Isomorphism of Skew-Holomorphic Harmonic Maass-Jacobi Forms and Certain Weak Harmonic Maass Forms

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Abstract

Recently Bringmann, Raum and Richter generalised the definition of Jacobi forms and Skoruppa's skew-holomorphic Jacobi forms by intertwining with harmonic Maass forms. We prove the isomorphism of the Kohnen's plus space analogue of harmonic Maass forms of weight $k-1/2$ for $\Gamma_0(4m)$ and the space of these skew-holomorphic harmonic Maass-Jacobi forms of weight k and index m for k odd and $m=1$ or a prime.

1 Introduction

Let k be an integer, and m a positive integer. Let $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ and $M^!_{k-\frac{1}{2}}(\Gamma_0(4m))$ be the space of holomorphic and weakly holomorphic modular forms respectively, of weight $k-\frac{1}{2}$ for $\Gamma_0(4m)$. Let $M^+_{k-\frac{1}{2}}(\Gamma_0(4m))$ be the Kohnen's plus space which is a subspace of $M_{k-\frac{1}{2}}(\Gamma_0(4m))$ defined as

$$M^+_{k-\frac{1}{2}}(\Gamma_0(4m)) := \left\{ f \in M_{k-\frac{1}{2}}(\Gamma_0(4m)) \mid c_f(n) = 0 \text{ unless } (-1)^k n \equiv 0, 1 \bmod 4m \right\}.$$

Eichler and Zagier systematically developed the theory of Jacobi forms in [5]. They also proved that the Kohnen's plus space of holomorphic modular forms of weight $k-1/2$ for $\Gamma_0(4m)$ is isomorphic to the space of holomorphic Jacobi forms of weight k and index m for k even and $m=1$ or a prime. They did so by decomposing the Fourier expansion of Jacobi forms in terms of theta series and showing that the coefficients transform like modular forms. For k odd, Skoruppa showed that the plus space is isomorphic to the space of a new kind of Jacobi form which is real analytic in variable τ and holomorphic in z. These forms were called the skew-holomorphic Jacobi forms [6, 7].

In 1920, Ramanujan, in his last letter to Hardy [10] listed 17 functions which he called mock theta functions. These functions led to huge developments in the theory of automorphic forms. After about eight decades, Zwegers [9] in 2002 came up with a systematic framework for mock theta functions and linked it to harmonic Maass forms. Motivated by this, the theory of Jacobi forms was generalised to harmonic Maass-Jacobi forms by Bringmann and Richter [2]. With this generalisation, Cho defined the analogue of Kohnen's plus space for harmonic Maass forms and proved Zagier type isomorphism between the plus space of harmonic Maass forms and the space of harmonic Maass-Jacobi form [8]. To be precise they proved the isomorphism of plus space of harmonic Maass forms and the space of vector valued harmonic Maass forms.

Recently the theory of skew holomorphic Jacobi forms was generalised to include harmonic properties with respect to some differential operator by Bringmann, Raum and Richter [3]. Several different generalisations and structure results have been proved in [3], but we will only consider a subspace of the generalisation containing Skoruppa's skew-holomorphic Jacobi forms. The aim of the present article is to prove the isomorphism of the plus space of harmonic Maass forms and skew-holomorphic harmonic Maass-Jacobi forms combining the results of [3, 8], in the spirit of Skoruppa. To describe the main results, we first introduce notations and definitions.

2 Preliminaries

2.1 Harmonic Maass Forms

In this section, we will briefly review harmonic Maass forms. The reader is referred to [4] for details.
Let $\mathbb{H}$ denote the usual upper half space. We write $\tau \in \mathbb{H}$ as $\tau = u + iv$ and $z \in \mathbb{C}$ as $z = x + iy$. Put $q = e(\tau) = e^{2\pi i\tau}$ and $\zeta = e(z) = e^{2\pi iz}$. Define

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Define the weight-k ($\in \mathbb{R}$) hyperbolic Laplacian

$$\Delta_k := -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right) = -4v^2 \frac{\partial}{\partial \tau}\frac{\partial}{\partial \bar{\tau}} + 2ikv\frac{\partial}{\partial \bar{\tau}}.$$

We define Harmonic Maass forms following Bruiner and Funke [1].

