

Number theoretic aspects of black holes in type IIB string theory on $K3 \times T^2$

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Outline

- 1 Black Holes Microstate Counting
- 2 Relation To Mock Modularity
- 3 Limitations
- 4 Generating Function of Single-Centered Degeneracy
- 5 Analytic Properties
- 6 Questions For Future

Black Hole Microstate Counting

Background

- One of the great successes of string theory: microscopic description of black holes. [Strominger-Vafa (1996), Dijkgraaf-Verlinde-Verlinde (1996), Shih-Strominger-Yin (2005), Gaiotto (2005), Shih-Yin (2005), Jatkar-Sen (2005), David-Jatkar-Sen (2006)].
- $S_{BH} = \frac{A}{4G_N} = \ln d_{\text{micro}} = S_{\text{stat}}.$

Macroscopic description

Find the Bekenstein-Hawking entropy of $\frac{1}{4}$ -BPS black holes carrying electric and magnetic charges $(\vec{Q}, \vec{P})$ in Type IIB supergravity compactified on $\mathcal{M} \times S^1 \times \tilde{S}^1$ with $\mathcal{M} = K3, T^4$.

Microscopic description

Count the number of $\frac{1}{4}$ -BPS configurations of various extended objects wrapping cycles in $\mathcal{M} \times S^1 \times \tilde{S}^1$ in Type IIB string theory at weak coupling.

- Perfect match in large charge approx! [Reviews: Sen (2007), Mandal-Sen (2010)]

Black Holes and Number Theory

- Microstate counting in Type IIB on $K3 \times T^2$ has fascinating connections to number theory.
- U-duality classes of black holes with charges $(\vec{Q}, \vec{P})$ and fixed $Q^2 P^2 - (Q \cdot P)^2 = D > 0$ are in one-to-one correspondence with elements of class group of binary quadratic forms with discriminant $-D$ [Moore (1998)].
- Degeneracy of $\frac{1}{4}$ -BPS states with charges $(\vec{Q}, \vec{P})$ with $Q^2 = 2n, P^2 = 2m, Q \cdot P = \ell$ are coefficients of mock Jacobi forms of weight -10 and index m . [Dabholkar-Murthy-Zagier (DMZ) (2012)]
- Generalization of both these connections to more general string compactifications called CHL models [Work-in-progress, RKS],[Bhand-RKS-Sen (2023), Banerjee-Bhutra-RKS (2025)]

CHL Models

- $\mathbb{Z}_N$ orbifold of Type IIB string theory compactified on $\mathcal{M} \times \widetilde{S}^1 \times S^1$, $\mathcal{M} = K3, T^4$.
- $\mathcal{N} = 4$ supersymmetry in 4d [Chaudhuri-Hockney-Lykken (1995)].
- Charge quantization of $\frac{1}{4}$ -BPS states [Jatkar-Sen (2005)]:

$$\frac{Q^2}{2} \in \frac{1}{N}\mathbb{Z}, \quad \frac{P^2}{2} \in \mathbb{Z}, \quad P \cdot Q \in \mathbb{Z} .$$

- Moduli space parametrized by the axio-dilaton moduli valued in $\mathbb{H}$ and an $r \times r$ matrix valued scalar field Ξ , r is an integer depending on N .
- For $N = 1, r = 22$. $N = 1, \mathcal{M} = K3$ is the DMZ case.

Degeneracy of $\frac{1}{4}$ -BPS single centered black holes

Degeneracy formula

For charges Q, P with $Q^2 = 2m > 0, P^2 = 2n/N > 0, Q \cdot P = \ell$ with $4mn - N\ell^2 > 0$, the degeneracy is given by

$$d(m, n/N, \ell) = (-1)^{\ell+1} \int_{\mathcal{C}_{m,n,\ell}} d\tau d\sigma dz \frac{e^{-2\pi i(m\tau + n\sigma/N + \ell z)}}{\tilde{\Phi}_k(\tau, z, \sigma)}$$

where $\tilde{\Phi}_k$ is a given meromorphic function, the contour $\mathcal{C}_{m,n,\ell}$ is given by

