Spinor Helicity Formalism

Tabasum Rahnuma & Ranveer Kumar Singh

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1 Introduction

Spinor-helicity formalism is a technique which makes scattering amplitude calculations incredibly simple. The tremendous power of this technique shows up when we try to compute gluon scattering amplitudes in Ynag-Mills theory. We will describe an example in next subsection. Table 1 is enough to motivate the need of techniques other than the standard Feynman rules to compute amplitudes and the spinor-helicity formalism does the same. After long and tedious computations involving thousands of terms, the final

# particles	# Feynman Diagrams
4	4
5	25
6	220
7	2485
8	34300
9	559405
10	10525900

Table 1: Number of Feynman Diagrams for given number of particles

amplitude squared turns out to be simple. This hints towards a method that could bypass the redundancies in each Feynman diagram. In particular, standard Feynman rules for massless particles have gauge redundancies which increase in complications as the number of external legs increase. Spinor helicity formalism bypasses these complications by expressing all kinematic variables in terms of what are called helicity spinors. We will mostly follow Chapter 27 of [1] adding stuff wherever necessary.

1.1 An Example from Yang-Mills Theory

We begin with an example from Yang-Mills theory. The YM Lagrangian of N fermions $\{\psi_i\}$, N scalar fields $\{\phi_i\}$ interacting with N^2-1 vector gauge bosons $\{A^a_\mu\}$ is

$$\begin{aligned}\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \left(\partial^\mu \bar{c}^a\right)\left(\delta^{ac}\partial_\mu + g f^{abc}A^b\right)c^c + \bar{\psi}^i\left(\delta_{ij} i\not{\partial} + g\not{A}^a_\mu T^a_{ij} - m\delta_{ij}\right)\psi_i \\ + \left[\left(\delta_{ki}\partial_\mu - igA^a_\mu T^a_{ij}\right)\phi_i\right]^\star\left[\left(\delta_{ki}\partial^\mu - igA^{a\mu}T^a_{ij}\right)\phi_i\right] - M^2\phi^\star_i\phi_i,\end{aligned}$$

where

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$

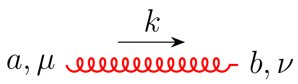
$\not{A}^a$ is $\gamma^\mu A^a_\mu$ with γ^μ being the Dirac gamma matrices, $\{T^a\}_{a=1}^{N^2-1}$ is the basis of the Lie algebra su(N) of SU(N), f^{abc} are the structure constants of SU(N) defined as

$$[T^a, T^b] = i f^{abc} T^c,$$

$\xi, c^a, \bar{c}^a$ are the Fadeev-Popov ghosts and g is the coupling constant. The kinetic term in the Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{kin}} =& -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \bar{\psi}_i(i\not{\partial} - m)\psi_i \\ & -\phi^*_i\left(\Box + M^2\right)\phi_i - \bar{c}^a \Box c^a.\end{aligned}$$

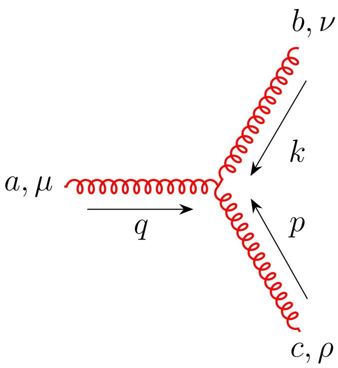
We can read of the gluon propagator (we will mostly be dealing with gluon scattering amplitudes) using our knowledge of photon propagator:



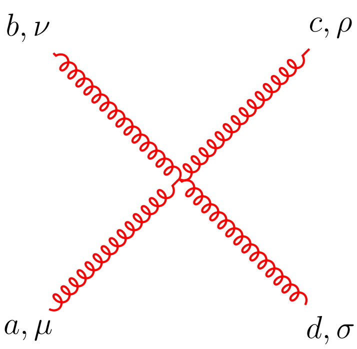
$$\equiv i\frac{-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2}}{k^2 + i\varepsilon}\delta^{ab}$$

For the demonstration of the spinor-helicity formalism, we just need gluon propagator and its three point and four point vertex. The corresponding interaction term in the Lagrangian is,

$$\mathcal{L}_{\text{int}} = -g f^{abc}\left(\partial_\mu A^a_\nu\right) A^b{}_\mu A^c_\nu - \tfrac{1}{4} g^2\, f^{eab} A^a_\mu A^b_v\, f^{ecd} A^c_\mu A^d_\nu \tag{1.1}$$



$$\equiv g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$



$$\equiv \begin{aligned}-ig^2\big[& f^{abe} f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu p}\right) + \\ & f^{ace} f^{bde}\left(g^{\mu\nu}g^{p\sigma} - g^{\mu\sigma}g^{\nu p}\right) + \\ & f^{ade} f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\big]\end{aligned}$$

With this vertex, let us see what is the scattering amplitude of $gg \to gg$ process. We will get four diagrams: s, t, u channel and a 4−vertex as above. They are shown in Figure 1.

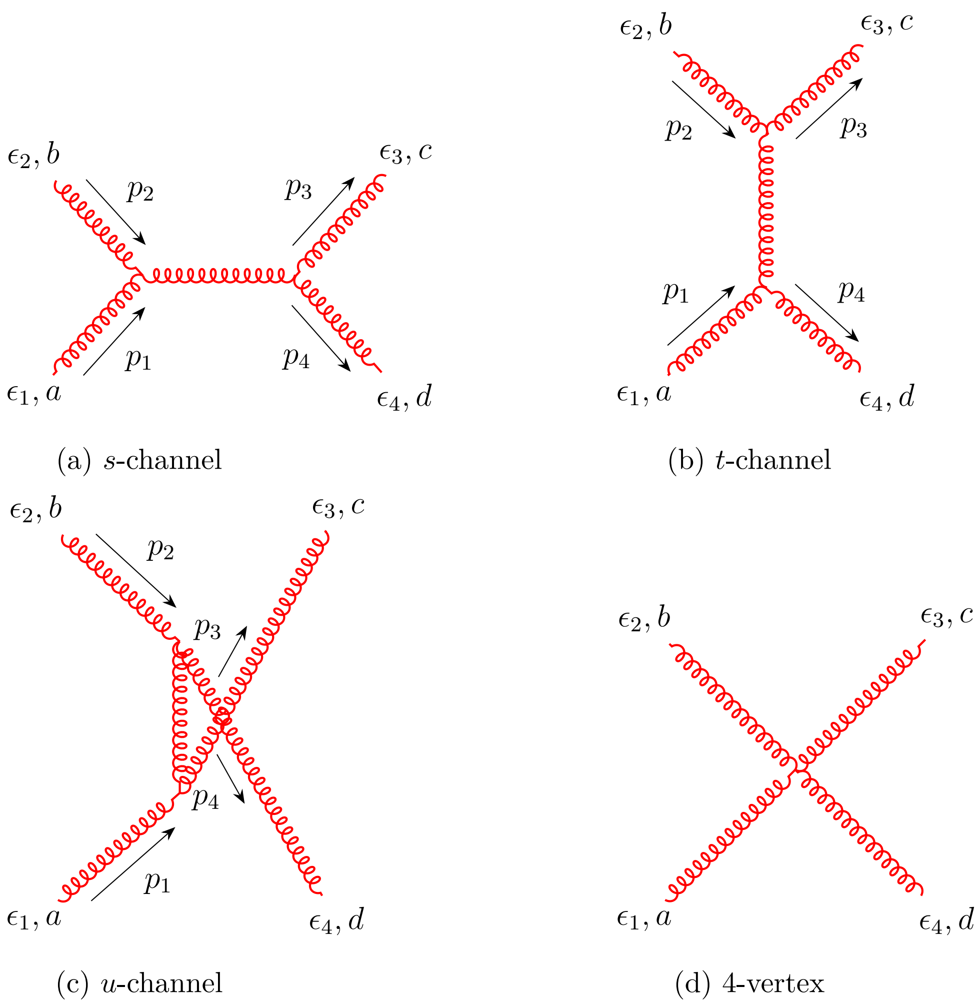


Figure 1: $gg \to gg$ Feynmann diagrams

The s-channel gives the contribution (in $\xi = 1$ gauge)

$$\begin{aligned} & i\mathcal{M}_s \left(p_1 p_2 \longrightarrow p_3 p_4\right) \\ & = -i\frac{g^2}{s} f^{abe} f^{cde} \left[\left(\epsilon_1 \cdot \epsilon_2\right)\left(p_1 - p_2\right)^\mu + \epsilon_2^\mu \left(p_2 + q\right) \cdot \epsilon_1 + \epsilon_1^\mu \left(-q - p_1\right) \cdot \epsilon_2\right] \\ & \times \left[\left(\epsilon_4^\star \cdot \epsilon_3^\star\right)\left(p_4 - p_3\right)^\mu + \epsilon_3^{\star\mu} \left(p_3 + q\right) \cdot \epsilon_4^\star + \epsilon_4^{\star\mu} \left(-q - p_4\right) \cdot \epsilon_3^\star\right] \end{aligned} \tag{1.2}$$

where as usual $s = q^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2$. We can use the transversality condition $p_i \cdot \epsilon_i = 0$ to simplify the above amplitude but the expansion of the product is still a mess. If we compute all the four contributions, square it and sum over polarisation and colors, then we get (Warning! Don't try by hand, it has more than 1000 terms) :

$$\frac{1}{256}\sum_{\substack{\text{polarisations}\\ \text{color}}}|\mathcal{M}|^2 = g^4\,\frac{9}{2}\left[3-\frac{tu}{s^2}-\frac{su}{t^2}-\frac{st}{u^2}\right]$$

which is incredibly simple. Here s, t, u are the Mandelstram variables (s is given above) given by

$$\begin{aligned} s &\equiv (p_1+p_2)^2 = (p_3+p_4)^2\,, \\ t &\equiv (p_2-p_3)^2 = (p_1-p_4)^2\,, \\ u &\equiv (p_2-p_4)^2 = (p_1-p_3)^2\,. \end{aligned}$$

We will see that spinor-helicity formalism will bypass the difficult computations. Although we can do computations on software but it also gets saturated very soon. For example $gg \to ggg$ has over 10000 terms. Thus spinor-helicity formalism comes in handy for these computations. Note that the complications appeared because of the polarisations coming from fields A^a_μ which required gauge invariance to be massless. The spinor-helicity formalism gets rid of the gauge fields A^a_μ by writing them in terms of spinors. This can be done since vector fields are $(\frac{1}{2}, \frac{1}{2})$ representations of SO$(1,3)$ and spinors are $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$. With this we will show that for $gg \to gg$ scattering, there are only two non-vanishing contributions:

$$\begin{aligned} \widetilde{\mathcal{M}}\,(1^-2^-3^+4^+) &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} \\ \widetilde{\mathcal{M}}\,(1^-2^+3^-4^+) &= \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \end{aligned} \tag{1.3}$$

The notations will be clear in the coming sections.

2 Dirac Equation and Spinors

We begin by discussing spinor representations.

2.1 Spinor Representations

Spinors are are spin $\frac{1}{2}$ representations of SO$(1,3)$. Let J^i, K^i be rotation and boost generators. Then the Lorentz algebra is

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J^k \\ [J_i, K_j] &= i\epsilon_{ijk}K^k \\ [k_i, k_j] &= -i\epsilon_{ijk}J^k. \end{aligned}$$

In terms of raising and lowering operators J_i^+, J_i^- :

$$J_i^+ = \frac{J_i + iK_i}{2}, \quad J_i^- = \frac{J_i - iK_i}{2},$$

the algebra is

$$\begin{aligned}\left[J_i^+, J_j^+\right] &= i\epsilon_{ijk} J_k^+ \\ \left[J_i^-, J_j^-\right] &= i\epsilon_{ijk} J_k^- \\ \left[J_i^-, J_j^+\right] &= 0.\end{aligned}$$

We easily identify that $\mathrm{so}(1,3) = \mathrm{su}(2) \oplus \mathrm{su}(2)$ where $\mathrm{so}(1,3)$ and $\mathrm{su}(2)$ is the Lie algebra of $\mathrm{SO}(1,3)$ and $\mathrm{SU}(2)$ respectively. Since $\mathrm{su}(2) \oplus \mathrm{su}(2) = \mathrm{sl}(2,\mathbb{R})$ which is the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ which is simply connected, $\mathrm{SL}(2,\mathbb{C})$ is the universal cover of $\mathrm{SO}(1,3)$. Spinor representations of $\mathrm{SO}(1,3)$ are fundamental and antifundamental representations of $\mathrm{SL}(2,\mathbb{C})$:

$$\begin{aligned}\mathrm{SL}(2,\mathbb{C}) \ni M &\longmapsto T_M : V_{Fund} \to V_{Fund}, \quad V_{Fund} \cong \mathbb{C}^2; \\ \begin{pmatrix} z \\ \omega \end{pmatrix} &\mapsto M \begin{pmatrix} z \\ \omega \end{pmatrix} \qquad \text{Fundamental} \\ M &\longmapsto T_{M^\star} : V_{Antifund} \to V_{Antifund}, \quad V_{Antifund} \cong \mathbb{C}^2; \\ \begin{pmatrix} z \\ \omega \end{pmatrix} &\mapsto M^\star \begin{pmatrix} z \\ \omega \end{pmatrix} \qquad \text{Antifundamental,}\end{aligned}$$

where $M^\star$ is the complex conjugate of the matrix M. The fundamental representation is $\left(\frac{1}{2}, 0\right)$ and the vectors $\psi_L \in V_{Fund}$ are called Left handed Weyl spinors and the antifundamental representation is $\left(0, \frac{1}{2}\right)$ and the vectors $\psi_R \in V_{Antifund}$ are called right handed Weyl spinors. We combine these to write the Dirac spinor as a 4 component vector $\{\psi_a\}_{a=1}^4$ (that is we consider the direct sum $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$) :

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}.$$

The Dirac spinor ψ satisfies the Dirac equation:

$$\left(i\gamma^\mu \partial_\mu - m\right) \psi'(x) = 0,$$

where the γ-matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

We also define $\bar{\psi} = \psi^\dagger \gamma^0 = \left(\psi_1^\dagger, \psi_2^\dagger, \psi_3^\dagger, \psi_4^\dagger\right)$. We will very often use the slash notation

$$\not{v} = \gamma^\mu v_\mu,$$

so that

$$\not{v}\not{v} = v^2 \mathbb{1}.$$

Indeed

$$\begin{aligned}
\not{v}\not{v} &= v^\mu \gamma_\mu v^\nu \gamma_\nu \\
&= \frac{1}{2}\left(v^\mu v^\nu \gamma_\mu \gamma_\nu + v^\nu v^\mu \gamma_\nu \gamma_\mu\right) \\
&= \frac{1}{2} v^\mu v^\nu \left\{\gamma_\mu, \gamma_\nu\right\} \\
&= \frac{1}{2} v^\mu v^\nu 2 g_{\mu\nu} \mathbb{1} \\
&= v^2 \mathbb{1}.
\end{aligned}$$

So the Dirac equation is $(i\not{\partial} - m)\psi = 0$. To see how $\psi(x)$ transforms under Lorentz transformation, we need the generators of the representation. With the Dirac γ-matrices, we construct the generators:

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right].$$

Then using the Clifford algebra, we can show that $\Sigma^{\mu\nu}$ satisfies Lorentz algebra:

$$\left[\Sigma^{\mu\nu}, \Sigma^{\rho,\sigma}\right] = i\left(g^{\nu\rho}\Sigma^{\mu\sigma} - g^{\mu\rho}\Sigma^{\nu\sigma} - g^{\nu\sigma}\Sigma^{\mu\mu} + g^{\mu\sigma}\Sigma^{\nu\rho}\right)$$

Thus $\psi(x)$ transforms as

$$\psi(x) \longrightarrow S_\Lambda \psi\left(\Lambda^{-1} x\right)$$

where

$$S_\Lambda = \exp\left(i\omega_{\mu\nu}\Sigma^{\mu\nu}\right) \tag{2.1}$$

and $\omega_{\mu\nu}$ are the rotations and boosts of Λ :

$$\omega_{0i} = -\omega_{i0} = \beta_i \qquad \text{(boosts)}$$

$$\omega_{ij} = \epsilon_{ijk}\theta^k \qquad \text{(rotations)}$$

Weyl representation. If we pick γ-matrices to be

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

where $\sigma^\mu = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ then

$$\begin{aligned}
\omega_{\mu\nu}\Sigma^{\mu\nu} &= \omega_{0i}\Sigma^{0i} + \omega_{i0}\Sigma^{i0} + \omega_{ij}\Sigma^{ij} \\
&= 2\beta_i\Sigma^{0i} + \epsilon_{ijk}\theta^k\Sigma^{ij} \\
&= 2\beta_i\left(-\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}\right) + \frac{1}{2}\epsilon_{ijk}\theta^k\epsilon^{ikl}\begin{pmatrix}\sigma_\ell & 0 \\ 0 & \sigma_\ell\end{pmatrix} \\
&= -i\beta_i\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} + \frac{2}{2}\delta_k^\ell\theta^k\begin{pmatrix}\sigma_\ell & 0 \\ 0 & \sigma_\ell\end{pmatrix} \\
&= -i\beta_i\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix} + \theta_i\begin{pmatrix}\sigma^i & 0 \\ 0 & \sigma^i\end{pmatrix},
\end{aligned}$$

where we used

$$\Sigma^{0i} = -\frac{i}{2}\begin{pmatrix}\sigma^i & 0 \\ 0 & -\sigma^i\end{pmatrix}, \quad \Sigma^{ij} = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$$

and $\epsilon_{ijk}\epsilon^{ij\ell} = 2\delta_k^\ell$. Thus we get

$$\begin{aligned}
\exp\left(i\omega_{\mu\nu}\Sigma^{\mu\nu}\right) &= \exp\left(i\left(-i\beta_i\begin{pmatrix}\sigma^i & 0 \\ 0 - \sigma^i & \end{pmatrix} + \theta_k\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}\right)\right) \\
&= \begin{pmatrix}\exp\left[i\theta_i\sigma^i + \beta i\sigma^i\right] & 0 \\ 0 & \exp\left[-\beta_i\sigma^i + i\theta_i\sigma^i\right]\end{pmatrix}
\end{aligned}$$

So if we write $\psi(x) = \begin{pmatrix}\psi_L(x) \\ \psi_R(x)\end{pmatrix}$, then $\psi(x) \longrightarrow S_\Lambda\psi\left(\Lambda^{-1}x\right)$ gives

$$\begin{aligned}
\psi_L &\longrightarrow \exp(\vec{z}\cdot\vec{\sigma})\psi_L \\
\psi_R &\longrightarrow \exp\left(-\vec{z}^\star\cdot\vec{\sigma}\right)\psi_R,
\end{aligned} \tag{2.2}$$

where $\vec{z} = \vec{\beta} + i\vec{\theta}$ is composed of the three rotation vector $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ and boost vector $\vec{\beta} = (\beta_1, \beta_2, \beta_3)$. Clearly the map $\vec{z} \mapsto \exp(\vec{z}\cdot\vec{\sigma})$ is a map from $\mathrm{SO}(1,3) \longrightarrow \mathrm{SL}(2,\mathbb{C})$. To see this, note that

$$\det(\exp(\vec{z}\cdot\vec{\sigma})) = \exp(\mathrm{tr}(\vec{z}\cdot\vec{\sigma})) = e^0 = 1,$$

where we used the fact that $\mathrm{tr}\left(\sigma_i\right) = 0$. So

$$M(\vec{z}) := \exp(\vec{z}\cdot\vec{\sigma}) \in \mathrm{SL}(2,\mathbb{C}).$$

Next observe that if $\epsilon = i\sigma_2$ then ϵ is the 2×2 antisymmetric symbol with $\epsilon^2 = -\mathbb{1}$.