Definition 2.1. Let $k \in \frac{1}{2}\mathbb{Z}$. A smooth function (in real sense) $f : \mathbb{H} \to \mathbb{C}$ is called a weight-k harmonic Maass form on $\Gamma_0(N)$ ($4|N$ if $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$) if

(i) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $\tau \in \mathbb{H}$, we have

$$f\left(\frac{az+b}{cz+d} \right) = \begin{cases} (cz+d)^k f(\tau) & \text{if } k \in \mathbb{Z} \\ (\frac{c}{d})\varepsilon_d^{-2k}(cz+d)^k f(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

Here $\left(\frac{c}{d}\right)$ is the Jacobi symbol and $\varepsilon_d = \sqrt{\left(\frac{-1}{d}\right)}$. Here $\sqrt{\cdot}$ denotes the principal branch of square root.

(ii) $\Delta_k(f) = 0$.

(iii) There exists a polynomial $P_f(\tau) \in \mathbb{C}[q^{-1}]$ such that $f(\tau) - P_f(\tau) = O(e^{-\varepsilon v})$ as $v \to \infty$ for some $\varepsilon > 0$. Similar conditions hold at other cusps.

If the third condition in the above definition is replaced by $f(\tau) = O(e^{\varepsilon v})$, then f is said to be a harmonic Maass form of *manageable growth*. Space of harmonic Maass forms of weight k is denoted by $H_k(\Gamma_0(N))$ and that of harmonic Maass forms of manageable growth is denoted by $H_k^!(\Gamma_0(N))$.

Remark 2.2. One easily sees that $M_k^!(\Gamma_0(N)) \subset H_k(\Gamma_0(N)) \subset H_k^!(\Gamma_0(N))$.

$f \in H_k^!(\Gamma_0(N))$ has a Fourier expansion [1] of the form

$$f(\tau) = f(u+iv) = \sum_{n>>-\infty} c_f^+(n)q^n + c_f^-(0)v^{1-k} + \sum_{\substack{n<<\infty \\ n\neq 0}} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n. \tag{1}$$

where $\Gamma(s,z)$ is the incomplete gamma function defined as

$$\Gamma(s,z) := \int_z^\infty e^{-t}t^s \frac{dt}{t}. \tag{2}$$

The notation $\sum\limits_{n>>-\infty}$ means $\sum\limits_{n=\alpha_f}^{\infty}$ for some $\alpha_f \in \mathbb{Z}$. $\sum\limits_{n<<\infty}$ is defined similarly. If $f \in H_k(\Gamma)$ then

$$f(\tau) = f(u+iv) = \sum_{n>>-\infty} c_f^+(n)q^n + \sum_{n<0} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n. \tag{3}$$

We call

$$f^+(\tau) = \sum_{n>>-\infty} c_f^+(n)q^n$$

the *holomorphic part* of f and

$$f^-(\tau) = c_f^-(0)v^{1-k} + \sum_{\substack{n<<\infty \\ n\neq 0}} c_f^-(n)\Gamma(1-k, -4\pi nv)q^n$$

the *nonholomorphic part* of f.

2.2 Vector Valued Harmonic Maass Forms

Let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic two-fold cover of $\mathrm{SL}_2(\mathbb{R})$ consisting of elements of the form (A, φ) where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\varphi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that $\varphi^2(\tau) = c\tau + d$. The operation in $\mathrm{Mp}_2(\mathbb{R})$ is defined as

$$(A, \varphi)(B, \psi) = (AB, \varphi(B\tau)\psi(\tau)).$$

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. One can show that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $\widetilde{T} := \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$ and $\widetilde{S} := \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$.

Let V be a rational vector space over $\mathbb{Q}$ with a non-degenerate quadratic form Q. Let $(x, y) = Q(x + y) - Q(x) - Q(y)$ be the associated bilinear form of signature (p, q). Let $L \subset V$ be an even lattice with dual L^*. We denote the standard basis elements of the group algebra $\mathbb{C}[L^*/L]$ by $\mathfrak{e}_\gamma$ for $\gamma \in L^*/L$. Let ϱ_L be the Weil representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L^*/L]$, defined by

$$\begin{aligned} \varrho_L(\widetilde{T})(\mathfrak{e}_\gamma) &:= e(Q(\gamma))\mathfrak{e}_\gamma \\ \varrho_L(\widetilde{S})(\mathfrak{e}_\gamma) &:= \frac{e((q-p)/8)}{\sqrt{|L^*/L|}} \sum_{\delta \in L^*/L} e(-(\gamma, \delta))\mathfrak{e}_\delta. \end{aligned}$$

