Introduction to String Theory

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Note To The Reader

These notes are a work in progress and will be updated frequently. The updated notes can be accessed at the link

https://ranveer14.github.io/String_Theory_notes.pdf

These notes are based on several references, but mostly I am following [8, 9, 2, 13]. I have also included discussions from standard references when discussing other topics. For example, in conformal field theory, I have included topics from [4, 3]. Specific references are indicated in the text wherever used. I greatly benefited from the very detailed calculations and articulate explanations in [12].

If you find typos/corrections, please send them to my email ranveersfl@gmail.com.

Thank you
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Chapter 1

The Free Relativistic Particle

In this chapter, we describe the various methods of quantising a free particle. The discussion is based on the video lectures of Shiraz Minwalla [7]. Let $\mathbb{R}^{1,D-1}$ denote the $D-$dimensional spacetime. We denote a typical vector $X^\mu \in \mathbb{R}^{1,D-1}$ by

$$X^\mu = (X^0, X^i)$$

where $(X^i) \in \mathbb{R}^{D-1}$. We sometimes write $X^\mu = (X^0, \vec{X})$. Our signature for the Minkowski space is

$$\eta_{\mu\nu} = \text{diag}(-1, 1, \dots, 1).$$

We use natural units $\hbar = c = 1$. Consider a free particle moving in $D-$dimensional spacetime. We want to describe its dynamics. We want our theory to be relativistic, meaning that the theory that we develop must be invariant under Lorentz transformation.

1.1 The Action of a Free Relativistic Particle

We begin by writing down a Lorentz invariant action for the free particle. The most natural choice for a Lorentz invariant action is choosing a Lorentz scalar and the cannonical choice is to chose the length of the world line traced by the particle in spacetime. Put

$$S = \int dt\ L = -m \int dt \sqrt{1 - \dot{\vec{X}}^2}, \quad X^0 = t, \dot{\vec{X}} = \frac{d\vec{X}}{dt}, \tag{1.1.1}$$

where m is a parameter which we will identify with the mass of the particle. We can now compute the conjugate momentum of the system in the usual way. We have

$$P^i = \frac{\delta S}{\delta \dot{X}^i} = \frac{-m(2\dot{X}^i)}{-2\sqrt{1 - \dot{\vec{X}}^2}} = \frac{m\dot{X}^i}{\sqrt{1 - \dot{\vec{X}}^2}}.$$

The Hamiltonian of the system is given by

$$H = \vec{P} \cdot \dot{\vec{X}} - L = \frac{m\dot{\vec{X}}^2}{\sqrt{1-\dot{\vec{X}}^2}} + m\sqrt{1-\dot{\vec{X}}^2} = \frac{m}{\sqrt{1-\dot{\vec{X}}^2}},$$

which we recognise as the usual relativistic energy of a free particle and hence m is identified as the mass of the particle. Note that the action is not manifestly Lorentz invariant as we are treating the first component of the spacetime vector differently from the remaining components. But we want an action which is manifestly Lorentz invariant. One way to obtain such an action is to promote t to be an independent variable and then parametrize the spacetime coordinates by some other parameter say τ. So put $t = X^0$ and parametrize $X^\mu = (X^0, X^i)$ as

$$X^\mu = X^\mu(\tau).$$

By a simple application of chain rule, we have

$$dt = \frac{dX^0}{d\tau} d\tau.$$

The action can then be written as

$$\begin{aligned} S = -m \int d\tau \sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{dt}\right)^2 \left(\frac{dX^0}{d\tau}\right)^2} &= -m \int d\tau \sqrt{-\left[-\left(\frac{dX^0}{d\tau}\right)^2 + \left(\frac{d\vec{X}}{d\tau}\right)^2\right]} \\ &= -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}. \end{aligned}$$

Remark 1.1.1. It seems that we have added a new degree of freedom to our system, namely X^0. Later we will see that this is not the case as our system will have *reparametrization invariance* also called *diffeomorphism invariance* which will make one of the degree of freedom redundant.

It is now clear that S can be interpreted as the length of the worldline it traces in spacetime.

1.2 Symmetries of the Action

Let us now look at the symmetries of our system:

1.2.1 Poincaré invariance

This is a manifest global symmetry of the system.

$$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + \xi^\mu, \quad \Lambda^\mu{}_\nu \in \mathrm{SO}(1, D-1), \xi^\mu \in \mathbb{R}^{1,D-1},$$

where $\mathrm{SO}(1, D-1)$ denotes the Lorentz group. We have

$$\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}_\mu}{d\tau} = \eta_{\mu\nu}\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}^\nu}{d\tau} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\frac{dX^\rho}{d\tau}\Lambda^\nu{}_\sigma\frac{dX^\sigma}{d\tau} = \left(\Lambda^\mu{}_\rho\eta_{\mu\nu}\Lambda^\nu{}_\sigma\right)\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau} = \eta_{\rho\sigma}\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau} = \frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau},$$

where we used the property of Lorentz transformations

$$\Lambda^T\eta\Lambda = \eta. \tag{1.2.1}$$

This implies that

$$\widetilde{S} = -m\int d\tau\sqrt{-\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}_\mu}{d\tau}} = -m\int d\tau\sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}} = S.$$

We could have directly concluded this by the fact that the action is the length of a curve and hence a Lorentz scalar. So it does not transform under Lorentz transformations.

1.2.2 Diffeomorphism Invariance

We can reparametrize the world line by changing the parameter τ:

$$\tau \to \widetilde{\tau} = \widetilde{\tau}(\tau).$$

where $\widetilde{\tau}(\tau)$ is a monotonic function[1] of τ. The integration measure changes according to the usual change of variable rule. Next under reparametrization we have

$$\widetilde{X}^\mu(\widetilde{\tau}(\tau)) = X^\mu(\tau).$$

Hence we see that the transformed action is

$$S = -m\int d\widetilde{\tau}\left|\frac{d\tau}{d\widetilde{\tau}}\right|\sqrt{-\frac{d\widetilde{X}^\mu}{d\widetilde{\tau}}\frac{d\widetilde{X}_\mu}{d\widetilde{\tau}}\left(\frac{d\widetilde{\tau}}{d\tau}\right)^2} = -m\int d\widetilde{\tau}\left|\frac{d\tau}{d\widetilde{\tau}}\frac{d\widetilde{\tau}}{d\tau}\right|\sqrt{-\frac{d\widetilde{X}^\mu}{d\widetilde{\tau}}\frac{d\widetilde{X}_\mu}{d\widetilde{\tau}}} = \widetilde{S}.$$

This is a local symmetry of the theory - a gauge symmetry as it depends on the the local coordinates of the spacetime. It is also a continuous symmetry of the action. As is well know, gauge symmetries are not really symmetries in the sense that we do not have an associated conserved charge, rather it is a redundancy in the description of the theory which we need to fix when we go to quantum theory by a process called gauge fixing. We now return to the

[1] monotonicity is a techinical requirement for reparametrization. Basically what we need is that as we increase τ, we should traverse the worldline in one given direction and not flip between positions. Generally, $\widetilde{\tau}$ is assumed to be increasing so that we travel the worldline in the same direction as in the original parametrization.

resolution of Remark 1.1.1. Since the time component of spacetime vector is monotonically increasing, we can reparametrize the worldline in such a way that

$$\widetilde{\tau} = X^0(\tau) = t.$$

Fixing the redundancy of the system we get back to our original action. This shows that we have not increased the number of degrees of freedom of the theory by introducing a parameter.