Lemma 2.1. *With the notations as above, we have* *(i)* $\epsilon\vec{\sigma}\epsilon = \vec{\sigma}^\star = \vec{\sigma}^T$ *where* $\vec{\sigma}^T = (\sigma_1^T, \sigma_2^T, \sigma_3^T)$,

(ii) $M(\vec{z})^{-1} = -\epsilon M(\vec{z})^T \epsilon$.

Proof. The proof of (i) is straightforward. To prove (ii), observe that by (i), we have

$$\epsilon\vec{\sigma}^T\epsilon = \vec{\sigma}.$$

So

$$\begin{aligned}-\epsilon M(\vec{z})^T\epsilon &= \epsilon^{-1}\exp\left(\vec{z}\cdot\vec{\sigma}^T\right)\epsilon \\ &= \exp\left(\vec{z}\cdot\left(-\epsilon\vec{\sigma}^T\epsilon\right)\right) \\ &= \exp(-\vec{z}\cdot\vec{\sigma}) \\ &= M(\vec{z})^{-1}.\end{aligned}$$

□

Lemma 2.2. *Let* $\psi(x) = \begin{pmatrix}\psi_L(x)\\ \psi_R(x)\end{pmatrix}$ *be a Dirac spinor. Then under Lorentz transformation* Λ *with angle and boost parameters* θ_i *and* β_i, $\psi_R(x)$ *transforms as*

$$\epsilon\psi_R(x) \to M(\vec{z})^\star\epsilon\psi_R(\Lambda^{-1}x),$$

where $\vec{z} = \vec{\beta} + i\vec{\theta}$ *as usual.*

Proof. By Eq. 2.2, we have

$$\psi_R \longrightarrow \exp\left(-\vec{z}^\star\cdot\vec{\sigma}\right)\psi_R.$$

So

$$\begin{aligned}\epsilon\psi_R \longrightarrow & \epsilon\exp\left(-\vec{z}^\star\cdot\vec{\sigma}\right)\psi_R \\ &= \epsilon\exp\left(-\vec{z}^\star\cdot\vec{\sigma}\right)\epsilon^{-1}\epsilon\psi_R \\ &= \exp\left(-\vec{z}^\star\cdot\epsilon\vec{\sigma}\epsilon^{-1}\right)\in\psi_R \\ &= \exp\left(\vec{z}^\star\cdot\epsilon\vec{\sigma}\epsilon\right)\epsilon V_R \\ &= \exp\left(\vec{z}^\star\cdot\vec{\sigma}^\star\right)\epsilon\psi_R \\ &= M(z)^\star\epsilon\psi_R.\end{aligned}$$

□

Thus in a Dirac spinor $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ in the weyl representation, ψ_L transforms in the the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ and $\epsilon\psi_R$ transforms in the antifundamental representation of $\mathrm{SL}(2,\mathbb{C})$. So it is conventional to denote a Dirac spinor in Weyl representation by

$$\psi = \begin{pmatrix} \chi^\alpha \\ \widetilde{\psi}_{\dot\beta} \end{pmatrix}$$

where $\widetilde{\psi}_{\dot\beta} = \epsilon_{\dot\alpha\dot\beta}\widetilde{\psi}^{\dot\alpha}$. Under Lorentz transformation $\vec{z} = \vec{\beta} + i\vec{\theta}$,

$$\chi^\alpha \longrightarrow (M(\vec{z})\chi)^\alpha$$

and

$$\widetilde{\psi}_{\dot\beta} \longrightarrow \left(\left(M(\vec{z})^\dagger\right)^{-1}\widetilde{\psi}\right)_{\dot\beta}$$

Indeed note that

$$\left(M(\vec{z})^\dagger\right)^{-1} = \left(M(\vec{z})^{-1}\right)^\dagger = \left(-\epsilon M(\vec{z})^T\epsilon\right)^\dagger = -\epsilon M(\vec{z})^\star\epsilon$$

where we used Lemma 2.1(ii). So we get that

$$\begin{aligned}\epsilon\widetilde{\psi} \longrightarrow & M(\vec{z})^\star\epsilon\widetilde{\psi} \\ &= \epsilon\left(M(\vec{z})^\dagger\right)^{-1}\epsilon^{-1}\epsilon\widetilde{\psi} \\ &= \epsilon\left(M(\vec{z})^\dagger\right)^{-1}\widetilde{\psi}.\end{aligned}$$

This gives

$$\widetilde{\psi}_{\dot\beta} \longrightarrow \left(\left(M(\vec{z})^\dagger\right)^{-1}\widetilde{\psi}\right)_{\dot\beta}.$$

Remark 2.3. Left and right handed spinor indices α and $\dot\beta$ respectively are lowered and raised using the antisymmetric symbol ϵ:

$$\varepsilon^{\alpha\beta} = -\varepsilon_{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = -\varepsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 2.4. *The quantities $\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta$ and $\epsilon^{\dot\alpha\dot\beta}\widetilde{\psi}_\alpha\widetilde{\chi}_\beta$ are Lorentz invariant.*

Proof. We have

$$\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta = \psi_L^T\epsilon\chi_L \quad \text{where} \quad \psi_L = \begin{pmatrix}\psi^1 \\ \psi^2\end{pmatrix}$$

Thus

$$\begin{aligned}\epsilon^{\alpha\beta}\psi_\alpha\chi_\beta \longrightarrow & \left(M(\vec{z})\psi_L\right)^T \epsilon M(\vec{z})\chi_L \\ &= \psi_L^T M(\vec{z})^T \epsilon M(\vec{z})\chi_L \\ &= \psi_L^T \epsilon M(\vec{z})^{-1} M(\vec{z})\chi_L \\ &= \psi_L^T \epsilon \chi_L\end{aligned}$$

where we used $\epsilon^{-1}M(\vec{z})^T\epsilon = M(\vec{z})^{-1}$ (Lemma 2.1(ii)). Similarly $\epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\psi}_{\dot{\alpha}}\widetilde{\chi}_\beta = \widetilde{\psi}_R^T\epsilon\widetilde{\chi}_R = \left(\epsilon\widetilde{\psi}_R\right)^T\widetilde{\chi}_R$, so that $\epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\psi}_{\dot{\alpha}}\widetilde{\chi}_{\dot{\beta}} \longrightarrow \widetilde{\psi}_R^T\epsilon\widetilde{\chi}_R$. □

Contractions. For two left handed spinors χ, ψ, we define

$$\begin{aligned}\chi\psi \equiv \chi_\alpha\psi^\alpha = \chi_\alpha\epsilon^{\alpha\beta}\psi_\beta = -\psi_\beta\epsilon^{\alpha\beta}\chi_\alpha &= -\psi_\beta\left(-\epsilon^{\beta\alpha}\right)\chi_\alpha \\ &= \psi_\beta\chi^\beta = \psi\chi,\end{aligned}$$

where we used the Grassmannian nature of the components of the spinor, that is the components anticommute. Hence the above contraction is symmetric. For right handed spinors $\widetilde{\chi}$ and $\widetilde{\psi}$, we define

$$\begin{aligned}\widetilde{\chi}\widetilde{\psi} \equiv \widetilde{\chi}^{\dot{\alpha}}\widetilde{\psi}_{\dot{\alpha}} = \widetilde{\chi}^{\dot{\alpha}}\epsilon_{\dot{\alpha}\dot{\beta}}\widetilde{\psi}^{\dot{\beta}} = -\widetilde{\psi}^{\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}\widetilde{\chi}^{\dot{\alpha}} &= -\widetilde{\psi}^{\dot{\beta}}\left(\epsilon_{\dot{\beta}\dot{\alpha}}\right)\widetilde{\chi}^{\dot{\alpha}} \\ &= \widetilde{\psi}^{\dot{\beta}}\widetilde{\chi}_{\dot{\beta}} = \widetilde{\psi}\widetilde{\chi},\end{aligned}$$

where we again used the Grasmannian nature of components of a spinor.

2.2 Helicity Spinors

We now define the protagonists of spinor-helicity formalism – the helicity spinors.

Definition 2.5. (Helicity spinors) We define helicity spinors as vectors of $\mathbb{C}^2$ which transform in $\left(\frac{1}{2},0\right)$ or $\left(0,\frac{1}{2}\right)$ representation of SO(1,3). The $\left(\frac{1}{2},0\right)$ helicity spinor is called *left-handed* and is denoted by χ and the $\left(0,\frac{1}{2}\right)$ helicity spinors is called *right-handed* and denoted by $\widetilde{\chi}$.

We can interpret helicity spinors as coming from massless Dirac equation:

$$i\partial\!\!\!/\psi(x) = 0.$$

We can consider plane wave solution:

$$\psi_p(x) = u(p)e^{-ipx} + v(p)e^{ipx}, \qquad \text{with } p^2 = 0.$$

The Dirac equation then decouples into a set of two equations if we write $\psi = \begin{pmatrix} u_L \\ u_R \end{pmatrix}$:

$$\begin{aligned}\left(E_p + \vec{p}\cdot\vec{\sigma}\right)u_L(\vec{p}) &= 0 \\ \left(E_p - \vec{p}\cdot\vec{\sigma}\right)u_R(\vec{p}) &= 0.\end{aligned}$$

Thus we see that we do not need all the four components of a Dirac field. We can constrain the upper and lower two component to be zero. Thus we have two decoupled solutions:

$$\chi(x) = \begin{pmatrix} \chi^\alpha(x) \\ 0 \end{pmatrix}, \quad \eta(x) = \begin{pmatrix} 0 \\ \eta_{\dot\alpha}(x) \end{pmatrix}, \tag{2.3}$$

where

$$\begin{aligned}
\chi^\alpha(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \lambda^\alpha(p) \left(a_p e^{-ipx} + b_p^\dagger e^{ipx} \right) \\
\eta_{\dot\alpha}(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{L}{\sqrt{2E_p}} \lambda_{\dot\alpha}(p) \left(a_p^\dagger e^{-ipx} + b_p e^{ipx} \right),
\end{aligned}$$

where $a_p, a_p^\dagger$ and $b_p, b_p^\dagger$ are the creation and annihilation operators of the particle and antiparticle. Then λ^α and $\lambda_{\dot\alpha}$ are helicity spinors.

We will now show that the left handed and right handed helicity spinors have definite helicity. The Dirac equation is

$$i\partial\!\!\!/\psi(x) = 0$$

which in Weyl representation is

$$\begin{pmatrix} 0 & i\left(\partial_0 + \vec{\sigma}\cdot\nabla\right) \\ i\left(\partial_0 - \vec{\sigma}\cdot\nabla\right) & 0 \end{pmatrix} \begin{pmatrix} \psi_L(x) \\ \psi_R(x) \end{pmatrix} = 0.$$

With the field decomposition for $\chi(x)$ and $\eta(x)$, we get

$$\begin{aligned}
&\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p \left(-iE_p - i\vec{p}\cdot\vec{\sigma}\right) e^{-ipx} + b_p^\dagger \left(iE_p + i\vec{p}\cdot\vec{\sigma}\right) e^{ipx} \right) \lambda^\alpha(p) = 0 \\
&\implies \left(E_p + \vec{p}\cdot\vec{\sigma}\right)\lambda^\alpha(p) = 0.
\end{aligned}$$

Similarly $\lambda^{\dot\alpha}(p)$ must satisfy

$$\left(E_p - \vec{p}\cdot\vec{\sigma}\right)\lambda_{\dot\alpha}(p) = 0.$$

Since $E_p = |\vec{p}|$, thus we get

$$(\hat{p}\cdot\vec{\sigma})\lambda_\alpha(p) = \lambda_{\dot\alpha}(p), \quad (\hat{p}\cdot\vec{\sigma})\lambda^\alpha(p) = -\lambda^\alpha(p)$$

where

$$\hat{p} = \frac{\vec{p}}{|\vec{p}|}.$$

Thus left handed spinors have helicity -1 and right-handed spinors have helicity 1.

With this understanding, we define the following contractions:

$$\begin{aligned}
\langle\lambda\chi\rangle &\equiv \epsilon^{\alpha\beta}\lambda_\alpha\chi_\beta = \lambda_\alpha\chi^\alpha = -\lambda^\alpha\chi_\alpha = -\langle\chi\lambda\rangle \\
[\lambda\chi] &\equiv \epsilon_{\dot\alpha\beta}\widetilde{\lambda}^\alpha\widetilde{\chi}^\alpha = \widetilde{\lambda}^\alpha\widetilde{\chi}_\alpha = -\widetilde{\lambda}_\alpha\widetilde{\chi}^\alpha = -[\lambda\chi].
\end{aligned} \tag{2.4}$$

Wherever we have angular brackets, we understand that it is the contraction of left handed spinor whereas the square bracket is the contraction of the right handed spinor. In particular $\langle\lambda\lambda\rangle = [\lambda\lambda] = 0$.

2.3 4-Vectors in terms of Helicity Spinors

We can represent 4-vector p^μ as a *bispinor* using helicity spinor as follows:

$$p^{\alpha\dot{\alpha}} = p^\mu \sigma_\mu^{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}.$$

The indices of the bispinor $p^{\alpha\dot{\alpha}}$ associated to p^μ can be lowered using $\epsilon_{\alpha\beta}, \epsilon_{\dot{\alpha}\dot{\beta}}$:

$$p_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} p^{\beta\dot{\beta}}.$$

The following lemma is straightforward verification:

Lemma 2.6. *We have that*

(i) $g^{\mu\nu}\sigma_\mu^{\alpha\beta}\sigma_\nu^{\dot{\alpha}\dot{\beta}} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}$

(ii) $\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\sigma^{\mu\beta\dot{\beta}} = \bar{\sigma}^\mu_{\alpha\dot{\alpha}}$.

We have the following proposition:

Proposition 2.7. *The following are true:*

(i) $p_{\alpha\dot{\alpha}} = p_\mu \bar{\sigma}^\mu_{\alpha\dot{\alpha}}$.

(ii) $p^\mu = \frac{1}{2}\sigma^{\mu\alpha\dot{\alpha}} p_{\dot{\alpha}\alpha} = \frac{1}{2}\bar{\sigma}^\mu_{\dot{\alpha}\alpha} p^{\alpha\dot{\alpha}}$, *where* $\sigma^{\mu\alpha\dot{\alpha}} = (\delta^{\alpha\dot{\alpha}}, \vec{\sigma}^{\alpha\dot{\alpha}})$.

Proof. To prove (i), we directly lower the indices. We have

$$\begin{aligned} p_{\alpha\dot{\alpha}} &= \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} p^{\beta\dot{\beta}} \\ &= \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} p^\mu \sigma_\mu^{\beta\dot{\beta}} \\ &= g_{\mu\nu} p^\mu \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \sigma^{\nu\beta\dot{\beta}} \\ &= g_{\mu\nu} p^\mu \bar{\sigma}^{\nu\alpha\dot{\alpha}} \\ &= p^\mu \bar{\sigma}_\mu^{\alpha\dot{\alpha}}, \end{aligned}$$

where we used Lemma 2.6 (ii) in the second last step. The proof of (ii) follows by a similar application of Lemma 2.6 (i) and hence we omit the details. $\square$

Remark 2.8. Proposition 2.7 (ii) gives a recipe to recover the 4-vector from its bispinor representation. Hence the assignment of a 4-vector to a bispinor is a one to one correspondence.