$$\begin{aligned} \mathcal{C}_{m,n,\ell} : \operatorname{Im}(\tau) &= \frac{P^2}{N\varepsilon} \equiv \frac{2n}{N\varepsilon}, \quad \operatorname{Im}(\sigma) = \frac{NQ^2}{\varepsilon} \equiv \frac{2m}{\varepsilon}, \\ \operatorname{Im}(z) &= -\frac{P \cdot Q}{\varepsilon} \equiv -\frac{\ell}{\varepsilon}, \quad 0 \leq \operatorname{Re}(\tau), \operatorname{Re}(z) < 1, \quad 0 \leq \operatorname{Re}(\sigma) < N, \end{aligned}$$

and $\varepsilon > 0$ is a small real number.

Degeneracy of $\frac{1}{4}$ -BPS single centered black holes

- The choice $\mathcal{C}_{m,n,\ell}$ makes sure that only the single-centered black holes contribute to the degeneracy.
- At generic point in the moduli space, single and two-centered black holes exist, the choice $\mathcal{C}_{m,n,\ell}$ corresponds to a chamber of the moduli space where no two-centered black holes exist. More on this later.

Relation to Mock Modularity

Degeneracy formula revisited

- The function $\tilde{\Phi}_k(\tau, z, \sigma)$ satisfies
 $\tilde{\Phi}_k(\tau + 1, z, \sigma) = \tilde{\Phi}_k(\tau, z + 1, \sigma) = \tilde{\Phi}_k(\tau, z, \sigma + N)$.
- Fourier-Jacobi expansion [Bhand-Sen-RKS (2023)]

$$\frac{1}{\tilde{\Phi}_k(\tau, z, \sigma)} = \sum_{m=-\tilde{\alpha}}^{\infty} \phi_m(\sigma, z) e^{2\pi i m \tau}, \quad \text{Im}(\tau) \rightarrow \infty$$

$$\frac{1}{\tilde{\Phi}_k(\tau, z, \sigma)} = \sum_{n=-\tilde{\gamma}N}^{\infty} \psi_n(\tau, z) e^{2\pi i n \sigma / N}, \quad \text{Im}(\sigma) \rightarrow \infty,$$

$$\tilde{\alpha}, \tilde{\gamma}N \in \mathbb{Z}.$$

- Goal: express $d(m, n/N, \ell)$ in terms of integrals of ϕ_m, ψ_n . Need to deform the contour $\mathcal{C}_{m,n,\ell}$ to $\text{Im}(\tau) \rightarrow \infty$ or $\text{Im}(\sigma) \rightarrow \infty$.

Poles of $\tilde{\Phi}_k^{-1}$

Poles of $\tilde{\Phi}_k^{-1}$ [David-Jatkar-Sen (2006)]

$\tilde{\Phi}_k^{-1}$ has double poles on the hypersurface

$$n_2 (\sigma\tau - z^2) + jz + n_1\sigma - m_1\tau + m_2 = 0$$

where

$$m_1 \in N\mathbb{Z}, \quad n_1 \in \mathbb{Z}, \quad j \in 2\mathbb{Z}+1, \quad m_2, n_2 \in \mathbb{Z}, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}.$$

The behaviour near $z = 0$ is

$$\tilde{\Phi}_k(\tau, z, \sigma) = (e^{i\pi z} - e^{-i\pi z})^{-2} f^{(k)}(N\tau) g^{(k)}(\sigma/N),$$

for some modular forms $f^{(k)}, g^{(k)}$.

DMZ case: $N = 1$, $\tilde{\Phi}_k = \Phi_{10}$ (Igusa cusp form), $f^{(k)} = g^{(k)} = \eta^{-24}$.

