomological conditions. This is again done using the $\mathcal{N} = 2$ conformal supergravity background. To then get the twisted action, we change variables from untwisted fields to twisted fields. We will directly write the twisted action with background fields. To this end, let

$$V_{\mathcal{N}=2}^u = \left(\mathcal{A}_\mu^u, \phi^u, \tilde{\phi}_u, \psi_\mu^u, \chi_{\alpha\dot{\beta}}^u, \eta^u, D_{\alpha\dot{\beta}}^u \right), \quad u = 1, \dots, \text{rank}(\mathbb{T}^f), \tag{D.4.178}$$

be the background vector multiplets. The $\mathcal{Q}$ -transformations on these fields is given by

$$\begin{aligned}
[\mathcal{Q}, \phi^u] &= 0, \quad [\mathcal{Q}, \tilde{\phi}_u] = 0, \quad \{\mathcal{Q}, \eta^u\} = [\phi^u, \tilde{\phi}_u] \quad (\text{no sum over } u) \\
[\mathcal{Q}, \mathcal{A}_\mu^u] &= \psi_\mu^u, \quad \{\mathcal{Q}, \psi_\mu^u\} = D_\mu \phi^u, \\
\{\mathcal{Q}, \chi_{\alpha\dot{\beta}}^u\} &= F_{\alpha\dot{\beta}}^{+u} - D_{\alpha\dot{\beta}}^u - 2i \widetilde{M}_{u(\dot{\alpha}} M_{\beta)}^u \quad (\text{no sum over } u) \\
[\mathcal{Q}, D_{\alpha\dot{\beta}}^u] &= 2(d_{\mathcal{A}^u} \psi^u)_{\alpha\dot{\beta}}^+ - 2i \left(\widetilde{M}_{u(\dot{\alpha}} \tilde{\mu}_{\beta)}^u + \tilde{\lambda}_{u(\dot{\alpha}} M_{\beta)}^u \right) \quad (\text{no sum over } u),
\end{aligned} \tag{D.4.179}$$

where we have taken the generator of $\text{U}(1)$ to be i . We now write the twisted action with these fields with the fermions as well as their $\mathcal{Q}$ -variation in the background vector multiplets set to 0 while the field strength $F_{\mu\nu}^u$ is set to first Chern class of the $\text{U}(1)/p_{3,u}(C^{\text{phys}})$ -bundle:

$$\left[\frac{F^u}{2\pi} \right] = c_1(u), \tag{D.4.180}$$

where $c_1(u)$ is as in (4.4.18). The scalars are set to the mass parameters of the hypermultiplets:

$$\phi^u = -m_u, \quad \tilde{\phi}_u = \frac{m_u}{2}. \tag{D.4.181}$$

Then the mass terms in the action is obtained from

$$\widetilde{M}_{\dot{\alpha}u}\{\phi, \widetilde{\phi}\}M^{\dot{\alpha}i} + \chi_u\phi\lambda^i - 2\widetilde{\lambda}_u\widetilde{\phi}\widetilde{\mu}^u \quad (\text{D.4.182})$$

by replacing one or more $\phi, \widetilde{\phi}$ by $-m_u, \frac{m_u}{2}$. We obtain

$$\begin{aligned} \widetilde{M}_{\dot{\alpha}u}\{\phi, \widetilde{\phi}\}M^{\dot{\alpha}u} + \mu_u\phi\lambda^u - 2\widetilde{\lambda}_u\widetilde{\phi}\widetilde{\mu}^u &\longrightarrow -2m_u\widetilde{M}_{\dot{\alpha}u}\widetilde{\phi}M^{\dot{\alpha}u} + m_u\widetilde{M}_{\dot{\alpha}u}\phi M^{\dot{\alpha}u} \\ &\quad - m_u^2\widetilde{M}_{\dot{\alpha}u}M^{\dot{\alpha}u} - m_u\mu_u\lambda^u - m_u\widetilde{\lambda}_u\widetilde{\mu}^u, \end{aligned} \quad (\text{D.4.183})$$

which are all the mass terms. All covariant derivatives are covariant with respect to the $\mathbb{T}^f$ symmetry and they appear explicitly in (4.7.12)–(4.7.18). The extra terms with background field strength, monopole fields and auxiliary fields is

$$\begin{aligned} \int d^4x \sqrt{g} \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} &\left(\frac{1}{4g^2} F_{\mu\nu}^u F_u^{\mu\nu} + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u - \frac{1}{g^2} \widetilde{M}_{v\dot{\alpha}} F_u^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}^v - \frac{1}{4g^2} D_{\dot{\alpha}\dot{\beta}}^u D_u^{\dot{\alpha}\dot{\beta}} \right. \\ &\quad \left. - \frac{1}{g^2} M^{u(\dot{\alpha}} M^{\dot{\beta})u} \widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u \right). \end{aligned} \quad (\text{D.4.184})$$

The $\mathcal{Q}$ -closed equation $\{\mathcal{Q}, \chi_{\dot{\alpha}\dot{\beta}}^u\} = 0$ gives

$$D_{\dot{\alpha}\dot{\beta}}^u = F_{\dot{\alpha}\dot{\beta}}^{+u} - 2i\widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u. \quad (\text{D.4.185})$$

We have

$$\begin{aligned} \frac{1}{4} D_{\dot{\alpha}\dot{\beta}}^u D_u^{\dot{\alpha}\dot{\beta}} &= \frac{1}{4} \left(F_{\dot{\alpha}\dot{\beta}}^{+u} - 2i\widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u \right) \left(F_u^{\dot{\alpha}\dot{\beta}} - 2i\widetilde{M}_u^{(\dot{\alpha}} M^{\dot{\beta})u} \right) \\ &= \frac{1}{4} F_{\dot{\alpha}\dot{\beta}}^u F_u^{\dot{\alpha}\dot{\beta}} - i F_{\dot{\alpha}\dot{\beta}}^u \widetilde{M}_u^{(\dot{\alpha}} M^{\dot{\beta})i} - \widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u \widetilde{M}_u^{(\dot{\alpha}} M^{\dot{\beta})u} \end{aligned} \quad (\text{D.4.186})$$

The extra term becomes

$$\begin{aligned} \int d^4x \sqrt{g} \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} &\left(\frac{1}{4g^2} F_{\mu\nu}^u F_u^{\mu\nu} + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u - \frac{1}{g^2} \widetilde{M}_{u\dot{\alpha}} F_u^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}^u - \frac{1}{4g^2} D_{\dot{\alpha}\dot{\beta}}^u D_u^{\dot{\alpha}\dot{\beta}} \right. \\ &\quad \left. - \frac{1}{g^2} M^{u(\dot{\alpha}} M^{\dot{\beta})u} \widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u \right) \\ &= \int d^4x \sqrt{g} \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \left(\frac{1}{4g^2} F_{\mu\nu}^u F_u^{\mu\nu} + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u - \frac{1}{g^2} i F_{\dot{\alpha}\dot{\beta}}^u M_u^{(\dot{\alpha}} M^{\dot{\beta})u} - \frac{1}{4g^2} F_{\dot{\alpha}\dot{\beta}}^u F_u^{\dot{\alpha}\dot{\beta}} \right. \\ &\quad \left. + \frac{1}{g^2} i F_{\dot{\alpha}\dot{\beta}}^u \widetilde{M}_u^{(\dot{\alpha}} M^{\dot{\beta})u} + \frac{1}{g^2} \widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u \widetilde{M}_u^{(\dot{\alpha}} M^{\dot{\beta})u} - \frac{1}{g^2} M^{u(\dot{\alpha}} M^{\dot{\beta})u} \widetilde{M}_{u(\dot{\alpha}} M_{\dot{\beta})}^u \right) \\ &= \int d^4x \sqrt{g} \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \left(\frac{1}{4g^2} F_{\mu\nu}^u F_u^{\mu\nu} + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u - \frac{1}{4g^2} F_{\dot{\alpha}\dot{\beta}}^u F_u^{\dot{\alpha}\dot{\beta}} \right). \end{aligned} \quad (\text{D.4.187})$$

Now

$$\begin{aligned}
F_{\dot{\alpha}\dot{\beta}}^{+u} F_u^{+\dot{\alpha}\dot{\beta}} &= (\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}} \dot{F}_{\mu\nu}^u (\tilde{\sigma}^{\rho\sigma})^{\dot{\alpha}\dot{\beta}} F_{\rho\sigma}^u \\
&= -(\tilde{\sigma}^{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} (\tilde{\sigma}^{\rho\sigma})^{\dot{\alpha}}_{\dot{\beta}} F_{\mu\nu}^u F_{\rho\sigma}^u \\
&= F_{\mu\nu}^u F_{\rho\sigma}^u \left(\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} + \frac{1}{2} \delta^{\mu\rho} \delta^{\nu\sigma} - \frac{1}{2} \delta^{\mu\sigma} \delta^{\nu\rho} \right) \\
&= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u + \frac{1}{2} F_{\mu\nu}^u F_u^{\mu\nu} - \frac{1}{2} F_{\mu\nu}^u F_u^{\nu\mu} \\
&= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u + F_{\mu\nu}^u F_u^{\mu\nu} .
\end{aligned} \tag{D.4.188}$$

Thus the extra term becomes

$$\begin{aligned}
\sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \left(-\frac{1}{8g^2} + \frac{i\Theta}{64\pi^2} \right) \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u &= \frac{i}{32\pi} \left(\frac{4\pi}{g^2} i + \frac{\Theta}{2\pi} \right) \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u \\
&= \frac{i\tau}{32\pi} \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u ,
\end{aligned} \tag{D.4.189}$$

which is a topological term. The full action with background fields is thus written as

$$\begin{aligned}
S_{\mathcal{N}=2+\text{back}}^{\text{twisted}} &= \frac{1}{g^2} \int d^4x \sqrt{g} \left[\text{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_{\mu\nu} D^{\mu\nu} - 2\chi_{\mu\nu} D^\mu \psi^\nu - \frac{1}{2} \chi_{\mu\nu} [\chi^{\mu\nu}, \phi] - \eta D_\mu \psi^\mu \right. \right. \\
&\quad \left. \left. - \tilde{\phi} [\psi_\mu, \psi^\mu] + \tilde{\phi} D_\mu D^\mu \phi - \frac{1}{2} \phi [\eta, \eta] - \frac{1}{2} [\phi, \tilde{\phi}]^2 \right) \right. \\
&\quad \left. + \frac{\Theta i}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr}(F_{\mu\nu} F_{\rho\sigma}) + \frac{i\tau}{32\pi} \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^u F_{\rho\sigma}^u \right] \\
&\quad + \frac{1}{g^2} \int d^4x \sqrt{g} \left[-\tilde{h}_u^\alpha \sigma_{\dot{\alpha}\dot{\beta}}^\mu D_\mu M^{\dot{\alpha}u} - D_\mu \tilde{M}_u^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^\mu h^{\alpha u} - \tilde{h}_{\alpha u} h^{\alpha u} + \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} \left(-m_u^2 \tilde{M}_{\dot{\alpha}u} M^{\dot{\alpha}u} \right. \right. \\
&\quad \left. \left. + m_u \tilde{M}_{\dot{\alpha}u} \phi M^{\dot{\alpha}u} - 2m_u \tilde{M}_{\dot{\alpha}u} \tilde{\phi} M^{\dot{\alpha}i} - m_u \mu_u \lambda^u - m_u \tilde{\lambda}_u \tilde{\mu}^u \right) \right. \\
&\quad \left. + \tilde{\lambda}_{\dot{\alpha}u} (\tilde{\sigma}^{\mu\nu})^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}^u \chi_{\mu\nu} + \eta \tilde{\lambda}_{\dot{\alpha}u} M^{\dot{\alpha}u} - \tilde{M}_u^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\dot{\beta}}^{\mu\nu} \tilde{\mu}^{\dot{\beta}u} \chi_{\mu\nu} \right. \\
&\quad \left. - \eta \tilde{M}_{\dot{\alpha}u} \tilde{\mu}^{\dot{\alpha}u} + \lambda_u \sigma^\mu D_\mu \tilde{\lambda}^u - \tilde{M}_{\dot{\alpha}u} (\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} \lambda_{\alpha u} \psi_\mu + \mu_u^\alpha \sigma_{\dot{\alpha}\dot{\beta}}^\mu M^{\dot{\beta}i} \psi_\mu \right. \\
&\quad \left. + \mu_u \sigma^\mu D_\mu \tilde{\mu}^u + \mu_u \phi \lambda^u - 2\tilde{\lambda}_u \tilde{\phi} \tilde{\mu}^u + \tilde{M}_{\dot{\alpha}u} \{ \phi, \tilde{\phi} \} M^{\dot{\alpha}u} - M_u^{(\dot{\alpha}} F_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta})u} \right. \\
&\quad \left. + \tilde{M}_{u(\dot{\alpha}} D^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}^u \right] .
\end{aligned} \tag{D.4.190}$$

We have already shown that this action is $\mathcal{Q}$ -exact up to a topological term. More precisely

$$S_{\mathcal{N}=2+\text{back}}^{\text{twisted}} = \{ \mathcal{Q}, V \} + \frac{i\tau}{16\pi} \int \left(\text{Tr}(F \wedge F) + \sum_{u=1}^{\text{rank}(\mathbb{T}^f)} F^u \wedge F^u \right) , \tag{D.4.191}$$

where V is still given by (D.4.158) but now the covariant derivatives in V are given as in (4.7.12)–(4.7.18).