1.3 Quantisation

We will now try to quantise the system. We will illustrate four different methods of quantisation, each with its own advantage. This will help us when we go to the string action.

1.3.1 First Method

We quantise our original action in (1.1.1) directly using the Dirac prescription. The conjugate momentum and the Hamiltonian was calculated to be

$$P^i = \frac{m\dot{X}^i}{\sqrt{1-\dot{\vec{X}}^2}}, \quad H = \frac{m}{\sqrt{1-\dot{\vec{X}}^2}}.$$

where the dot represents derivative with respect to t. We promote the fields to operators with the standard substitution $P^i = -i\partial_i$ where $\partial_i = \frac{\partial}{\partial X^i}$ and introduce the wavefunction which satisfies the Schrödinger equation with the above Hamiltonian. Let $\phi(t, X^i)$ be the wavefunction. Then the Schroödinger equation is given by

$$i\frac{\partial \phi}{\partial t} = H\phi.$$

This implies that

$$-\frac{\partial^2 \phi}{\partial t^2} = H^2\phi.$$

Next, one can easily check that

$$H^2 = \vec{P}^2 + m^2.$$

Thus the Schroödinger equation becomes

$$-\frac{\partial^2 \phi}{\partial t^2} = (-\partial_i^2 + m^2)\phi$$

which implies

$$(\partial_\mu \partial^\mu - m^2)\phi = 0. \tag{1.3.1}$$

We can now solve (1.3.1) and get all the quantum dynamics of the system.

Remark 1.3.1. We can recognise (1.3.1) with the Klein-Gordan equation in field theory. There is one crucial difference in our case and the field theory Klein-Gordan equation. In field theory we quantise quantum fields while in our case (relativistic quantum mechanics), we quantise wavefunctions.

1.3.2 Second Method

We will now denote the τ derivative by dot. That is

$$\dot{X}^\mu = \frac{dX^\mu}{d\tau}.$$

Momentum conjugate to X^μ is

$$P^\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}.$$

One easily sees that

$$P^\mu P_\mu + m^2 = 0. \tag{1.3.2}$$

Eq. (1.3.1) is a constraint. Note that we have not yet appealed to the equation of motion of the action to derive Eq. (1.3.1). Such constraints which follow directly from the definition of the conjugate momenta are called *primary constraints*. The number of primary constraints in a system is equal to the number of zero eigenvalues of the Hessian matrix

$$\frac{\partial P^\mu}{\partial \dot{X}^\nu} = \frac{\partial^2 L}{\partial \dot{X}^\mu \dot{X}^\nu}.$$

Note that by the Inverse Function Theorem we need that all eigenvalues of $\frac{\partial P^\mu}{\partial \dot{X}^\nu}$ be nonzero if we want to express P^μ as a function of $\dot{X}^\mu$. Hence in a system with primary constraint, we cannot express P^μ as functions of $\dot{X}^\mu$.

Remark 1.3.2. Any system with "τ"$-$reparametrization invariance has primary constraints.

The Hamiltonian for the system is

$$H = P^\mu \dot{X}_\mu - L = \frac{m\dot{X}^\mu \dot{X}_\mu}{\sqrt{-\dot{X}^\nu \dot{X}_\nu}} + m\sqrt{-\dot{X}^\nu \dot{X}_\nu} = 0.$$

This is not surprising. Vanishing Hamiltonian signals that nothing changes if we pick another parametrization. To quantise the system, we follow Dirac prescription. We promote the fields to operators and the constraint to an operator equation and demand that the wavefunction $|psi(X)$ satisfy the operator equation:

$$(P^\mu P_\mu + m^2)\Psi(X) = 0.$$

The Schrödinger equation is

$$i\frac{\partial \Psi}{\partial \tau} = H\Psi = 0.$$

This simply implies that the wavefunction does not depend on the parametrization - something that we expected. After the standard substitution

$$P^\mu = -i\partial_\mu, \quad \text{where } \partial_\mu = \frac{\partial}{\partial X^\mu},$$

the operator equation (1.3.1) becomes

$$(\partial_\mu \partial^\mu - m^2)\Psi = 0. \tag{1.3.3}$$

This is the same equation that we got in the first method. Hence we again get the same dynamics.

1.3.3 Third Method - Introducing Einbein

Note that both the equivalent actions above have squareroots which makes it difficult to quantise when we go to path-integral quantisation. So, we somehow want to get rid of the squareroot. Moreover the two previous actions cannot be generalised to massless particles due to the m factor in front of the action. Both these problems can be fixed on the expense of introducing another auxiliary field - an einbein in the action which will be fixed by its equation of motion in the classical theory. To be more precise, consider the action

$$S_e = \frac{1}{2}\int d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{e} - em^2\right),$$

where $e = e(\tau)$ is the auxiliary einbein field. Varying the action with respect to e gives

$$\delta S = \frac{1}{2}\int d\tau \left(-\frac{\dot{X}^\mu \dot{X}_\mu}{e^2} - m^2\right)\delta e.$$

Thus $\delta S = 0$ implies

$$e = \frac{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}{m}. \tag{1.3.4}$$

Not that the equation of motion of e is an algebraic equation and hence the field e is not dynamical. If we now plug the expression for e from (1.3.4) in the action S_e we get

$$\begin{aligned} S_e = \frac{1}{2}\int d\tau \left(-m\frac{\dot{X}^\mu \dot{X}_\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}} - m^2\frac{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}{m}\right) &= -\frac{2m}{2}\int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \\ &= S. \end{aligned}$$

So both the actions are really the same. Thus the two actions are equivalent classically and give the same dynamics. We now want to quantise this action. The conjugate momentum corresponding to e is

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0.$$

The momentum conjugate to X^μ is

$$P^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \frac{2}{2e}\dot{X}^\mu \implies \dot{X}^\mu = eP^\mu. \tag{1.3.5}$$

The Hamiltonian of the system is given by

$$H = \dot{X}^\mu P_\mu - L = eP^\mu P_\mu - \frac{m}{2e}e^2 P^\mu P_\mu + \frac{m^2}{2}e = \frac{e}{2}(P^\mu P_\mu + m^2),$$

where we used (1.3.5). The Poisson bracket

$$\{P_e, H\}_{P.B.} = \frac{\partial P_e}{\partial P_e}\frac{\partial H}{\partial e} = \frac{1}{2}(P^\mu P_\mu + m^2).$$

But since $P_e = 0$, we get

$$H = \frac{e}{2}(P^\mu P_\mu + m^2) = 0. \tag{1.3.6}$$

The next step in the quantisation process is to promote fields to operators and use the standard operator substitution for P^μ. Suppose the wavefunction of the system is $\psi = \psi(X, e)$. Then the operator eqaution corresponding to $P_e = 0$ implies

$$-i\frac{\partial \psi}{\partial e} = 0.$$

This means that the wavefunction is independent of the einbein - again something that we expected physically. The operator equation corresponding to (1.3.6) gives

$$(P^\mu P_\mu + m^2)\psi = 0 \implies (\partial_\mu \partial^\mu - m^2)\psi = 0.$$

Thus we get the same quantum dynamics as in the previous two methods.