Degeneracy formula revisited

- There is a special set of charges A, B such that the contour $\mathcal{C}_{m,n,\ell}$ can be deformed to $\text{Im}(\tau) \rightarrow \infty$, $\text{Im}(\sigma) \rightarrow \infty$ respectively without picking up poles [Bhand-Sen-RKS (2023)]
- Doing some straightforward calculation after deforming the contour [Dabholkar-Murty-Zagier (2012), Bhand-Sen-RKS (2023)]

$$d(m, n/N, \ell) = (-1)^{\ell+1} \int d\sigma \int dz e^{-2\pi i \frac{n}{N} \sigma} e^{-2\pi i \ell z} \phi_m^F(\sigma, z), \quad (m, n, \ell) \in A$$

where

$$\phi_m^F(\sigma, z) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell^2 \sigma / 4m} e^{2\pi i \ell z} h_{\ell}^{(-\frac{\ell}{2m}\sigma)}(\sigma),$$

and

$$h_{\ell}^{(-\frac{\ell}{2m}\sigma)}(\sigma) := e^{-2\pi i \ell^2 \sigma / 4m} \int_{-\frac{\ell}{2m}\sigma}^{-\frac{\ell}{2m}\sigma+1} dz e^{-2\pi i \ell z} \phi_m(\sigma, z).$$

Degeneracy formula revisited

For $(m, n, \ell) \in B$

$$d(m, n/N, \ell) = (-1)^{\ell+1} \int d\tau \int dz e^{-2\pi i m \tau} e^{-2\pi i \ell z} \psi_n^F(\tau, z) ,$$

where

$$\psi_n^F(\tau, z) = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell^2 N \tau / 4n} e^{2\pi i \ell z} \tilde{h}_\ell^{(-\frac{\ell N}{2n} \tau)}(\tau) ,$$

and

$$\tilde{h}_\ell^{(-\frac{\ell N}{2n} \tau)}(\tau) := e^{-2\pi i \tau \frac{\ell^2 N}{4n}} \int_{-\frac{\ell N}{2n} \tau}^{-\frac{\ell N}{2n} \tau + 1} \psi_n(\tau, z) e^{-2\pi i \ell z} dz .$$

Conclusion

- For $(m, n, \ell) \in A$, the degeneracy $d(m, n/N, \ell)$ is the corresponding Fourier coefficient of ϕ_m^F .
- For $(m, n, \ell) \in B$, the degeneracy $d(m, n/N, \ell)$ is the corresponding Fourier coefficient of ψ_n^F .

Generalization to $\mathbb{Z}_M$ -invariant states in $\mathbb{Z}_N$ CHL models [BSB (2025)].

Modularity

$\tilde{\Phi}_k$ is a Siegel modular form of weight k for some subgroup $G \subset \mathrm{Sp}(2, \mathbb{Z})$: for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$, and $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in \mathbb{H}_2$ (defined by requiring $\tau_2, \sigma_2 > 0, \tau_2\sigma_2 - z_2^2 > 0$)

$$\tilde{\Phi}_k((A\Omega + B)(C\Omega + D)^{-1}) = [\det(C\Omega + D)]^k \tilde{\Phi}_k(\Omega) .$$

Using this, one can prove that

$$\begin{aligned} \phi_m \left(\frac{a\sigma + b}{c\sigma + d}, \frac{z}{c\sigma + d} \right) &= (c\sigma + d)^{-k} e^{2\pi i \left(\frac{mcz^2}{c\sigma + d} \right)} \phi_m(\sigma, z) , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^1(N) \\ \psi_n \left(\frac{a\sigma + b}{c\sigma + d}, \frac{z}{c\sigma + d} \right) &= (c\sigma + d)^{-k} e^{2\pi i \left(\frac{ncz^2}{N(c\sigma + d)} \right)} \psi_n(\sigma, z) , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N) \end{aligned}$$

These are meromorphic Jacobi forms of weight $-k$ and index $m, n/N$ respectively.

Mock Modularity

ϕ_m^F, ψ_n^F are holomorphic **but not** modular. But there exists **nonholomorphic functions** ϕ_m^{F*}, ψ_n^{F*} such that the completion

$$\widehat{\phi}_m^F = \phi_m^F + \phi_m^{F*}, \quad \widehat{\psi}_n^F = \psi_n^F + \psi_n^{F*},$$

transform as a Jacobi form $\implies$ mock Jacobi forms [DMZ (2012), Bhand-Sen-RKS (2023)].