Next observe that

$$\det(p^{\alpha\dot{\alpha}}) = p_0^2 - p_1^2 - p_2^2 - p_3^2 = p_\mu^2 = m^2.$$

If p^μ is massless then $\det(p^{\alpha\dot{\alpha}}) = 0$. The following lemma is easy to prove:

Lemma 2.9. *Let A be a 2×2 complex matrix with vanishing determinant. Then it can be written as a product of a column and a row in order.*

Proof. We first note that such matrix A can only have the following forms:

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & \lambda a \\ b & \lambda b \end{pmatrix}, \quad a,b,c,d \in \mathbb{C}, \quad \lambda \in \mathbb{C}\backslash\{0\}.$$

We explicitly provide the decomposition for the first and the last case and others are similar. Indeed we can write

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} \begin{pmatrix} 1 & b/a \end{pmatrix}, \quad \begin{pmatrix} a & \lambda a \\ b & \lambda b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} 1 & \lambda \end{pmatrix}.$$

$\square$

In particular, for a null vector p^μ we can write:

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}}.$$

Indeed, we take

$$\lambda^\alpha = \frac{z}{\sqrt{p^0 - p^3}} \begin{pmatrix} p^0 - p^3 \\ -p^1 - ip^2 \end{pmatrix}, \quad \widetilde{\lambda}^{\dot{\alpha}} = \frac{z^{-1}}{\sqrt{p^0 - p^3}} \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \end{pmatrix} \tag{2.5}$$

with

$$p^0 = \sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2} \quad \text{and} \quad z \in \mathbb{C}\backslash\{0\}.$$

We will generalise to massless complex momenta in later sections which will be useful in analytic continuation. If the momentum is real then

$$\begin{aligned} &\left(\lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}}\right)^\dagger = \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}} \implies \lambda^\alpha = \left(\widetilde{\lambda}^{\dot{\alpha}}\right)^\dagger \implies z = (z^\star)^{-1} \implies |z|^2 = 1 \\ &\implies z \in S^1 := \{z \in \mathbb{C} \mid |z| = 1\}. \end{aligned}$$

We now show that if p^μ transforms as 4-vector then λ^α, $\widetilde{\lambda}^{\dot{\alpha}}$ in $p^{\alpha\dot{\alpha}} = \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}}$ transform as left-handed and right-handed spinors respectively. We begin by proving the following lemma:

Lemma 2.10. *Let p^μ transform as a 4-vector: $p^\mu \to \Lambda^\mu{}_\nu p^\nu$. Then $p^{\alpha\dot{\alpha}}$ transforms as $p^{\alpha\dot{\alpha}} \longrightarrow M(\vec{z})^\alpha{}_\beta \, p^{\beta\dot{\beta}} \left(M(\vec{z})^\dagger\right)^{\dot{\alpha}}{}_{\dot{\beta}}$ where $M(\vec{z}) = \exp(\vec{z}\cdot\vec{\sigma})$ and $\vec{z} = \vec{\beta} + i\vec{\theta}$ with $\vec{\beta}$ and $\vec{\theta}$ being boost and rotation associated to $\Lambda^\mu{}_\nu$.*

Proof. We know that the Dirac matrices satisfy

$$S_\Lambda^{-1}\,\gamma^\mu\,S_\Lambda = \Lambda^\mu{}_\nu\gamma^\nu$$

where $S_\Lambda = \exp\left(i\omega_{\mu\nu}\Sigma^{\mu\nu}\right)$ (see Eq. 2.1). In Weyl representation

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

and we calculated S_Λ:

$$S_\Lambda = \begin{pmatrix} \exp(\vec{z}\cdot\vec{\sigma}) & 0 \\ 0 & \exp\left(-\vec{z}^\star\cdot\vec{\sigma}\right) \end{pmatrix}.$$

So we have that

$$\exp(-\vec{z}\cdot\vec{\sigma})\;\sigma^\mu\;\exp\left(-\vec{z}^\star\cdot\vec{\sigma}\right) = \Lambda^\mu{}_\nu\sigma^\nu. \tag{2.6}$$

We know that

$$\Lambda^T g\Lambda = g \implies g\Lambda^T\,g = \Lambda^{-1}.$$

In index notation, we have

$$g_{\mu\nu}\left(\Lambda^T\right)^\nu{}_\rho\,g^{\rho\sigma} = \left(\Lambda^{-1}\right)^\sigma{}_\mu \quad\implies \left(\Lambda^T\right)_{\mu\rho}g^{\rho\sigma} = \left(\Lambda^{-1}\right)^\sigma{}_\mu \quad\implies \Lambda_{\rho\mu}g^{\rho\sigma} = \left(\Lambda^{-1}\right)^\sigma{}_\mu.$$

Using this, we get

$$\begin{aligned}(\Lambda p)^{\alpha\dot{\alpha}} &= \Lambda^\mu{}_\nu p^\nu\sigma_\mu^{\alpha\dot{\alpha}} = p^\nu\Lambda_{\mu\nu}g^{\mu\rho}\sigma_p^{\alpha\dot{\alpha}}\\ &= p^\nu\left(\Lambda^{-1}\right)^\rho{}_\nu\,\sigma_p^{\alpha\dot{\alpha}}\\ &= p^\nu\left[\exp(\vec{z}\cdot\vec{\sigma})\sigma_\nu\exp\left(\vec{z}^\star\cdot\vec{\sigma}\right)\right]^{\alpha\dot{\alpha}},\end{aligned}$$

where we used Eq. 2.6 for Λ^{-1}. Thus we see that

$$(\Lambda p)^{\alpha\dot{\alpha}} = \left(M(\vec{z})pM(\vec{z})^\dagger\right)^{\alpha\dot{\alpha}},$$

where $p = p^{\alpha\dot{\alpha}}$. □

Proposition 2.11. *Let $p^{\alpha\dot{\alpha}} = \lambda^\alpha\widetilde{\lambda}^{\dot{\alpha}}$ be the decomposition of a 4-vector p^μ into product of row $\widetilde{\lambda}^{\dot{\alpha}}$ and column λ^α. Then λ^α and $\widetilde{\lambda}^{\dot{\alpha}}$ transforms as left-handed spinor and right-handed spinor respectively.*

Proof. By Lemma 2.10, under Lorentz transformation, $p = p^{\alpha\dot{\alpha}}$ transforms as

$$p \longrightarrow M(\vec{z})pM(\vec{z})^{\dagger}.$$

Thus if we write $p = \lambda\widetilde{\lambda}$ then we have

$$\lambda\widetilde{\lambda} \longrightarrow M(\vec{z})\lambda\widetilde{\lambda}M(\vec{z})^{\dagger}.$$

Thus under Lorentz transformation,

$$\lambda \longrightarrow M(\vec{z})\lambda$$

which is correct transformation of left-handed spinor. Next

$$\widetilde{\lambda} \longrightarrow \widetilde{\lambda}M(\vec{z})^{\dagger}.$$

Thus

$$\epsilon_{\dot{\alpha}\dot{\beta}}\widetilde{\lambda}^{\dot{\beta}} \longrightarrow \epsilon_{\dot{\alpha}\dot{\beta}}\left(\widetilde{\lambda}M(\vec{z})^{\dagger}\right)^{\dot{\beta}}.$$

Note that L.H.S is a 2 component vector, and hence this can be written as a matrix transformation:

$$\begin{aligned}
\epsilon\lambda^{T} \longrightarrow & \epsilon\left(\widetilde{\lambda}M(\vec{z})^{\dagger}\right)^{T} \\
=& \epsilon\left(M(\vec{z})^{\dagger}\right)^{T}\widetilde{\lambda}^{T} \\
=& -\epsilon M(\vec{z})^{\star}\epsilon\epsilon\widetilde{\lambda}^{T} \\
=& \left(M(\vec{z})^{\dagger}\right)^{-1}\epsilon\widetilde{\lambda}^{T},
\end{aligned}$$

Thus we have

$$\widetilde{\lambda}_{\dot{\alpha}} \longrightarrow \left(\left(M(\vec{z})^{\dagger}\right)^{-1}\widetilde{\lambda}\right)_{\dot{\alpha}}$$

which is the right-handed spinorial transformation rule. □

2.3.1 Momentum Conservation and Schouten's Identity

Lemma 2.12. *Let* $p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\widetilde{\lambda}^{\dot{\alpha}}, \quad q^{\alpha\dot{\alpha}} = \chi^{\alpha}\widetilde{\chi}^{\dot{\alpha}}$, *then we have*

$$p \cdot q = \frac{1}{2}\langle\lambda\chi\rangle[\chi\lambda].$$

Proof. By definition, we have

$$\begin{aligned}
p \cdot q &= g_{\mu\nu} p^\mu q^\nu \\
&= \frac{1}{4} g_{\mu\nu} \sigma^\mu_{\alpha\dot{\alpha}} \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}} \sigma^\nu_{\beta\dot{\beta}} \chi^\beta \widetilde{\chi}^{\dot{\beta}} \\
&= \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}} \chi^\beta \widetilde{\chi}^{\dot{\beta}} \\
&= \frac{1}{2} \lambda^\alpha \chi_\alpha \widetilde{\lambda}^{\dot{\alpha}} \widetilde{\chi}_{\dot{\alpha}} \\
&= \frac{1}{2} \langle \lambda \chi \rangle [\chi \lambda],
\end{aligned}$$

where we used Proposition 2.7 (ii) in first step and Lemma 2.6 (i) in second step. □

Remark 2.13. By Lemma 2.12, it is easy to see that

$$p^2 = \frac{1}{2} \langle \lambda\lambda \rangle [\lambda\lambda] = 0 = \frac{1}{2} \langle \chi\chi \rangle [\chi\chi] = q^2.$$

Notations. If $p^{\alpha\dot{\alpha}} = \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}}$ then we will write

$$\begin{aligned}
\lambda^\alpha &= |p\rangle \quad \widetilde{\lambda}^\alpha = [p| \\
\lambda_\alpha &= \langle p| \quad \widetilde{\lambda}_\alpha = [p|.
\end{aligned}$$

So we have

$$p^{\alpha\dot{\alpha}} = |p\rangle\, [p|, \quad p_{\alpha\dot{\alpha}} = \langle p||p]$$

With this notation, we can forget about the computations in Lemma 2.12 and quickly write

$$p \cdot q = p^\mu q_\mu = \frac{1}{2} \langle pq \rangle [qp].$$

We record a small result which will be useful in future computations.

Lemma 2.14. *For real momenta p and q, we have that*

$$\langle pq \rangle^\star = [qp].$$

Proof. Suppose $p^{\alpha\dot{\alpha}} = \lambda^\alpha \widetilde{\lambda}^{\dot{\alpha}}, q^{\alpha\dot{\alpha}} = \chi^\alpha \widetilde{\chi}^{\dot{\alpha}}$. We have

$$\begin{aligned}
\langle pq \rangle^\star &= \left(\epsilon^{\alpha\beta} \lambda_\alpha \chi_\beta\right) \\
&= (\epsilon^{\alpha\beta})^\star (\lambda_\alpha)^\star (\chi_\beta)^\star \\
&= -\epsilon_{\dot{\alpha}\dot{\beta}} \widetilde{\lambda}^{\dot{\alpha}} \widetilde{\dot{\beta}} \\
&= -[pq] \\
&= [qp],
\end{aligned}$$

where we used Remark 2.3 and Eq. 2.5 in third step and Eq. 2.4 in last step. □

We now apply above results to massless momenta. Suppose we have n incoming momenta p_i. Then momentum conservation implies

$$\sum_{i=1}^{n} p_i = 0.$$

Converting the above zero four vector into bispinor using σ^μ, we see that this gives

$$\sum_{i=1}^{n} |p_i\rangle[p_i| = 0,$$

where $|p_i\rangle[p_i|$ is the bispinor of p_i. To make notations more light, we write $|i\rangle[i|$ for the bispinor of the momentum p_i. Thus the momentum conservation can be written as

$$|1\rangle[1| + |2\rangle[2| + \cdots + |n\rangle[n| = 0.$$

We can cast the momentum conservation equation into more useful form by contracting above equation with arbitrary spinors $\langle i|$ and $|k]$ from the left and right respectively to get:

$$\sum_{j=1}^{n} \langle ij\rangle[jk] = 0. \tag{2.7}$$

Remark 2.15. The above simple looking equation us terribly useful when we simplify scattering amplitudes as we will see soon in next sections. For example, for four momentas, contracting with $\langle 1|$ and $|2]$ from the left and right respectively, we get

$$\langle 11\rangle[12] + \langle 12\rangle[22] + \langle 13\rangle[32] + \langle 14\rangle[42] = 0,$$

which gives

$$\langle 13\rangle[32] = -\langle 14\rangle[42]. \tag{2.8}$$

Contracting with other combinations, we get many more such equations, for example contracting with $\langle 2|$ and $|3]$ from the left and right respectively gives

$$\langle 21\rangle[13] = -\langle 24\rangle[43].$$

We will use such equations very often.

We now prove an important identity called *Schouten's identity.*

Theorem 2.16. *Let $\{|i\rangle, \langle i|\}$ be the left-handed spinors coming from bispinor representation of four massless momentums p_i. Then we have*

$$\langle 23\rangle\langle 41\rangle + \langle 31\rangle\langle 42\rangle + \langle 12\rangle\langle 43\rangle = 0.$$

Proof. Since the three spinors $|1\rangle, |2\rangle, |3\rangle$ are 2 component vectors, thus all the three cannot be linearly independent. This gives

$$|1\rangle = a|2\rangle + b|3\rangle, \quad \text{for some} \quad a, b \in \mathbb{C}.$$

Taking inner product with $\langle 2|$ we get

$$\langle 21\rangle = b\langle 23\rangle \implies b = \frac{\langle 21\rangle}{\langle 23\rangle}.$$

Similarly

$$\langle 31\rangle = a\langle 32\rangle \implies a = \frac{\langle 31\rangle}{\langle 32\rangle}.$$

This gives

$$|1\rangle = \frac{\langle 31\rangle}{\langle 32\rangle}|2\rangle + \frac{\langle 21\rangle}{\langle 23\rangle}|3\rangle.$$

Contracting with the spinor $\langle 4|$, we get

$$\begin{aligned}\langle 41\rangle &= \frac{\langle 31\rangle\langle 42\rangle}{\langle 32\rangle} + \frac{\langle 21\rangle}{\langle 23\rangle}\langle 43\rangle \\ \implies \langle 41\rangle &= -\frac{\langle 31\rangle}{\langle 23\rangle}\langle 42\rangle - \frac{\langle 12\rangle}{\langle 23\rangle}\langle 43\rangle.\end{aligned}$$

Rearranging gives the required identity. □

2.4 Polarisation in terms of Helicity-Spinors

Spin 1 massless representation of Poincaregroup has 2 degree of freedom. This can be embedded into a vector field A^μ along with *gauge invariance*. The spin 1 free Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. $\mathcal{L}$ is invariant under the gauge transformation

$$A_\mu(x) \longrightarrow A_\mu + \partial_\mu \Lambda(x)$$

for some scalar function $\lambda(x)$. Equation of motion is

$$\Box A_\mu - \partial_\mu\left(\partial_\nu A^\nu\right) = 0,$$

where $\Box = \partial^2$ is the D'Alembertian operator. In Coloumb gauge, $A^0 = 0 = \partial_i A^i$. So eqution of motion becomes $\Box A^i = 0$. We can write plane wave solution:

$$A^\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\epsilon^\mu(p)e^{ipx}.$$

Equation of motion becomes $p^2 = 0$ and $p^i\epsilon_i = 0$ and $\epsilon^0 = 0$ by gauge choice. For $p^\mu = (E, 0, 0, E)$, we can choose a basis for polarisations ϵ^μ as

$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}(0, 1, i, 0), \quad \epsilon_-^\mu = \frac{1}{\sqrt{2}}(0, 1, -i, 0)$$

These are called *transverse polarisations*. The polarisations are normalised so that

$$\epsilon^{\star\mu}\epsilon_\mu = -1.$$

But note that $\epsilon^\mu\epsilon_\mu = 0$, that is the polarisations are null. Thus their bispinors have vanishing determinant and hence we can also write ϵ^μ in terms of spinors. We also see that $\epsilon^{+\mu}\epsilon^-_\mu = -1$. We will now write polarisation vectors with arbitrary momenta in terms of spinors. To do this let r^μ be a null vector with $r \cdot p \neq 0$. For a given momenta, write

$$p^{\alpha\dot{\alpha}} = |p\rangle[p|, \quad r^{\alpha\dot{\alpha}} = |r\rangle[r|.$$

The vector r^μ is arbitrary except the restriction $r^2 = 0$ and $r \cdot p \neq 0$. It is called the *reference vector*.

Lemma 2.17. *We can write an arbitrary polarisation with momenta p as*

$$\left[\epsilon_p^-(r)\right]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{|p\rangle[r|}{[pr]}, \quad \left[\epsilon_p^+(r)\right]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{|r\rangle[p|}{\langle rp\rangle}.$$

Proof. We just need to check three conditions:

(a) $\epsilon_p^+(r) \cdot \epsilon_p^-(r) = -1$,

(b) $\epsilon_p^+ \cdot \epsilon_p^+ = \epsilon_p^- \cdot \epsilon_p^- = 0$,

(c) $\epsilon_p^\pm \cdot p = 0$.

We check the first one and leave the rest for the reader to verify as they are similar. By Lemma 2.12, we have

$$\begin{aligned}
\epsilon_p^-(r) \cdot \epsilon_p^+(r) &= \frac{1}{2}\frac{2}{[pr]\langle rp\rangle}\langle pr\rangle[pr] \\
&= -\frac{1}{[pr]\langle rp\rangle}[pr]\langle rp\rangle \\
&= -1.
\end{aligned}$$

□

Remark 2.18. We will often take r^μ to be momentum of another gluon in the problem. If the gluons are labelled by i, then we can write $\epsilon_i(j)$ for the polarisations of the gluon with momentum p_i^μ and reference momentum p_j^μ. Thus gluon scattering for any massless scattering can be expressed in terms of $[ij]$ and $\langle ij\rangle$.