1.3.4 Fourth Method - Gauge Fixing

We begin by observing that S_e has diffeomorphism symmetry. Indeed if we choose another parametrization $\widetilde{\tau} = \widetilde{\tau}(\tau)$, then

$$X^\mu(\tau) \to \widetilde{X}^\mu(\widetilde{\tau}(\tau)) = X^\mu(\tau)$$

and

$$\frac{\partial X^\mu}{\partial \tau} = \frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{\tau}}{\partial \tau}$$

and using (1.3.4)

$$e(\tau) = \sqrt{-\frac{\partial \widetilde{X}^\mu}{\partial \tau}\frac{\partial \widetilde{X}_\mu}{\partial \tau}} = \sqrt{-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}}\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right| = \widetilde{e}(\widetilde{\tau})\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|.$$

Thus we see that

$$\begin{aligned} S_e &= \frac{1}{2}\int d\widetilde{\tau}\left|\frac{\partial \tau}{\partial \widetilde{\tau}}\right|\left(-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}\left(\frac{\partial \widetilde{\tau}}{\partial \tau}\right)^2\frac{1}{\widetilde{e}(\widetilde{\tau})}\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|^{-1} - m^2\widetilde{e}(\widetilde{\tau})\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|\right) \\ &= \frac{1}{2}\int d\widetilde{\tau}\left(-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}\frac{1}{\widetilde{e}(\widetilde{\tau})} - m^2\widetilde{e}(\widetilde{\tau})\right) \\ &= \widetilde{S}_e. \end{aligned}$$

So before we go on quantising the system, we will fix a gauge. We choose a reparametrization $\widetilde{\tau}(\tau)$ such that

$$\widetilde{e}(\widetilde{\tau}) = 1.$$

With this gauge choice, when we go to the quantum theory, we will have to take care of the equation of motion of the einbein and impose it as operator equation with the chosen gauge. We follow the standard procedure of quatization by promoting fields to operators. The equation of motion for e with the above gauge choice becomes:

$$\widetilde{e}(\widetilde{\tau})^2 = -\frac{1}{m^2}\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}} = 1.$$

Using (1.3.5), the equation of motion of einbein becomes

$$P^\mu P_\mu + m^2 = 0. \tag{1.3.7}$$

With the chosen gauge, the action becomes

$$S = \frac{1}{2}\int d\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2\right),$$

where we removed the tildes for brevity. Using (1.3.6) and the gauge choice along with (1.3.7), the Hamiltonian is given by

$$H = \frac{1}{2}(P^\mu P_\mu + m^2) = 0.$$

With the standard substitution for the momentum operator $P_\mu = -i\partial_\mu$, the wavefunction ψ of the system satisfies

$$(\partial_\mu \partial^\mu - m^2)\psi = 0.$$

Hence we again get the same dynamics.

Chapter 2

The Relativistic String

We now want to write an action of the a free relativistic string - the fundamental objects in string theory. As we discussed in the previous chapter, we need to start with a Lorentz invariant action. Since the string is a two dimensional object, it traces a surface called the *worldsheet* in the spacetime. The most natural choice of the action would then be the surface area of the worldsheet traced by the string. We begin by deriving the action of the relativistic string.

2.1 Nambu-Goto Action

The surface traced by the string can be parametrized by two parameters (σ, τ). Let the worldsheet coordinates be $X^\mu(\sigma, \tau)$. To calculate the area of the worldsheet, we will use the worldsheet coordinates. Infinitesimal change in the parameters σ and τ along the worldsheet coordinates is

$$\delta\sigma = \frac{\partial X^\mu}{\partial \sigma} d\sigma, \quad \delta\tau = \frac{\partial X^\mu}{\partial \tau} d\tau.$$

Note that the area of the parallelogram determined by two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \|\mathbf{A}\|\|\mathbf{B}\| \sin\theta = \|\mathbf{A}\|\|\mathbf{B}\|\sqrt{1-\cos^2\theta} &= \sqrt{\mathbf{A}^2\mathbf{B}^2 - \frac{(\mathbf{A}\cdot\mathbf{B})^2}{\mathbf{A}^2\mathbf{B}^2}\mathbf{A}^2\mathbf{B}^2} \\ &= \sqrt{(\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2} \\ &= \left(\det\begin{bmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} \\ \mathbf{A}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{B}\end{bmatrix}\right)^{\frac{1}{2}}, \end{aligned}$$

where $\|\mathbf{A}\|^2 = \mathbf{A}\cdot\mathbf{A} = \mathbf{A}^2$. So the infinitesimal area of the parallelogram on the worldsheet determined by the vectors $\delta\sigma$ and $\delta\tau$ is

$$d\text{Area} = \left[-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)\right]^{\frac{1}{2}}, \qquad \boldsymbol{\sigma} = \sigma^\alpha \equiv (\sigma, \tau), \quad \alpha = 1, 2.$$

The minus sign indicates the fact that one of the vectors is timelike ($X^2 < 0$). The Nambu-Goto action is then defined by

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma d\tau \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)}, \tag{2.1.1}$$

where M is the surface traced by the string, α' is called the *Regge slope*. The reason for this name will be evident in later chapters. We often write

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu.$$

The action can then be written as

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma d\tau \mathcal{L}_{NG}, \quad \mathcal{L}_{NG} = \left[-\det(h_{\alpha\beta})\right]^{\frac{1}{2}}.$$

The worldsheet is in general a curved manifold embedded in spacetine. In the language of differential geometry, $h_{\alpha\beta}$ is called the *pullback metric* from the ambient spacetime. The factor of $\frac{1}{2\pi\alpha'}$ can be interpreted as string tension.

2.1.1 Symmetries of the Nambu-Goto action

S_{NG} has global symmetries as well as local symmetries. Let us look at them more closely.

Reparametrization Invariance

If we choose another parametrization for the worldsheet $\widetilde{\tau}(\sigma,\tau), \widetilde{\sigma}(\sigma,\tau)$ then the Jacobian of the variable change is

$$J = \det\begin{bmatrix} \frac{\partial\widetilde{\sigma}}{\partial\sigma} & \frac{\partial\widetilde{\sigma}}{\partial\tau} \\ \frac{\partial\widetilde{\tau}}{\partial\sigma} & \frac{\partial\widetilde{\tau}}{\partial\tau} \end{bmatrix}$$

and the worldsheet coordinates changes as

$$\frac{\partial X^\mu}{\partial\sigma^\alpha} = \frac{\partial\widetilde{X}^\mu}{\partial\widetilde{\sigma}^\beta}\frac{\partial\widetilde{\sigma}^\beta}{\partial\sigma^\alpha}.$$

This gives

$$h_{\alpha\beta} = \frac{\partial\widetilde{X}^\mu}{\partial\widetilde{\sigma}^\gamma}\frac{\partial\widetilde{X}_\mu}{\partial\widetilde{\sigma}^\delta}\frac{\partial\widetilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\widetilde{\sigma}^\delta}{\partial\sigma^\beta} = \widetilde{h}_{\alpha\beta}\frac{\partial\widetilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\widetilde{\sigma}^\delta}{\partial\sigma^\beta}.$$

Thus we have

$$\det\left(h_{\alpha\beta}\right) = \det\left(\widetilde{h}_{\alpha\beta}\right) J^2,$$

where we used the fact that $J = \det\left(\frac{\partial\widetilde{\sigma}^\alpha}{\partial\sigma^\beta}\right)$. Plugging everything in the action, we see that