Action with General Coupling Matrix

The twisted action (D.4.190) can be generalized by replacing the complexified coupling τ by a general symmetric coupling matrix τ_{IJ} where $I = (a, u)$ are the Lie algebra indices for $G \times \mathbb{T}^f$. In general τ_{IJ} are taken to be field dependent and the action is written in terms of a complex function $\mathcal{F}$, called the prepotential. The potential satisfies the following *homogeneity relations*:

$$2\bar{\mathcal{F}} = \bar{\mathcal{F}}_I \tilde{\phi}^I, \quad \bar{\mathcal{F}}_{IJ} \tilde{\phi}^J = \bar{\mathcal{F}}_I, \quad \bar{\mathcal{F}}_{IJK} \tilde{\phi}^K = 0, \quad (\text{D.4.192})$$

$$2\mathcal{F} = \mathcal{F}_I \phi^I, \quad \mathcal{F}_{IJ} \phi^J = \mathcal{F}_I, \quad \mathcal{F}_{IJK} \phi^K = 0, \quad (\text{D.4.193})$$

where

$$\mathcal{F}_{IJK\dots} := \frac{\partial}{\partial \phi^I} \frac{\partial}{\partial \phi^J} \frac{\partial}{\partial \phi^K} \dots \mathcal{F}, \quad \bar{\mathcal{F}}_{IJK\dots} := \frac{\partial}{\partial \tilde{\phi}^I} \frac{\partial}{\partial \tilde{\phi}^J} \frac{\partial}{\partial \tilde{\phi}^K} \dots \bar{\mathcal{F}}. \quad (\text{D.4.194})$$

The coupling matrix is then identified as

$$\mathcal{F}_{IJ} = \tau_{IJ}, \quad \bar{\mathcal{F}}_{IJ} = \bar{\tau}_{IJ}. \quad (\text{D.4.195})$$

The hypermultiplet part of the action (D.4.150) does not depend on the prepotential and hence is unchanged, while the vector multiplet part of the action takes the form [CMRS23]:

$$\begin{aligned} \mathcal{S}_{\text{vec}}^{\text{general}} := & \frac{i}{16\pi} \int_{\mathbb{X}} d^4x \sqrt{g} \left[\frac{1}{2} (\bar{\tau}_{IJ} F_{\mu\nu}^{+I} F^{\mu\nu+J} - \tau_{IJ} F_{\mu\nu}^{-I} F^{\mu\nu-J}) + i (\text{Im} \tau_{IJ}) D_{\mu\nu}^I D^{\mu\nu,J} \right. \\ & + 2\bar{\mathcal{F}}_I D_\mu D^\mu \phi^I - 2\mathcal{F}_I D_\mu D^\mu \tilde{\phi}^I - 2\tau_{IJ} (D_\sigma \eta^I) \psi^{\sigma J} - 2\bar{\tau}_{IJ} \eta^I D_\sigma \psi^{\sigma,J} \\ & + 2\tau_{IJ} (D_\rho \chi_\sigma^{\rho I}) \psi^{\sigma J} + 2\bar{\tau}_{IJ} (D_{[\mu} \psi_{\nu]}^I) \chi^{\mu\nu J} \\ & + \frac{1}{2} \bar{\mathcal{F}}_{IJK} \eta^I (F_{\mu\nu}^{+J} + D_{\mu\nu}^J) \chi^{\mu\nu K} + \mathcal{F}_{IJK} (F^{\mu\nu-I} - D^{\mu\nu,I}) \psi_\mu^J \psi_\nu^K \\ & + 2\mathcal{F}_I [\eta, \eta]^I + 2\bar{\tau}_{IJ} \eta^I [\eta, \phi]^J + 2\mathcal{F}_I [\tilde{\phi}, [\phi, \tilde{\phi}]]^I - 2\bar{\mathcal{F}}_I [\phi, [\phi, \tilde{\phi}]]^I \\ & + \frac{i}{12} \bar{\mathcal{F}}_{IJKL} \eta^I \chi_\mu^{\rho,J} \chi^{\mu\sigma,K} \chi_{\rho\sigma}^L + \frac{i}{4} \bar{\mathcal{F}}_{IJK} (F_{\mu\rho}^{+I} - D_{\mu\rho}^I) \chi^{\mu\sigma,J} \chi_\sigma^{\rho K} \\ & - \frac{1}{2} \bar{\tau}_{IJ} [\phi, \chi_{\mu\nu}]^I \chi^{\mu\nu,J} + \frac{1}{2} \tau_{IJ} [\phi, \chi_{\mu\nu}]^I \chi^{\mu\nu J} - 2\bar{\mathcal{F}}_I [\psi_\mu, \psi^\mu]^I + 2\tau_{IJ} \psi_\mu^I [\psi^\mu, \tilde{\phi}]^J \Big] \\ & + \frac{i}{192\pi} \int_{\mathbb{X}} d^4x \mathcal{F}_{IJKL} \varepsilon^{\mu\nu\rho\sigma} \psi_\mu^I \psi_\nu^J \psi_\rho^K \psi_\sigma^L, \end{aligned} \quad (\text{D.4.196})$$

where

$$\text{Im} \tau_{IJ} := \frac{\mathcal{F}_{IJ} - \bar{\mathcal{F}}_{IJ}}{2i}. \quad (\text{D.4.197})$$

We can decompose this action into dynamical and background. $\text{Ad}(G \times \mathbb{T}^f)$ -invariance of τ_{IJ} implies that

$$\tau_{au} = \tau_{ua} = 0. \quad (\text{D.4.198})$$

Noting that in the background vector multiplets, the fermions are set to zero and the scalars are set to the mass parameters, we can use the homogeneity relations to get

$$\begin{aligned}
\bar{\mathcal{F}}_I D_\mu D^\mu \phi^I - \mathcal{F}_I D_\mu D^\mu \tilde{\phi}^I &= -2i(\text{Im}\tau_{ab})\tilde{\phi}^b D_\mu D^\mu \phi^a \\
\mathcal{F}_I [\tilde{\phi}, [\phi, \tilde{\phi}]]^I - \bar{\mathcal{F}}_I [\phi, [\phi, \tilde{\phi}]]^I &= i(\text{Im}\tau_{ab})[\phi, \tilde{\phi}]^a [\phi, \tilde{\phi}]^b \\
\mathcal{F}_I [\eta, \eta]^I + \bar{\tau}_{IJ} \eta^J [\eta, \phi]^I &= i(\text{Im}\tau_{ab})\phi^a [\eta, \eta]^b \\
\tau_{IJ} \psi_\mu^I \left[\psi^\mu, \tilde{\phi} \right]^J - \bar{\mathcal{F}}_I [\psi_\mu, \psi^\mu]^I &+ = 2i(\text{Im}\tau_{ab})\tilde{\phi}^a [\psi_\mu, \psi^\mu] .
\end{aligned} \tag{D.4.199}$$

Then we see that

$$\mathcal{S}_{\text{vec}}^{\text{general}} = \mathcal{S}_{\text{vec}}^{\text{dynamical}} + \mathcal{S}_{\text{vec}}^{\text{background}} , \tag{D.4.200}$$

where

$$\mathcal{S}_{\text{vec}}^{\text{background}} := \frac{i}{16\pi} \int_{\mathbb{X}} d^4x \sqrt{g} \left(\frac{1}{2} \bar{\tau}_{uv} F_{\mu\nu}^{+u} F^{\mu\nu+v} - \frac{1}{2} \tau_{uv} F_{\mu\nu}^{-u} F^{\mu\nu-v} + i(\text{Im}\tau_{uv}) D_{\mu\nu}^u D^{\mu\nu,v} \right) . \tag{D.4.201}$$

and

$$\begin{aligned}
\mathcal{S}_{\text{vec}}^{\text{dynamical}} &:= \frac{\text{Im}\tau_{ab}}{4\pi} \int_{\mathbb{X}} d^4x \sqrt{g} \left[\frac{1}{4} (F_{\mu\nu}^{+a} F^{\mu\nu+b} - D_{\mu\nu}^a D^{\mu\nu,b}) + \tilde{\phi}^a D_\mu D^\mu \phi^b - \eta^a D_\mu \psi^{\mu,b} \right. \\
&\quad + 2 (D_\mu \psi_\nu)^a \chi^{\mu\nu,b} - \tilde{\phi}^a [\psi_\mu, \psi^\mu]^b + \frac{1}{2} \chi_{\mu\nu}^a [\phi, \chi^{\mu\nu}]^b - \frac{1}{2} \phi^a [\eta, \eta]^b - \frac{1}{2} [\phi, \tilde{\phi}]^a [\phi, \tilde{\phi}]^b \Big] \\
&\quad + \frac{i}{16\pi} \int_{\mathbb{X}} d^4x \sqrt{g} \left[\frac{1}{2} \bar{\mathcal{F}}_{abc} \eta^a (F_{\mu\nu}^{+b} + D_{\mu\nu}^b) \chi^{\mu\nu,c} + \mathcal{F}_{abc} (F^{\mu\nu-a} - D^{\mu\nu,a}) \psi_\mu^b \psi_\nu^c \right. \\
&\quad + \frac{i}{12} \bar{\mathcal{F}}_{abcd} \eta^a \chi_\mu^{\rho,b} \chi^{\mu\sigma,c} \chi_{\rho\sigma}^d + \frac{i}{4} \bar{\mathcal{F}}_{abc} (F_{\mu\rho}^{+a} - D_{\mu\rho}^a) \chi^{\mu\sigma,b} \chi_\sigma^{\rho,c} \Big] \\
&\quad + \frac{i}{192\pi} \int_{\mathbb{X}} d^4x \mathcal{F}_{abcd} \varepsilon^{\mu\nu\rho\sigma} \psi_\mu^a \psi_\nu^b \psi_\rho^c \psi_\sigma^d + \frac{i}{32\pi} \tau_{ab} \int_{\mathbb{X}} F^a \wedge F^b ,
\end{aligned} \tag{D.4.202}$$

where we used the relation

$$\int_{\mathbb{X}} d^4x \sqrt{g} \left(\frac{1}{2} \bar{\tau}_{ab} F_{\mu\nu}^{+a} F^{\mu\nu+b} - \frac{1}{2} \tau_{ab} F_{\mu\nu}^{-a} F^{\mu\nu-b} \right) = \frac{1}{2} \tau_{ab} \int_{\mathbb{X}} F^a \wedge F^b - \int_{\mathbb{X}} d^4x \sqrt{g} i(\text{Im}\tau_{ab}) F_{\mu\nu}^{+a} F^{\mu\nu+b} . \tag{D.4.203}$$