We now list various Lorentz contractions which is easy to check:

$$\begin{aligned}
\epsilon_1^-(i)\cdot\epsilon_2^-(j) &= \frac{\langle 12\rangle[ji]}{[1i][2j]}, \quad \epsilon_1^-(i)\cdot\epsilon_2^+(j) = \frac{\langle 1j\rangle[2i]}{[1i]\langle j2\rangle}\\
\epsilon_1^+(i)\cdot\epsilon_2^+(j) &= \frac{\langle ij\rangle[21]}{\langle i1\rangle\langle j2\rangle}, \quad \epsilon_1^-(i)\cdot p_3 = \frac{1}{\sqrt{2}}\frac{\langle 13\rangle[3i]}{[1i]},\\
\epsilon_1^+(i)\cdot p_3 &= \frac{1}{\sqrt{2}}\frac{[13]\langle 3i\rangle}{\langle i1\rangle}, \quad p_1\cdot p_2 = \frac{1}{2}\langle 12\rangle[21].
\end{aligned} \tag{2.9}$$

Note that flipping helicity flips $\langle\cdots\rangle \leftrightarrow [\cdots]$. This is called *parity conjugation symmetry.*

2.4.1 Little Group Covariance

Let $p^{\alpha\dot\alpha} = |p\rangle[p|$ be a massless momentum. Recall that the little group of p are those Lorentz transformations which leave the momentum invariant. Note that $p^{\alpha\dot\alpha}$ is invariant under scalings:

$$|p\rangle \longmapsto z|p\rangle, \quad [p| \longmapsto z^{-1}[p|, \quad z\in\mathbb{C}\backslash\{0\}.$$

Thus little group of p^μ should be rescalings of this form. If we have a polarisation with momentum p, then under little group rescaling of p, we see that

$$\begin{aligned}
\epsilon_p^-(r) &= \sqrt{2}\frac{|p\rangle[r|}{[pr]} \longrightarrow z^2\epsilon_p^-(r),\\
\epsilon_p^+(r) &= \sqrt{2}\frac{|r\rangle[p|}{\langle rp\rangle} \longrightarrow z^{-2}\epsilon_p^+(r).
\end{aligned}$$

Note that polarisations are independent of rescaling associated to the reference vector r as it should be. Little group covariance puts strong restrictions on amplitudes. It will be particularly useful when we generalise to complex momenta. For now, we content ourselves with the consistency conditions for amplitudes using little group covariance. For example consider the scattering of two positive and two negative helicity gluons. The kinematic factor in the amplitude can be

$$\widetilde{\mathcal{M}}\left(1^-,2^-,3^+,4^+\right) = \frac{\langle 21\rangle[34]^2}{[21][14]\langle 41\rangle}, \text{ or } \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

but cannot be $\langle 12\rangle\langle 34\rangle$ as it is not invariant under little group scaling. A nice compact rule to determine if an amplitude is invariant under little group scaling is the following remark:

Remark 2.19. (Little group invariance check) The total power of $|i\rangle$ and $\langle i|$ minus the total power of $|i]$ and $[i|$ must be 2 for negative helicity gluon and -2 for positive helicity gluon. Here we count the power of a spinor to be 1 if it is in numerator and -1 if it is in denominator.

2.5 Dirac Spinors

As we saw in Eq. 2.3, solutions to massless Dirac equaition can be described by two component vectors. So by slight abuse of notation, it is natural to define the following notation: in Weyl representation, we write

$$|p\rangle = \begin{pmatrix} \lambda^\alpha \\ 0 \end{pmatrix}, \quad |p] = \begin{pmatrix} 0 \\ \widetilde{\lambda}_{\dot\alpha} \end{pmatrix}, \quad [p| = \begin{pmatrix} 0 & \widetilde{\lambda}^{\dot\alpha} \end{pmatrix}, \quad \langle p| = (\lambda_\alpha \quad 0)\,.$$

It will be clear from context whether $\langle p|, |p]$ represent four component Dirac spinor (with lower or upper two components zero) or a helicity-spinor. We also have

$$\gamma^\mu_{\alpha\dot\alpha} = \begin{pmatrix} 0 & \sigma^{\mu\alpha\dot\alpha} \\ \bar\sigma^\mu_{\alpha\dot\alpha} & 0 \end{pmatrix}.$$

We have the following lemma:

Lemma 2.20. *Assume the above notation for Dirac spinors. For massless momentas p, q, r and s, the following statements hold:*

(a) $[p\gamma^\mu q] = 0 = \langle p\gamma^\mu q\rangle$.

(b) $\langle p\gamma^\mu q] = \langle p\sigma^\mu q] = [q\bar\sigma^\mu_{\dot\alpha\alpha} p\rangle = [q\gamma^\mu p\rangle\,.$

(c) $\langle p\gamma^\mu q]\,\langle r\gamma_\mu s] = 2\langle pr\rangle[sq]$.

(d) $\langle p\not{r} q] = \langle pr\rangle[rq]$.

Proof. (a) We have

$$[p\gamma^\mu q] = (* \quad 0)\begin{pmatrix} 0 \\ * \end{pmatrix} = 0.$$

The other case of (a) follows similarly.
(b) We have

$$\begin{aligned}\langle p\gamma^\mu q] &= (\lambda_\alpha \quad 0)\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \widetilde{\chi}_{\dot\alpha} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \lambda_\alpha\sigma^\mu \end{pmatrix}\begin{pmatrix} 0 \\ \widetilde{\chi}_{\dot\alpha} \end{pmatrix} \\ &= \lambda_\alpha\sigma^{\mu\alpha\dot\alpha}\widetilde{\chi}_{\dot\alpha}.\end{aligned}$$

So first equality is clear. Next

$$\lambda_\alpha \sigma^{\mu\alpha\dot{\alpha}} \widetilde{\chi}_{\dot{\alpha}} = \lambda^\beta \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \sigma^{\mu\alpha\dot{\alpha}} \widetilde{\chi}^{\dot{\beta}}$$
$$= \lambda^\beta \bar{\sigma}_{\beta\dot{\beta}} \widetilde{\chi}^{\dot{\beta}}.$$

So the second and third equality follow.
(c) We have

$$\langle p\gamma^\mu q] \langle r\gamma_\mu s] = g_{\mu\nu} \langle p\sigma^\mu q] \langle r\sigma^\nu s]$$
$$= 2\langle pr\rangle[sq],$$

where we used Lemma 2.6 (i) and the definition of angle and square brackets.
(d) We have

$$\langle p\not{r}q] = \langle p(r_\mu \gamma^\mu)q] = \langle p(|r\rangle[r|)q] = \langle pr\rangle[rq].$$

□

3 First Examples: Yukawa Theory and QED

Although the real power of spinor helicity formalism comes into play when we compute gluon scattering amplitude, we will anyway demonstrate its applicability in compute the scattering amplitudes in some simple theories. SInce we formulated the momentum conservation condition in terms of spinors taking all momentas to be incoming, thus our convention will be that we will take all momentas to be incoming in our computations. The Mandelstram variables with our convention become:

$$\begin{aligned} s &\equiv (p_1+p_2)^2 = 2p_1\cdot p_2 = \langle 12\rangle[21] = (p_3+p_4)^2 = 2p_3\cdot p_4 = \langle 34\rangle[43], \\ t &\equiv (p_2+p_3)^2 = 2p_2\cdot p_3 = \langle 23\rangle[32] = (p_1+p_4)^2 = 2p_1\cdot p_4 = \langle 14\rangle[41], \\ u &\equiv (p_2+p_4)^2 = 2p_2\cdot p_4 = \langle 24\rangle[42] = (p_1+p_3)^2 = 2p_1\cdot p_3 = \langle 13\rangle[31]. \end{aligned} \tag{3.1}$$

With this notation, we see that

$$s+t+u=0. \tag{3.2}$$

Indeed since all momenta are massless, we have

$$s+t+u = 2(p_1\cdot p_2 + p_1\cdot p_4 + p_1\cdot p_3) = 2p_1\cdot(p_2+p_3+p_4) = -2p_1\cdot p_1 = 0,$$

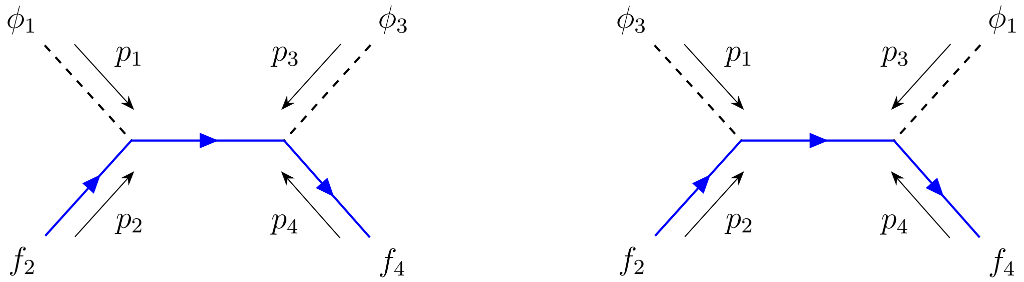
where we used momentum conservation $p_1+p_2+p_3+p_4=0$.

3.1 Yukawa Theory: $\phi f \longrightarrow \phi f$

In Yukawa theory, we have fermions and scalars. The Lagrangian is:

$$\mathcal{L}_{\text{Yukawa}} = i\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}\partial^2\phi + g\phi\bar{\psi}\psi.$$

We will compute the scattering amplitude of $\phi f \longrightarrow \phi f$ via fermion exchange where ϕ and f denote a scalar and a fermion respectively. Two diagrams contribute to the amplitude. They are shown below.



(a) $\phi_1 f_2 \longrightarrow \phi_3 f_4$ (b) $\phi_3 f_2 \longrightarrow \phi_1 f_4$

Figure 2: $\phi f \longrightarrow \phi f$ Feynman diagrams. The dashed line denotes scalar and the bold lines denote fermions.

We will make a choice of helicities of the two fermions and then we will have to sum over all possible helicities. First notice that both the fermions cannot have same helicity since $\langle p\gamma^\mu q\rangle = [p\gamma^\mu q] = 0$ by Lemma 2.20 (a) and the amplitude would be zero. Suppose f_2 is left-handed (negative helicity) $\langle 2|$ and f_4 is right-handed (positive helicity) $|4]$. The amplitude for the two diagrams is

$$\begin{aligned} i\mathcal{M}\left(\phi_1 f_2^- \phi_3 f_4^+\right) &= (ig)^2 \frac{\langle 2| - \gamma^\mu i\,(p_1+p_2)_\mu\,|4]}{(p_1+p_2)^2} + (1 \leftrightarrow 3) \\ &= ig^2 \frac{\langle 2\,(p\!\!\!/_1 + p\!\!\!/_2)\,4]}{(p_1+p_2)^2} + (1 \leftrightarrow 3) \\ &= ig^2 \frac{\langle 2p\!\!\!/_1 4] + \langle 2p\!\!\!/_2 4]}{(p_1+p_2)^2} + (1 \leftrightarrow 3). \end{aligned}$$

Using Lemma 2.20 (d) and Lemma 2.12, we get

$$\begin{aligned}
i\mathcal{M}\left(\phi_1 f_2^- \phi_3 f_4^+\right) &= g^2 i\left(\frac{\langle 21\rangle[14]}{2p_1\cdot p_2}\right) + (1\leftrightarrow 3)\\
&= ig^2\left(\frac{\langle 21\rangle[14]}{\langle 21\rangle[12]}\right) + (1\leftrightarrow 3)\\
&= ig^2\frac{[14]}{[12]} \qquad\quad + (1\leftrightarrow 3)\\
&= ig^2\frac{[14]}{[12]} + ig^2\frac{[34]}{[32]}\\
&= ig^2\left(\frac{[14]}{[12]} + \frac{[34]}{[32]}\right),
\end{aligned}$$

where we used $\langle 22\rangle = 0$. Next

$$\begin{aligned}
\left|i\mathcal{M}\left(\phi_1 f_2^- \phi_3 f_4^+\right)\right|^2 &= g^4\left(\frac{[14]}{[12]} + \frac{[34]}{[32]}\right)\left(\frac{[14]}{[12]} + \frac{[34]}{[32]}\right)^\star\\
&= g^4\left(\frac{[14]}{[12]} + \frac{[34]}{[32]}\right)\left(\frac{\langle 41\rangle}{\langle 21\rangle} + \frac{\langle 43\rangle}{\langle 23\rangle}\right)\\
&= g^4\left(\frac{[14]\langle 41\rangle}{[12]\langle 21\rangle} + \frac{[14]\langle 43\rangle}{[12]\langle 23\rangle} + \frac{[34]\langle 41\rangle}{[32]\langle 21\rangle} + \frac{[34]\langle 43\rangle}{[32]\langle 23\rangle}\right)\\
&= g^4\left(\frac{t}{s} + \frac{[14]\langle 43\rangle[32]\langle 21\rangle + [34]\langle 41\rangle[12]\langle 23\rangle}{[12]\langle 21\rangle\langle 23\rangle[32]} + \frac{s}{t}\right)\\
&= g^4\left(\frac{t}{s} + \frac{s}{t} - \frac{\langle 23\rangle[32]\langle 21\rangle[12] + [34]\langle 43\rangle[32]\langle 23\rangle}{[12]\langle 21\rangle\langle 23\rangle[32]}\right)\\
&= g^4\frac{(s-t)^2}{ts}
\end{aligned}$$

where we used $\langle 21\rangle[14] = -\langle 23\rangle[34]$ and $\langle 41\rangle[12] = -\langle 43\rangle[32]$ which is easy to see from momentum conservation equation (see Remark 2.15). Using parity conjugation symmetry, we have

$$\begin{aligned}
\left|i\mathcal{M}\left(\phi_1 f_2^+ \phi_3 f_4^-\right)\right|^2 &= g^4\left(\frac{[14]\langle 41\rangle}{[12]\langle 21\rangle} + \frac{[14]\langle 43\rangle}{[12]\langle 23\rangle} + \frac{[34]\langle 41\rangle}{[32]\langle 21\rangle} + \frac{[34]\langle 43\rangle}{[32]\langle 23\rangle}\right)\\
&= g^4\frac{(s-t)^2}{st}.
\end{aligned}$$

So total amplitude is

$$\sum_{\text{spin}}|\mathcal{M}(\phi_1 f_2 \phi_3 f_4)|^2 = 2g^4\frac{(s-t)^2}{st}.$$

3.2 QED: $e^+e^- \longrightarrow \mu^+\mu^-$

Consider the process $e^+e^- \longrightarrow \mu^+\mu^-$ via photon exchange in QED. We will find the amplitude of the process using spinor helicity formalism in high energy limit when we can take the fermions to be massless. First observe that in QED, the contribution to this process comes only from s-channel due to the vertices involving same particle and antiparticle. The Feynman diagram is shown below. Now if we take the electron to be

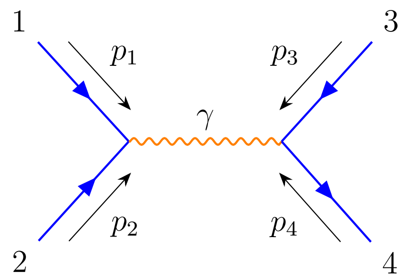


Figure 3: $e^+e^- \longrightarrow \mu^+\mu^-$

left-handed, the positron has to be right handed. To see this, we will do the following: Let us take the electron to be right handed and denote it by $|1]$. Since $[2\gamma^\mu \perp] = 0$ thus the positron has to be left banded. Similarly take μ^- to be $\langle 3|$ then μ^+ is $4]$. Since helicities of left and right handed spinors are opposite and we take all momentas incoming, this choice of spinors corresponds to the amplitude $i\mathcal{M}(1^+2^-3^+4^-)$ where the superscripts indicate the helicities. We have

$$i\mathcal{M}(1^+2^-3^+4^-) = (-ie)^2\langle 2\gamma^\mu 1]^2 \frac{-ig_{\mu\nu}}{s}\langle 3\gamma_\nu 4] = 2\frac{ie^2}{s}[41]\langle 23\rangle,$$

where we used Lemma 2.20 (c). Amplitude squared is then given by

$$\left|\mathcal{M}\left(1^-2^+3^-4^+\right)\right|^2 = 4e^4\frac{[41]\langle 14\rangle\langle 23\rangle[32]}{s^2}$$
$$== 4e^4\frac{t^2}{s^2}.$$

Other helicity combinations are $\mathcal{M}\left(1^+2^-3^+4^-\right), \mathcal{M}\left(1^-2^+3^+4^-\right)$ and $\mathcal{M}\left(1^+2^-3^-4^+\right)$. By parity conjugation symmetry

$$\mathcal{M}\left(1^+2^-3^+4^-\right) = 2\frac{ie^2}{s}\langle 41\rangle[23].$$

So

$$\left|\mathcal{M}\left(1^+2^-3^+4^-\right)\right|^2 = 4\frac{e^4}{s^2}\langle 41\rangle[14][23]\langle 32\rangle = 4e^2\frac{u^2}{s^2}.$$

The other amplitude $\mathcal{M}\left(1^-2^+3^+4^-\right)$ can be computed just by $1 \leftrightarrow 2$ exchange. We get

$$\left|\mathcal{M}\left(1^-2^+3^+4^-\right)\right|^2 = 4e^4\frac{s^2}{u^2},$$

where we used the fact that under $1 \leftrightarrow 2$ exchange $t \leftrightarrow u$. Thus we get

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\frac{t^2+u^2}{s^2}$$

which matches with the QED calculation with $m_e = m_\mu = 0$.

4 QCD: Gluon Scattering

We will now consider gluon scattering in QCD and YM theory. Since each gluon comes with a polarisation in the amplitude, thus we will have polarisation contractions in the amplitude. We use the freedom of reference vector in polarisation vector to prove strong constraints on scattering amplitude. Take r^μ to be the reference vector of all polarisation. Then we see that

$$\begin{aligned}\epsilon_i^+(r)\cdot\epsilon_j^+(r) &= \frac{\langle rr\rangle[ji]}{\langle ri\rangle\langle rj\rangle} = 0\\ \epsilon_i^-(r)\cdot\epsilon_j^-(r) &= \frac{\langle ij\rangle[rr]}{[2r][jr]} = 0\end{aligned}$$

This has the following consequence:

Theorem 4.1. *The scattering amplitude of gluons with all of either positive or negative helicity vanishes at tree level for any number of external leg.*

Proof. First observe that any contraction of polarisations or its complex conjugate or a mix of the two, with same helicity vanishes (as reference vector is same by our choice). Moreover every amplitude should not have any free Lorentz index (by Lorentz invariance). Thus a polarisation can either contract to a polarisation or a momenta from a vertex. Note that the 4-vertex does not contribute any momenta. Since the number of vertices is always less than external legs. So there is atleast one contraction of polarisations. The proof is complete. □

Theorem 4.2. *Tree level amplitude of any number of gluons greater than 3 with all but one positive helicity or all but one negative helicity vanishes at tree level.*

Proof. Assume that all but one gluon has positive helicity. Let p_1^μ be the momentum of the negative helicity gluon. Choose the reference vector for $p_{i\neq 1}^\mu$ polarisation to be p_1^μ. Then we still have

$$\epsilon_i^+\cdot\epsilon_j^\dagger = 0 \quad \forall i,j\neq 1.$$

Next, we have

$$\epsilon_i^+(1)\cdot\epsilon_1^-(r)=\frac{[ir]\langle 11\rangle}{[1i][1r]}=0,$$

for any reference vector r^μ for p_1^μ polarisation. Thus all polarisation contractions vanish. Thus we have our statement. The other case is similar. $\square$

This theorem does not work for three gluons because the reference vector r^μ satisfies $r\cdot p_i\neq 0$. But for $r=p_1$, we have $p_1\cdot p_3=\frac{1}{2}\left(p_1+p_3\right)^2=\frac{1}{2}p_2^2=0$. Nevertheless the restriction on the number of gluons in the previous theorem is not very important. For real momenta, the amplitude for three gluon scattering is anyway trivial. To see this, suppose $p_i^\mu, i=1,2,3$ are the momentas of the three gluons, then we have $p_i^2=0;\quad i=1,2,3$. At the vertex, we can assume that

$$p_1^\mu+p_2^\mu+p_3^\mu=0$$

Thus $p_3^\mu=(p_1^\mu+p_2^\mu)$. This gives

$$0=p_3^2=p_1^2+p_2^2+2p_1^\mu p_{2\mu}\Rightarrow p_1^\mu p_{2\mu}=0.$$

Similarly $p_2\cdot p_3=p_1\cdot p_3=0$. Thus all contractions are trivial. We can now show that p_1,p_2,p_3 are collinear. Indeed, we can show that

$$p_1^\mu=\alpha p_3^\mu,\quad p_2^\mu=(1+\alpha)p_3^\mu$$

for some $\alpha>0$. To see this, note that combining

$$\left(p_i^0\right)^2=\left(p_i^1\right)^2+\left(p_i^2\right)^2+\left(p_i^3\right)^2,\quad i=1,2,3$$

and $p_i^\mu p_{j\mu}=0$ for $i,j=1,2,3$ gives $p_1^\mu=\alpha p_3^\mu,\ p_2^\mu=\beta p_3^\mu$. Momentum conservation gives $\beta=1+\alpha$. Thus the three momentum are collinear and hence such an interaction is not possible.