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\widetilde{\sigma} d\widetilde{\tau} J^{-1}\left(-\det\left(\widetilde{h}_{\alpha\beta}\right)\right)^{\frac{1}{2}} J = \widetilde{S}_{NG}.$$

Reparametrization invariance is also called *diffeomorphism* invariance and is a *gauge symmetry* of the action. We can write the infinitesimal version of the reparametrization as follows:

$$\sigma^\alpha \to \widetilde{\sigma}^\alpha = \sigma^\alpha + \xi^\alpha + O(\xi^2). \tag{2.1.2}$$

Under this change in parameter we have

$$X^\mu(\sigma^\alpha) \to \widetilde{X}^\mu(\widetilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha).$$

We have

$$\widetilde{X}^\mu(\widetilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha) = X^\mu(\widetilde{\sigma}^\alpha - \xi^\alpha) = X^\mu(\widetilde{\sigma}^\alpha) - \xi^\alpha \partial_\alpha X^\mu,$$

where we used Taylor's theorem. This gives

$$\delta X^\mu = \widetilde{X}^\mu(\widetilde{\sigma}^\alpha) - X^\mu(\widetilde{\sigma}^\alpha) = -\xi^\alpha \partial_\alpha X^\mu \tag{2.1.3}$$

Poincaré Invariance

The worldsheet coordinates transform under the Poincaré transformation as follows:

$$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu, \tag{2.1.4}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and c^μ is a constant vector. This is a manifest symmetry of the action. Poincaré invariance is a global symmetry of the action. The infinitesimal version is often calculated in a first course in quantum field theory. We will record it here for later use.

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu, \quad (a_{\mu\nu} = -a_{\nu\mu}). \tag{2.1.5}$$

2.1.2 Equations of Motion

We begin by expanding out the determinant in the action. We get

$$\det(h_{\alpha\beta}) = \det(\partial_\alpha X^\mu \partial_\beta X_\mu) = X'^2 \dot{X}^2 - (X' \cdot \dot{X})^2,$$

where

$$X' = \frac{\partial X}{\partial \sigma} \quad ; \quad \dot{X} = \frac{\partial X}{\partial \tau} \quad \& \quad X^2 = X^\mu X_\mu.$$

So we have

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[-X'^2 \dot{X}^2 + \left(X' \cdot \dot{X} \right)^2 \right]^{\frac{1}{2}},$$

where we have written $X \equiv X^\mu$. The conjugate momenta are given by

$$\Pi^\tau_\mu = \frac{\partial L_{NG}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{\left(\dot{X} \cdot X' \right) X'_\mu - (X'^2) X_\mu}{\sqrt{\left(X' \cdot \dot{X} \right)^2 - (X'^2 \dot{X}^2)}} \right]$$

$$\Pi^\sigma_\mu = \frac{2\mathcal{L}_{NG}}{\partial X'^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{\left(\dot{X} \cdot X' \right) \dot{X}_\mu - (X'^2) \dot{X}_\mu}{\sqrt{\left(X' \cdot \dot{X} \right)^2 - \left(X'^2 \dot{X}^2 \right)}} \right].$$

Observe that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot \dot{X}^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} \dot{X}^\nu = 0,$$
$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot X'^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} X'^\nu = 0.$$

So the Hessian $\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$ has two zero eigenvalues with eigenvectors $\dot{X}^\mu, X'^\mu$ must have two constraints. We can check that

$$\Pi^\tau_\mu X'^\mu = 0, \quad \Pi^\tau_\mu \Pi^{\tau\mu} + \frac{1}{4\pi^2 \alpha'^2} X'^\mu X'_\mu = 0. \tag{2.1.6}$$

These are one set of constraints. Another set of constraints arise from the fact that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^2} \dot{X}^\nu = 0 \quad \text{and} \quad \frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^\nu} X'^\nu = 0.$$

The resulting constraints are

$$\Pi^\sigma_\mu \dot{X}^\mu = 0, \quad \Pi^\sigma_\mu \Pi^{\sigma\mu} + \frac{1}{4\pi^2 \alpha^2} \dot{X}^\mu \dot{X}_\mu = 0. \tag{2.1.7}$$

The Hamiltorian

$$\mathcal{H}^\sigma = \Pi^\sigma_\mu X'^\mu - \mathcal{L}_{NG} = 0; \quad \& \quad \mathcal{H}^\tau = \Pi^\tau_\mu \dot{X}^\mu - \mathcal{L}_{NG} = 0.$$

So the dynamics is determined by constraints. The equation of motion is given by

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0. \tag{2.1.8}$$

We can also write the equation of motion in another way. Recall that

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\sigma d\tau \sqrt{-h}; \quad h = \det h_{\alpha\beta}.$$

From general relativity we have

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h} h^\alpha \beta \delta h_{\alpha\beta}$$

So

$$\frac{\delta \mathcal{L}_{NG}}{\delta\left(\partial_\alpha X^\mu\right)} = -\frac{1}{2\pi\alpha'}\left(\frac{1}{2}\sqrt{-h} h^{\alpha\beta}\left(2\partial_\beta X_\mu\right)\right)$$

So equation of motion is

$$\partial_\alpha\left(\frac{\partial \mathcal{L}_{NG}}{\partial\left(\partial_\alpha X^\mu\right)}\right) = 0$$

which gives

$$\partial_\alpha\left(\sqrt{-h} h^{\alpha\beta}\left(\partial_\beta X_\mu\right)\right) = 0.$$

2.2 The Polyakov Action

The final goal of studying string action is to quantise the action and analyse the spectrum that we obtain. The first challenge that we face when we try to quantise the Nambu-Goto action is the squareroot in the action. It is generally tricky to quantise such complicated actions when we go to path integral quantisation. This is why we will use the fourth method of quantisation introduced in Chapter 1. To this end, consider the following action:

$$S_p = -\frac{1}{4\pi\alpha'}\int_M d\sigma d\tau \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{2.2.1}$$

where $g = \det(g^{\alpha\beta})$ and $g^{\alpha\beta}$ is an auxiliary background field which plays the role of the einbein in the fourth method of quantisation. This action is called the Polyakov action. The auxiliary field $g_{\alpha\beta}$ is a dynamical metric on the world-sheet with Lorentzian signature $(-,+)$. Thus the action S_P can be viewed as a bunch of scalar fields $X^\mu(\sigma,\tau)$ coupled to a $2d$ gravity theory.