To prove this, note that

$$\begin{aligned}
&\frac{1}{2} \tau_{ab} \int_{\mathbb{X}} F^a \wedge F^b - \int_{\mathbb{X}} d^4x \sqrt{g} i(\text{Im}\tau_{ab}) F_{\mu\nu}^{+a} F^{\mu\nu+b} \\
&= \frac{1}{2} \tau_{ab} \int_{\mathbb{X}} (F^{+a} + F^{-a}) \wedge (F^{+b} + F^{-b}) - \frac{\tau_{ab} - \bar{\tau}_{ab}}{2} \int_{\mathbb{X}} F^{+a} \wedge \star F^{+a} \\
&= \frac{1}{2} \bar{\tau}_{ab} \int_{\mathbb{X}} F^{+a} \wedge \star F^{+b} - \frac{1}{2} \tau_{ab} \int_{\mathbb{X}} F^{-a} \wedge \star F^{-b} \\
&= \int_{\mathbb{X}} d^4x \sqrt{g} \left(\frac{1}{2} \bar{\tau}_{ab} F_{\mu\nu}^{+a} F^{\mu\nu+b} - \frac{1}{2} \tau_{ab} F_{\mu\nu}^{-a} F^{\mu\nu-b} \right) .
\end{aligned} \tag{D.4.204}$$

Similarly, one can prove

$$\int_{\mathbb{X}} d^4x \sqrt{g} \left(\frac{1}{2} \bar{\tau}_{uv} F_{\mu\nu}^{+u} F^{\mu\nu+v} - \frac{1}{2} \tau_{uv} F_{\mu\nu}^{-u} F^{\mu\nu-v} \right) = \frac{1}{2} \tau_{uv} \int_{\mathbb{X}} F^u \wedge F^v - \int_{\mathbb{X}} d^4x \sqrt{g} i (\text{Im} \tau_{uv}) F_{\mu\nu}^{+u} F^{\mu\nu+v} . \quad (\text{D.4.205})$$

Finally using the $\mathcal{Q}$ -closed background configuration $D_{\mu\nu}^u = F_{\mu\nu}^{+u}$ which comes from $\{\mathcal{Q}, \chi_{\mu\nu}^u\} = 0$, we get

$$\begin{aligned} \mathcal{S}_{\text{vec}}^{\text{background}} &= \frac{i}{32\pi} \tau_{uv} \int_{\mathbb{X}} F^u \wedge F^v + \frac{1}{16\pi} \text{Im} \tau_{uv} \int_{\mathbb{X}} d^4x \sqrt{g} (F_{\mu\nu}^{+u} F^{\mu\nu+v} - D_{\mu\nu}^u D^{\mu\nu,v}) \\ &= \frac{i}{32\pi} \tau_{uv} \int_{\mathbb{X}} F^u \wedge F^v . \end{aligned} \quad (\text{D.4.206})$$

Collecting all these calculations together, we can write

$$\mathcal{S}_{\text{vec}}^{\text{general}} = \{\mathcal{Q}, V_{\text{vec}}\} + \mathcal{S}^{\text{topological}} + \mathcal{S}^{\text{extra}} , \quad (\text{D.4.207})$$

where

$$\begin{aligned} V_{\text{vec}} &:= \frac{\text{Im} \tau_{ab}}{4\pi} \int_{\mathbb{X}} d^4x \sqrt{g} \left(\frac{1}{4} \chi^{\mu\nu,a} (F_{\mu\nu}^+ + D_{\mu\nu})^b - \frac{1}{4} \eta^a [\phi, \tilde{\phi}]^b + \frac{1}{2} \psi^{\mu,a} D_{\mu} \tilde{\phi}^b \right) \\ \mathcal{S}^{\text{topological}} &:= \frac{i}{32\pi} \tau_{ab} \int_{\mathbb{X}} F^a \wedge F^b + \frac{i}{32\pi} \tau_{uv} \int_{\mathbb{X}} F^u \wedge F^v + \frac{i}{192\pi} \int_{\mathbb{X}} d^4x \mathcal{F}_{abcd} \varepsilon^{\mu\nu\rho\sigma} \psi_{\mu}^a \psi_{\nu}^b \psi_{\rho}^c \psi_{\sigma}^d \\ \mathcal{S}^{\text{extra}} &:= \frac{i}{16\pi} \int_{\mathbb{X}} d^4x \left[\frac{i}{12} \bar{\mathcal{F}}_{abcd} \eta^a \chi_{\mu}^{\rho,b} \chi^{\mu\sigma,c} \chi_{\rho\sigma}^d + \frac{i}{4} \bar{\mathcal{F}}_{abc} (F_{\mu\rho}^{+a} - D_{\mu\rho}^a) \chi^{\mu\sigma,b} \chi_{\sigma}^{\rho,c} \right] . \end{aligned} \quad (\text{D.4.208})$$

Thus the full $\mathcal{N} = 2$ action can be written as

$$\mathcal{S}_{\mathcal{N}=2}^{\text{general}} = \{\mathcal{Q}, V_{\text{general}}\} + \mathcal{S}^{\text{topological}} + \mathcal{S}^{\text{extra}} , \quad (\text{D.4.209})$$

where

$$\begin{aligned} V_{\text{general}} &:= V_{\text{vec}} + \int_{\mathbb{X}} d^4x \sqrt{g} \left(D_{\mu} \tilde{M}^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^{\mu} \lambda^{\alpha} + D_{\mu} M^{\dot{\alpha}} \tilde{\sigma}_{\dot{\alpha}\alpha}^{\mu} \mu^{\alpha} + \frac{1}{2} m \tilde{\lambda}_{\dot{\alpha}} M^{\dot{\alpha}} - \frac{1}{2} \mu_{\alpha} h^{\alpha} - \frac{1}{2} m \tilde{M}^{\dot{\alpha}} \tilde{\mu}_{\dot{\alpha}} \right. \\ &\quad \left. + \frac{1}{2} \tilde{h}_{\alpha} \lambda^{\alpha} + \frac{1}{2} \left(\tilde{M}^{\dot{\alpha}} \tilde{\phi} \tilde{\mu}_{\dot{\alpha}} + \tilde{\lambda}^{\dot{\alpha}} \tilde{\phi} M_{\dot{\alpha}} \right) - \tilde{M}^{(\dot{\alpha}} \chi_{\dot{\alpha}\dot{\beta}} M^{\dot{\beta})} \right) , \end{aligned} \quad (\text{D.4.210})$$

where we suppressed the flavor index on the hypermultiplet part of V . This agrees with the action (D.4.191) when we specialize to the quadratic prepotential with constant coupling proportional to identity:

$$\mathcal{F} = \frac{1}{2} \tau \phi^2, \quad \bar{\mathcal{F}} = \frac{1}{2} \bar{\tau} \tilde{\phi}^2 . \quad (\text{D.4.211})$$

We have thus derived (4.7.8)–(4.7.11).

D.5 Superconformal Multiplets

In this section, we will briefly review superconformal multiplets for the $d = 4$ $\mathcal{N} = 2$ superconformal algebra $\mathfrak{su}(2, 2|2)$.¹¹ The reader is referred to [CDI16b, RR14, DO02, Ebe20] for more details. The superconformal algebra admits a bosonic subalgebra,

$$\mathfrak{su}(2, 2) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R \subset \mathfrak{su}(2, 2|2) . \quad (\text{D.5.1})$$

Note that $\mathfrak{su}(2, 2) \cong \mathfrak{so}(4, 2)$, and there is an $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ symmetry, we will denote the charge under the R-symmetry by $(R; r)$ where $R \in \mathbb{Z}_{\geq 0}$ is the Dynkin label for the $\mathfrak{su}(2)_R$ and $r \in \mathbb{R}$ is the charge under $\mathfrak{u}(1)_R$. The scaling dimension will be denoted by Δ and the Dynkin labels for the representation under the (spin lift of) the 4d rotation group $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$ will be denoted by $(j_1; j_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. An operator with these eigenvalues under the generators of the superconformal algebra will then be denoted by $[j_1; j_2]_{\Delta}^{(R; r)}$. In particular, the supercharges transform in the following representations:

$$\mathcal{Q} \in [1; 0]_{\frac{1}{2}}^{(1; -1)}, \quad \overline{\mathcal{Q}} \in [0; 1]_{\frac{1}{2}}^{(1; 1)} . \quad (\text{D.5.2})$$

The basic generating multiplets are the long multiplets and short multiplets satisfying certain unitarity bounds [Min97]. Short multiplets are obtained from the long multiplet by imposing certain shortening conditions. These multiplets are listed in Tables D.1 and D.2.

Name	Primary	Unitarity Bound
L	$[j_1; j_2]_{\Delta}^{(R; r)}$	$\Delta > 2 + j_1 + R - \frac{1}{2}r$
A_1	$[j_1; j_2]_{\Delta}^{(R; r)}, \quad j_1 \geq 1$	$\Delta = 2 + j_1 + R - \frac{1}{2}r$
A_2	$[0; j_2]_{\Delta}^{(R; r)}$	$\Delta = 2 + R - \frac{1}{2}r$
B_1	$[0; j_2]_{\Delta}^{(R; r)}$	$\Delta = R - \frac{1}{2}r$

Table D.1: Multiplets with $\mathcal{Q}$ -shortening conditions, adapted from [CDI16b, Table 13].

¹¹This is the real form appropriate to $d = 4$ Lorentzian signature, which is implicit in standard treatments of radial quantization. The appropriate real form for $d = 4$ Euclidean signature is $\mathfrak{su}^*(4|2)$, which would lead to a noncompact summand in the Euclidean R-symmetry group. This distinction will be unimportant for us, however, see [CMRS23] for a discussion in the context of topological twisting.

Name	Primary	Unitarity Bound
$\bar{L}$	$[j_1; j_2]_{\Delta}^{(R;r)}$	$\Delta > 2 + j_2 + R + \frac{1}{2}r$
$\bar{A}_1$	$[j_1; j_2]_{\Delta}^{(R;r)}, \quad j_2 \geq 1$	$\Delta = 2 + j_2 + R + \frac{1}{2}r$
$\bar{A}_2$	$[j_1; 0]_{\Delta}^{(R;r)}$	$\Delta = 2 + R + \frac{1}{2}r$
$\bar{B}_1$	$[j_1; 0]_{\Delta}^{(R;r)}$	$\Delta = R + \frac{1}{2}r$

Table D.2: Multiplets with $\bar{\mathcal{Q}}$ -shortening conditions, adapted from [CDI16b, Table 14].

Full multiplets are two-sided, i.e., they satisfy both $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ -shortening conditions which leads to restrictions on some of the quantum numbers. The consistent two-sided multiplets are listed below.¹² We refer the reader to [CDI16b, Table 15] for details on the restrictions on the quantum numbers.¹³

$$\begin{aligned}
\mathcal{A}_{R,r(j_1,j_2)}^{\Delta} &= L\bar{L}[j_1; j_2]_{\Delta}^{(R;r)}, \\
\mathcal{C}_{R,r(j_1,j_2)} &= A_{\ell}\bar{L}[j_1; j_2]_{\Delta}^{(R;r)}, & \mathcal{B}_{R,r(0,j_2)} &= B_1\bar{L}[0; j_2]_{\Delta}^{(R>0;r)}, \\
\mathcal{E}_{r(0,j_2)} &= B_1\bar{L}[0; j_2]_{\Delta}^{(0;r)}, & \hat{\mathcal{C}}_{R(j_1,j_2)} &= A_{\ell}\bar{A}_{\bar{\ell}}[j_1; j_2]_{\Delta}^{(R;j_1-j_2)}, \\
\mathcal{D}_{R(0,j_2)} &= B_1\bar{A}_{\bar{\ell}}[0; j_2]_{\Delta}^{(R;-j_2-2)}, & \hat{\mathcal{B}}_R &= B_1\bar{B}_1[0; 0]_{\Delta}^{(R;0)}.
\end{aligned} \tag{D.5.3}$$

where $\ell, \bar{\ell} = 1, 2$. The conjugate multiplets $\bar{\mathcal{C}}_{R,r(j_1,j_1)}, \bar{\mathcal{B}}_{R,r(j_1,0)}, \bar{\mathcal{E}}_{r(j_1,0)}$, and $\bar{\mathcal{D}}_{R(j_1,0)}$ can be obtained by complex conjugation. The full operator content of each of these multiplets is tabulated in [CDI16b].