Compare with the simplest example:

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n \tau}.$$

Transformation

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{6c(c\tau + d)}{\pi i}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

But if we define $\widehat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}$, then

$$\widehat{E}_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \widehat{E}_2(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}).$$

Limitations

Limitations

- The single-centered degeneracy of all charges is not captured by mock Jacobi forms. For example in $N = 1$ case, $(10, 14, 5), (13, 19, 7) \notin A, B$ [Bhand-RKS-Sen (2023)]
- S-duality invariance of the degeneracy is obscure.
- Goal: construct the generating function of single-centered degeneracy which is manifestly duality invariant.

Generating Function of Single-Centered Degeneracy

The generating function

We want to construct the function

$$F(\tau, \sigma, z) = \sum_{\substack{m, n, \ell \in \mathbb{Z} \\ m, n \geq 0, 4mn - N\ell^2 \geq 0}} d(m, n/N, \ell) e^{2\pi i(m\tau + n\sigma/N + \ell z)},$$

where $\tau, \sigma \in \mathbb{H}$, $z \in \mathbb{C}$, $\tau_2\sigma_2 - z_2^2 > 0$ (Siegel upper half space $\mathbb{H}_2$)

- Obstacle: $d(m, n, \ell)$ is not defined when $4mn - N\ell^2 \leq 0$.
- When $4mn - N\ell^2 = 0$, deform the contour as follows [Bhand-Sen-RKS (2025)]

$$\operatorname{Im}(\tau) = \frac{2n + \varepsilon_1}{N\varepsilon}, \quad \operatorname{Im}(\sigma) = \frac{2m + \varepsilon_2}{\varepsilon}, \quad \operatorname{Im}(z) = -\frac{\ell + \varepsilon_3}{\varepsilon},$$

with $0 < |\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \ll 1$.

- Gives unambiguous answer independent of the way we deform the contour.
- Goal: show that this generating function converges in some domain of $\mathbb{H}_2$.

Strategy

- Relate $F(\tau, \sigma, z)$ to $\tilde{\Phi}_k$ and use analytic continuation.
- Subtract from the total degeneracy, the contribution of two-centered configurations.
- Focus on DMZ $N = 1$ case.

Two-centered configurations and wall crossing

- Type IIB on $K3 \times T^2$ has a moduli space parametrized by ρ, Ξ .
- For a given charge (m, n, ℓ) , the moduli space can be divided into chambers separated by codimension 1 walls called *walls of marginal stability* [Sen (2007)].
- The degeneracy of black holes jumps across these walls of marginal stability due to (dis)appearance of two centered black holes.
- Two centered solutions can be interpreted as bound state of two small black holes – black holes with charges $(Q, 0)$ or $(0, P)$.
- In terms of BPS states, two-centered black holes correspond to those $\frac{1}{4}$ -BPS states which are unstable against decay into half-BPS states.
- All two-centered configurations have charges of the form

$$Q = aM + cN, \quad P = bM + dN, \quad ad - bc = 1,$$

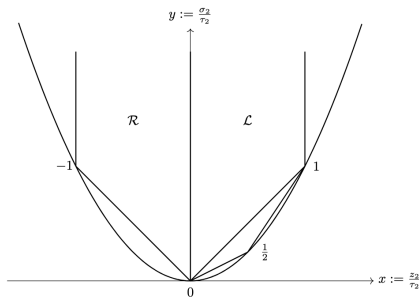
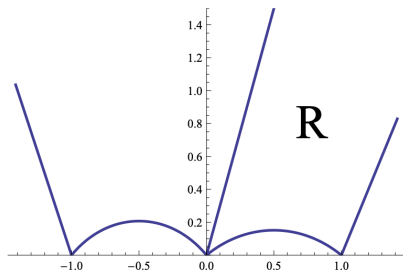
corresponding to the bound state of half BPS states with charges (aM, bM) and (cN, dN) which are S-duality equivalent to small black holes with charges $(M, 0), (0, N)$ [Dabholkar-Gaiotto-Nampuri (2007), Sen (2007)].