2.2.1 Equivalence of S_P and S_{NG}

Let us find the equations of motion of $g_{\alpha\beta}$. Varying S_P with respect to $g_{\alpha\beta}$ gives two terms. We get

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \left[\sqrt{-g}\delta g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X\mu - \frac{1}{2}\sqrt{-g} g_{\alpha\beta}\delta g^{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X\mu\right].$$

So

$$\delta S_P = 0 \implies \sqrt{-g}\delta g^{\alpha\beta}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X_\mu\right) = 0.$$

Here we used

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\alpha\beta}\delta g^{\alpha\beta} = \frac{1}{2}\sqrt{-g} g^{\alpha\beta}\delta g_{\alpha\beta}.$$

So the equation of motion of $g_{\alpha\beta}$ is

$$\partial\alpha X^\mu \partial\beta X_\mu = \frac{1}{2} g_{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X_\mu.$$

Or

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\partial^c X^\mu \partial_c X_\mu. \tag{2.2.2}$$

Taking determinant both sides we get

$$\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right) = \det\left(\frac{1}{2} g_{\alpha\beta}\partial^c X^\mu \partial_c X_\mu\right).$$

Since $g_{\alpha\beta}$ is 2×2, we get

$$\begin{aligned}\det\left(\partial\alpha X^\mu \partial_\beta X_\mu\right) &= \frac{1}{4}\left(\partial^c X^\mu \partial_c X_\mu\right)^2 g \\ \implies \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)} &= \frac{1}{2}\sqrt{-g}\left(\partial^c X^\mu \partial_c X_\mu\right).\end{aligned}$$

Substituting this in S_P gives S_{NG}. Thus we see that S_P and S_{NG} are equivalent classically. These two actions presumably gives same quantum dynamics but a rigorous proof is lacking. Indeed path-integral quantisation of S_{NG} is rather difficult to perform due to squareroot and manipulating it to get results involves similar tricks as we have used in the $S_{NG} \longrightarrow S_P$ transition.

2.2.2 Equation of Motion

Let us vary the action with respect to X^μ with $\delta X^\mu(\sigma, \tau_0) = \delta X^\mu(\sigma, \tau_1) = 0$ for some initial and final value τ_0, τ_1 respectively of the parameter τ. Assuming that the string length is ℓ, we have

$$
\begin{aligned}
\delta S_P &= -\frac{1}{4\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left[2\partial_\alpha X^\mu \left(\partial^\alpha \delta X_\mu\right)\right] \\
&= -\frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left[\partial_\alpha \left(\partial^\alpha X^\mu \delta X_\mu\right) - \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu\right] \\
&= \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu - \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\sigma \left(\partial^\sigma X^\mu \delta X_\mu\right) - \partial_\tau \left(\partial^\tau X^\mu \delta X_\mu\right)\right) \\
&= \frac{1}{2\pi\alpha} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu + \underbrace{\frac{1}{2\pi\alpha'} \int_0^\ell d\sigma \left(\partial^\tau X^\mu\right) \delta X_\mu \Big|_{\tau_0}^{\tau_1}}_{= 0 \text{ as } \delta X^\mu(\sigma, \tau_0) = \delta X^\mu(\sigma, \tau_1) = 0} \\
&\qquad + \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \underbrace{\left(\partial^\sigma X^\mu \delta X_\mu\right) \Big|_0^\ell}_{\text{surface term}} .
\end{aligned}
$$

To get the equations of motion, we need the surface term to go to zero. Physically we distinguish between two cases - the closed string and the open string. We will deal with the two cases separately.

Closed Strings

We normalise the string length so that $\ell = 2\pi$. Closed string then means that the ends of the string are joined together in a smooth fashion to form a loop. This means that $X^\mu(\sigma, \tau)$ are periodic in σ with period 2π :

$$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau).$$

This implies that $\delta X_\mu(0,\tau) = \delta X_\mu(2\pi,\tau) = 0$. Thus the equation of motion for closed strings is

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{2.2.3}$$

Open Strings

The ends of open string are free and so we need to impose boundary conditions on the ends of the open string. We normalise the length of the string to $\ell = \pi$. We impose the boundary condition such that the surface term vanishes. There are three ways for this to happen - atleast one of the two $\partial^\sigma X^\mu$ and δX_μ or the combination $\partial^\sigma X^\mu \delta X_\mu$ must be zero at $\sigma = 0$ and $\sigma = \pi$. Hence we have three different bondary condition:

1. **Dirichlet boundary condition:** $\delta X_\mu = 0 \quad$ at $\sigma = 0, \pi$.
2. **Neumann boundary condition:** $\partial_\sigma X_\mu = 0$ at $\sigma = 0, \pi$.
3. **Robin boundary condition:** $\partial_\sigma X_\mu \delta X^\mu = 0$ at $\sigma = 0, \pi$.

The first two boundary conditions have been studied in detail in literature and we will also analyse each boundary condition along with mixed boundary condition in detail as we progress in our study.

2.2.3 Symmetries of S_P

As with S_{NG}, we can directly read off two obvious symmetries of S_P:

Reparametrization Invariance

If we transform the parameters as $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha = \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$ then the scalar fields X^μ transform as

$$X^\mu(\sigma,\tau) \longrightarrow \widetilde{X}^\mu\left(\widetilde{\sigma}^\alpha\right) = X^\mu\left(\sigma^\alpha\right)$$

and the world-sheet metric $g_{\alpha\beta}$ transforms in the usual way

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta}\left(\tilde{\sigma}^\alpha\right) = \frac{\partial \sigma^\gamma}{\partial \widetilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \widetilde{\sigma} \beta} g_{\gamma\delta}(\sigma).$$

We can find the infinitesimal transformation under $\sigma^\alpha \longrightarrow \sigma^\alpha = \sigma^\alpha - \eta^\alpha$, where η^α is small, using Lie derivative. Indeed under infinitesimal transformation

$$\delta g_{\alpha\beta} = \mathcal{L}_\eta g_{\alpha\beta} = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha,$$

where ∇_α is the Levi-Civita covariant derivative with the usual Levi-Civita connection

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\rho\gamma}\right)$$

Also $\sqrt{-g}$ changes as $\delta(\sqrt{-g}) = \partial_\alpha\left(\eta^\alpha \sqrt{-g}\right)$. The Polyakov action S_P is easily seen to be invariant under reparametrizations. This is a gauge symmetry of the action.

Poincaré Invariance

This is a global symmetry of the action.

$$X^\mu \longrightarrow \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + \xi^\mu$$

for some constant ξ^μ. The infinitesimal version of this transformation is given in Subsection 2.1.1.

Weyl Invariance

There is another gauge invariance called Weyl symmetry. Under this $X^\mu \longrightarrow X^\mu$ and the metric transforms as

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \Omega^2(\boldsymbol{\sigma}) g_{\alpha\beta}$$

or infinitesimally if $\Omega^2(\boldsymbol{\sigma}) = e^{2\phi(\boldsymbol{\sigma})}$ then

$$\delta g_{\alpha\beta} = 2\phi(\boldsymbol{\sigma}) g_{\alpha\beta}.$$

To see that this is a symmetry of the action, note that $\sqrt{-g} \longrightarrow \Omega^2(\boldsymbol{\sigma})\sqrt{-g}$ as

$$\det\left(\Omega^2 g_{\alpha\beta}\right) = \Omega^4(\boldsymbol{\sigma}) \det\left(g_{\alpha\beta}\right)$$

and $g^{\alpha\beta} \longrightarrow (\Omega(\boldsymbol{\sigma}))^{-2} g^{\alpha\beta}$. Thus factors from $\sqrt{-g}$ and $g^{\alpha\beta}$ cancel.

Remark 2.2.1. Weyl transformation is not a coordinate transformation. Rather it is a local change of scale under which the theory is invariant. More precisely, this scale change preserves angles between as the metric transforms conformally.

Remark 2.2.2. Weyl transformation is unique to two dimensions since $\sqrt{-g}g^{\alpha\beta}$ remain invariant under $g_{\alpha\beta} \longrightarrow \Omega^2 g_{\alpha\beta}$ only in two dimensions.