Vectormultiplets realize the short multiplet $A_2\bar{B}_1[0; 0]_{\Delta}^{(0;1)}$ and hypermultiplets realize the short multiplet $B_1\bar{B}_1[0; 0]_{\Delta}^{(1;0)}$. Superconformal multiplets containing currents for symmetries are called conserved current multiplets. Here we are only interested in the flavor current multiplet and the stress tensor multiplet (also called the energy-momentum multiplet). See [CDI16b] for other conserved current multiplets.

Flavor current multiplet.

The $R = 2$ multiplet $B_1\bar{B}_1[0; 0]_{\Delta}^{(2;0)}$ is a flavor current multiplet containing eight bosonic and eight fermionic operators. The full flavor multiplet [CDI16b] has the structure shown in Fig. D.1.

¹²The notation on the LHS of (D.5.3) is borrowed from [DO02].

¹³Note that the scaling dimensions of the short multiplets are determined by shortening conditions; this is not indicated in the notation.

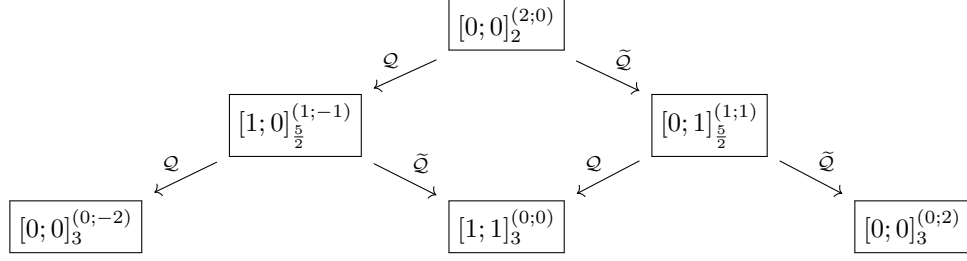


Figure D.1: The structure of the flavor multiplet, adapted from [Ebe20, Fig. 5].

Let us denote the bottom component by $\mu^{((AB))a}$, where $A = 1, 2$ is the index for the fundamental representation of $\text{SU}(2)_R$ and a is the index for the adjoint representation of the flavor group.¹⁴ Then the full multiplet can be constructed by acting on the bottom component by the supercharges:

$$[\mathcal{Q}_{A\alpha}, \mu^{((BC))a}] = \delta_A^{((B}\psi_\alpha^{C))a} \in [1; 0]_{\frac{5}{2}}^{(1;-1)}, \quad (\text{D.5.4})$$

$$[\tilde{\mathcal{Q}}_{A\dot{\alpha}}, \mu^{((BC))a}] = \delta_A^{((B}\tilde{\psi}_{\dot{\alpha}}^{C))a} \in [0; 1]_{\frac{5}{2}}^{(1;1)}, \quad (\text{D.5.5})$$

$$\{\tilde{\mathcal{Q}}_{A\dot{\alpha}}, \psi_\alpha^{Ba}\} = \delta_A^B \tilde{\sigma}_{\dot{\alpha}\beta}^\mu J_\mu^a \in [1; 1]_3^{(0;0)}, \quad (\text{D.5.6})$$

$$\{\mathcal{Q}_{A\alpha}, \tilde{\psi}_{\dot{\beta}}^{Ba}\} = \delta_A^B \sigma_{\alpha\dot{\beta}}^\mu J_\mu^a \in [1; 1]_3^{(0;0)}. \quad (\text{D.5.7})$$

The current J_μ^a is the conserved current for the flavor symmetry of the theory.

Stress tensor multiplet.

The multiplet $A_2\bar{A}_2[0; 0]_2^{(0;0)}$ is the stress tensor multiplet whose top component is the stress tensor of the theory. The generators of the $d = 4$ $\mathcal{N} = 2$ superconformal algebra must be integrals of local currents over a spatial slice. The bosonic generators are integrals of the stress tensor $T_{\mu\nu}$ and the $\mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_R$ R-symmetry currents $(j_\mu^{(R)})^A, j_\mu^{(R)}$. The fermionic generators are integrals of conserved fermionic supercurrents $G_{\mu\alpha}^A$ and $\tilde{G}_{\mu\dot{\alpha}}^A$. The superconformal algebra dictates that the stress tensor, R-symmetry current, and supercurrent sit in the same multiplet, called the stress tensor multiplet. In particular,

$$\{\mathcal{Q}_{\alpha A}, \tilde{G}_{\mu\dot{\beta}}^B\} = \sigma_{\alpha\dot{\beta}}^\nu \delta_A^B T_{\mu\nu}, \quad \{\tilde{\mathcal{Q}}_{\dot{\alpha} A}, G_{\mu\beta}^B\} = \tilde{\sigma}_{\dot{\alpha}\beta}^\nu \delta_A^B T_{\mu\nu}. \quad (\text{D.5.8})$$

The full stress tensor multiplet has the form [CDI16b, Ebe20] shown in Fig. D.2.

¹⁴The notation $T^{((AB))}$ denotes a symmetric traceless rank-2 tensor transforming in the adjoint (**3**) of $\text{SU}(2)_R$.

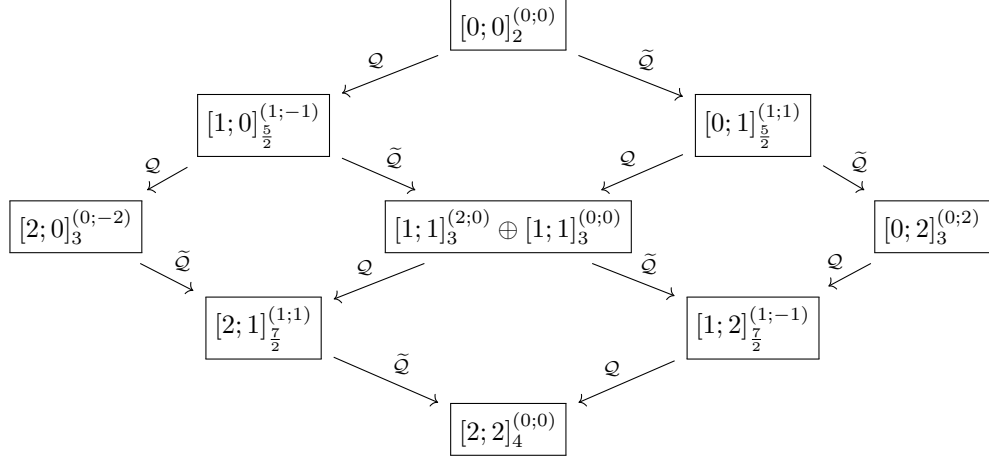


Figure D.2: The structure of the stress tensor multiplet, adapted from [Ebe20, Fig. 6].

The top component of the supermultiplet is the stress tensor (which is, in fact, at the bottom of the diagram), and the middle components (with $\Delta = 3$) are the R-symmetry currents. The operators with $\Delta = \frac{7}{2}$ are the supercurrents.

D.6 The Superconformal Index

In this appendix, we review the superconformal index. For recent reviews, see [RR14, RR16, Gad20]. First, let us review the supersymmetric index as a trace over the Hilbert space in the Hamiltonian quantization of a generic supersymmetric theory in d spacetime dimensions, coupled to conserved currents of some global symmetries (i.e., to background flavor bundles). To compute it, we pick a supercharge¹⁵ $\mathcal{Q}$ with an adjoint $\mathcal{Q}^+$ where $\{\mathcal{Q}, \mathcal{Q}^+\} = 2\mathbf{H}$, and $\mathcal{Q}^2 = 0$. In a unitary theory, this implies that the spectrum of $\mathbf{H}$ is positive semidefinite. The index is simply a weighted supertrace of states in $\text{Ker } \mathbf{H}$, i.e., states in the cohomology of $\mathcal{Q}$ and $\mathcal{Q}^+$, or equivalently, the Hilbert space of a radially quantized theory on $S^{d-1} \times \mathbb{R}$. Concretely, the index has a trace formula of the form

$$\mathcal{I}(\{\mu_i\}) = \text{Tr}_{\text{Ker } \mathbf{H}}(-1)^F e^{-\sum_i \mu_i T_i} e^{-2\beta\{\mathcal{Q}, \mathcal{Q}^+\}} , \quad (\text{D.6.1})$$

¹⁵In terms of background supergravity, this is a choice of generalized Killing spinor, i.e., a solution to the differential constraints obtained by demanding the vanishing of the supersymmetry variations of the gravitino, in a bosonic background where the auxiliary fields of the supergravity multiplet may have nontrivial profiles [FS11, KTZ12, CKM⁺12, KZ13, KR88, CMRS23].

where $\{T_i\}$ is a complete set of generators that commute with $\mathcal{Q}$ and with each other, $\{\mu_i\}$ is a set of “fugacities,” or “chemical potentials” for the associated global symmetries¹⁶ and F is the fermion number. As is the case for the original Witten index [Wit82], only the states with $H = 0$ contribute to the index, and consequently the index does not depend on β . The index so defined is a function of the fugacities that couple to the conserved charges of the corresponding global symmetries.¹⁷ There are various refinements of the index, and specializations that are useful in different contexts, especially for probing enhanced supersymmetry [GRRY11b].

Expressed as a trace over the full Hilbert space $\mathcal{H}$ of states on S^3 the $d = 4$ $\mathcal{N} = 2$ superconformal index has the generic form

$$\mathcal{I}(p, q, t; \{a_i\}) := \text{Tr}_{\mathcal{H}}(-1)^F p^{\frac{1}{3}(\Delta + \frac{1}{2}j_1 - R - r) + \frac{1}{2}j_2} q^{\frac{1}{3}(\Delta + \frac{1}{2}j_1 - R - r) - \frac{1}{2}j_2} t^{\frac{1}{2}(R + r)} e^{-\beta(\Delta - j_1 - R + \frac{1}{2}r)} \prod_{i=1}^{\text{rk } G^f} a_i^{f_i}. \quad (\text{D.6.2})$$

Recall that the index is written with respect to a choice of supercharge¹⁸ $\mathcal{Q}$, which determines an $\mathcal{N} = 1$ subalgebra of an $\mathcal{N} = 2$ algebra. The commutant of $\mathcal{Q}$ in the superconformal algebra $\mathfrak{su}(2, 2|2)$ is $\mathfrak{su}(2, 1|1)$, which has rank 3. Here (p, q, t) are the three superconformal fugacities, and $\{a_i\}$ is a set of fugacities associated with global symmetries G^f that commute with each other and with the superconformal algebra (and $\{f_i\}$ are the conserved flavor charges), Δ is the dilatation generator (a.k.a. the conformal Hamiltonian in radial quantization), j_1 and j_2 are the Cartan generators of the $\text{SU}(2)_1 \times \text{SU}(2)_2$ isometry group of the spatial S^3 , and R and r are the Cartan generators of $\text{SU}(2)_R$ and $\text{U}(1)_R$ respectively. As previously stated, the trace formula makes a choice of supercharge. The formula (D.6.2) is written for a supercharge with quantum numbers $\Delta = \frac{1}{2}R = -r = \frac{1}{2}$, $(j_1, j_2) = (0, -1)$.¹⁹

The superconformal index counts short multiplets modulo an equivalence relation that sets to zero

¹⁶At the moment, we are including superconformal fugacities (finitely many in number) as well as fugacities associated with flavor symmetries (which could be arbitrarily many in number, and which commute with all generators of the superconformal algebra). We will shortly refine this definition.

¹⁷Sometimes, the factor $e^{-\sum_i \mu_i T_i}$ is replaced by $\prod_i \tilde{\mu}_i^{C_i}$ with $\tilde{\mu}_i$ referred to as the fugacities and C_i the conserved charges.