Remark 4.3. A more physical[1] argument to show that three gluon scattering is trivial is the following: let us go to the center of mass (COM) frame[2]. In COM, the momenta of two massive incoming particle can be chosen to be $p_1^\mu=(E_1,0,0,p)\,,p_2^\mu=(E_2,0,0,-p)$. So $p_3^\mu=(E_1+E_2,0,0,0)$. Thus p_3^μ cannot be massless as it is at rest. If p_1^μ and p_2^μ are massless then in COM frame $p_1^\mu=-p_2^\mu$. So $p_3^\mu=0$ which again cannot be massless.

The following result follows using the parity symmetry of QCD.

[1]This argument was suggested by Raj Patil.

[2]This argument generalises to two massive, one massless scattering.

Theorem 4.4. *Amplitudes at tree level is invariant under parity, which flips helicities $h_i \longrightarrow -h_i$.*

Proof. Note that under parity, $\vec{p} \longrightarrow -\vec{p}$, thus helicity flips. Now if the theory is parity invariant, so is the scattering amplitude under $\vec{p} \to -\vec{p}$. Since QCD is parity invariant, thus the amplitude is invariant under $\vec{p} \longrightarrow -\vec{p}$. This amounts to the invariance under helicity flip. $\square$

Remark 4.5. From above theorems, leading non vanishing amplitudes must have atleast two negative or two positive helicities. Those with exactly two positive or negative helicity are called *maximum helicity violating* (MHV) amplitudes.

4.1 Color Factors

Recall that $\{T^\alpha\}_{a=1}^{N^2-1}$ are the generators (in fundamental representation) of SU(N) and the structure constants f^{abc} are defined as

$$\left[T^a, T^b\right] = i f^{abc} T^c, \tag{4.1}$$

where there is sum over repeated indices. Moreover the generators are normalised as

$$\operatorname{tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}. \tag{4.2}$$

Multiplying Eq. 4.1 by T^c on the right, taking trace and using 4.2, we get

$$\operatorname{tr}\left(\left[T^a, T^b\right] T^c\right) = i f^{abd} \operatorname{tr}\left(T^d T^c\right) = \frac{i}{2} f^{abd}\delta^{cd}.$$

Thus we have

$$f^{abc} = -2i \operatorname{tr}\left(\left[T^a, T^b\right] T^c\right). \tag{4.3}$$

We have the following identities for the generators :

Proposition 4.6. *For arbitrary $n \times n$ complex matrices A and B, we have*

(a) $T^a_{ij} T^a_{k\ell} = \frac{1}{2}\left(\delta_{i\ell}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{k\ell}\right)$.

(b) $\operatorname{tr}\left(T^a A\right) \operatorname{tr}\left(T_a B\right) = \frac{1}{2}\left(\operatorname{tr}(AB) - \frac{1}{N}\operatorname{tr}(A)\operatorname{tr}(B)\right)$.

(c) $\operatorname{tr}\left(A T^a B T_a\right) = \frac{1}{2}\left(\operatorname{tr}(A)\operatorname{tr}(B) - \frac{1}{N}\operatorname{tr}(AB)\right)$.

Proof. (a) We begin by observing that any $N \times N$ complex matrix M can be written as a *complex* linear combination[3] of the $N \times N$ identity matrix and the T^a

$$M = M_0 \mathbb{1} + M_a T^a, \quad M_0, M_a \in \mathbb{C}.$$

[3] If we take real linear combinations of T^a then we get the Lie algebra su(N) of the Lie group SU(N).

Using the fact that $\mathrm{tr}(T^a) = 0$, we get

$$\mathrm{tr}(M) = M_0 \mathrm{tr}(\mathbb{1}) + 0 \implies M_0 = \frac{1}{N}\mathrm{tr}(M).$$

Using Eq. 4.2, it easily follows that

$$M_a = 2\,\mathrm{tr}\left(MT^a\right).$$

This gives

$$M = \frac{1}{N}(\mathrm{tr}(M))\mathbb{1} + 2\,\mathrm{tr}\left(MT^a\right)T^a.$$

Writing the above equation in matrix indices, we get

$$M_{ij} = \frac{1}{N}M_{kk}\delta_{ij} + 2M_{\ell k}T^a_{k\ell}T^a_{ij}$$

where there is a sum over repeated indices. Writing $M_{ij} = \delta_{jk}\delta_{i\ell}M_{\ell k}$, $M_{kk} = \delta_{\ell k}M_{\ell k}$, we get

$$\delta_{i\ell}\delta_{jk}M_{\ell k} = \left(\frac{1}{N}\delta_{ij}\delta_{k\ell} + 2T^a_{ij}T^a_{k\ell}\right)M_{\ell k}.$$

Since M was arbitrary, we get

$$T^a_{ij}T^a_{k\ell} = \frac{1}{2}\left(\delta_{i\ell}\delta_{kj} - \frac{1}{N}\delta_{ij}\delta_{k\ell}\right).$$

(b) This follows upon contracting (a) with A_{ji} and $B_{\ell k}$.

(c) This follows by contracting with $A_{\ell i}$ and B_{jk}. $\square$

Using Proposition 4.6, we can manipulate the color factors that appear in the amplitudes.

Proposition 4.7. *The following relations hold.*

(a) $\mathrm{tr}(T^aT^a) = \frac{N^2-1}{2}$, $\mathrm{tr}(T^aT^bT^aT^b) = \frac{1-N^2}{4N}$, $\mathrm{tr}(T^aT^bT^cT^d)\mathrm{tr}(T^aT^bT^cT^d) = \frac{N^4+2N^2-3}{16N^2}$, $\mathrm{tr}(T^aT^bT^cT^d)\mathrm{tr}(T^dT^cT^bT^a) = \frac{N^6-4N^4+6N^2-3}{16N^2}$.

(b) $f^{abe}f^{cde} = \mathrm{tr}\left([T^a,T^b][T^c,T^d]\right)$.

(c) $(f^{abe}f^{cde})^2 = N^2(N^2-1)$.

(d) $\left(f^{abe}f^{cde}\right)\left(f^{acg}f^{bdg}\right) = \frac{1}{2}N^2\left(N^2-1\right)$.

Proof. (a) Take $A = B = \mathbb{1}$ in Proposition 4.6 (b) to get the first relations. Others are also similar with added complication due to four factors.

(b) Using Eq. 4.3, we have

$$\begin{aligned} f^{abe}f^{cde} &= -4\mathrm{tr}\left(\left[T^a,T^b\right]T^e\right)\mathrm{tr}\left(\left[T^c,T^d\right]T^e\right) \\ &= -2\left[\mathrm{tr}\left(\left[T^a,T^b\right]\left[T^c,T^d\right]\right) - \frac{1}{N}\mathrm{tr}\left(\left[T^a,T^b\right]\right)\mathrm{tr}\left(\left[T^c,T^d\right]\right)\right] \\ &= -2\,\mathrm{tr}\left\{\left[T^a,T^b\right]\left[T^c,T^d\right]\right\}, \end{aligned}$$

where we used Proposition 4.6 (b) in second step and the cyclic property of trace to get rid of the second term in third step.
(c) We have

$$\begin{aligned}
(f^{abe}f^{cde})^2 &= (f^{abe}f^{cde})(f^{abg}f^{cdg}) \\
&= 4\mathrm{tr}\left(\left[T^a,T^b\right]\left[T^c,T^d\right]\right)\mathrm{tr}\left(\left[T^a,T^b\right]\left[T^c,T^d\right]\right) \\
&= 4\left[\mathrm{tr}\left(T^aT^bT^cT^d - T^aT^bT^dT^c - T^bT^aT^cT^d + T^bT^aT^dT^c\right)\right]\times \\
&\quad\ \left[\mathrm{tr}\left(T^aT^bT^cT^d - T^aT^bT^dT^c - T^bT^aT^cT^d + T^bT^aT^dT^c\right)\right].
\end{aligned}$$

Next we multiply out to get 16 terms of the form $\mathrm{tr}(T^aT^bT^cT^d)\mathrm{tr}(T^{\pi(a)}T^{\pi(b)}T^{\pi(c)}T^{\pi(d)})$ where π is a permutation of $\{a,b,c,d\}$. Using analogous result as in (a) and tedious calculation gives

$$(f^{abe}f^{cde})^2 = N^2(N^2-1).$$

(d) Similar to (c). □

4.2 $gg \longrightarrow gg$ scattering amplitude

In this subsection, we will see the full fledged application of spinor helicity formalism. We will compute four gluon scattering amplitude mentioned in Subsection 1.1.

By Remark 4.5, since there are only four gluon, we know that only MHV amplitudes contribute. We will compute $\mathcal{M}(1^-2^-3^+4^+)$ and other helicity combinations will be related by crossing symmetry. Note that all momentas are incoming and hence physically $\mathcal{M}\,(1^-2^-3^+4^+)$ corresponds to all neegative helicities. For ϵ_1, ϵ_2 we choose p_4 as the reference vector and for ϵ_3, ϵ_4 we choose p_1 as the reference vector. We easily see that with our choice of helicities

$$\begin{aligned}
&\epsilon_1^-(4)\cdot\epsilon_2^-(4) = 0, \quad \epsilon_3^+(1)\cdot\epsilon_4^+(1) = 0 \\
&\epsilon_1^-(4)\cdot\epsilon_4^+(1) = \frac{\langle 11\rangle[44]}{[14]\langle 14\rangle} = 0, \quad \epsilon_2^-(4)\cdot\epsilon_3^+(1) = \frac{\langle 21\rangle[34]}{[21]\langle 13\rangle} \neq 0.
\end{aligned} \tag{4.4}$$

We can also check that

$$\epsilon_1\cdot p_4 = \epsilon_2\cdot p_4 = \epsilon_3\cdot p_1 = \epsilon_4\cdot p_1 = 0.$$

Also $\epsilon_i \cdot p_i = 0$ is the gauge choice. Note that with our convention the Mandelstram variables are as in Eq. 3.1. We will now compute the contribution of each of the four Feynman diagram. We begin with the 4-vertex.

4-vertex: Since the 4-vertex does not contribute any momentum, thus only polarisations contract. Since only non-vanishing polarisation contraction is $\epsilon_2 \cdot \epsilon_3$, thus 4-vertex amplitude is necessarily zero.

s-channel: The s-channel amplitude is

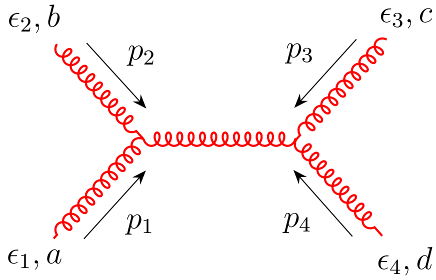


Figure 4: $gg \longrightarrow gg$: s-channel

$$
\begin{aligned}
i\mathcal{M}_s = \frac{-ig_s^2}{s} f^{abe} f^{cde} &\left[(\epsilon_1 \cdot \epsilon_2)(p_1 - p_2)^\mu + 2\epsilon_2^\mu (p_2 \cdot \epsilon_1) - 2\epsilon_1^\mu (p_1 \cdot \epsilon_2)\right] \times \\
&\left[(\epsilon_3 \cdot \epsilon_4)(p_3 - p_4)^\mu + 2\epsilon_4^\mu (p_4 \cdot \epsilon_3) - 2\epsilon_3^\mu (p_3 \cdot \epsilon_4)\right]
\end{aligned}
$$

Note that polarisations of 3 and 4 are not complex conjugated because the momentas are incoming. Thus for tour choice of helicities $--++$ only one term survives (see Eq. 4.4). We get

$$
\begin{aligned}
\mathcal{M}_s\left(1^-2^-3^+4^+\right) &= \frac{4g_s^2}{s} f^{abe} f^{cde} \left(\epsilon_2^- \cdot \epsilon_3^+\right)\left(p_2 \cdot \epsilon_1^-\right)\left(p_3 \cdot \epsilon_4^+\right) \\
&= \frac{4g_s^2}{s} f^{abe} f^{cde} \frac{1}{2} \left(\frac{\langle 12\rangle[24]}{[14]}\right)\left(\frac{[43]\langle 31\rangle}{\langle 14\rangle}\right)\left(\frac{\langle 21\rangle[34]}{[24]\langle 13\rangle}\right) \\
&= \frac{2g_s^2}{\langle 12\rangle[21]} f^{abe} f^{cde} \frac{\langle 21\rangle[24][43]\langle 31\rangle\langle 21\rangle[34]}{[14]\langle 14\rangle[24]\langle 13\rangle} \\
&= -2g_s^2 f^{abe} f^{cde} \frac{\langle 21\rangle[34]^2}{[21][14]\langle 41\rangle} \\
&= -2g_s^2 f^{abe} f^{cde} \frac{\langle 21\rangle[34]^2}{[21][23]\langle 32\rangle}
\end{aligned}
$$

where we used $[14]\langle 41\rangle = [23]\langle 32\rangle$ which easily follows from $(p_1 + p_4)^2 = (p_2 + p_3)^2$. Similarly $(p_1 + p_2)^2 = (p_3 + p_4)^2$ gives $[21]\langle 12\rangle = [34]\langle 43\rangle$ and momentum conservation gives $\langle 12\rangle[23] = -\langle 14\rangle[43]$ (see Remark 2.15). Using these we get

$$
\frac{[34]}{[21]} = \frac{\langle 12\rangle}{\langle 43\rangle}, \qquad \frac{[34]}{[23]} = \frac{\langle 12\rangle}{\langle 14\rangle}
$$

Thus we get

$$
\begin{aligned}
\mathcal{M}_s\left(1^-2^-3^+4^+\right) &= -2g_s^2 f^{abe} f^{cde} \frac{\langle 12\rangle^2\langle 21\rangle}{\langle 32\rangle\langle 43\rangle\langle 14\rangle} \\
&= -2g_s^2 f^{abc} f^{cde} \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}
\end{aligned}
$$

This is a special case of the *Taylor-Parke formula* which we will discuss in the next subsection.

t-channel: We get this by $2 \leftrightarrow 4$ and $b \leftrightarrow d$ crossing of the s-channel amplitude.