Chapter 3

The Closed String

In Chapter 2, we found the correct action for relativistic strings namely the Polyakov action. We also found the equations of motion arising from the action and depending on the type of string - open or closed, we imposed boundary conditions. In this chapter, we will solve the classical equations of motion for the closed string and also quantise the theory using two different procedures. We will also analyse the closed string spectrum.

3.1 The Closed Classical String

As we saw in the previous chapter, the Polyakov action has two gauge symmetries. Hence to find the equations of motion, we first need to fix a gauge. This means that we should make an appropriate choice of the background metric using our gauge symmetries.

3.1.1 Fixing a Gauge

We have two diffeomorphism invariance namely for σ, τ and three independent metric components. Write

$$g_{\alpha\beta} = \begin{pmatrix} g_{\sigma\sigma} & g_{\sigma\tau} \\ g_{\tau\sigma} & g_{\tau\tau} \end{pmatrix} \qquad \text{then} \qquad g_{\sigma\tau} = g_{\tau\sigma}.$$

Now since $g_{\alpha\beta}$ has signature $(-,+)$, locally one out of $g_{\sigma\sigma}$ and $g_{\tau\tau}$ must be positive since $\mathrm{Tr} g_{\alpha\beta} = g_{\sigma\sigma} + g_{\tau\tau} = 0$. Under diffeomorphism we have

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \frac{\partial\sigma^\gamma}{\partial\widetilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\widetilde{\sigma}^\beta} g_{\gamma\delta}.$$

This gives

$$\begin{aligned}
\widetilde{g}_{\sigma\sigma} &= \left(\frac{\partial\sigma}{\partial\widetilde{\sigma}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\sigma}} g_{\sigma\tau} \\
\widetilde{g}_{\tau\tau} &= \left(\frac{\partial\sigma}{\partial\widetilde{\tau}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\tau}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\tau}}\frac{\partial\tau}{\partial\widetilde{\tau}} g_{\sigma\tau}
\end{aligned}$$

and

$$\widetilde{g}_{\sigma\tau} = \widetilde{g}_{\tau\sigma} = \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\sigma\sigma} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\tau}g_{\tau\tau} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\tau\sigma} + \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau}.$$

Now suppose in a neighbourhood of (σ, τ), $g_{\tau\tau} > 0$ then we put $\widetilde{g}_{\sigma\tau} = \widetilde{g}_{\tau\sigma} = 0$ and $\widetilde{g}_{\sigma\sigma} = -g_{\tau\tau}$. Thus we have a system of two first order partial differential equations to solve for two function $\widetilde{\sigma}(\sigma, \tau)$ and $\widetilde{\tau}(\sigma, \tau)$ that is we need to solve for $\widetilde{\sigma}(\sigma, \tau)$ and $\widetilde{\tau}(\sigma, \tau)$ from

$$\begin{gathered}\left(\frac{\partial\sigma}{\partial\widetilde{\sigma}}\right)^2 g_{\gamma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\sigma}}g_{\sigma\tau} = -g_{\tau\tau}\\ \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\sigma\sigma} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\tau}g_{\tau\tau} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\tau\sigma} + \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau} = 0.\end{gathered}$$

Solution to this exists atleast locally by Cauchy-Kowalevski theorem (see [5, Chapter 3] for a proof) since the coefficient functions are real analytic. Thus we have transformed $g_{\alpha\beta}$ to $g_{\tau\tau}\eta_{\alpha\beta}$ using the two diffeomorphisms. Since $g_{\tau\tau} = e^{\phi(\boldsymbol{\sigma})}$ for some function ϕ, thus we now use Weyl rescaling to transform

$$g_{\alpha\beta} \longrightarrow e^{-\phi(\boldsymbol{\sigma})}g_{\alpha\beta} = \eta_{\alpha\beta}.$$

This gauge is called *Conformal gauge.*

Remark 3.1.1. Any $2d$ metric can be made flat using Wely invariance: Suppose $g'_{\alpha\beta} = e^{\phi(\boldsymbol{\sigma})}g_{\alpha\beta}$ then one can easily check that

$$\sqrt{-g'}R' = \sqrt{g}\left(R - \nabla^2\phi\right).$$

If we choose ϕ such that $\nabla^2\phi = R$ then $R' = 0$. But in $2d$ vanishing Ricci scalar implies that Riemann curvature tensor is zero since in $2d$ one can show that

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right).$$

Hence the metric is flat.

Remark 3.1.2. Can the world-sheet metric be made flat globally? Depends on the topology of the space. Locally the metric can be made flat using the three gauge symmetries. Suppose we could extend this locally flat metric to whole worldsheet. This means that the whole worldsheet is covered by a coordinate chart which is flat. This in turn means that the Ricci scalar identically vanishes on the worldsheet. Topologically since in $2d$, the Euler characteristic χ of a manifold satisfies

$$\chi \propto \int_M R.$$

Thus a necessary condition of the extension to be possible is that $\chi = 0$.

We have fixed a gauge. Now we need to find the equation of motion of $g_{\alpha\beta}$ and impose it as a constraint on the classical system after substituting $g_{\alpha\beta} = \eta_{\alpha\beta}$. We have already calculated the equation of motion in Subsection 2.2.1 but we can recast it in terms of energy momentum tensor which is often more useful. We begin by writing the gauge fixed action:

$$S_P^{gf} = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \partial_\alpha X^\mu \partial^\alpha X_\mu. \tag{3.1.1}$$

The equation of motion for X^μ is

$$\partial^\alpha \partial_\alpha X^\mu = 0. \tag{3.1.2}$$

Next we have

$$\frac{\delta S}{\delta g^{\alpha\beta}} = -\frac{1}{4\pi\alpha'}\left[-\frac{\sqrt{-g}}{2} g_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu + \sqrt{-g}\partial_\alpha X^\mu \partial_\beta X_\mu\right].$$

Define the energy momentum tensor[1] by

$$T_{\alpha\beta} = -4\pi\alpha' \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}.$$

We get

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu.$$

So that

$$T_{\alpha\beta}\big|_{g_{\alpha\beta}=\eta_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu.$$

The equation of motion for $g_{\alpha\beta}$ was

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\partial_c X^\mu \partial' X_\mu.$$

So our constraint is $T_{\alpha\beta} = 0$. Written in terms of components:

$$T_{01} = \dot{X}^\mu X'_\mu = 0 \quad T_{11} = T_{00} = \dot{X}^2 - \frac{1}{2}\left(-\left(-\dot{X}^2 + X'^2\right)\right) = \frac{1}{2}\left(\dot{X}^2 + X'^2\right).$$

So we have to impose two constraints

$$\dot{X}^\mu X'_\mu = 0, \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0. \tag{3.1.3}$$

So the equation of motion is a wave equation along with the two constraints. We will now solve it.

[1] note that this is not the usual definition of energy momentum tensor. In general relativity (GR) we have different normalisation. In GR the energy momentum tensor is given by $T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}$.