Definition 2.3. Let $k \in \frac{1}{2}\mathbb{Z}$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}[L^*/L]$ is called a weakly holomorphic modular form of weight k and type ϱ_L for the group $\mathrm{Mp}_2(\mathbb{Z})$ if it satisfies:

1. $f(M\tau) = \varphi(\tau)^{2k} \varrho_L(M, \varphi) f(\tau)$ for all $(M, \varphi) \in \mathrm{Mp}_2(\mathbb{Z})$
2. f has a Fourier expansion of the form

$$f(\tau) = \sum_{\gamma \in L^*/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n \gg -\infty}} c_f(\gamma, n) e(n\tau)\mathfrak{e}_\gamma.$$

The space of these $\mathbb{C}[L^*/L]$ -valued weakly holomorphic modular forms is denoted by $M^!_{k,\varrho_L}$. Similarly we define holomorphic modular forms of weight k and type ϱ_L and denote the corresponding space by M_{k,ϱ_L}.

Definition 2.4. A smooth function $f : \mathbb{H} \to \mathbb{C}[L^*/L]$ is called a harmonic Maass form of weight k and type ϱ_L for the group $\mathrm{Mp}_2(\mathbb{Z})$ if it satisfies:

1. $f(M\tau) = \varphi(\tau)^{2k} \varrho_L(M, \varphi) f(\tau)$ for all $(M, \varphi) \in \mathrm{Mp}_2(\mathbb{Z})$;
2. $\Delta_k f = 0$;
3. There is a polynomial $P_f(\tau) = \sum_{\gamma \in L^*/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ -\infty \ll n \leq 0}} c_f^+(\gamma, n) e(n\tau)\mathfrak{e}_\gamma$ such that $f(\tau) = P_f(\tau) + O\left(e^{-\varepsilon v}\right)$ as $v \to \infty$ for some $\varepsilon > 0$.

We denote by H_{k,ϱ_L} the space of these $\mathbb{C}[L^*/L]$-valued harmonic Maass forms. One can similarly define harmonic Maass forms of weight k and type ϱ_L of manageable growth by replacing Condition 3 above by $f(\tau) = O(e^{\varepsilon v})$. The space of such forms is denoted by $H^!_{k,\varrho_L}$.

Remark 2.5. It can easily be seen that $M^!_{k,\varrho_L} \subset H_{k,\varrho_L} \subset H^!_{k,\varrho_L}$.

Define the following subspace of $H^!_{k-\frac{1}{2}}(\Gamma_0(4m))$

$$H^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m)) := \{f \in H^!_{k-\frac{1}{2}}(\Gamma_0(4m)) \mid c_f^\pm(n) = 0 \text{ unless } (-1)^k n \equiv 0, 1 (\mathrm{mod}\ 4m)\}.$$

Similarly, one can define $H^{+}_{k-\frac{1}{2}}(\Gamma_0(4m))$. Note that $M^{+}_{k-\frac{1}{2}}(\Gamma_0(4m)) \subset M^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m)) \subset H^{+}_{k-\frac{1}{2}}(\Gamma_0(4m)) \subset H^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m))$.

Now for some $m \in \mathbb{Z}_{>0}$, suppose $(L^*/L, Q) = (\mathbb{Z}/2m\mathbb{Z}, Q)$ where $Q(\gamma) = \gamma^2/4m$ for $\gamma \in L^*/L$. In this case the level of L is $4m$ and we have $p - q \equiv 1(\text{mod } 8)$. One easily sees that for this lattice and quadratic form, the Weil representation is given by

$$\begin{aligned}\varrho_m(\widetilde{T})\mathfrak{e}_\ell &:= e\left(\tfrac{\ell^2}{4m}\right)\mathfrak{e}_\ell\\ \varrho_m(\widetilde{S})\mathfrak{e}_\ell &:= \tfrac{1}{\sqrt{2im}}\sum_{\ell'(\text{ mod } 2m)} e\left(-\tfrac{\ell\ell'}{2m}\right)\mathfrak{e}_{\ell'}.\end{aligned}$$

With these notations, given an $f \in H^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m))$ we define a $\mathbb{C}[L^*/L]$-valued function $F = \sum_{\gamma\in\mathbb{Z}/2m\mathbb{Z}} F_\gamma \mathfrak{e}_\gamma$ by

$$F_\gamma(\tau) := \frac{1}{s(\gamma)}\sum_{n\in\mathbb{Z}} c_f(n, y/4m)q^{n/4m}$$

where

$$c_f(n,y) := \begin{cases} c_f^+(n) + c_f^-(n)\Gamma\left(\frac{3}{2} - k, 4\pi|n|y\right) & \text{if } n \neq 0\\ c_f^+(n) + c_f^-(n)v^{\frac{3}{2}-k} & \text{if } n = 0\end{cases}$$

and $s(\gamma) = 1$ if $\gamma \equiv 0, m \bmod 2m$, and 2 otherwise. Cho proved the following Theorem [8].