Walls of marginal stability and chambers

- For fixed value of Ξ , the walls of marginal stability in ρ -plane has nice form [Sen (2007)].
- On the other hand, the poles of Φ_{10}^{-1} with $n_2 = 0$ divides $\mathbb{H}_2$ into chambers:

$$-m_1 + n_1 (\sigma_2/\tau_2) + j (z_2/\tau_2) = 0, \quad m_1, n_1, j \in \mathbb{Z}, \quad m_1 n_1 + j^2/4 = 1/4$$

- There is a one-one correspondence between chambers in τ plane and chambers in $\mathbb{H}_2$ [Sen (2011)]



Bound state metamorphosis (BSM)

- Given a charge (Q, P) , there are precise conditions under which it exists as a bound state of half BPS states at a given point in the moduli space [Sen (2011)].
- When $Q^2 = -2$ or $P^2 = -2$, two apparently different bound state configurations corresponding to these charges must be identified [Sen (2011)].
- For $P^2 = -2, Q^2 \geq 0$ bound state of $(Q, 0), (0, P)$ must be identified with $(Q + (Q \cdot P)P, 0), (-(Q \cdot P)P, P)$.
- For $Q^2 = -2, P^2 \geq 0$ bound state of $(Q, 0), (0, P)$ must be identified with $(0, P + (Q \cdot P)Q), (Q, -(Q \cdot P)Q)$.
- For $P^2 = Q^2 = -2$, infinitely many identifications.
- These bound states exist only in the intersection of the chambers in which the individual bound states exist.
- Justification: Supergravity analysis [Chowdhuri-Lal-Saha-Sen (2012)].

Degeneracy of two-centered configuration

- $f(\tau) = \eta^{-24}(\tau) = \sum_{p=-1}^{\infty} f_p e^{2\pi i p \tau}$ is the generating function for the degeneracy of half BPS states.
- At a given point in the moduli space, if a two centered black hole with charge (Q, P) exists as a bound state of $(aM, bM), (cN, dN)$ then the degeneracy is thus given by

$$d^{\text{two}}(Q, P) = (-1)^{M \cdot N + 1} |M \cdot N| f_{M^2/2} f_{N^2/2} .$$

- The generating function in a given chamber is then given by

$$(e^{\pi i z} - e^{-\pi i z})^{-2} f(\tau) f(\sigma) .$$

Naive guess for generating function

To get only single-centered contributions, naively one would subtract $(e^{\pi iz} - e^{-\pi iz})^{-2} f(\tau) f(\sigma)$ and its $\mathrm{SL}(2, \mathbb{Z})$ -images from $1/\Phi_{10}$:

$$\frac{1}{\Phi_{10}} - \frac{1}{2} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z})} \left(e^{\pi i(ac\tau + bd\sigma + (ad+bc)z)} - e^{-\pi i(ac\tau + bd\sigma + (ad+bc)z)} \right)^{-2} \\ \times f(a^2\tau + b^2\sigma + 2abz) f(c^2\tau + d^2\sigma + 2cdz)$$

where $1/2$ compensates for the invariance of the summand under $(a, b, c, d) \rightarrow (-c, -d, a, b)$.



The subseries corresponding to $(a, b, c, d) = (1, 0, c, 1), c \in \mathbb{Z}$ has a divergent contribution:

$$\sum_{c \in \mathbb{Z}} (e^{\pi i(c\tau + z)} - e^{-\pi i(c\tau + z)})^{-2} e^{-2\pi i(c^2\tau + \sigma + 2cz)} f(\tau) .$$

Recognizing the two-centered contributions

Using the expansion

$$(e^{\pi iz} - e^{-\pi iz})^{-2} = \sum_{r=1}^{\infty} r (H(z_2) e^{2\pi i r z} + H(-z_2) e^{-2\pi i r z}) ,$$

we can write the subtraction term as

$$\begin{aligned} & -\frac{1}{2} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r > 0} r \left\{ e^{\pi i r (ac\tau + bd\sigma + (ad+bc)z)} H(ac\tau_2 + bd\sigma_2 + (ad+bc)z_2) \right. \\ & \quad \left. e^{-\pi i r (ac\tau + bd\sigma + (ad+bc)z)} H(-ac\tau_2 - bd\sigma_2 - (ad+bc)z_2) \right\} \\ & \quad \sum_{p, q \geq -1} f_p f_q e^{2\pi i p (a^2\tau + b^2\sigma + 2abz)} e^{2\pi i q (c^2\tau + d^2\sigma + 2cdz)} \end{aligned}$$

Focus on $p, q \geq 0$ terms.