3.1.2 Solving the Equation of Motion: Mode Expansion

We will use the lightcone coordinates to solve the equations of motion (3.1.2) subject to constraints (3.1.3). Introduce the lightcone coordinates

$$\sigma^{\pm} = \tau \pm \sigma,$$

then

$$\partial_+ = \partial_\tau + \partial_\sigma, \quad \partial_- = \partial_\tau - \partial_\sigma.$$

With this, the equation of motion $\partial_\alpha \left(\partial^\alpha X^\mu\right) = 0$ reduces to

$$\partial_+ \partial_- X^\mu = 0. \tag{3.1.4}$$

Indeed we have

$$\partial_+ \partial_- X^\mu = \partial_+ \left(\partial_\tau X^\mu - \partial_\sigma X^\mu\right) = \partial_{\tau\tau} X^\mu - \partial_{\tau\sigma} X^\mu - \partial_{\sigma\tau} X^\mu - \partial_{\sigma\sigma} X^\mu = 0.$$

The most general solution to $\partial_+ \partial_- X^\mu = 0$ is given by

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-) \tag{3.1.5}$$

for arbitrary functions X_L and X_R. For closed strings, we have the periodicity condition

$$X^\mu(\sigma + 2\pi, \tau) = \quad X^\mu(\sigma, \tau).$$

This implies that X^μ can be written as a Fourier series. More precisely, we have

$$\begin{aligned} X_L^\mu\left(\sigma^+\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu\left(\sigma^-\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}. \end{aligned} \tag{3.1.6}$$

The functions X_L^μ are called left movers and X_R^μ are called right movers.

Remark 3.1.3.

1. The factors $\alpha', \frac{1}{n}$ have been chosen for convenience when we quantise the system.
2. X_L^μ and X_R^μ are not periodic due to the linear term σ^+, σ^- but the combination $X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$ is periodic as σ cancels from the combination $\sigma^+ + \sigma^- = 2\tau$.
3. The quantities x^μ and p^μ are the position and momentum of the center of mass of the string. We will prove this explicitly. Observe that for the Polyakov action,

$$\Pi_\mu^\tau = \frac{\partial \mathcal{L}_P}{\partial \dot{X}^\mu} = -\frac{1}{4\pi\alpha'} \frac{\partial}{\partial \dot{X}^\mu} \left[-\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right] = \frac{1}{2\pi\alpha'} \dot{X}_\mu.$$

So

$$P^\mu = \int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'} \dot{X}^\mu = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \dot{X}_L^\mu\left(\sigma^+\right) + \dot{X}_R^\mu\left(\sigma^-\right) = \frac{1}{2\pi\alpha'} 2\pi\alpha' p^\mu = p^\mu,$$

and

$$q^\mu = \frac{1}{2\pi}\int_0^{2\pi} d\sigma X^\mu = \frac{1}{2\pi}\int_0^{2\pi} d\sigma X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right) = \frac{1}{2\pi}\left[2\pi x^\mu + 2\pi\alpha' p^\mu \tau\right] = x^\mu + \alpha' p^\mu \tau.$$

So we see that p^μ is indeed the momentum and x^μ is the position of center of mass of the string at $\tau = 0$.

4. The coordinate functions X^μ is real. So $(X_L^\mu)^\star = X_L^\mu$ and $(X_R^\mu)^\star = X_R^\mu$. This means that the coefficients α_n^μ and $\widetilde{\alpha}_n^\mu$ satisfy

$$(\alpha_n^\mu)^\star = \alpha_{-n}^\mu \quad \text{and} \quad (\widetilde{\alpha}_n^\mu)^\star = \alpha_{-n}^\mu \quad \forall n \in \mathbb{Z}\backslash\{0\}.$$

Recall that we had two constraints

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0.$$

In Light-cone coordinates, these transform to

$$\begin{aligned}
&\left(\frac{\partial_+ + \partial_-}{2}\right) X^\mu \left(\frac{\partial_+ - \partial_-}{2}\right) X_\mu = 0 \\
\implies &\left(\partial_+ X^\mu + \partial_- X^\mu\right)\left(\partial_+ X_\mu - \partial_- X_\mu\right) = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 - \left(\partial_- X^\mu\right)^2 = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 = \left(\partial_- X^\mu\right)^2.
\end{aligned}$$

The second constrain becomes

$$\begin{aligned}
&\left(\left(\frac{\partial_+ + \partial_-}{2}\right) X^\mu\right)^2 + \left(\left(\frac{\partial_+ - \partial_-}{2}\right) X^\mu\right)^2 = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 + \left(\partial_- X^\mu\right)^2 + 2\partial_+ X^\mu \partial_- X_\mu + \left(\partial_+ X^\mu\right)^2 + \left(\partial_- X^\mu\right)^2 - 2\partial_+ X^\mu \partial_- X_\mu = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 + \left(\partial_- X^\mu\right)^2 = 0.
\end{aligned}$$

Combining these two we get the constraint

$$\left(\partial_+ X^\mu\right)^2 = 0 = \left(\partial_- X^\mu\right). \tag{3.1.7}$$

We now impose this constraint on the Fourier modes. We have

$$\begin{aligned}
\partial_- X^\mu = \partial_- X_R^\mu &= \frac{\alpha' p^\mu}{2} + \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}\backslash\{0\}} \alpha_n^\mu e^{-in\sigma^-} \\
&= \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-in\sigma^-},
\end{aligned}$$

where we have defined

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu.$$

The constraint $(\partial_- X^\mu)^2 = 0$ gives

$$\left(\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_{n\mu} e^{-in\sigma^-}\right) \left(\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \alpha_m^\mu e^{-im\sigma^-}\right) = \frac{\alpha'}{2} \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}\right) e^{-in\sigma^-} = 0$$

where we used Cauchy product formula and $\boldsymbol{\alpha}_k \equiv \alpha_k^\mu$. If we define

$$L_n := \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}, \tag{3.1.8}$$

then the constraint becomes $L_n = 0$ for every $n \in \mathbb{Z}$. Similarly the constraint $(\partial_+ X^\mu)^2 = 0$ gives $\widetilde{L}_n = 0$ for every $n \in \mathbb{Z}$ where

$$\widetilde{L}_n := \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{n-k}, \tag{3.1.9}$$

and $\widetilde{\alpha}_0^\mu = \alpha_0^\mu$. The quantities L_n and $\widetilde{L}_n$ are called *classical Virasoro generators*. The constraints $L_0 = 0 = \widetilde{L}_0$ are particularly interesting as they contain information about the physical degrees of freedom of the string - the string momentum. We have

$$L_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{-k}, \quad \widetilde{L}_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{-k}.$$

In relativistic mechanics, we know that

$$p^\mu p_\mu = -M^2$$

where M is the rest mass of the particle. Since

$$p^\mu p_\mu = \frac{2}{\alpha'} \boldsymbol{\alpha}_0^2 = \frac{2}{\alpha'} \widetilde{\boldsymbol{\alpha}}_0^2,$$

we see that the constraints $L_0 = \widetilde{L}_0 = 0$ implies

$$\frac{1}{2} \sum_{n\neq 0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n - \frac{\alpha'}{4} M^2 = 0 = \frac{1}{2} \sum_{n\neq 0} \widetilde{\boldsymbol{\alpha}}_{-n} \cdot \widetilde{\boldsymbol{\alpha}}_n - \frac{\alpha'}{4} M^2.$$

This gives

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n = \frac{4}{\alpha'} \sum_{n>0} \widetilde{\boldsymbol{\alpha}}_{-n} \cdot \widetilde{\boldsymbol{\alpha}}_n. \tag{3.1.10}$$

This is called the *level matching condition* and will be crucial when we analyse the spectrum of the quantised theory.