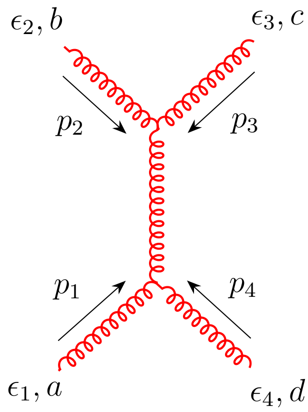


Figure 5: $gg \longrightarrow gg$: t-channel

The t-channel amplitude is

$$i\mathcal{M}_t = \frac{-ig_s^2}{t} f^{ade} f^{cbe} \left[(\epsilon_1 \cdot \epsilon_4)(p_1 - p_4)^\mu + 2\epsilon_4^\mu (p_4 \cdot \epsilon_1) - 2\epsilon_1^\mu (p_1 \cdot \epsilon_4)\right] \times$$
$$\left[(\epsilon_3, \epsilon_2)(p_3 - p_2)^\mu + 2\epsilon_2^\mu (p_2 \cdot \epsilon_3) - 2\epsilon_3^\mu (p_3 \cdot \epsilon_2)\right].$$

With our choice of helicities $--++$, we see that this amplitude vanishes identically:

$$i\mathcal{M}_t \left(1^- 2^- 3^+ 4^+\right) = 0.$$

u-channel: We get this by $2 \leftrightarrow 3$ and $b \leftrightarrow c$ crossing of the s-channel amplitude. The u-channel amplitude is

$$i\mathcal{M}_u = \frac{-ig_s^2}{u} f^{ace} f^{bde} \times \left[(\epsilon_i \cdot \epsilon_3)(p_1 - p_3)^\mu + 2\epsilon_3^\mu (p_3 \cdot \epsilon_1) - 2\epsilon_1^\mu (p_1 \cdot \epsilon_3)\right] \times$$
$$\Big[(\epsilon_3 \cdot \epsilon_4)(p_2 - p_4)^\mu + 2\epsilon_4^\mu (p_4 \cdot \epsilon_2) - 2\epsilon_2^4 (p_2 \cdot \epsilon_4)\Big].$$

For helicities $--++$, we have

$$\begin{aligned}
\mathcal{M}_u \left(1^- 2^- 3^+ 4^+\right) &= \frac{g_s^2}{u} f^{ace} f^{bde} 4 \left(\epsilon_3^+ \cdot \epsilon_2^-\right)\left(p_3 \cdot \epsilon_1^-\right)\left(p_2 \cdot \epsilon_4^+\right) \\
&= \frac{1}{2} \frac{4g_s^2}{\langle 31\rangle[34]} f^{ace} f^{bde} \left(\frac{\langle 21\rangle[34]}{[24]\langle 13\rangle}\right)\left(\frac{\langle 13\rangle[31]}{[14]}\right)\left(\frac{[42]\langle 21\rangle}{\langle 14\rangle}\right) \\
&= -2g_s^2 f^{ace} f^{bde} \frac{\langle 21\rangle^2 \langle 13\rangle [34]^2 [42]}{\langle 13\rangle^2 [13][24][14]\langle 14\rangle} \\
&= 2g_s^2 f^{ace} f^{bde} \frac{\langle 21\rangle^2 [34]^2}{[13]\langle 13\rangle[14]\langle 14\rangle}.
\end{aligned}$$

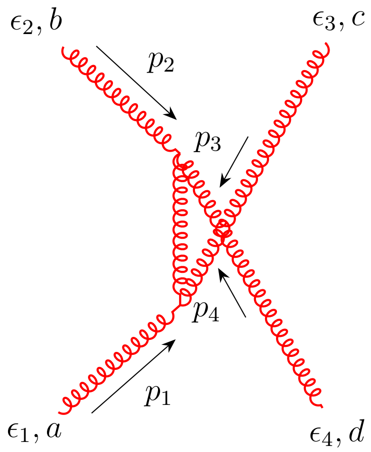


Figure 6: $gg \longrightarrow gg$: u-channel

Now we use momentum conservation to get

$$\langle 12\rangle[23] = -\langle 14\rangle[43], \quad \langle 21\rangle[13] = -\langle 24\rangle[43]$$

Also $(p_1+p_4)^2 = (p_2+p_3)^2$ gives $\langle 14\rangle[41] = \langle 23\rangle[32]$. We thus have

$$\begin{aligned}\mathcal{M}_u\left(1^-2^-3^+4^+\right) &= 2g_s^2 f^{ace} f^{bde} \frac{\langle 21\rangle^2[34]^2}{[13]\langle 13\rangle[14]\langle 14\rangle}\left(-\frac{\langle 12\rangle[23]}{\langle 14\rangle[43]}\right)\left(-\frac{\langle 21\rangle[13]}{\langle 24\rangle[43]}\right)\left(\frac{\langle 14\rangle[41]}{\langle 23\rangle[32]}\right)\\ &= -2g_s^2 f^{ace} f^{bde}\left(\frac{\langle 21\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\right).\end{aligned}$$

We now compute the total amplitude. The total amplitude is

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u + \mathcal{M}_{\text{4-vertex}}.$$

Thus

$$\mathcal{M}\left(1^-2^-3^+4^+\right) = -2g_s^2\left[f^{abe} f^{cde}\left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right) + f^{ace} f^{bde}\left(\frac{\langle 21\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\right)\right]$$

We need to find $|\mathcal{M}|^2$. There are three terms in the square. Let us simplify each term separately. We have

$$\begin{aligned}\left|\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right|^2 &= \left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)\left(\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)^\star\\ &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[21]^4}{[21][32][43][14]}\\ &= \frac{(\langle 12\rangle[21])^4}{(\langle 12\rangle[21])(\langle 23\rangle[32])(\langle 34\rangle[43])(\langle 41\rangle[14])}\\ &= \frac{s^4}{s^2t^2} = \frac{s^2}{t^2}.\end{aligned}$$

$$\left|\frac{\langle 21\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\right|^2 = \frac{\langle 21\rangle^4}{\langle 14\rangle\langle 42\rangle\langle 23\rangle\langle 31\rangle}\frac{[12]^4}{[41][24][32][13]} = \frac{s^4}{t^2u^2},$$
$$\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\frac{[12]^4}{[41][24][32][13]} = \frac{s^3}{t^2u}.$$

Using Proposition 4.7 (c), (d), we get

$$\begin{aligned}\sum_{\substack{\text{polarisation}\\ \text{sum}}} |\mathcal{M}\left(1^-2^-3^+4^+\right)|^2 &= 4g_s^4 N^2\left(n^2-1\right)\left\{\frac{s^2}{t^2}+\frac{s^4}{t^2u^2}+\frac{s^3}{t^2u}\right\},\\ &= 4g_s^u N^2\left(N^2-1\right)\left(\frac{s^4}{t^2u^2}+\frac{s^2(s+u)}{t^2u}\right)\\ &= 4g_s^4 N^2\left(N^2-1\right)\left(\frac{s^4}{t^2u^2}-\frac{s^2}{ut}\right),\end{aligned}$$

where we used $s+t+u=0$. Other helicity combination is related to these by crossing symmetry, for example $\mathcal{M}\left(1^-2^+3^-4^+\right)$ is given by $\mathcal{M}\left(1-2^-3^+4^+\right)$ with $2\leftrightarrow 3$ which means $s\leftrightarrow u$. Thus the six non-vanishing amplitude correspond to six permutations of s,t,u. Summing all of these gives

$$\begin{aligned}\sum_{\substack{\text{color sum}\\ \text{polarisation}\\ \text{sum}}} |\mathcal{M}|^2 &= 4g_s^4 N^2\left(N^2-1\right)\left\{\left(\frac{s^4}{t^2u^2}-\frac{s^2}{ut}\right)+(\text{permutations of} s,t,u)\right\}\\ &= 4g_s^4 N^2\left(N^2-1\right)\left\{\left(\frac{s^4}{t^2u^2}-\frac{s^2}{tu}\right)+\left(\frac{u^4}{t^2s^2}-\frac{u^2}{ts}\right)+\left(\frac{s^4}{t^2u^2}-\frac{s^2}{tu}\right)+\right.\\ &\qquad\left.\left(\frac{t^4}{s^2u^2}-\frac{t^2}{su}\right)+\left(\frac{t^4}{u^2s^2}-\frac{t^2}{us}\right)+\left(\frac{u^4}{t^2s^2}-\frac{u^2}{ts}\right)\right\}\\ &= 4g_s^4 N^2\left(N^2-1\right)\left\{2\left(\frac{s^4}{t^2u^2}+\frac{u^4}{t^2s^2}+\frac{t^4}{s^2u^2}\right)-2\left(\frac{s^2}{tu}+\frac{t^2}{su}+\frac{u^2}{ts}\right)\right\}\\ &= 8g_s^4 N^2\left(N^2-1\right)\left\{\frac{s^6+t^6+u^6}{s^2t^2u^2}-\frac{s^3+t^3+u^3}{stu}\right\}\\ &= 8g_s^4 N^2\left(N^2-1\right)\left\{\frac{\left(s^3+t^3+u^3\right)^2-2\left(s^3t^3+u^3t^3+s^3u^3\right)}{(stu)^2}-3\right\}\\ &= 8g_s^4 N^2\left(N^2-1\right)\left\{6-2\left(\frac{st}{u^2}+\frac{ut}{s^2}+\frac{su}{t^2}\right)\right\}\\ &= 16g_s^4 N^2\left(N^2-1\right)\left\{3-\frac{st}{u^2}-\frac{ut}{s^2}-\frac{su}{t^2}\right\}\end{aligned}$$

where we used the fact that $s+t+u=0 \Rightarrow s^3+t^3+u^3=3stu$. Number of initial states are $4\times\left(N^2-1\right)^2$ which correspond to N^2-1 number of color for each helicity and there are four such gluons. Averaging over them, we get

$$\frac{1}{4(N^2-1)^2}\sum_{\substack{\text{color sum}\\ \text{polarisation}\\ \text{sum}}}|\mathcal{M}|^2 = 16g_s^4\frac{N^2}{4\left(N^2-1\right)}\left(3-\frac{st}{u^2}-\frac{ut}{s^2}-\frac{su}{t^2}\right).$$

For QCD, plugging in $N=3$ in above result gives the standard four gluon scattering amplitude:

$$\frac{1}{256}\sum_{\substack{\text{polarisations}\\ \text{color}}}|\mathcal{M}|^2 = g^4\,\frac{9}{2}\left[3-\frac{tu}{s^2}-\frac{su}{t^2}-\frac{st}{u^2}\right].$$

4.3 Color Ordering and Taylor-Parke Formula

To simplify computations in YM theory, we separate the kinematic factors from colour factors. We define the *colour stripped amplitude* to be the amplitude coming from the same Feynman rules but without a factor of $\sqrt{2}ig_s f^{abc}$ and we denote the color stripped amplitude with a tilde. For example the colour stripped amplitude for $gg\to gg$ scattering s-channel (ignoring helicity) is

$$\begin{aligned}\widetilde{\mathcal{M}}_s(1234) =& \frac{1}{2s}\left[\left(\epsilon_1\cdot\epsilon_2\right)\left(p_1-p_2\right)^\mu+2\epsilon_2^\mu\left(p_2\cdot\epsilon_1\right)-2\epsilon_1^\mu\left(p_1\cdot\epsilon_2\right)\right]\times\\ &\left[\left(\epsilon_3\cdot\epsilon_4\right)\left(p_3-p_4\right)^\mu+2\epsilon_4^\mu\left(p_4\cdot\epsilon_3\right)-2\epsilon_3^\mu\left(p_3\cdot\epsilon_4\right)\right]\end{aligned}$$

We immediately have the following relations:

$$\begin{aligned}\widetilde{\mathcal{M}}_s(1234) = -\widetilde{\mathcal{M}}_s(2134) = -\widetilde{\mathcal{M}}_s(1243) = \widetilde{\mathcal{M}}_s(2143).\\ \widetilde{\mathcal{M}}_s(1234) = \widetilde{\mathcal{M}}_s(3412).\end{aligned}\tag{4.5}$$

To take care of the color factor, we introduce the notation $T^a\to 1, T^b\to 2, T^c\to 3, T^d\to 4$. With this notation, the color factor in s-channel diagram is

$$\begin{aligned}f^{12a}f^{34a} &= -2\,\mathrm{tr}\{[1,2][3,4]\}\\ &= -2\,\mathrm{tr}\{1234-2134-1243+2143\}.\end{aligned}$$

So the amplitude for s-channel can be written as:

$$\begin{aligned}\mathcal{M}_s(1234) &= -2g_s^2(-2)\mathrm{tr}\{1234-2134-1243+2143\}\widetilde{\mathcal{M}}_s(1234)\\ &= 4g_s^2\Big[\mathrm{tr}\{1234\}\widetilde{\mathcal{M}}_s(1234)-\mathrm{tr}\{2134\}\widetilde{\mathcal{M}}_s(1234)-\\ &\qquad \mathrm{tr}\{1243\}\widetilde{\mathcal{M}}_s(1234)+\mathrm{tr}\{2143\}\widetilde{\mathcal{M}}_s(1234)\Big]\\ &= 4g_s^2\Big[\mathrm{tr}\{1234\}\widetilde{\mathcal{M}}_s(1234)+\mathrm{tr}\{2134\}\widetilde{\mathcal{M}}_s(2134)+\\ &\qquad \mathrm{tr}\{1243\}\widetilde{\mathcal{M}}_s(1243)+\mathrm{tr}\{2143\}\widetilde{\mathcal{M}}_s(2143)\Big],\end{aligned}$$

where we used Eq. 4.5. The t-channel and u-channel amplitude can be gotten by crossings of s-channel. For example, t-channel is $2 \leftrightarrow 4$ exchange of s-channel and u -channel is $2 \leftrightarrow 3$ exchange in s-channel. That is

$$\begin{aligned}\widetilde{\mathcal{M}}_t(1234) &= \widetilde{\mathcal{M}}_s(1432) \quad \text{(second-fourth slot exchange)},\\ \widetilde{\mathcal{M}}_u(1234) &= \widetilde{\mathcal{M}}_s(1324) \quad \text{(second-third slot exchange)}.\end{aligned} \tag{4.6}$$

Since the 4-vertex vanishes, the full amplitude is thus

$$\mathcal{M}(1234) = \mathcal{M}_s(1234) + \mathcal{M}_t(1234) + \mathcal{M}_u(1234)$$

which has 12 terms, four each for s, t, u channels. Using the cyclic property of trace and the crossing relations between the s, t, u channel, all of the 12 terms can be written as $\mathrm{tr}\{ijk\ell\}\widetilde{\mathcal{M}}_s(ijkl)$ with appropriate summation. Indeed the following is true:

Proposition 4.8. *The four gluon amplitude (with helicities suppressed) can be written as a single trace sum:*

$$\mathcal{M}(1234) = 4g_s^2 \sum_{\sigma \in S_3} \mathrm{tr}\{1\sigma(2)\sigma(3)\sigma(4)\}\widetilde{\mathcal{M}}(1\sigma(2)\sigma(3)\sigma(4))$$

where $\widetilde{\mathcal{M}}(ijkl) = \widetilde{\mathcal{M}}_s(ijk\ell) + \widetilde{\mathcal{M}}_t(ijkl)$ *is called the* color-ordered partial amplitude.

Proof. From calculations in previous subsection, we have

$$\begin{aligned}\mathcal{M}_t(1234) &= -2ig_s^2 f^{14e} f^{32e} \widetilde{\mathcal{M}}_t(1234)\\ &= -2g_s^2(-2)\mathrm{tr}\left([1,4][3,2]\right)\widetilde{\mathcal{M}}_t(1234)\\ &= 4g_s^2 \mathrm{tr}\{1432 - 4132 - 1423 + 4123\}\widetilde{\mathcal{M}}_s(1432)\\ &= 4g_s^2\Big[\mathrm{tr}\{1432\}\widetilde{\mathcal{M}}_s(1432) + \mathrm{tr}\{4132\}\widetilde{\mathcal{M}}_s(4132) +\\ &\qquad \mathrm{tr}\{1423\}\widetilde{\mathcal{M}}_s(1423) + \mathrm{tr}\{4123\}\widetilde{\mathcal{M}}_s(4123)\Big],\end{aligned}$$

where we used Proposition 4.7 (b) (second step), Eq. (4.6) (third step) and Eq. (4.5) (fourth step). Similarly we have

$$\begin{aligned}\mathcal{M}_u(1234) = 4g_s^2\Big[&\mathrm{tr}\{1324\}\widetilde{\mathcal{M}}_s(1324) + \mathrm{tr}\{3124\}\widetilde{\mathcal{M}}_s(3124) +\\ &\mathrm{tr}\{1342\}\widetilde{\mathcal{M}}_s(1342) + \mathrm{tr}\{3142\}\widetilde{\mathcal{M}}_s(3142)\Big].\end{aligned}$$

Thus the three amplitudes are:

$$\begin{aligned}
\mathcal{M}_s(1234) = 4g_s^2 \Big[&\mathrm{tr}\{1234\}\widetilde{\mathcal{M}}_s(1234) + \mathrm{tr}\{2134\}\widetilde{\mathcal{M}}_s(2134) \\
&+\mathrm{tr}\{1243\}\widetilde{\mathcal{M}}_s(1243) + \mathrm{tr}\{2143\}\widetilde{\mathcal{M}}_s(2143)\Big], \\
\mathcal{M}_t(1234) = 4g_s^2 \Big[&\mathrm{tr}\{1432\}\widetilde{\mathcal{M}}_s(1432) + \mathrm{tr}\{4132\}\widetilde{\mathcal{M}}_s(4132) \\
&+\mathrm{tr}\{1423\}\widetilde{\mathcal{M}}_s(1423) + \mathrm{tr}\{4123\}\widetilde{\mathcal{M}}_s(4123)\Big], \\
\mathcal{M}_u(1234) = 4g_s^2 \Big[&\mathrm{tr}\{1324\}\widetilde{\mathcal{M}}_s(1324) + \mathrm{tr}\{3124\}\widetilde{\mathcal{M}}_s(3124) \\
&+\mathrm{tr}\{1342\}\widetilde{\mathcal{M}}_s(1342) + \mathrm{tr}\{3142\}\widetilde{\mathcal{M}}_s(3142)\Big].
\end{aligned}$$

Collecting the same color terms above and using cyclic property of trace along with Eq. (4.5), we get

$$\begin{aligned}
\mathcal{M}(1234) &= \mathcal{M}_s(1234) + \mathcal{M}_t(1234) + \mathcal{M}_u(1234) \\
&= 4g_s^2 \Big[\mathrm{tr}\{1234\}\Big(\widetilde{\mathcal{M}}_s(1234) + \widetilde{\mathcal{M}}_s(1432)\Big) + \mathrm{tr}\{1342\}\Big(\widetilde{\mathcal{M}}_s(1243) + \widetilde{\mathcal{M}}_s(1342)\Big) \\
&\qquad + \mathrm{tr}\{1243\}\Big(\widetilde{\mathcal{M}}_s(1243) + \widetilde{\mathcal{M}}_s(1342)\Big) + \mathrm{tr}\{1432\}\Big(\widetilde{\mathcal{M}}_s(1234) + \widetilde{\mathcal{M}}_s(1432)\Big) \\
&\qquad +\mathrm{tr}\{1324\}\Big(\widetilde{\mathcal{M}}_s(1423) + \widetilde{\mathcal{M}}_s(1324)\Big) + \mathrm{tr}\{1423\}\Big(\widetilde{\mathcal{M}}_s(1423) + \widetilde{\mathcal{M}}_s(1324)\Big)\Big].
\end{aligned}$$

Using the crossing in Eq. (4.6), we have

$$\begin{aligned}
\mathcal{M}(1234) &= 4g_s^2 \Big[\mathrm{tr}\{1234\}\Big(\widetilde{\mathcal{M}}_s(1234) + \widetilde{\mathcal{M}}_t(1234)\Big) + \mathrm{tr}\{1342\}\Big(\widetilde{\mathcal{M}}_t(1342) + \widetilde{\mathcal{M}}_s(1342)\Big) \\
&\qquad + \mathrm{tr}\{1243\}\Big(\widetilde{\mathcal{M}}_s(1243) + \widetilde{\mathcal{M}}_t(1243)\Big) + \mathrm{tr}\{1432\}\Big(\widetilde{\mathcal{M}}_t(1432) + \widetilde{\mathcal{M}}_s(1432)\Big) \\
&\qquad +\mathrm{tr}\{1324\}\Big(\widetilde{\mathcal{M}}_s(1324) + \widetilde{\mathcal{M}}_t(1324)\Big) + \mathrm{tr}\{1423\}\Big(\widetilde{\mathcal{M}}_s(1423) + \widetilde{\mathcal{M}}_t(1423)\Big)\Big] \\
&= 4g_s^2 \Big[\mathrm{tr}\{1234\}\widetilde{\mathcal{M}}(1234) + \mathrm{tr}\{1342\}\widetilde{\mathcal{M}}(1342) + \mathrm{tr}\{1243\}\widetilde{\mathcal{M}}(1243) \\
&\qquad + \mathrm{tr}\{1432\}\widetilde{\mathcal{M}}(1432) + \mathrm{tr}\{1324\}\widetilde{\mathcal{M}}(1324) + \mathrm{tr}\{1423\}\widetilde{\mathcal{M}}(1423)\Big] \\
&= 4g_s^2 \sum_{\sigma\in S_3} \mathrm{tr}\{1\sigma(2)\sigma(3)\sigma(4)\}\widetilde{\mathcal{M}}(1\sigma(2)\sigma(3)\sigma(4)).
\end{aligned}$$

□

Remark 4.9. The partial color-ordered amplitude $\widetilde{\mathcal{M}}(ijkl)$ is the sum over the amplitude coming only from the *planer* Feynman diagram. Intuitively, planer Feynman diagrams are those in which the legs are uncrossed. For example, the s, t channel diagrams are planer while the u channel diagram is non planer.