¹⁸The supercharges of the $d = 4$ $\mathcal{N} = 2$ superconformal algebra in radial quantization are $\mathcal{Q}_{A\alpha}, \tilde{\mathcal{Q}}_A^{\dot{\alpha}}, \mathcal{S}^{A\alpha} := \mathcal{Q}^{\dagger A\alpha}, \tilde{\mathcal{S}}_A^{\dot{\alpha}} := \tilde{\mathcal{Q}}_A^{\dagger \dot{\alpha}}$. Here A is an $\text{SU}(2)_R$ symmetry index, and α and $\dot{\alpha}$ are spinor indices, which take values $+, -$ and $\dot{+}, \dot{-}$ respectively. The supercharge used to define the index is $\mathcal{Q} := \mathcal{Q}_{1-}$, for which $\{\mathcal{Q}_{1-}, (\mathcal{Q}_{1-})^\dagger\} = \Delta - j_1 - 2R + \frac{1}{2}r$. This is the coefficient of $-\beta$ in (D.6.2). This $\mathcal{Q}$ should not be confused with the scalar supercharge in the bulk of the paper.

¹⁹As in the main text, we use the same symbols for the generators and their eigenvalues. The eigenvalues are normalized in accordance with Appendix D.5.

the contributions of short multiplets that can recombine to long ones [KMMR05].²⁰ Superconformal multiplets and their shortening conditions are briefly reviewed in Appendix D.5. (See [BLL⁺13, App. B] for further details).

The background $S^3 \times S^1$ has been studied extensively in the context of supersymmetric localization (see [P⁺16] for a review). $S^3 \times S^1$ is a non-Kähler complex manifold that admits a Hermitian metric, and admits nontrivial holomorphic line bundles classified by a rank one Dolbeault cohomology group $\mathcal{H}^{0,1}(S^3 \times S^1)$ (although complex line bundles on it are trivial since $H^2(S^3 \times S^1; \mathbb{Z}) = 0$). The parameter dependence of the partition function on background fields is discussed in [CDFK13, CDFK14], where it is argued that the path integral of a supersymmetric theory on a Hermitian manifold depends only on the complex structure deformations of the manifold.

While it is often remarked that the supersymmetric index is the partition function on $S^3 \times S^1$, a more precise statement is that it is proportional to the partition function up to a scheme-independent (and hence physical) multiplicative factor involving the supersymmetric Casimir energy [ACM14, LM14, ACDP⁺15]. This multiplicative factor is dependent on the central charges a_{4d} and c_{4d} and reduces to 1 for a conformal theory.

In general, the three-parameter superconformal index (D.6.2) is not known in closed analytic form as a function of (p, q, t) . There is at least one technique for determining it in terms of a series expansion in these variables by exploiting generalized S-duality and recasting the problem to one of diagonalizing a set of commuting difference operators known as the elliptic Ruijsenaars-Schneider Hamiltonians [GRR12, Raz13]. The eigenfunctions of these operators are functions of (p, q, t) and one can reconstruct the index as a sum of products of these “wavefunctions,” weighted with coefficient functions that encode data at the punctures in Class $\mathcal{S}$ theory. In principle, this algorithm can be used to determine a series expansion of $\mathcal{I}(p, q, t, \{a_i\})$ in all fugacities.²¹

In practice, however, it is preferable to have a specialization that is computable with less effort and nevertheless contains useful information. One such specialization is the so-called Schur limit of the superconformal index, with the resulting index known as the Schur index.²² It can be obtained from (D.6.2) by setting $q = t$ and noting that the p -dependence is $\mathcal{Q}$ -exact and drops out²³ leading

²⁰A consequence of the definition (D.6.2) is that the index is zero on long multiplets, and therefore, it *cannot* be used to distinguish between two short multiplets for which it evaluates to the same result. See [GPR09] for a discussion of this issue.

²¹See also [PP21] for some recent work on expressing the Schur index (with flavor fugacities) in closed form.

²²Its name stems from the existence of closed-form expressions for the “wavefunctions,” in terms of Schur polynomials.

²³One uses the fact that the index is independent of β , and in particular, counts states with

$$\Delta - j_1 - R + \frac{1}{2}r = 0 \, .$$

to

$$\mathcal{I}_{\text{Schur}}(q, \{a_i\}) = \text{Tr}_{\mathcal{H}(S^3)}(-1)^F q^{\Delta - \frac{R}{2}} \prod_{i=1}^{\text{rk } G^f} a_i^{f_i} . \quad (\text{D.6.3})$$

The states counted by this index – known as Schur operators (or equivalently, Schur states) – are harmonic representatives of cohomology classes of a nilpotent supercharge defined by the sum of a Poincaré supercharge $\mathcal{Q}$ and a conformal supercharge S .²⁴ The Schur index is, in fact, related to q -deformed 2d topological Yang-Mills theory [GPRR09, GRRY11a]. It is now known that Schur operators form the structure of a two-dimensional chiral algebra [BLL⁺13, BPRvR14, LP14], and the Schur index is the supercharacter of the vacuum module of this chiral algebra. The Schur index also has interesting modularity properties related to conformal anomalies [Raz12]. Finally, the Schur index is also conjecturally related to a wall-crossing invariant [CS15] defined in terms of the protected spin character of [GMN10]. So, despite its apparent simplicity relative to the “full” superconformal index, it encodes a lot of useful information about the protected operator spectrum of the theory.²⁵ For a superconformal $\mathcal{N} = 2$ theory, the index *is* the partition function on $S^3 \times S^1$ as stated above. A natural question is whether there is a supergravity background on which the partition function computes the superconformal index, with couplings to background $\mathcal{N} = 2$ conformal supergravity fields.²⁶ Such superconformal backgrounds have been discussed in [HH12, Pes14], with a detailed discussion in [PP19], where explicit expressions may be found for the metric, the generalized Killing spinors, and profiles of background fields of $\mathcal{N} = 2$ conformal supergravity (in suitable parametrizations, bases of fields, and spinors).²⁷ (For a recent quick introduction to $\mathcal{N} = 2$ conformal supergravity, see [CMRS23].) The background is an S^3 fibration over S^1 , where the fiber S^3 itself is a $\mathbb{T}^2$ fibration over an interval. In this formalism, one can turn on flat connections for background $\mathcal{N} = 2$ vectormultiplets for a maximal torus of the flavor symmetry group. These correspond to the flavor fugacities appearing in the index.

This determines j_1 for a given (Δ, r, R) . The exponents of p , q , and t in (D.6.2) are combinations of Cartan generators that commute with the supercharge $\mathcal{Q}$ used to define the index in the first place. The power of p in (D.6.2) becomes $\frac{1}{2} \{ \tilde{\mathcal{Q}}_{1,+}, (\tilde{\mathcal{Q}}_{1,+})^\dagger \} = \frac{1}{2} (\Delta + j_2 - R - \frac{1}{2}r)$ – see footnote 18. The p -dependence is therefore $\mathcal{Q}$ -exact, and the index counts states with $\Delta + j_2 - R - \frac{1}{2}r = 0$. This simplification leads to (D.6.3)

²⁴The Schur index is an intrinsically $\mathcal{N} = 2$ index.

²⁵See also [GV24] for a relation between pure $\text{SU}(2)$ SYM and the SYK model, where Schur indices play an important role.

²⁶For the $\mathcal{N} = 1$ index, a background based on new minimal supergravity was found in [FS11, Sec. 5].

²⁷It would be interesting to apply the methods of [CMRS23] to determine a background on which the partition function computes a “topologically twisted Schur index,” but as discussed in the main text, the interpretation of such an index would probably be appropriate only for a conformal theory. As is well known [Wit88], the trace of the energy-momentum tensor for topologically twisted $\mathcal{N} = 2$ SYM is a total derivative: the theory has global scale invariance, but not conformal invariance.

To our knowledge, a mass deformation of the Schur index is not known. On the one hand, it is quite natural – from the standpoint of $\mathcal{N} = 2$ supergravity – to consider couplings to background $\mathcal{N} = 2$ vectormultiplets with their scalars tuned to mass deformations (e.g., as we did in (4.7.6)). However, for non-conformal theories, one cannot perform a Weyl rescaling from $S^3 \times \mathbb{R}$ to $\mathbb{R}^4$, and there is no state-operator correspondence. The interpretation of the resulting mass-deformed index is therefore less clear.²⁸

D.7 Lagrangian Analysis Of The ffs Trinion Theory

In this appendix, we describe the ffs trinion theory as a Lagrangian theory of a free hypermultiplet and present the action. Let V be the N -dimensional defining representation of $SU(N)$. The field content of the theory is a hypermultiplet valued in the quaternionic space $E \oplus \bar{E}$ where E is the representation $V \otimes \bar{V}$ of $SU(N) \times SU(N) \times U(1)$ with $U(1)$ acting as scalars.²⁹

To write the Lagrangian, we write the hypermultiplet in terms of a pair of $\mathcal{N} = 1$ chiral multiplets. The field content is same as a chiral multiplet $Q^{m\bar{n}}$ and an anti-chiral multiplet $\tilde{Q}'^{m\bar{n}}$ both in $V \otimes \bar{V}$.³⁰ Let us write their component fields as³¹

$$Q^{m\bar{n}} = ((q^2)^{m\bar{n}}, (\psi_\alpha)^{m\bar{n}}) \ , \quad \tilde{Q}'^{m\bar{n}} = ((q^1)^{m\bar{n}}, (\tilde{\chi}_{\dot{\alpha}})^{m\bar{n}}) \ . \quad (\text{D.7.1})$$

Define complex scalars $\tilde{q}^{A, \bar{n}m}$ valued in $\bar{V} \otimes V$ via

$$\tilde{q}^{A, \bar{n}m} = (q_{A, m\bar{n}})^\dagger = \Omega^{m\bar{n}, \bar{n}'m'} \varepsilon^{AB} q_{B, m'\bar{n}'} \ , \quad (\text{D.7.2})$$

where Ω is a symplectic form³² on $V \otimes \bar{V} \oplus \bar{V} \otimes V$, and ε^{AB} is the completely antisymmetric invariant

²⁸The authors of [CS15] propose an extension of the Schur index to non-conformal Lagrangian $\mathcal{N} = 2$ theories (with G^{gauge} -vectormultiplets and hypermultiplets in some representation of G^{gauge}) in terms of a matrix integral over the maximal torus of G^{gauge} . However, they note that it is a priori unclear why this extension will lead to a protected observable and that further investigation is merited. Another non-conformal generalization of the Schur index which is based on isolating a subalgebra of the superconformal algebra that includes only the isometries of $S^3 \times \mathbb{R}$ and $SU(2)_R$ was proposed in [Dum16], but remains an unpublished proposal at the time of writing this.

²⁹In fact we could have chosen global symmetry group $U(N) \times U(N) \times U(1)$ but a subgroup of the center isomorphic to $U(1) \times U(1)$ acts trivially, so we choose to take $SU(N) \times SU(N) \times U(1)$ for simplicity. Otherwise, we would need to discuss non-effective group actions and stacks. This is beyond the scope of this paper.

³⁰An anti-chiral multiplet is distinguished from a chiral multiplet by having a Weyl spinor of the opposite chirality. In Euclidean signature, chiral and antichiral multiplets are independent and unrelated by complex conjugation: the complex conjugate of a chiral multiplet is another chiral multiplet.

³¹Here $m, n = 1, \dots, N$ are indices for V , and $\bar{m}, \bar{n} = 1, \dots, N$ are indices for $\bar{V}$. We use the convention that upper left (right) indices are for the first (second) factor and lower left (right) indices are for the dual space of the first (second) factor. We could equivalently have treated $(q^2)^{m\bar{n}}$ and $(\tilde{q}^1)^{\bar{n}m}$ as independent scalars.

³²There exists a choice of basis exhibiting $V \otimes \bar{V} \oplus \bar{V} \otimes V$ as a real vector space with real dimension $4N^2$. In this basis, Ω is a $2N^2 \times 2N^2$ matrix with a block diagonal form, with N^2 blocks each of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

tensor of $\mathfrak{su}(2)_R$, with $\varepsilon^{12} = 1$. The matrix

$$\begin{pmatrix} q^1 & q^2 \\ -\tilde{q}_1 & \tilde{q}_2 \end{pmatrix}, \quad (\text{D.7.3})$$

(with $V \otimes \bar{V}$ indices suppressed) is then a quaternion in the standard quaternionic structure on $V \otimes \bar{V} \oplus \bar{V} \otimes V$ and $\text{SU}(2)_R$ acts on the scalars by right multiplication by unit quaternions.