Theorem 2.6. *If k is odd and $m = 1$ or a prime, then the map $f \mapsto F$ defines an isomorphism of $H^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m))$ onto $H^!_{k-\frac{1}{2},\varrho_L}$.*

Remark 2.7. The isomorphism of Theorem 2.6 restricts to the following isomorphisms (cf. Remark 2,(2) of [8]):

1. $H^{+}_{k-\frac{1}{2}}(\Gamma_0(4m)) \simeq H_{k-\frac{1}{2},\varrho_L}$,
2. $M^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m)) \simeq M^!_{k-\frac{1}{2},\varrho_L}$,
3. $M^{+}_{k-\frac{1}{2}}(\Gamma_0(4m)) \simeq M^{+}_{k-\frac{1}{2},\varrho_L}$.

2.3 Skew-Holomorphic Harmonic Maass Jacobi Forms

We mostly follow the notation of [3] for this section. Let $k, m \in \mathbb{Z}, m > 0$. Let $\Gamma^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ be the Jacobi modular group. Define the following skew slash operator: For $A = (\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right), (\lambda, \mu))$

$$\left(\Phi|^{\text{sk}}_{k,m} A\right)(\tau, z) := \quad \Phi\left(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\overline{\tau} + d)^{1-k}|c\tau + d|^{-1}e^{2\pi im\left(-\frac{c(z+\lambda\tau+\mu)^2}{c\tau+d}+\lambda^2\tau+2\lambda z\right)} \tag{4}$$

Define the differential operator - the *skew Casimir element* by

$$\mathcal{C}^{\text{sk}}_{k,m} = -\frac{iv^2}{\pi m}v^{\frac{1}{2}-k} \circ \frac{\partial}{\partial\overline{\tau}} \circ v^{k-\frac{1}{2}}L_m, \tag{5}$$

where

$$L_m := 8\pi i m \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2}$$

is the heat operator.

Definition 2.8. A function $\Phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$ is a skew-holomorphic harmonic Maass-Jacobi form of weight k and index m if Φ is real-analytic in $\tau \in \mathbb{H}$ and holomorphic in $z \in \mathbb{C}$, and satisfies the following conditions:

(i) For all $A \in \Gamma^J$, $\left(\Phi|_{k,m}^{\mathrm{sk}} A\right) = \Phi$;

(ii) $\mathcal{C}_{k,m}^{\mathrm{sk}}(\Phi) = 0$;

(iii) $\varphi(\tau, z) = O(e^{\varepsilon v} e^{2\pi m y^2/v})$ as $v \to \infty$ for some $\varepsilon > 0$.

We denote the space of skew-holomorphic harmonic Maass-Jacobi form of weight k and index m by $\widehat{\mathbb{J}}_{k,m}^{\mathrm{sk}}$.

Remark 2.9. Let $J_{k,m}^{!\mathrm{sk}}$ (respectively $J_{k,m}^{\mathrm{sk}}$) denote the space of Skoruppa's skew-holomorphic weak (respectively holomorphic) Jacobi forms of weight k and index m. Then only easily sees that $J_{k,m}^{\mathrm{sk}} \subset J_{k,m}^{!\mathrm{sk}} \subset \widehat{\mathbb{J}}_{k,m}^{\mathrm{sk}}$.

Proposition 3.6 of [3] along with the growth condition (iii) implies that $\Phi \in \widehat{\mathbb{J}}_{k,m}^{\mathrm{sk}}$ has Fourier expansion of the form

$$\begin{aligned}\Phi(\tau, z) = v^{\frac{3}{2}-k} \sum_{\substack{n,r\in\mathbb{Z} \\ D=0}} c^0(n,r) q^n \zeta^r + \sum_{\substack{n,r\in\mathbb{Z} \\ D\ll\infty}} c^+(n,r) \exp\left(\frac{\pi D v}{m}\right) q^n \zeta^r \\ + \sum_{\substack{n,r\in\mathbb{Z} \\ D\gg-\infty}} c^-(n,r) \Gamma\left(\frac{3}{2} - k, \frac{\pi D v}{m}\right) \exp\left(\frac{\pi D v}{m}\right) q^n \zeta^r\end{aligned} \tag{6}$$