The above formula for 4-gluon scattering generalises easily to n-gluon scattering. We will state the result without proof and refer to [5, Subsection 2.2.2] for proof and detailed discussion.

Theorem 4.10. *The n-gluon amplitude (with helicities suppressed) can be written as a single trace sum:*

$$\mathcal{M}(12\cdots n) = -2\left(\sqrt{2}ig_s\right)^{n-2}\sum_{\sigma\in S_{n-1}}\operatorname{tr}\{1\sigma(2)\ldots\sigma(n)\}\widetilde{\mathcal{M}}(1\sigma(2)\ldots\sigma(n))$$

where $\widetilde{\mathcal{M}}(12\ldots n)$ is the sum over all planer diagrams.

For MHV amplitudes, the color-ordered partial amplitude is given by (see Theorem **??** for proof)

$$\widetilde{\mathcal{M}}\left(1^+2^+\ldots i^-\ldots j^-\ldots n^+\right) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n-1n\rangle\langle n1\rangle}.$$

This is called the *Taylor-Parke formula.* We will prove it in next subsection using the BCFW recursion relation.

4.4 Complex Momenta and BCFW Recursion Relation

We will now take our momentums to be complex. With this leap, the advantage is that we can invoke complex analysis to study amplitudes and get the analytic continuations. We can finally take real momentum limit to get the physical amplitude. This trick is used to derive powerful recursion relations called the Britto-Feng-Cachazo-Witten (BCFW) recursion relations.

4.4.1 3-gluon Amplitude

Let us begin with the three point amplitude. We will write the most general three gluon amplitude without Feynman rules using only the symmetries of the theory.[4] We begin by proving a constraint on the 3-gluon amplitude.

Proposition 4.11. *The 3-gluon amplitude with complex momenta has kinematic factor in terms of either the angular bracket of spinor contraction or the square bracket of spinor contraction coming from the momentas.*

Proof. The three gluon amplitude must depend on polarisation vectors ϵ_i and momenta p_i, or the spinors $[1|, [2|, [3|$ and $\langle 1, \langle 2|, \langle 3|$. Momentum conservation gives

$$\sum_{i=1}^{3}|i\rangle[i| = 0.$$

[4]This is called the bottom-up construction. A whole field of S-matrix bootstrap based on this approach is an active field of research.

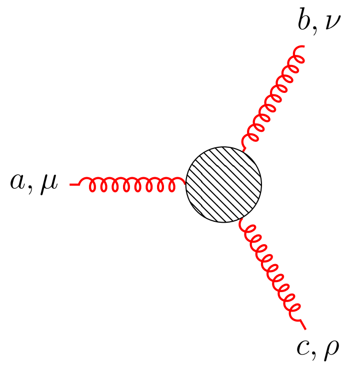


Figure 7: 3-gluon Feynman diagram

Contracting with $\langle 1|$ or $\langle 2|$ gives

$$\langle 12\rangle[2| = -\langle 13\rangle[3|, \quad \langle 21\rangle[1| = -\langle 23\rangle[3|.$$

Now since we have three 2 component spinors, all of them cannot be linearly independent. If $|1]$ and $|2]$ are proportional then above equation says that $|1], |2]$ and $|3]$ are proportional. Only two cases can occur, either $\langle 12\rangle = 0$ which in turn implies $\langle 23\rangle = \langle 31\rangle = 0$, or that all $|i]$ are proportional to each other, in which case $[12] = [23] = [31] = 0$. So either all angle brackets vanish or all square brackets vanish. Thus the answer must only be in terms of angles or squares. □

Remark 4.12. In the real momenta limit, $\langle ij\rangle^\star = [ji]$ (which is not true for complex momenta), so all inner products vanish and hence we have no nontrivial 3-gluon amplitude for real momenta (which we already concluded in Remark 4.3).

The most general 3-gluon amplitude is recorded in the following theorem:

Theorem 4.13. *In a renormalizable theory of massless spin 1 particles (gluons), the most general 3-gluon amplitude with complex momenta has the following forms:*

$$\mathcal{M}\left(1^+2^+3^+\right) = 0, \quad \mathcal{M}\left(1^+2^+3^-\right) = C^{abc}\frac{[12]^3}{[13][32]}, \quad \mathcal{M}\left(1^-2^-3^+\right) = C^{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle}.$$

Before we begin to prove this theorem, we need the following proposition on the mass dimension of amplitudes:

Proposition 4.14. *The mass dimension of an n-particle scattering amplitude $\mathcal{M}(p_i, \sigma_i, a_i)$*[5]

[5] pi, σ_i, a_i corresponding to a massive (massless) particle denotes the momentum, spin (helicity) and other internal indices respectively of the particle.

is

$$[\mathcal{M}] = 4 - n.$$

Proof. Recall that the scattering amplitude is related to the S-matrix by

$$S_{fi} = i(2\pi)^4 \delta^{(4)} \left(\sum_{i=1}^{N} p_i \right) \mathcal{M}\left(p_i, \sigma_i, a_i\right), \tag{4.7}$$

where S_{fi} is the scattering matrix of m $(m < n)$ incoming particles and $n - m$ outgoing particles:

$$S_{fi} = \langle \textbf{in } p_1, \ldots, p_m | \, p_{m+1}, \ldots, p_n \textbf{ out} \rangle \, .$$

The incoming and outgoing states are made of single particle states $|p_i, \sigma_i, a_i\rangle$ created from vacuum by creation operator

$$|p, \sigma, a\rangle = \sqrt{2E_p} a_{\mathbf{p}}^{\dagger\ \sigma, a} |0\rangle,$$

and are normalised as

$$\langle p, \sigma, a | p', \sigma', a' \rangle = 2E_p (2\pi)^3 \delta_{\sigma'\sigma} \delta_{a'a} \delta^{(3)}(\mathbf{p}' - \mathbf{p}).$$

Now observe that the mass dimension of the (one dimensional) Dirac delta function is the inverse of the dimension of its argument. To see this, note that for dimensionless x, we have

$$\delta(ax) = \frac{\delta(x)}{|a|} \implies |a| = [\delta(ax)] = [a]^{-1} = [ax]^{-1}.$$

Since $[E_p] = [\sqrt{\mathbf{p}^2 + m^2}] = 1$, and $[\delta^{(3)}(\mathbf{p}' - \mathbf{p})] = [\delta(p'_x - p_x)]^3$, thus the mass dimension of $\langle p, \sigma, a | p', \sigma', a' \rangle$ is $-3 + 1 = -2$. Hence

$$[|p, \sigma, a\rangle] = [\langle p, \sigma, a|] = -1.$$

Thus we have $[S_{fi}] = -n$. Plugging this into Eq. (4.7)

$$-n = -4 + [\mathcal{M}] \implies [\mathcal{M}] = 4 - n.$$

□

Proof of Theorem 4.13. By Proposition 4.11, we know that the answer must only be in terms of either angular product or square product. Next we invoke little group rescaling to get the most general 3-gluon amplitude with complex momenta. Let us make a choice of helicities. We take $1^+2^+3^+$. By Remark 2.19, the total power of $[1|$ minus the power of $\langle 1|$ should be 2, for 1^- it must be -2. For all other momenta, same rule holds. Hence under the above constraints for $+++$, the most general amplitude is

$$\mathcal{M}\left(1^+2^+3^+\right) = C^{abc}[12][23][31] \quad \text{or} \quad C^{abc}\frac{1}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

where C^{abc} is some color structure. Since in the limit of real momenta, the amplitude must vanish, therefore the first form is the only possibility as the second form is incompatible with this limit (as it diverges due to vanishing of the angle brackets). Since $\mathcal{M}(123)$ has mass dimension 1 (Proposition 4.14) and $[12][23][31]$ has mass dimension 3 (since $[|i\rangle, \langle i|] = [|i], [i|] = 1/2$ which is easy to see from $1 = [p][|p\rangle[p|])$, C^{abc} must have dimension -2. Thus, if we consider only renormalizable theories with dimensionless couplings, the only solution is $C^{abc} = 0$. For helicities $++-$ and $--+$, there are only two possibilities consistent with little group scaling. Since $\frac{\langle 12\rangle\langle 23\rangle\langle 31\rangle}{\langle 12\rangle^4}$ diverges in the limit of real momenta, the only possibility is

$$\mathcal{M}\left(1^+2^+3^-\right) = C^{abc}\frac{[12]^3}{[13][32]}$$

Similarly,

$$\mathcal{M}\left(1^-2^-3^+\right) = C^{abc}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle}.$$

□

4.4.2 BCFW Recursion Relations

The key idea of BCFW recursion relation is to relate the n-gluon amplitude with the amplitude of lesser number of gluons. This is achieved by introducing a complex variable in the momenta and then using Cauchy's residue theorem to get the physical amplitude. We now systematically describe the method.

Let z be a complex variable. Let us shift the spinors of gluons with momenta p_i and p_j in the following way:

$$[\hat{i}| = [i| + z[j|, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle, \quad |\hat{i}\rangle = |i\rangle, \quad [\hat{j}| = [j|.$$

Then the shifted momenta is

$$\hat{p}_i = |i\rangle[i| + z|i\rangle[j|, \quad \hat{p}_j = |j\rangle[j| - z|i\rangle[j|.$$

It is easy to see that the shifted momentas are still massless and satisfy the momentum conservation. Let $\mathcal{M}(z)$ be the amplitude with shifted momenta. Then the physical amplitude is obviously $\mathcal{M}(0)$. Consider the following integral:

$$\mathcal{I} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{d}z \frac{\mathcal{M}(z)}{z},$$

where the contour $\mathcal{C}$ is

$$\mathcal{C} = \lim_{R\to\infty} \mathcal{C}_R, \quad \mathcal{C}_R = \left\{z \in \mathbb{C} \mid z = Re^{i\theta}, \quad 0 \le \theta < 2\pi\right\}.$$

By Cauchy's residue theorem, we have

$$\mathcal{I} = \sum_{\text{poles } z_\alpha} \mathrm{Res}_{z=z_\alpha}\left(\frac{\mathcal{M}(z)}{z}\right).$$

It is easy to see that $z = 0$ is a pole with residue $\mathcal{M}(0)$. Thus we have

$$\mathcal{I} = \mathcal{M}(0) + \sum_{\substack{\text{poles}\\ z_\alpha \neq 0}} \mathrm{Res}_{z=z_\alpha}\left(\frac{\mathcal{M}(z)}{z}\right).$$

We have the following theorem:

Theorem 4.15. *Suppose* $\mathcal{M}(z) \longrightarrow 0$ *as* $|z| \longrightarrow \infty$. *Then*

$$\mathcal{I} = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathrm{d}z \frac{\mathcal{M}(z)}{z} = 0.$$

Proof. We begin by observing that

$$\mathcal{I} = \lim_{R\to\infty} \frac{1}{2\pi i}\oint_{\mathcal{C}_R} \mathrm{d}z \frac{\mathcal{M}(z)}{z}.$$

Now consider the function

$$f(z) = \mathcal{M}(1/z).$$

Since $\mathcal{M}(z)$ is meromorphic, there exists a neighbourhood U of $z = 0$ such that $f : U \longrightarrow \mathbb{C}$ is holomorphic. Moreover $f(0) = 0$ by assumption. Then by [2, Theorem 1.1. Page 73], there exists a neighbourhood $r_0 > 0$, a unique integer $n > 0$ and a nonvanishing holomorphic function[6] $g : \mathbb{D}_{r_0} \longrightarrow \mathbb{C}$ such that

$$f(z) = z^n g(z) \implies \mathcal{M}(z) = z^{-n} g(1/z),$$

[6] Actually [2, Theorem 1.1. Page 73] only gives a neighbourhood $V \subset U$ of 0 on which g is holomorphic, but we cal always choose r_0 small enough such that g is holomorphic on $\mathbb{D}_{r_0}$.

where $\mathbb{D}_{r_0} = \{z \in \mathbb{C} \mid |z| \leq r_0\}$. Since g is holomorphic on $\mathbb{D}_{r_0}$, it is in particular continuous. Since $\mathbb{D}_{r_0}$ is compact, by *extreme value theorem* [3, Theorem 27.4], we have

$$\sup_{z\in\mathbb{D}_{r_0}} |g(z)| < \infty.$$

This readily implies that

$$\lim_{R\to\infty} \sup_{z\in\mathcal{C}_R} |g(1/z)| < \infty.$$

Thus we get

$$\begin{aligned}\lim_{R\to\infty}\left|\frac{1}{2\pi i}\oint_{\mathcal{C}_R} \mathrm{d}z\frac{\mathcal{M}(z)}{z}\right| &= \lim_{R\to\infty}\frac{1}{2\pi}\left|\int_0^{2\pi}\mathrm{d}\theta(Re^{i\theta})\frac{g(R^{-1}e^{-i\theta})}{R^{n+1}e^{i(n+1)\theta}}\right| \\ &\leq \lim_{R\to\infty}\frac{\sup_{z\in\mathcal{C}_R}|g(1/z)|}{2\pi R^n}\int_0^{2\pi}\mathrm{d}\theta\mathrm{d}z \\ &= 0.\end{aligned}$$

The conclusion now follows. □

Assuming that $\mathcal{M}(z) \longrightarrow 0$ as $|z| \longrightarrow \infty$, Theorem 4.15 implies that the physical amplitude is given by sum over residues of $\mathcal{M}(z)$:

$$\mathcal{M}(0) = -\sum_{\substack{\text{poles}\\ z_\alpha\neq 0}} \mathrm{Res}_{z=z_\alpha}\left(\frac{\mathcal{M}(z)}{z}\right). \tag{4.8}$$

We will analyse this falloff condition of the amplitude at last and show that it is not such a restrictive requirement. Let us now analyze the pole structure of $\mathcal{M}(z)$. We can easily see that the only pole contribution in the amplitude comes from the propagators connecting vertices which are in turn connected to the legs i and j. Let this propagator have momentum $\hat{P}(z)$. Thus the pole is at z for which $\hat{P}(z)^2 = 0$. Thus we can split the Feynman diagram as shown in Figure 8. We label the gluons from 1 to n with gluons $a, a+1, \ldots, b$ lying on the right of $\hat{P}$ and the remaining gluons to the left of $\hat{P}$. Note that the propagator momentum $\hat{P}(z)$ is independent of z if the gluons i, j lie on the same side of the propagator. This is because the momentum $\hat{P}(z)$ is the sum of the momentas of gluons a to b. So we always take i and j to be on opposite sides of the propagator. The bispinor corresponding to $\hat{P}(z)$, which we also denote with $\hat{P}(z)$ without spinor indices is

$$\hat{P}(z) = \sum_{k=a}^{b} |k\rangle[k| - z|i\rangle[j|.$$

So $\hat{P}^2(z_{a,b}) = 0$ implies

$$(p_a + \cdots + p_b)^2 - z_{a,b}\sum_{k=a}^{b}\langle ik\rangle[kj] + \frac{(z_{a,b})^2}{2}\langle ii\rangle[jj] = 0,$$

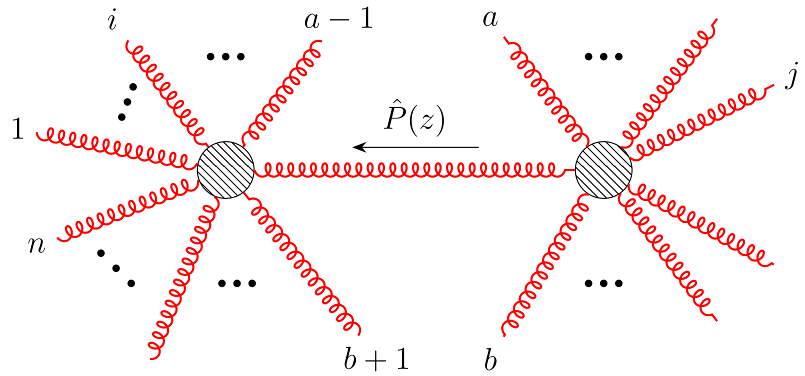


Figure 8: Partition of Feynman diagram across propagator with momentum $\hat{P}(z)$

which gives

$$z_{a,b} = \frac{(p_a + \cdots + p_b)^2}{\langle ia\rangle[aj] + \cdots + \langle ib\rangle[bj]} \tag{4.9}$$

where we used the fact that $[ii] = \langle ii\rangle = 0$. Thus for each such partition of the diagram by a, b, we will get a pole at $z_{a,b}$. The residue at $z_{a,b}$ can easily be calculated:

$$\begin{aligned} -\frac{1}{z_{a,b}} \mathrm{Res}_{z=z_{a,b}} &\left(\mathcal{M}_1(z) \frac{1}{(p_a + \cdots + p_b)^2 - z\sum\langle ik\rangle[kj]} \mathcal{M}_2(z) \right) \\ &= \mathcal{M}_1(z_{a,b}) \frac{1}{(p_a + \cdots + p_b)^2} \mathcal{M}_2(z_{a,b}), \end{aligned}$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are the amplitudes of the left and right side in the partition of the diagram. Substituting this in Eq. (4.8), we get

$$\begin{aligned} \mathcal{M}(1\ldots n) = \sum_{a,b,h} &\mathcal{M}(1, \ldots, a-1, b+1, \ldots, n \to \hat{P}^h) \\ &\times \frac{1}{(p_a + \cdots + p_b)^2} \mathcal{M}\left(\hat{P}^{-h} \to a, \ldots, b\right), \end{aligned} \tag{4.10}$$

where h denotes the helicity of the virtual particle in the propagator which we need to sum over. Note that the helicities of $\hat{P}$ is opposite on the two subdiagram due to opposite direction of the momenta relative to the two subdiagrams. This is the Britto-Feng-Cachazo-Witten (BCFW) recursion relation.