For later use, we also define conjugate spinors

$$\begin{aligned} \chi^{\alpha, \bar{n}m} &:= (\psi_{\alpha, m\bar{n}})^\dagger = \varepsilon^{\alpha\beta} \Omega^{m\bar{n}, \bar{n}'m'} \psi_{\beta, m'\bar{n}'} , \\ \tilde{\psi}^{\dot{\alpha}, \bar{n}m} &:= (\tilde{\chi}_{\dot{\alpha}, m\bar{n}})^\dagger = \varepsilon^{\dot{\alpha}\dot{\beta}} \Omega^{m\bar{n}, \bar{n}'m'} \tilde{\chi}_{\dot{\beta}, m'\bar{n}'} . \end{aligned} \quad (\text{D.7.4})$$

Let $\tilde{P}_T$ denote a principal $\tilde{G}_T$ bundle over $\mathbb{X}$, where

$$\tilde{G}_T := \text{Spin}(4) \times \text{SU}(2)_R \times \text{SU}(N) \times \text{SU}(N) \times \text{U}(1) . \quad (\text{D.7.5})$$

For this bundle to exist, $\mathbb{X}$ must be a spin manifold. We assume this is so for the moment. In the main text, we have discussed how this can be relaxed by defining the theory in terms of a $\tilde{G}_T/C_T$ bundle, for a suitable choice of subgroup C_T of the center of $\tilde{G}_T$.

To write the action in curved space, we follow the procedure outlined in Section 4.7 or [KR88, CMRS23], coupling the hypermultiplet theory to a bosonic background of off-shell $\mathcal{N} = 2$ conformal supergravity that preserves $\mathcal{N} = 2$ supersymmetry. Such a background can be defined by consistently setting the gravitinos and auxiliary spinor fields in the $\mathcal{N} = 2$ conformal supergravity multiplet to zero, as well as their supersymmetry variations to zero.³³ The supergravity background contains a connection for $\text{SU}(2)_R \times \text{U}(1)_R$ symmetry, but we treat the $\text{U}(1)_R$ symmetry as rigid and gauge its connection to zero (see [CMRS23] for details). The $\text{SU}(2)_R$ connection plays an important role in the specialization of twisting, discussed later. We also couple to background vectormultiplets for the aforementioned $\text{SU}(N) \times \text{SU}(N) \times \text{U}(1)$ flavor symmetry: to this end, we introduce connections $A_\mu^{\text{SU}(N)^{(1)}}$, $A_\mu^{\text{SU}(N)^{(2)}}$, $A_\mu^{\text{U}(1)}$. Although we write separate connections to clarify how they contribute to the covariant derivatives, it is useful to think of a composite background vectormultiplet for the $\text{SU}(N) \times \text{SU}(N) \times \text{U}(1)$ flavor symmetry. We denote the composite curvature for this background vectormultiplet by $F_{\mu\nu}^a$ where $a, b = 1, \dots, 2(N^2 - 1) + 1$ run over the generators of the Lie algebra of

³³For the untwisted theory, this leads to a differential constraint on the supersymmetry parameter. However, as is well known from [Wit88, KR88, CMRS23], upon twisting, there is a globally-defined scalar supersymmetry parameter corresponding to $\mathcal{Q}$ symmetry, and no differential constraints are needed. The bosonic auxiliary fields of the supergravity multiplet are allowed to have nontrivial profiles consistent with supersymmetry [FS11]. However, for simplicity we assume that the 2-form field in the $\mathcal{N} = 2$ conformal supergravity multiplet is set to zero: $T_{\mu\nu}^\pm = 0$, as this simplifies our partition function. For examples of more general backgrounds, see [HH12, Pes14].

the gauge group as usual. The background vectormultiplet also contains an adjoint-valued auxiliary field D_{AB}^a which transforms as a triplet of $\mathrm{SU}(2)_\mathrm{R}$.

The action of the free hypermultiplet coupled to background fields is

$$\begin{aligned} \mathcal{S}_\mathrm{Hyper} = \int_{\mathbb{X}} \mathrm{vol}(g) & \left[\nabla_\mu^f(\tilde{q}_A)^{\bar{n}m} \nabla_f^\mu (q^A)_{m\bar{n}} - \frac{1}{24} (\mathcal{R} + 3\mathcal{D})(\tilde{q}_A)^{\bar{n}m} (q^A)_{m\bar{n}} \right. \\ & \left. + (\tilde{q}_A)^{\bar{n}m} D^{AB} (q_B)_{m\bar{n}} + i\tilde{\psi}^{\bar{n}m} \not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^+ \psi_{m\bar{n}} - i\chi^{\bar{n}m} \not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^- \tilde{\chi}_{m\bar{n}} \right] \\ & + \frac{1}{4\pi} \int_{\mathbb{X}} \mathrm{vol}(g) \left[\mathrm{Im} \tau_{\underline{a}\underline{b}} \left(\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu,b} - \frac{1}{2} D_{AB}^a D^{AB,b} \right) + \frac{i}{32\pi} \tau_{\underline{a}\underline{b}} (F^a \wedge F^b) \right]. \end{aligned} \quad (\mathrm{D}.7.6)$$

Here, $\mathcal{D}$ is the auxiliary scalar in the $\mathcal{N} = 2$ conformal supergravity multiplet, $\mathcal{R}$ is the Ricci scalar curvature of $\mathbb{X}$, ∇_μ^f (or ∇_f^μ) are (metric + background gauge (flavor f) + $\mathrm{SU}(2)_\mathrm{R}$)-covariant derivatives, and $\not{D}^+$ denotes the chiral Dirac operator coupled to background gauge (flavor) and $\mathrm{SU}(2)_\mathrm{R}$ connections – we will momentarily write their explicit expressions.

We have made the following important specialization of the background vectormultiplet: we have set its scalars to zero (to preserve the full flavor symmetry) and we have set its fermions to zero as doing so eliminates coupling terms between spinors in the hypermultiplet and the spinors in the background vectormultiplet. However, this imposes an algebraic constraint on D_{AB}^a in terms of the supersymmetry parameter defining the $\mathcal{N} = 2$ supergravity background and the curvature $F_{\mu\nu}^a$.³⁴

The covariant derivatives appearing above are connections on associated bundles induced from the $\tilde{G}_T$ -connection ∇_T on $\tilde{P}_T$, and $\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^+$ denotes the Dirac operator coupled to the vector bundle $\mathcal{V} \otimes \bar{\mathcal{V}}$ associated to $\tilde{P}_T$ by the representation $V \otimes \bar{V}$ of $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{U}(1)$.

Let ω_μ , A_μ^R , $A_\mu^{\mathrm{SU}(N)^{(1)}}$, $A_\mu^{\mathrm{SU}(N)^{(2)}}$ and $A_\mu^{\mathrm{U}(1)}$ denote local coordinate representatives of the connections on vector bundles associated to $(p_1)_*(\nabla_T)$, $(p_2)_*(\nabla_T)$, $(p_3)_*(\nabla_T)$, $(p_4)_*(\nabla_T)$, and $(p_5)_*(\nabla_T)$

³⁴Upon further specialization to the twisted theory, this constraint will simplify to (D.7.17). The supersymmetry constraints on the background vectormultiplet can be easily deduced from the results of [CMRS23], but are not needed to follow the present discussion.

respectively. Here ω_μ is the spin connection. The covariant derivatives are

$$\begin{aligned} \nabla_\mu^f q^A_{m\bar{n}} &= \partial_\mu q^A_{m\bar{n}} + (A_\mu^R)^A{}_B q^B_{m\bar{n}} + (A_\mu^{\text{SU}(N)^{(1)}})^{m'}{}_m q^2_{m'\bar{n}} \delta_2^A + (A_\mu^{\text{SU}(N)^{(2)}})^{\bar{n}'}{}_{\bar{n}} q^1_{m\bar{n}'} \delta_1^A \\ &\quad + A_\mu^{\text{U}(1)} q^A_{m\bar{n}} , \end{aligned} \quad (\text{D.7.7})$$

$$\begin{aligned} \nabla_\mu^f \tilde{q}^{\bar{m}n}_A &= \partial_\mu \tilde{q}^{\bar{m}n}_A + (A_\mu^R)^B{}_A \tilde{q}^{\bar{m}n}_B + (A_\mu^{\text{SU}(N)^{(1)}})^{\bar{m}}{}_{\bar{m}'} \tilde{q}^{\bar{m}'n}_A \delta_2^A + (A_\mu^{\text{SU}(N)^{(2)}})^n{}_{n'} \tilde{q}^{\bar{m}n'}_A \delta_1^A \\ &\quad + A_\mu^{\text{U}(1)} \tilde{q}^{\bar{m}n}_A , \end{aligned} \quad (\text{D.7.8})$$

$$\begin{aligned} (D_{\mathcal{V} \otimes \bar{\mathcal{V}}})_\mu \psi_{\alpha m \bar{n}} &= \partial_\mu \psi_{\alpha m \bar{n}} - \frac{1}{2} \omega_\mu^{\underline{a}\underline{b}} (\sigma_{\underline{a}\underline{b}})^\beta{}_\alpha \psi_{\beta m \bar{n}} + (A_\mu^{\text{SU}(N)^{(1)}})^{m'}{}_m \psi_{\alpha m' \bar{n}} + (A_\mu^{\text{SU}(N)^{(2)}})^{\bar{n}'}{}_{\bar{n}} \psi_{\alpha m \bar{n}'} \\ &\quad + A_\mu^{\text{U}(1)} \psi_{\alpha m \bar{n}} , \end{aligned} \quad (\text{D.7.9})$$

$$\begin{aligned} (D_{\mathcal{V} \otimes \bar{\mathcal{V}}})_\mu \tilde{\chi}_{\dot{\alpha}}^{m\bar{n}} &= \partial_\mu \tilde{\chi}_{\dot{\alpha}}^{m\bar{n}} - \frac{1}{2} \omega_\mu^{\underline{a}\underline{b}} (\tilde{\sigma}_{\underline{a}\underline{b}})^{\dot{\beta}}{}_{\dot{\alpha}} \tilde{\chi}_{\dot{\beta}}^{m\bar{n}} + (A_\mu^{\text{SU}(N)^{(1)}})^m{}_{m'} \tilde{\chi}_{\dot{\alpha}}^{m' \bar{n}} + (A_\mu^{\text{SU}(N)^{(2)}})^{\bar{n}}{}_{\bar{n}'} \tilde{\chi}_{\dot{\alpha}}^{m \bar{n}'} \\ &\quad + A_\mu^{\text{U}(1)} \tilde{\chi}_{\dot{\alpha}}^{m\bar{n}} , \end{aligned} \quad (\text{D.7.10})$$

Recall that for chiral spinors valued in a vector bundle $E \rightarrow \mathbb{X}$ with connection, the chiral Dirac operators are maps

$$\begin{aligned} \not{D}_E^+ : \Gamma(S^+ \otimes E) &\rightarrow \Gamma(S^- \otimes E) , \\ \not{D}_E^- : \Gamma(S^- \otimes E) &\rightarrow \Gamma(S^+ \otimes E) , \end{aligned} \quad (\text{D.7.11})$$

where $S^\pm$ are chiral spinor bundles on $\mathbb{X}$.³⁵ The corresponding Dirac operator is

$$\not{D}_E = \begin{pmatrix} 0 & -i\not{D}_E^- \\ i\not{D}_E^+ & 0 \end{pmatrix} , \quad (\text{D.7.12})$$

in a basis determined by the chiral splitting. For us,

$$\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^+ := \tilde{\sigma}^\mu (D_{\mathcal{V} \otimes \bar{\mathcal{V}}})_\mu , \quad \not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^- := \sigma^\mu (D_{\mathcal{V} \otimes \bar{\mathcal{V}}})_\mu . \quad (\text{D.7.13})$$

The partition function of the theory is

$$\begin{aligned} \tilde{Z}_T &= \int [\mathcal{D}\tilde{q}\mathcal{D}q\mathcal{D}\tilde{\psi}\mathcal{D}\psi\mathcal{D}\tilde{\chi}\mathcal{D}\chi] e^{-S_{\text{Hyper}}} \\ &= \text{Det}(i\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^+) \text{Det}(-i\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^-) \int [\mathcal{D}\tilde{q}\mathcal{D}q] e^{-S_{\text{Hyper}}^{\text{Bosonic}}} , \end{aligned} \quad (\text{D.7.14})$$

where $S_{\text{Hyper}}^{\text{Bosonic}}$ is the bosonic part of the action (D.7.6), and $\text{Det}(i\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^+) \text{Det}(-i\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^-)$ is a section of the determinant line bundle

$$\text{DET}(i\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^+) \text{DET}(-i\not{D}_{\mathcal{V} \otimes \bar{\mathcal{V}}}^-) . \quad (\text{D.7.15})$$

Recall that the determinant line bundle of a chiral Dirac operator coupled to a bundle E with connection has fiber

$$\left(\Lambda^{\max} \text{Ker}(\not{D}_E^\pm) \right)^\vee \otimes \Lambda^{\max} \text{Cok}(\not{D}_E^\pm) \cong \left(\Lambda^{\max} \text{Ker}(\not{D}_E^\pm) \right)^\vee \otimes \Lambda^{\max} \text{Ker}(\not{D}_E^\pm)^\dagger , \quad (\text{D.7.16})$$

³⁵For a complex vector bundle E , the adjoint of D_E^+ is $(\not{D}_E^+)^\dagger = \not{D}_E^-$.

over the space of connections on $\tilde{P}_T$. Note that $\text{DET}(-i\mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^+)$ is dual to $\text{DET}(i\mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^-)$, therefore (D.7.15) is isomorphic to the trivial line bundle, and therefore, the partition function of a free hypermultiplet is just a c -number, as expected.