Recognizing the two-centered contributions: $p, q \geq 0$

The $p, q \geq 0$ terms can be reorganized as

$$\begin{aligned} & -\frac{1}{2} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r>0} \sum_{p, q \geq 0} r H(ac\tau_2 + bd\sigma_2 + (ad+bc)z_2) f_p f_q \\ & \times e^{2\pi i \{ (pa^2+qc^2+rac)\tau + (pb^2+qd^2+rbd)\sigma + (2pab+2qcd+r(ad+bc))z \}} \\ & -\frac{1}{2} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r>0} \sum_{p, q \geq 0} r H(-ac\tau_2 - bd\sigma_2 - (ad+bc)z_2) f_p f_q \\ & \times e^{2\pi i \{ (pa^2+qc^2-rac)\tau + (pb^2+qd^2-rbd)\sigma + (2pab+2qcd-r(ad+bc))z \}} \end{aligned}$$

Recognizing the two-centered contributions: $p, q \geq 0$

The terms containing red exponents can be recognized as the contribution coming from the bound state of half-BPS states carrying charges (aM, bM) and (cN, dN) satisfying

$$M^2 = 2p, \quad N^2 = 2q, \quad M \cdot N = r, \quad r > 0,$$

so that the total charge $(Q, P) = (aM + cN, bM + dN)$ satisfies

$$Q^2 = 2(pa^2 + qc^2 + rac), \quad P^2 = 2(pb^2 + qd^2 + rbd), \\ Q \cdot P = 2pab + 2qbd + r(ad + bc).$$

Such bound states exist in the chamber $ac\tau_2 + bd\sigma_2 + (ad + bc)z_2 > 0$ for $r > 0$ [Sen (2011)].

Recognizing the two-centered contributions: $p, q \geq 0$

The terms containing blue exponents can be recognized as the contribution coming from the bound state of half-BPS states carrying charges (aM, bM) and (cN, dN) satisfying

$$M^2 = 2p, \quad N^2 = 2q, \quad M \cdot N = -r, \quad r > 0,$$

so that the total charge $(Q, P) = (aM + cN, bM + dN)$ satisfies

$$\begin{aligned} Q^2 &= 2(pa^2 + qc^2 - rac), \quad P^2 = 2(pb^2 + qd^2 - rbd), \\ Q \cdot P &= 2pab + 2qbd - r(ad + bc). \end{aligned}$$

Such bound states exist in the chamber $ac\tau_2 + bd\sigma_2 + (ad + bc)z_2 < 0$ for $r > 0$ [Sen (2011)].

Thus these subtractions are correct.

Recognizing the two-centered contributions:

$$p = -1, q \geq 0 \text{ and } p \geq 0, q = -1$$

Reorganize the sum to write

$$\begin{aligned} & - \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r > 0} \sum_{p \geq 0} f_p f_{-1} r H(ac\tau_2 + bd\sigma_2 + (ad + bc)z_2) \\ & \times e^{2\pi i \{ (pa^2 - c^2 + rac)\tau + (pb^2 - d^2 + rbd)\sigma + (2pab - 2cd + r(ad + bc))z \}} \\ & - \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r > 0} \sum_{p \geq 0} f_p f_{-1} r H(-ac\tau_2 - bd\sigma_2 - (ad + bc)z_2) \\ & \times e^{2\pi i \{ (pa^2 - c^2 - rac)\tau + (pb^2 - d^2 - rbd)\sigma + (2pab - 2cd - r(ad + bc))z \}} \end{aligned}$$

Used $(a, b, c, d) \rightarrow (-c, -d, a, b)$ and $p \leftrightarrow q$ symmetry to drop the $1/2$ and $f_{-1}f_q$ terms.