Let us now turn to the falloff condition of the amplitude. With the shift in momentas, the corresponding polarisations of gluons i and j with respective reference vectors p_j and p_i is easily checked to be

$$\epsilon^-_{\hat{i}}(j) = \epsilon^-_i(j), \quad \epsilon^+_{\hat{i}}(j) = \epsilon^+_i(j) + z\sqrt{2}\frac{|j\rangle[j|}{\langle ji\rangle}$$

$$\epsilon^-_{\hat{j}}(i) = \epsilon^-_j(i) - z\sqrt{2}\frac{|i\rangle[i|}{[ji]}, \quad \epsilon^+_{\hat{j}}(i) = \epsilon^+_j(i).$$

This is easy to see from the form of polarisation vectors given in Lemma 2.17. Let us write $q^\mu = [|i\rangle[j|]^\mu$. Then it is easily checked that

$$q^{\star\mu} = |j\rangle[i| \implies p_{i,j} \cdot q = p_{i,j} \cdot q^\star = 0. \tag{4.11}$$

Then the shifted momentas are

$$\hat{p}_i^\mu = p_i^\mu + zq^\mu, \quad \hat{p}_j^\mu = p_j^\mu - zq^\mu, \tag{4.12}$$

and the polarisations are

$$\begin{aligned} \epsilon_{\hat{i}}^{-\mu} &= \frac{\sqrt{2}q^\mu}{[ij]}, \quad \epsilon_{\hat{i}}^{+\mu} = \frac{\sqrt{2}(q^{\star\mu} + zp_j^\mu)}{\langle ji\rangle} \\ \epsilon_{\hat{j}}^{-\mu} &= \frac{\sqrt{2}(q^{\star\mu} - zp_i^\mu)}{[ji]}, \quad \epsilon_{\hat{j}}^{+\mu} = \frac{\sqrt{2}q^\mu}{\langle ij\rangle}. \end{aligned} \tag{4.13}$$

Now it was argued by Arkani-Hamed and Kaplan in [4] that the general form of the amplitude $\mathcal{M}(z)$ is

$$\mathcal{M}(z) = \epsilon_{\hat{i}}^\mu \mathcal{M}_{\mu\nu}(z)\epsilon_{\hat{j}}^\nu,$$

where

$$\mathcal{M}_{\mu\nu}(z) = \left(c_1 z + c_0 + c_{-1}z^{-1} + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu}z^{-1} + \cdots,$$

where $A_{\mu\nu}$ is antisymmetric and the dots indicate terms of the order of z^{-2}. Before we delve into the falloff analysis, it is useful to see what the Ward identity gives. The Ward identity is given by

$$\hat{p}_i^\mu \mathcal{M}_{\mu\nu}(z)\epsilon_{\hat{j}}^\nu = 0 \implies q^\mu \mathcal{M}_{\mu\nu}(z)\epsilon_{\hat{j}}^\nu = -\frac{1}{z}p_i^\mu \mathcal{M}_{\mu\nu}(z)\epsilon_{\hat{j}}^\nu, \tag{4.14}$$

where we used Eq. (4.12). Similarly, the Ward identity for momenta $\hat{p}_j$ gives

$$\epsilon_{\hat{j}}^\mu \mathcal{M}_{\mu\nu}(z)q^\nu = \frac{1}{z}\epsilon_{\hat{j}}^\mu \mathcal{M}_{\mu\nu}(z)p_j^\nu$$

It is obvious that the falloff condition of the amplitude $\mathcal{M}(z)$ heavily depends on the helicities (h_i, h_j) of i and j. We separate the analysis in various cases.

(i) $(h_i, h_j) = (-, +)$.

Using Eq. (4.13), the amplitude is

$$\begin{aligned} \mathcal{M}(z)^{-+} &= \epsilon_{\hat{i}}^{-\mu}(j)\left[\left(c_1 z + c_0 + c_{-1}z^{-1} + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu}z^{-1} + \cdots\right]\epsilon_{\hat{j}}^{+\nu}(i) \\ &= \frac{2}{\langle ij\rangle[ij]}q^\mu\left[\left(c_1 z + c_0 + c_{-1}z^{-1} + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu}z^{-1} + \cdots\right] q^\nu \\ &= \frac{2}{z\langle ij\rangle[ji]}p_i^\mu\left[\left(c_1 z + c_0 + c_{-1}z^{-1} + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu}z^{-1} + \cdots\right] q^\nu, \end{aligned}$$

where we used Eq. (4.14). Using Eq. (4.11), we get

$$\mathcal{M}(z)^{-+} = -\frac{1}{z}\left(\frac{2}{p_i \cdot p_j} p_i^\mu A_{\mu\nu} q^\nu\right) + O(z^{-2}) \longrightarrow \frac{1}{z}.$$

(ii) $(h_i, h_j) = (-, -)$.
By similar computation as above, we get

$$\begin{aligned}
\mathcal{M}(z)^{--} &= \epsilon_{\hat{i}}^{-\mu}(j)\mathcal{M}_{\mu\nu}(z)\epsilon_{\hat{j}}^{-\nu}(i) \\
&= -\frac{2}{z[ji][ij]} p_i^\mu \left[\left(c_1 z + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu} z^{-1} + \cdots\right] (q^\star - z p_i)^\nu \\
&= -\frac{2}{z[ji][ij]} \left[p_i^\mu A_{\mu\nu} \left(q^\star - z p_i\right)^\nu - p_i^\mu B_{\mu\nu} z^{-1} \left(q^\star - z p_i\right)^\nu + \cdots\right] \\
&= \frac{1}{z}\left(\frac{2 p_i^\mu A_{\mu\nu} q^{\star\nu} + 2 p_i^\mu B_{\mu\nu} p_i^\nu + O\left(z^{-2}\right)}{[ij]^2}\right) \longrightarrow \frac{1}{z},
\end{aligned}$$

where we used the antisymmetry of $A_{\mu\nu}$ and $p_i^2 = 0$.

(iii) $(h_i, h_j) = (+, +)$.
This case is similar to (ii) if we use the Ward identity for $\hat{p}_j$. We get

$$\begin{aligned}
\mathcal{M}(z)^{++} &= \epsilon_{\hat{i}}^{+\mu}(j)\mathcal{M}_{\mu\nu}(z)\epsilon_{\hat{j}}^{+\nu}(i) \\
&= -\frac{2}{z\langle ji\rangle\langle ij\rangle} (q^\star + z p_j)^\mu \left[\left(c_1 z + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu} z^{-1} + \cdots\right] p_j^\nu \\
&= \frac{2}{z\langle ji\rangle^2} \left[(q^\star + z p_j)^\mu A_{\mu\nu} p_j^\nu - (q^\star + z p_j)^\mu B_{\mu\nu} z^{-1} p_j^\nu + \cdots\right] \\
&= \frac{1}{z}\left(\frac{2(q^\star + z p_j)^\mu A_{\mu\nu} p_j \nu + 2(q^\star + z p_j)^\mu B_{\mu\nu} p_j^\nu + O\left(z^{-2}\right)}{\langle ij\rangle^2}\right) \longrightarrow \frac{1}{z},
\end{aligned}$$

(iv) $(h_i, h_j) = (+, -)$.
We have

$$\begin{aligned}
\mathcal{M}(z)^{+-} &= \epsilon_{\hat{i}}^{+\mu}\mathcal{M}(z)_{\mu\nu}\epsilon_{\hat{j}}^{-\nu} \\
&= \frac{2\left(q^\star + z p_j\right)^\mu}{\langle ji\rangle} \left[\left(c_1 z + \cdots\right) g_{\mu\nu} + A_{\mu\nu} + B_{\mu\nu} z^{-1} + \cdots\right] \frac{2\left(q^\star - z p_i\right)^\nu}{[ji]} \\
&= c_1 z^3 + \mathcal{O}\left(z^2\right) \longrightarrow z^3.
\end{aligned}$$

Thus in this case, we cannot use the BCFW recursion relation with the assumned momentum shift. An easy way out is to reverse the shifting. That is we use the shifts in Eq. (4.12) with i and j reversed and we get back to case (i).

As it was remarked earlier, the falloff condition is not that restrictive.

4.4.3 Example

Let us work out the four gluon partial amplitude for the helicity combination $--++$ *i,e,* $\widetilde{\mathcal{M}}(1^-2^-3^+4^+)$. We have already done this computation using Feynman rules. Let us now do this using the BCFW relation. There are two diagrams contributing to this amplitude – the s,t-channel. We choose $i=1, j=4$ which is $(-,+)$ combination and hence the falloff condition necessary for BCFW is satisfied. First observe that first and fourth gluon must be on opposite sides of the propagator to get a pole. This immediately rules out the t-channel. Thus t-channel does not contribute. Moreover, there is only one partition of the s-channel diagram corresponding to $a=3, b=4$ as shown below. The

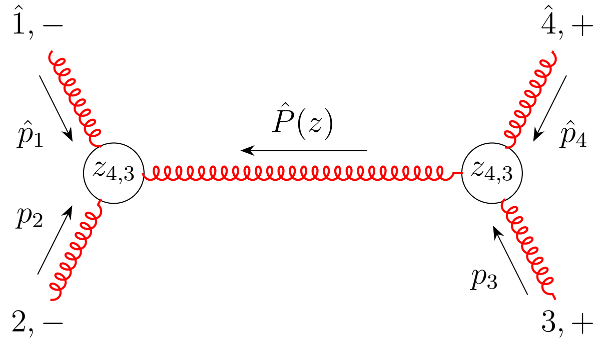


Figure 9: s-channel partition for BCFW

propagator momenta is $\hat{P}(z) = -\hat{p}_1 - p_2$ and using Eq. (4.9), the pole is at

$$z_{4,3} = \frac{s}{\langle 13\rangle[34] + \langle 14\rangle[44]} = \frac{\langle 34\rangle}{\langle 31\rangle}.$$

By BCFW, we have

$$\widetilde{\mathcal{M}}\left(1^-2^-3^+4^+\right) = \sum_h \widetilde{\mathcal{M}}\left(\hat{1}^-2^-\hat{P}^h\right)\frac{1}{\langle 12\rangle[21]}\widetilde{\mathcal{M}}\left([-\hat{P}^{-h}]3^+\hat{4}^+\right),$$

where

$$[\hat{1}| = [1| + z_{4,3}[4| \quad \text{ and } |\hat{4}\rangle = |4\rangle - z_{4,3}|1\rangle.$$

We see that $\widetilde{\mathcal{M}}\left(\hat{1}^-2^-\hat{P}^-\right) = 0$ by Theorem 4.13. Thus we must choose $h=+$. Again making use of Theorem 4.13, we get

$$\widetilde{\mathcal{M}}\left(1^-2^-3^+4^+\right) = -\frac{\langle\hat{1}2\rangle^3}{\langle\hat{1}\hat{P}\rangle\langle\hat{P}2\rangle}\frac{1}{\langle 12\rangle[21]}\frac{[3\hat{4}]^3}{[3\hat{P}][\hat{P}\hat{4}]}.$$

Using the fact that $\hat{P}(z) = \hat{p}_4 + p_3$, we get

$$\begin{aligned}|\hat{P}\rangle[\hat{P}| &= |\hat{4}\rangle[\hat{4}| + |3\rangle[3| \\ &= \left(|4\rangle - z_{4,3}|1\rangle\right)[4| + |3\rangle[3| \\ &= |4\rangle[4| - \frac{\langle 34\rangle}{\langle 31\rangle}|1\rangle[4| + |3\rangle[3|.\end{aligned}$$

This gives

$$\langle \hat{1}\hat{P}\rangle[\hat{P}3] = \langle 1\hat{P}\rangle[\hat{P}3]$$
$$= \langle 14\rangle[43].$$

Similarly

$$\langle 2\hat{P}\rangle[\hat{P}4] = \langle 23\rangle[34].$$

Thus we get

$$\widetilde{\mathcal{M}}\left(1^-2^-3^+4^+\right) = -\frac{\langle 12\rangle^3[34]^3}{\langle 14\rangle[43]\langle 23\rangle[34]\langle 12\rangle[21]}$$
$$= \frac{\langle 12\rangle^3[34]}{\langle 14\rangle\langle 23\rangle\langle 12\rangle[21]}.$$

By momentum conservation, we have $\langle 43\rangle[34] = \langle 12\rangle[21]$, which gives

$$\frac{[34]}{[21]} = \frac{\langle 12\rangle}{\langle 43\rangle}.$$

Using this, we get

$$\widetilde{\mathcal{M}}\left(1^-2^-3^+4^+\right) = \frac{\langle 12\rangle^3\langle 12\rangle}{\langle 14\rangle\langle 23\rangle\langle 12\rangle\langle 43\rangle}$$
$$= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}.$$

This agrees with our previous calculation.

4.5 Taylor-Parke Formula

We will give an inductive proof of the Taylor-Parke formula using BCFW recursion relation.

Theorem 4.16. *The n-gluon MHV amplitude is given by*

$$\widetilde{\mathcal{M}}\left(1^+2^+\dots i^-\dots j^-\dots n^+\right) = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\cdots\langle n-1n\rangle\langle n1\rangle}.$$

Proof. For $n = 4$, we have already proved the result in Subsection 4.2. We assume that the formula is true for all $m < n$ and we will prove it for n gluons. Since there are only two $-$ helicities, thus we can always renumber the gluons so that the amplitude we need to compute is $\widetilde{\mathcal{M}}\left(1^-\dots k^-\dots n^+\right)$ and all dots are positive helicity gluons. We choose $i = 1$ and $j = n$ in BCFW which is $(-,+)$ combination and hence appropriate. By Theorem 4.1 and Theorem 4.2, we see that the subamplitudes in BCFW are nonvanishing only for one partition which is shown below. We have

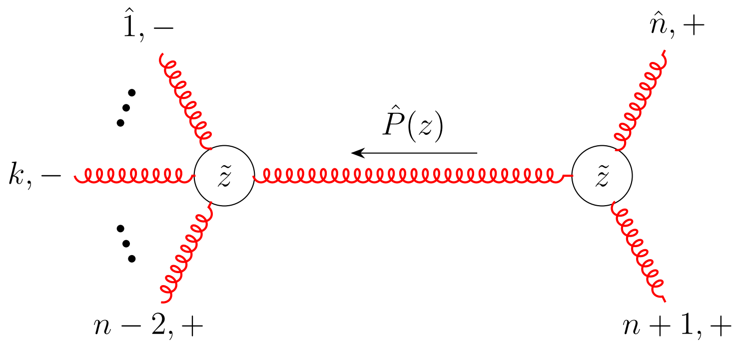


Figure 10: Partition of amplitude for MHV amplitudes

$$\tilde{z} = z_{n,n-1} = \frac{\langle (n-1)n\rangle}{\langle (n-1)1\rangle}.$$

By BCFW, we get

$$\begin{aligned}
\widetilde{\mathcal{M}}\left(1^- \dots k^- \dots n^+\right) &= \widetilde{\mathcal{M}}\left(1^- \dots k^- \dots (n-2)^+ \hat{P}^h\right) \frac{1}{\langle n(n-1)\rangle [(n-1)n]} \\
&\qquad\qquad\qquad \times \widetilde{\mathcal{M}}\left([-\hat{P}^{-h}](n-1)^+ n^+\right) \\
&= \frac{\langle \hat{1}k\rangle^4}{\langle \hat{1}2\rangle \cdots \langle (n-2)\hat{P}\rangle \langle \hat{P}1\rangle} \frac{1}{\langle n(n-1)\rangle [(n-1)n]} \frac{-[(n-1)\hat{n}]^3}{[\hat{n}\hat{P}][\hat{P}(n-1)]} \\
&= \frac{\langle 1k\rangle^4}{\langle 12\rangle \cdots \langle (n-2)\hat{P}\rangle \langle \hat{P}1\rangle} \frac{1}{\langle n(n-1)\rangle [(n-1)n]} \frac{-[(n-1)n]^3}{[n\hat{P}][\hat{P}(n-1)]},
\end{aligned}$$

where we used Theorem 4.13 to conclude that only nonvanishing contribution comes from $h = +$ and then used the induction hypothesis and again Theorem 4.13. Since $\hat{P} = \hat{p}_n + p_{n-1}$, we have

$$\begin{aligned}
|\hat{P}\rangle[\hat{P}| &= |\hat{n}\rangle[\hat{n}| + |n-1\rangle[n-1| \\
&= \left(|n\rangle - z_{n,n-1}|1\rangle\right)[n| + |n-1\rangle[n-1| \\
&= |n\rangle[n| - \frac{\langle (n-1)n\rangle}{\langle (n-1)1\rangle}|1\rangle[n| + |n-1\rangle[n-1|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\langle (n-2)\hat{P}\rangle[\hat{P}n] &= \langle (n-2)(n-1)\rangle[(n-1)n], \\
\langle 1\hat{P}\rangle[\hat{P}(n-1)] &= \langle 1n\rangle[n(n-1)].
\end{aligned}$$

Using this, we get

$$\begin{aligned}
&\widetilde{\mathcal{M}}\left(1^- \dots k^- \dots n^+\right) \\
&\quad = -\frac{\langle 1k\rangle^4 [(n-1)n]^3}{\langle 12\rangle \cdots \langle n(n-1)\rangle [(n-1)n] \langle (n-2)(n-1)\rangle [(n-1)n] \langle 1n\rangle [n(n-1)]} \\
&\quad = \frac{\langle 1k\rangle^4}{\langle 12\rangle \cdots \langle (n-2)(n-1)\rangle \langle (n-1)n\rangle \langle n1\rangle},
\end{aligned}$$

which is what we wanted to prove. □

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