D.7.1 Topological Twisting Of The ffs Theory

Upon twisting, q_A becomes $M_{\dot{\alpha}}$ and $\tilde{q}_A$ becomes $\tilde{M}_{\dot{\alpha}}$. Note that $M_{\dot{\alpha}} \in \Gamma(S^- \otimes \mathcal{V} \otimes \bar{\mathcal{V}})$ and $\tilde{M}_{\dot{\alpha}} \in \Gamma(S^- \otimes \bar{\mathcal{V}} \otimes \mathcal{V})$. After topological twisting, for $\mathcal{Q}$ -symmetry, and in particular, for $\mathcal{S}_{\text{Hyper}}$ to define a $\mathcal{Q}$ -exact energy-momentum tensor – as we have already noted in Section 4.7 – the fermions in the background vectormultiplet must be set to zero, and therefore, their $\mathcal{Q}$ -variations must be required to vanish for consistency. In particular, the vanishing of the $\mathcal{Q}$ -variation of $\chi_{\mu\nu}$ yields the condition

$$D_{\mu\nu} = F_{\mu\nu}^+, \text{ or equivalently } D_{\dot{\alpha}\dot{\beta}} = F_{\dot{\alpha}\dot{\beta}}, \quad (\text{D.7.17})$$

where $F_{\mu\nu}$ is the composite curvature for the background $\text{SU}(\text{N}) \times \text{SU}(\text{N}) \times \text{U}(1)$ connection (which, for purposes of writing the covariant derivatives above, was split into three connections for the three individual factors) and $D_{\mu\nu}$ is the self-dual 2-form auxiliary field of the background vectormultiplet. We also set

$$\mathcal{D} = \frac{1}{6}\mathcal{R}, \quad (\text{D.7.18})$$

to specialize to the twisted background.³⁶

To obtain the twisted action $S_{\text{hyp}}^{\text{twisted}}$ from the untwisted action (D.7.6) we use the constraints (D.7.17) and (D.7.18) and the Lichnerowicz identity [Lic63]³⁷

$$\mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^+ \mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^- M^{\dot{\alpha}} = \nabla_{\mu}^f \nabla_f^{\mu} M^{\dot{\alpha}} - \frac{1}{16} \mathcal{R} M^{\dot{\alpha}} + F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}. \quad (\text{D.7.19})$$

to combine terms in the twisted action containing $M_{\dot{\alpha}}$ and $\tilde{M}_{\dot{\alpha}}$ as follows:

$$\begin{aligned} & \int_{\mathbb{X}} \text{vol}(g) \left(\nabla_{\mu}^f \tilde{M}_{\dot{\alpha}} \nabla_f^{\mu} M^{\dot{\alpha}} - \frac{1}{16} \mathcal{R} \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} + \tilde{M}_{\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \right) \\ & \xrightarrow[\text{by parts}]{\text{integrate}} \int_{\mathbb{X}} \text{vol}(g) \left(\tilde{M}_{\dot{\alpha}} \nabla_{\mu}^f \nabla_f^{\mu} M^{\dot{\alpha}} - \frac{1}{16} \mathcal{R} \tilde{M}_{\dot{\alpha}} M^{\dot{\alpha}} + \tilde{M}_{\dot{\alpha}} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}} \right) \\ & \xrightarrow[\text{(D.7.19)}]{\text{Lichnerowicz}} \int_{\mathbb{X}} \text{vol}(g) \tilde{M} \mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^+ \mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^- M. \end{aligned} \quad (\text{D.7.20})$$

³⁶See [KR88, CMRS23] for a derivation of this condition.

³⁷In standard mathematical literature, and indeed if the conventions/normalizations of [CMRS23] were used, the Lichnerowicz identity would have the more familiar form

$$\mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^+ \mathcal{D}_{\mathcal{V}\otimes\bar{\mathcal{V}}}^- M^{\dot{\alpha}} = \nabla_{\mu}^f \nabla_f^{\mu} M^{\dot{\alpha}} - \frac{1}{4} \mathcal{R} M^{\dot{\alpha}} + \frac{1}{2} F^{\dot{\alpha}\dot{\beta}} M_{\dot{\beta}}.$$

However, the conventions of this paper differ slightly from [CMRS23], leading to (D.7.19).

The path integral of the twisted theory is expressed as

$$Z_T^{\text{twisted}} = \exp \left(-\frac{i}{16\pi} \int_{\mathbb{X}} \tau_{\underline{a}\underline{b}} (F^a \wedge F^b) \right) \int [\mathcal{D}\text{HM}] e^{-S_{\text{hyp}}^{\text{twisted}}}, \quad (\text{D.7.21})$$

where $S_{\text{hyp}}^{\text{twisted}}$ is the twisted version of the hypermultiplet part of the action of equation (4.7.10).

Thus the Gaussian path integral in (D.7.21) yields

$$\int [\mathcal{D}\text{HM}] e^{-S_{\text{hyp}}^{\text{twisted}}} = \left(\text{Det} \left(\not{D}_{\nu \otimes \bar{\nu}}^+ \not{D}_{\nu \otimes \bar{\nu}}^- \right) \right)^{-1} \text{Det}(\text{i} \not{D}_{\nu \otimes \bar{\nu}}^+) \text{Det}(-\text{i} \not{D}_{\nu \otimes \bar{\nu}}^-) = 1. \quad (\text{D.7.22})$$

Because of $\mathcal{Q}$ -symmetry, the determinants cancel. The prefactor in (D.7.21) is a topological invariant, and in fact,

$$Z_T^{\text{twisted}} = \exp \left(-\frac{i}{16\pi} \int_{\mathbb{X}} \tau_{\underline{a}\underline{b}} (F^a \wedge F^b) \right). \quad (\text{D.7.23})$$

D.7.2 Central Action And Extension Of Domain

Since the two determinant line bundles in (D.7.15) are dual, the action of the global symmetry associated to the center of $\tilde{G}_T$ acts trivially on the product. Therefore we cannot use that expression to derive C_T^{max} . However, a further extension of the domain allows us to derive this group. Note that equation (D.7.6) just involves massless free bosons and fermions coupled to external fields. For trivial background fields, the flavor symmetry of the action is in fact larger than the product of flavor symmetries associated to the separate punctures (as often happens in theories of class $\mathcal{S}$). Let

$$G_f := \text{SU}(N) \times \text{SU}(N) \times \text{U}(1). \quad (\text{D.7.24})$$

There is (classically) a $G_f^+ \times G_f^-$ symmetry where G_f^+ acts trivially on $\tilde{\chi}, \chi$ and acts on $\tilde{\psi}, \psi$ in the standard way. Similarly, G_f^- acts trivially on $\tilde{\psi}, \psi$. We introduce a background bundle with connection for $G_f^+ \times G_f^-$. The coupling to the bosonic fields breaks $\text{SU}(2)_{\text{R}}$ symmetry by choosing a complex structure for the quaternionic target space: We couple G_f^+ to the chiral multiplets $Q^{m\bar{n}}$ and $(\tilde{Q}^{m\bar{n}})^\dagger$, and G_f^- to the anti-chiral multiplets $(\tilde{Q}^{m\bar{n}})^\dagger$ and $\tilde{Q}^{m\bar{n}}$. Then our full symmetry group is extended to:

$$\tilde{G}_T^{\text{extended}} = \text{Spin}(4) \times \text{SU}(2)_{\text{R}} \times G_f^+ \times G_f^-, \quad (\text{D.7.25})$$

Now we can consider the chiral actions of subgroups of the center. Elements of C_T^+ are of the form

$$(\zeta, 1, \zeta, \omega^{k_1^+} \mathbf{1}_N, \omega^{k_2^+} \mathbf{1}_N, \xi^+, 1_{G_f^-}), \quad (\text{D.7.26})$$

while elements of C_T^- are of the form

$$(1, \zeta, \zeta, 1_{G_f^+}, \omega^{k_1^-} \mathbf{1}_N, \omega^{k_2^-} \mathbf{1}_N, \xi^-), \quad (\text{D.7.27})$$

The action on the partition function is:

$$(\zeta, 1, \zeta, \omega^{k_1^+} \mathbf{1}_N, \omega^{k_2^+} \mathbf{1}_N, \xi^+, 1_{G_f^-}) \cdot \tilde{Z}_T(\tilde{P}_T, \tilde{\nabla}_T) = (\zeta \omega^{k_1^+} \bar{\omega}^{k_2^+} \xi^+)^{\text{Ind}(\mathcal{D}_{\mathcal{V} \otimes \mathcal{V}}^+)} \tilde{Z}_T(\tilde{P}_T, \tilde{\nabla}_T) . \quad (\text{D.7.28})$$

$$(1, \zeta, \zeta, 1_{G_f^+}, \omega^{k_1^-} \mathbf{1}_N, \omega^{k_2^-} \mathbf{1}_N, \xi^-,) \cdot \tilde{Z}_T(\tilde{P}_T, \tilde{\nabla}_T) = (\zeta \omega^{k_1^-} \bar{\omega}^{k_2^-} \xi^-)^{\text{Ind}(\mathcal{D}_{\mathcal{V} \otimes \mathcal{V}}^-)} \tilde{Z}_T(\tilde{P}_T, \tilde{\nabla}_T) . \quad (\text{D.7.29})$$

Demanding that these actions are trivial implies:

$$\xi^\pm = \zeta \omega^{-k_1^\pm + k_2^\pm} , \quad (\text{D.7.30})$$

and we thus produce invariance groups $C_T^{\pm, \max}$ isomorphic to $C_T^{\max}$ in (4.8.17). We can, of course, restrict to the vectorlike couplings used above. In general, we expect it is important to use the maximal symmetry group to find the most stringent conditions for defining $C_T^{\max}$.

D.8 Examples Of Cohomological Conditions For A_{N-1} Class $\mathcal{S}$ Theories With Full And Simple Punctures

In this section, we will derive cohomological conditions for the existence of weakly coupled descriptions of A_{N-1} class $\mathcal{S}$ theories on general non-spin manifolds following the discussion in Section 4.8. These cohomological conditions depend on the chosen pants decomposition for the weakly coupled description of the theory. For simplicity, we will restrict to the theories constructed from trinions with ffs and fff punctures and only full punctures being gauged.

Let us briefly recall the notation from Section 4.8. Choose a pants decomposition of $C_{g,n}$ into a set $\mathcal{T}$ of trinions. The external cutting curves are denoted by c_i and internal cutting curves are denoted by c_α . We will denote the external cutting curve bounding a full (resp. simple) puncture by c_i^f (resp. c_i^s).