where $D = r^2 - 4mn$. We also consider $\Phi \in \widehat{\mathbb{J}}_{k,m}^{\mathrm{sk}}$ which have Fourier expansion of the form

$$\begin{aligned}\Phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ D\ll\infty}} c^+(n,r) \exp\left(\frac{\pi D v}{m}\right) q^n \zeta^r \\ + \sum_{\substack{n,r\in\mathbb{Z} \\ D>0}} c^-(n,r) \Gamma\left(\frac{3}{2} - k, \frac{\pi D v}{m}\right) \exp\left(\frac{\pi D v}{m}\right) q^n \zeta^r.\end{aligned} \tag{7}$$

We denote this subspace by $\widehat{\mathbb{J}}_{k,m}^{\mathrm{sk,cusp}}$. With the above Definition and notations we have the following Theorem:

Theorem 2.10. *Let k be odd, and $m = 1$ or a prime. Then*

1. $\widehat{\mathbb{J}}_{k,m}^{sk} \simeq H_{k-\frac{1}{2}}^{!+}(\Gamma_0(4m))$,

2. $\widehat{\mathbb{J}}_{k,m}^{sk,cusp} \simeq H_{k-\frac{1}{2}}^{+}(\Gamma_0(4m))$.

3 Proof of Theorem 2.10

We will only prove 2 since 1 is similar. Let $\Phi \in \widehat{\mathbb{J}}^{\mathrm{sk,cusp}}_{k,m}$. Then we have that

$$\Phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z}\\ D\ll\infty}} c^{+}(n,r)\exp\left(\frac{\pi D v}{m}\right) q^n\zeta^r \\ + \sum_{\substack{n,r\in\mathbb{Z}\\ D>0}} c^{-}(n,r)\Gamma\left(\frac{3}{2}-k, \frac{\pi D v}{m}\right)\exp\left(\frac{\pi D v}{m}\right) q^n\zeta^r.$$

Using the transformation of Φ for $A = (I, (\lambda, \mu))$, we have that

$$\Phi(\tau, z+\lambda\tau+\mu) = e\left(-m\left(\lambda^2\tau + 2\lambda z\right)\right)\Phi(\tau, z).$$

Using similar arguments as in Theorem 2.2 of [5], one can deduce that if $r' \equiv r \bmod 2m$ and $D' = D$ with $D' := r'^2 - 4n'm$, then

$$c^{+}(n', r') = c^{+}(n,r), \quad c^{-}(n', r') = c^{-}(n,r), \quad \Gamma\left(\frac{3}{2}-k, \frac{\pi D' y}{m}\right) = \Gamma\left(\frac{3}{2}-k, \frac{\pi D y}{m}\right)$$

Hence, we can decompose $\Phi(\tau, z)$ as a linear combination of the theta functions as

$$\Phi(\tau, z) = \sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}} h_\ell(\tau)\vartheta_{m,\ell}(\tau, z)$$

where

$$h_\ell(\tau) = \sum_{N\gg-\infty} c^{+}\left(\frac{N+r^2}{4m}, r\right)\exp\left(-\frac{\pi N v}{m}\right) q^{N/4m} \\ + \sum_{N<0} c^{-}\left(\frac{N+r^2}{4m}, r\right)\Gamma\left(\frac{3}{2}-k, -\frac{\pi N y}{m}\right)\exp\left(-\frac{\pi N v}{m}\right) q^{N/4m}$$

with any $r \in \mathbb{Z}, r \equiv \ell \bmod 2m$, and

$$\vartheta_{m,\ell}(\tau, z) := \sum_{\substack{r\in Z\\ r\equiv\ell \bmod 2m}} q^{r^2/4m}\zeta^r.$$

Put $g_\ell(\tau) = h_\ell(-\overline{\tau})$. Then using similar arguments as in Theorem 5.1 of [5], one can show that the $2m$ tuple $(g_\ell)_{\ell(\mathrm{mod}\ 2m)}$ satisfies the transformation property of vector valued harmonic Maass form. It remains to show that $\Delta_{k-\frac{1}{2}} g_\ell(\tau) = 0$. Using the fact that $\vartheta_{m,\ell}$ is annihilated by the heat operator L_m, it is easy to check that

$$\mathcal{C}^{\mathrm{sk}}_{k,m}(h_\ell\vartheta_{m,\ell}) = 2\left[2iv\left(k-\frac{1}{2}\right)\frac{\partial h_\ell}{\partial\tau} + 4v^2\frac{\partial^2 h_\ell}{\partial\tau\overline{\tau}}\right]\vartheta_{m,\ell}.$$