Recognizing the two-centered contributions:

$$p = -1, q \geq 0 \text{ and } p \geq 0, q = -1$$

These terms represent subtraction of the contribution from the bound states of half-BPS states carrying charges $(aM, bM), (cN, dN)$ satisfying

$$M^2 = 2p, \quad N^2 = -2, \quad M \cdot N = \textcolor{red}{r}, \textcolor{blue}{-r}, \quad r > 0,$$

so that the total charge $(Q, P) = (aM + cN, bM + dN)$ satisfies

$$Q^2 = 2(\textcolor{red}{p}a^2 - c^2 + \textcolor{red}{r}ac), \quad P^2 = 2(\textcolor{red}{p}b^2 - d^2 + \textcolor{red}{r}bd),$$

$$Q \cdot P = \textcolor{red}{2}pab - 2bd + \textcolor{red}{r}(ad + bc),$$

$$Q^2 = 2(\textcolor{blue}{p}a^2 - c^2 - \textcolor{blue}{r}ac), \quad P^2 = 2(\textcolor{blue}{p}b^2 - d^2 - \textcolor{blue}{r}bd),$$

$$Q \cdot P = \textcolor{blue}{2}pab - 2bd - \textcolor{blue}{r}(ad + bc).$$

Effect of BSM: $p = -1, q \geq 0; p \geq 0, q = -1$

- Bound state metamorphosis dictates that several terms in $p = -1, q \geq 0, p \geq 0, q = -1$ and $p = q = -1$ must be identified since they represent the same physical state.
- In the $p = -1, q \geq 0, p \geq 0, q = -1$ sum, changing (a, b, c, d) to $(a, b, c - ar, d - br)$ in the blue term gives the same exponent as the red term.
- BSM tells us to identify these two contributions and it is nontrivial only if it exists in both the chambers corresponding to (a, b, c, d) and $(a, b, c - ar, d - br)$.
- Thus we must replace the $p = -1, q \geq 0, p \geq 0, q = -1$ sum by

$$\begin{aligned}
 & - \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r > 0} r \sum_{p \geq 0} f_p f_{-1} H(ac\tau_2 + bd\sigma_2 + (ad + bc)z_2) \\
 & \times H(-ac\tau_2 - bd\sigma_2 - (ad + bc)z_2 + ra^2\tau_2 + rb^2\sigma_2 + 2rabz_2) \\
 & e^{2\pi i \{ (pa^2 - c^2 + rac)\tau + (pb^2 - d^2 + rbd)\sigma + (2pab - 2cd + r(ad + bc))z \}}
 \end{aligned}$$

Recognizing the two-centered contributions: $p = q = -1$

The subtraction term can be written as

$$-f_{-1}^2 \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r > 0} r H(ac\tau_2 + bd\sigma_2 + (ad + bc)z_2) \\ \times e^{2\pi i \{ (-a^2 - c^2 + rac)\tau + (-b^2 - d^2 + rbd)\sigma + (-2ab - 2cd + r(ad + bc))z \}}.$$

These terms represent subtraction of the contribution from the bound states of half-BPS states carrying charges $(aM, bM), (cN, dN)$ satisfying

$$M^2 = -2, \quad N^2 = -2, \quad M \cdot N = r, \quad r > 0,$$

so that the total charge $(Q, P) = (aM + cN, bM + dN)$ satisfies

$$Q^2 = 2(-a^2 - c^2 + rac), \quad P^2 = 2(-b^2 - d^2 + rbd), \\ Q \cdot P = -2ab - 2bd + r(ad + bc).$$

Effect of BSM: $p = q = -1$

- The exponent in the $p = q = -1$ sum remains unchanged under $(a, b, c, d) \rightarrow (a_n, b_n, c_n, d_n)$, where

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -r \end{pmatrix}^n \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad n \in \mathbb{Z}.$$

- So BSM tells us to identify these contributions in the sum.
- These contributions are nontrivial only if these bound states exist in all corresponding chambers.

Thus we must replace the sum by

$$\begin{aligned} & -f_{-1}^2 \sum_{r>0} r \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_r \setminus \mathrm{PSL}(2, \mathbb{Z})} \prod_{n=-\infty}^{\infty} H(a_n c_n \tau_2 + b_n d_n \sigma_2 + (a_n d_n + b_n c_n) z_2) \\ & \times e^{2\pi i \{ (-a^2 - c^2 + rac)\tau + (-b^2 - d^2 + rbd)\sigma + (-2ab - 2cd + r(ad + bc))z \}}, \end{aligned}$$

where G_r is the cyclic group generated by $\begin{pmatrix} 0 & -1 \\ 1 & -r \end{pmatrix}$.