For the cutting curves bounding a trinion T , we will write $c_{T,i}$, $c_{T,\alpha}$. The simply connected cover of the flavor group at an external cutting curve c_i is denoted by $\tilde{G}_{c_i}$ and the simply connected cover of the gauge group at an internal cutting curve c_α is denoted by $\text{SU}(N)_{c_\alpha}$. Define the group

$$G_S = \left(\text{Spin}(4) \times \text{SU}(2)_R \times \prod_i \tilde{G}_i \times \prod_\alpha \text{SU}(N)_\alpha \right) / C''_{\mathcal{T}} , \quad (\text{D.8.1})$$

where $C''_{\mathcal{T}}$ is defined below (4.8.46).

The cohomological conditions for the theory are obtained from the existence of the principal G_S bundle $P_S \rightarrow \mathbb{X}$. To determine these cohomological conditions, we will follow the same strategy as in Section 4.4. Let $p_1, p_2, p_{3,c_i}, p_{4,c_\alpha}$ be projection maps to various factors:

$$\begin{array}{ccccc}
 & & G_S & & \\
 & \swarrow p_1 & & \searrow p_{4,c_\alpha} & \\
 \text{Spin}(4) & & & & \text{SU}(N)_{c_\alpha} \\
 & \swarrow p_2 & \searrow p_{3,c_i} & & \\
 & \text{SU}(2)_R & \tilde{G}_{c_i} & &
 \end{array} \quad (\text{D.8.2})$$

Let us also denote the transfer of structure groups of P_S along various projections by

$$\begin{aligned} \text{Fr}(\mathbb{X}) &:= (p_1)_*(P_S) , \\ P^R &:= (p_2)_*(P_S) , \\ P^{f, c_i} &:= (p_{3, c_i})_*(P_S) , \\ P^{\text{gauge}, c_\alpha} &:= (p_{4, c_\alpha})_*(P_S) . \end{aligned} \tag{D.8.3}$$

Define a set of positive integers $n_{c_i^f}$ by

$$p_{3, c_i^f}(C''_{\mathcal{T}}) \cong \mathbb{Z}/n_{c_i^f}\mathbb{Z} . \tag{D.8.4}$$

Let us write

$$C_{c_i^f} := p_{3, c_i^f}(C''_{\mathcal{T}}) \subseteq \mathbb{Z}_N , \quad C_{c_\alpha} := p_{4, c_\alpha}(C''_{\mathcal{T}}) \subseteq \mathbb{Z}_N . \tag{D.8.5}$$

We can then introduce flat $C_{c_i^f}$, C_{c_α} -gerbes $b_{c_i^f}$, b_{c_α} respectively. Their characteristic classes

$$\mu(b_{c_i^f}) \in H^2(\mathbb{X}; C_{c_i^f}) , \quad \mu(b_{c_\alpha}) \in H^2(\mathbb{X}; C_{c_\alpha}) , \tag{D.8.6}$$

can be identified with the obstruction to lifting the structure groups of P^{f, c_i^f} and $P^{\text{gauge}, c_\alpha}$ respectively from $\text{SU}(N)/C_{i, f}^{\text{flavor}}$ and $\text{SU}(N)/C_\alpha^{\text{grb}}$, to $\text{SU}(N)$. In particular, $\mu(b_{c_\alpha})$ can be identified with the 't Hooft flux of the $\text{SU}(N)_{c_\alpha}$ factor of the gauge group. Now choose transition functions

$$t_{pq}^S := [(\tilde{t}_{pq}^+, \tilde{t}_{pq}^-, \tilde{t}_{pq}^R, \tilde{t}_{pq}^f, \tilde{t}_{pq}^g)] : U_{pq} \longrightarrow G_S , \tag{D.8.7}$$

for the G_S -bundle $P_S \rightarrow \mathbb{X}$. The cocycle on triple overlaps is given by

$$t_{pq}^S t_{qr}^S t_{rp}^S = [(\zeta_{pqr}^+, \zeta_{pqr}^-, \zeta_{pqr}^R, \zeta_{pqr}^f, \zeta_{pqr}^g)] . \tag{D.8.8}$$

Thus the cocycle condition is given by

$$(\zeta_{pqr}^+, \zeta_{pqr}^-, \zeta_{pqr}^R, \zeta_{pqr}^f, \zeta_{pqr}^g) \in C''_{\mathcal{T}} . \tag{D.8.9}$$

As before, the cocycles $\zeta_{pqr}^\pm$ and ζ_{pqr}^R can be identified with the second Stiefel-Whitney class of $\mathbb{X}$ and P^R respectively:

$$\begin{aligned} w_2(\mathbb{X}) &= [\zeta_{pqr}^\pm] \in H^2(\mathbb{X}, \mathbb{Z}_2) , \\ w_2(P^R) &= [\zeta_{pqr}^R] \in H^2(\mathbb{X}, \mathbb{Z}_2) . \end{aligned} \tag{D.8.10}$$

The projections of the cocycles ζ_{pqr}^f and ζ_{pqr}^g can be identified with representatives of the characteristic classes $\mu(b_{i, f})$ and $\mu(b_\alpha)$ by

$$\begin{aligned} \mu(b_{c_i^f}) &= [p_{3, c_i^f}(\zeta_{pqr}^f)] \in H^2(\mathbb{X}; C_{c_i^f}) , \\ \mu(b_{c_\alpha}) &= [p_{4, c_\alpha}(\zeta_{pqr}^g)] \in H^2(\mathbb{X}; C_{c_\alpha}) . \end{aligned} \tag{D.8.11}$$

Finally, let $c_1(c_i^s) \in H^2(\mathbb{X}; \mathbb{Z})$ be the first Chern class of the $U(1)$ -bundle obtained by transfer of structure group of P^{f, c_i^s} along the isomorphism (see footnote 22)

$$U(1)/\mathbb{Z}_{n_{c_i^s}} \xrightarrow{\cong} U(1) . \quad (\text{D.8.12})$$

The cocycle $p_{3, c_i^s}(\zeta_{pqr}^R)$ represents the $\text{mod } n_{i, s}$ reduction of $c_1(c_i^s)$:

$$[p_{3, c_i^s}(\zeta_{pqr}^R)] = r_{n_{c_i^s}}(c_1(c_i^s)) . \quad (\text{D.8.13})$$

We are now equipped to write the cohomological conditions. Due to the definition of $C''_{\mathcal{T}}$, we must have

$$\zeta_{pqr}^- = \zeta_{pqr}^+ = \zeta_{pqr}^R , \quad (\text{D.8.14})$$

which implies

$$w_2(\mathbb{X}) = w_2(P^R) . \quad (\text{D.8.15})$$

Next, if two trinions $T, T' \in \mathcal{T}$ are bounded by a common internal cutting curve $c_{T, \alpha} = c_{T', \alpha}$, then we must have

$$p_{4, c_{T, \alpha}}(\zeta_{pqr}^g) = p_{4, c_{T', \alpha}}(\zeta_{pqr}^g) , \quad (\text{D.8.16})$$

which implies the cohomological condition

$$\mu(b_{c_{T, \alpha}}) = \mu(b_{c_{T', \alpha}}) . \quad (\text{D.8.17})$$

Next, we derive cohomological conditions for the two types of trinions.

fff Trinion.

For every trinion $T \in \mathcal{T}$ of type fff with two external cutting curves $c_{T, i}^f, c_{T, j}^f$ and one internal cutting curve $c_{T, \alpha}^f$, equations (4.8.41), (4.8.44) and the explicit form (4.8.30) implies that

$$(\zeta_{pqr}^{\pm}) p_{3, c_{T, i}^f}(\zeta_{pqr}^f) p_{3, c_{T, j}^f}(\zeta_{pqr}^f) p_{4, c_{T, \alpha}^f}(\zeta_{pqr}^f) = 1 , \quad N \text{ even} , \quad (\text{D.8.18})$$

$$p_{3, c_{T, i}^f}(\zeta_{pqr}^f) p_{3, c_{T, j}^f}(\zeta_{pqr}^f) p_{4, c_{T, \alpha}^f}(\zeta_{pqr}^f) = 1 , \quad N \text{ odd} . \quad (\text{D.8.19})$$

This gives the cohomological condition

$$\rho_s(w_2(\mathbb{X})) + \iota(\mu(b_{c_{T, i}^f})) + \iota(\mu(b_{c_{T, j}^f})) + \iota(\mu(b_{c_{T, \alpha}^f})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_{[2, N]}) , \quad N \text{ even} , \quad (\text{D.8.20})$$

$$\iota(\mu(b_{c_{T, i}^f})) + \iota(\mu(b_{c_{T, j}^f})) + \iota(\mu(b_{c_{T, \alpha}^f})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_N) , \quad N \text{ odd} , \quad (\text{D.8.21})$$

where $[2, N] = \text{lcm}(2, N)$, ρ_s is the map in cohomology induced from the inclusion map $\rho_s : \mathbb{Z}_2 \rightarrow \mathbb{Z}_{[2, N]}$, and ι is the map in cohomology obtained from inclusions in (D.8.5). If the trinion is bounded

by two internal cutting curves $c_{T,\alpha}^f, c_{T,\beta}^f$ and one external cutting curve $c_{T,i}^f$, the cohomological condition we get is

$$\rho_s(w_2(\mathbb{X})) + \iota(\mu(b_{c_{T,i}^f})) + \iota(\mu(b_{c_{T,\alpha}^f})) + \iota(\mu(b_{c_{T,\beta}^f})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_{[2,N]}) , \quad N \text{ even} , \quad (\text{D.8.22})$$

$$\iota(\mu(b_{c_{T,i}^f})) + \iota(\mu(b_{c_{T,\alpha}^f})) + \iota(\mu(b_{c_{T,\beta}^f})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_N) , \quad N \text{ odd} . \quad (\text{D.8.23})$$

If the trinion is bounded by three internal cutting curves $c_{T,\alpha}^f, c_{T,\beta}^f$ and $c_{T,\gamma}^f$, the cohomological condition we get is

$$\rho_s(w_2(\mathbb{X})) + \iota(\mu(b_{c_{T,\alpha}^f})) + \iota(\mu(b_{c_{T,\beta}^f})) + \iota(\mu(b_{c_{T,\gamma}^f})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_{[2,N]}) , \quad N \text{ even} , \quad (\text{D.8.24})$$

$$\iota(\mu(b_{c_{T,\alpha}^f})) + \iota(\mu(b_{c_{T,\beta}^f})) + \iota(\mu(b_{c_{T,\gamma}^f})) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_N) , \quad N \text{ odd} . \quad (\text{D.8.25})$$

ffs Trinion.

For every trinion $T \in \mathcal{T}$ of type ffs with two external cutting curves $c_{T,i}^f, c_{T,j}^s$ and one internal cutting curve $c_{T,\alpha}^f$, equations (4.8.41), (4.8.44) and the explicit form (4.8.17) implies that

$$(\zeta_{pqr}^\pm) \left(p_{3,c_{T,i}^f}(\zeta_{pqr}^f) \right)^{-1} p_{4,c_{T,\alpha}^f}(\zeta_{pqr}^g) p_{3,c_{T,j}^s}(\zeta_{pqr}^f) = 1 . \quad (\text{D.8.26})$$

This gives the cohomological condition

$$\rho_s(w_2(\mathbb{X})) - \iota(\mu(b_{c_{T,i}^f})) + \iota(\mu(b_{c_{T,\alpha}^f})) + \iota \circ r_{n_{c_{T,j}^s}}(c_1(c_j^s)) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_{[2,N]}) , \quad (\text{D.8.27})$$

where r_n denotes reduction $\bmod n$. If the trinion is bounded by two internal cutting curves $c_{T,\alpha}^f, c_{T,\beta}^f$ and one external cutting curve $c_{T,i}^s$, the cohomological condition we get is

$$\rho_s(w_2(\mathbb{X})) + \iota(\mu(b_{c_{T,\alpha}^f})) - \iota(\mu(b_{c_{T,\beta}^f})) + \iota \circ r_{n_{c_{T,i}^s}}(c_1(c_i^s)) = 0 \in H^2(\mathbb{X}; \mathbb{Z}_{[2,N]}) . \quad (\text{D.8.28})$$

Note that in this case, we cannot have all three internal cutting curves.

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