Thus $\mathcal{C}^{\text{sk}}_{k,m}(\Phi)=0$ gives

$$2iv\left(k-\frac{1}{2}\right)\frac{\partial h_\ell}{\partial\tau}+4v^2\frac{\partial^2 h_\ell}{\partial\tau\overline{\tau}}=0. \tag{8}$$

for each ℓ. Next observe that

$$\Delta_{k-\frac{1}{2}}g_\ell(\tau)=\Delta_{k-\frac{1}{2}}h_\ell(-\overline{\tau}).$$

Using the expression for the hyperbolic Laplacian we have

$$\Delta_{k-\frac{1}{2}}h_\ell(-\overline{\tau})=-4v^2\frac{\partial^2 h_\ell(-\overline{\tau})}{\partial\tau\overline{\tau}}+2ikv\frac{\partial h_\ell(-\overline{\tau})}{\partial\overline{\tau}}$$

Substituting $T=-\overline{\tau}$ and using chain rule we get

$$\Delta_{k-\frac{1}{2}}h_\ell(-\overline{\tau})=-4(v^2)\frac{\partial^2 h_\ell(T)}{\partial T\overline{T}}+2i\left(k-\frac{1}{2}\right)(-v)\frac{\partial h_\ell(T)}{\partial T}.$$

Observing that $V=\operatorname{Im}(-\overline{\tau})=v$, we get

$$\Delta_{k-\frac{1}{2}}h_\ell(-\overline{\tau})=-\left[4V^2\frac{\partial^2 h_\ell(T)}{\partial T\overline{T}}+2i\left(k-\frac{1}{2}\right)V\frac{\partial h_\ell(T)}{\partial T}\right]=0$$

where we used Eq. (8). Conversely, given $2m$ tuple $(g_\ell)_{\ell (\text{mod } 2m)}$ of eigenfunctions of the hyperbolic Laplacian, having growth condition as in (iii) of Definition 2.1 and satisfying the transformation rule of a vector valued harmonic Maass form, it is easy to show that the function

$$\Phi(\tau,z)=\sum_{\ell\in\mathbb{Z}/2m\mathbb{Z}}h_\ell(\tau)\vartheta_{m,\ell}(\tau,z)$$

where $h_\ell(\tau)=g_\ell(-\overline{\tau})$, satisfies the transformation property of skew-holomorphic harmonic Maass-Jacobi form. The annihilation by the skew Casimir element follows easily by a similar calculation as in Eq. (8). Finally the Fourier expansion follows from the Fourier expansion (3) of $g_\ell(\tau)$. It follows that we have an isomorphism

$$\widehat{\mathbb{J}}^{\text{sk,cusp}}_{k,m}\simeq H_{k-\frac{1}{2},\varrho_L}.$$

We conclude by noting that Theorem 2.6 implies that

$$\widehat{\mathbb{J}}^{\text{sk,cusp}}_{k,m}\simeq H_{k-\frac{1}{2},\varrho_L}\simeq H^{+}_{k-\frac{1}{2}}(\Gamma_0(4m)).$$

Remark 3.1. In view of Remark 2.7, we see that the isomorphism of Theorem 2.10 restricts to isomorphisms of smaller subspaces. To be precise, we have the following isomorphisms: For k odd and $m=1$ or a prime, we have

1. $H^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m))\simeq H^{!}_{k-\frac{1}{2},\varrho_L}\simeq\widehat{\mathbb{J}}^{\text{sk}}_{k,m}$,
2. $H^{+}_{k-\frac{1}{2}}(\Gamma_0(4m))\simeq H_{k-\frac{1}{2},\varrho_L}\simeq\widehat{\mathbb{J}}^{\text{sk,cusp}}_{k,m}$,
3. $M^{!+}_{k-\frac{1}{2}}(\Gamma_0(4m))\simeq M^{!}_{k-\frac{1}{2},\varrho_L}\simeq J^{!\text{sk}}_{k,m}$,
4. $M^{+}_{k-\frac{1}{2}}(\Gamma_0(4m))\simeq M_{k-\frac{1}{2},\varrho_L}\simeq J^{\text{sk}}_{k,m}$.

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