Final expression

$$\tilde{F}(\tau, \sigma, z) = \frac{1}{\Phi_{10}} - S(\tau, \sigma, z) \text{ where [Bhand-Sen-RKS (2025)]}$$

$$S(\tau, \sigma, z) = \frac{1}{2} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} f_+(a^2\tau + b^2\sigma + 2abz) f_+(c^2\tau + d^2\sigma + 2cdz) \\ \times \left(e^{\pi i \{ac\tau + bd\sigma + (ad+bc)z\}} - e^{-\pi i \{ac\tau + bd\sigma + (ad+bc)z\}} \right)^{-2}$$

$$+ \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})} \sum_{r>0} r \sum_{p \geq 0} f_p f_{-1} H(ac\tau_2 + bd\sigma_2 + (ad+bc)z_2) \\ \times H(-ac\tau_2 - bd\sigma_2 - (ad+bc)z_2 + ra^2\tau_2 + rb^2\sigma_2 + 2rabz_2) \\ \times e^{2\pi i \{(pa^2 - c^2 + rac)\tau + (pb^2 - d^2 + rbd)\sigma + (2pab - 2cd + r(ad+bc))z\}}$$

$$+ f_{-1}^2 \sum_{r>0} \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gr \setminus \text{PSL}(2, \mathbb{Z})} \prod_{n=-\infty}^{\infty} H(a_n c_n \tau_2 + b_n d_n \sigma_2 + (a_n d_n + b_n c_n) z_2) \\ \times e^{2\pi i \{(-a^2 - c^2 + rac)\tau + (-b^2 - d^2 + rbd)\sigma + (-2ab - 2cd + r(ad+bc))z\}}$$

Analytic Properties

Properties of $\tilde{F}$

Theorem (Convergence of the sum) [Bhand-Sen-RKS (2025)]

The sum over a, b, c, d, r in S converges absolutely and uniformly on compact subsets of $\mathbb{H}_2$ except on the hypersurface

$$m_1\tau - n_1\sigma + m_2 + jz = 0, \quad m_1, n_1, m_2 \in \mathbb{Z}, \quad m_1n_1 + \frac{j^2}{4} = \frac{1}{4},$$

where it has double poles with the same residue as Φ_{10}^{-1} .

Holomorphicity of $\tilde{F}$ [Bhand-Sen-RKS (2025)]

$\tilde{F}$ is holomorphic in the domain $\det \operatorname{Im} \Omega > \frac{1}{4}$, $\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$ and has double poles on the hypersurface

$$n_2(\tau\sigma - z^2) + m_1\tau - n_1\sigma + m_2 + jz = 0,$$

$$m_1, n_1, m_2, n_2 \in \mathbb{Z}, \quad n_2 \geq 1, \quad m_1n_1 + m_2n_2 + \frac{j^2}{4} = \frac{1}{4}.$$

Main Theorem

Analytic continuation of F [Bhand-Sen-RKS (2025)]

We have $F = \tilde{F}$ and hence the generating function $F(\tau, \sigma, z)$ admits a meromorphic continuation to $\mathbb{H}_2$.

The generating function is thus manifestly S-duality invariant:

$$F(\gamma\Omega\gamma^t) = F(\Omega) \ , \quad \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in \mathbb{H}_2 \ , \quad \gamma \in \mathrm{SL}(2, \mathbb{Z}) \ .$$

Questions For Future

- Does there exist a nonholomorphic function F^* such that $F + F^*$ transforms as a Siegel modular form for $\mathrm{Sp}(2, \mathbb{Z})$?
- Is $F + F^*$ harmonic with respect to some Laplacian?
- If these two statements are true, then following the standard terminology of mock modular forms, we may call F a mock Siegel modular form.
- Generalize the story to CHL models $N > 1$.

Thank You