3.2 Quantisation of Closed String

There are two ways to quantise the Polyakov action. One is the cannonical quantisation using Dirac's presciption. The other is Feynman's path integral quantisation. The cannonical quantisation procedure involves two ways as we are dealing with a gauge theory:

- **Covariant quantisation:** Change cannonical Poisson brackets to commutators and impose the constraint obtained by fixing a gauge as an operator equation to be satisfied by the states X^μ which are now operators. This method is manifestly Lorentz invariant but gives rise to negative norm states called *ghosts*. These decouple from the theory in the critical dimension $D = 26$.
- **Lightcone quantisation:** In this method we first solve the constraints to classify all classically distinct states and then we quantise the physical states. We break Lorentz invariance in the process and later obtain the same critical dimension $D = 26$ to ensure Lorentz invariance.

We will look at both of these quantisation scheme in detail now.

3.3 Covariant Quantisation

We have D scalar fields $X^\mu, \mu = 0, 1, \ldots, D-1$ and two constraints

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0.$$

3.3.1 Poisson Brackets

Let us begin by computing the classical Poisson brackets.

(i) Equal τ Poisson bracket $\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{P.B.} = 0$.

Proof. For Polyakov action, we have $\Pi^\tau_\mu \sim \dot{X}_\mu$. We will use the notation $\Pi_\mu := \Pi^\tau_\mu$ everywhere unless stated explicitly. Thus this P.B. is obvious. □

(ii) Equal τ Poisson bracket $\{\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} = 0$.

Proof. Obvious from the fact that $\Pi^\tau_\mu \sim \dot{X}_\mu$. □

(iii) Equal τ Poisson bracket $\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} = \eta^{\mu\nu}\delta(\sigma-\sigma')$.

Proof. By definition

$$
\begin{aligned}
\{X^\mu(\sigma,\tau),\Pi^\nu(\sigma',\tau)\}_{P.B.} &= \eta^{\rho\lambda}\frac{\partial X^\mu(\sigma,\tau)}{\partial X^\rho(\sigma,\tau)}\frac{\partial \Pi^\nu(\sigma',\tau)}{\partial \Pi^\lambda(\sigma,\tau)} \\
&= \eta^{\rho\lambda}\delta^\mu_\rho\delta^\nu_\lambda\delta(\sigma-\sigma') \\
&= \eta^{\mu\nu}\delta(\sigma-\sigma')
\end{aligned}
$$

□

From these Poisson brackets, we can easily calculate the Poisson brackets for $x^\mu, p^\mu, \alpha^\mu_n, \widetilde{\alpha}^\mu_n$ We have

$$
\begin{aligned}
&\{x^\mu, p^\nu\}_{P.B.} = \eta^{\mu\nu}, \quad \{\widetilde{\alpha}^\mu_m, \alpha^\nu_n\}_{P.B.} = 0 \\
&\{\alpha^\mu_m, \alpha^\nu_n\}_{P.B.} = \{\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n\} = -im\eta^{\mu\nu}\delta_{m+n,0}.
\end{aligned}
\tag{3.3.1}
$$

Using these Poisson brackets, we can get a algebra satisfied by the Virasoro generators.

Lemma 3.3.1. *The classical Virasoro generators satisfy the Witt algebra:*

$$
\{L_n, L_m\}_{P.B.} = 2(m-n)L_{m+n}, \quad \left\{\widetilde{L}_n, \widetilde{L}_m\right\}_{P.B.} = i(m-n)\widetilde{L}_{n+m}, \quad \left\{\widetilde{L}_n, L_m\right\}_{P.B.} = 0.
\tag{3.3.2}
$$

Proof. We have

$$
\begin{aligned}
\{L_n, L_m\}_{P.B.} &= \left\{\sum_{l\in\mathbb{Z}}\boldsymbol{\alpha}_{n-l}\cdot\boldsymbol{\alpha}_l, \sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k\right\}_{P.B.} \\
&= \sum_{l,k\in\mathbb{Z}}\left\{\eta_{\mu\nu}\alpha^\mu_{n-l}\alpha^v_l, \eta_{\rho\sigma}\alpha^\rho_{m-k}\alpha^\sigma_k\right\}_{P.B.}
\end{aligned}
$$

Using

$$
\begin{aligned}
\{AB, CD\}_{P.B.} &= \{A, CD\}_{P.B.}B + A\{B, CD\}_{P.B.} \\
&= C\{A, D\}_{P.B.}B + \{A, C\}_{P.B.}DB + AC\{B, D\}_{P.B.} + A\{B, C\}_{P.B.}D,
\end{aligned}
$$

we get

$$
\begin{aligned}
\{L_n, L_m\}_{P.B.} = &\sum_{l,k\in\mathbb{Z}}\eta_{\mu\nu}\eta_{\rho\sigma}\big[\alpha^\mu_{m-k}\left\{\alpha^\mu_{n-l}, \alpha^\sigma_k\right\}_{P.B.}\alpha^\nu_l + \left\{\alpha^\mu_{n-l}, a^\rho_{m-k}\right\}_{P.B.}\alpha^\nu_l\alpha^\sigma_k \\
&\qquad + \alpha^\mu_{n-l}\alpha^\rho_{m-k}\left\{\alpha^v_k, \alpha^\sigma_k\right\}_{P.B.} + \alpha^\mu_{n-k}\left\{\alpha^\nu_l, \alpha^\rho_{m-k}\right\}_{P.B.}\alpha^\sigma_k\big] \\
= &\sum_{l,k\in\mathbb{Z}}\eta_{\mu\nu}\eta_{\rho\sigma}\big[-\alpha^\rho_{m-k}\eta^{\mu\sigma}i(n-l)\delta_{n-l+k,0}\alpha^\nu_l - i(n-l)\eta^{\mu\rho}\delta_{n-l+m-k,0}\alpha^\nu_l\alpha^\sigma_k \\
&\qquad - il\delta_{l+k,0}\eta^{\nu\sigma}\alpha^\mu_{n-l}\alpha^\rho_{m-k} - il\eta^{\nu\rho}\delta_{l+m-k,0}\alpha^\mu_{n-k}\alpha^\sigma_k\big] \\
= &-i\sum_{k\in\mathbb{Z}}\big[\eta_{\nu\rho}\alpha^\rho_{m-k}\alpha^\nu_{n+k}k + \eta_{\nu\sigma}\alpha^\nu_{n+m-k}\alpha^\sigma_k(k-m) + \\
&\qquad + \eta_{\mu\rho}\alpha^\mu_{n+k}\alpha^\rho_{m-k}k + \eta_{\mu\sigma}(k-m)\alpha^\mu_{n+m-k}\alpha^\sigma_k\big],
\end{aligned}
$$

where we used (3.3.1). Replacing $m-k$ by k in first and k by $k-n$ in third sum we get

$$\begin{aligned}
\{L_n, L_m\}_{P.B.} &= -i\sum_{k\in\mathbb{Z}} \big[\eta_{\nu\rho}\alpha_k^\rho \alpha_{n+m-k}^\nu (m-k) + \eta_{\nu\sigma}\alpha_{n+m-k}^\nu \alpha_k^\sigma (k-m) + \\
&\qquad\qquad + \eta_{\mu\rho}\alpha_k^\mu \alpha_{m+n-k}^\rho (n-k) + \eta_{\mu\sigma}(k-m)\alpha_{n+m-k}^\mu \alpha_k^\sigma\big] \\
&= -i\sum_{k\in\mathbb{Z}} \eta_{\mu\nu}\alpha_{n+m-k}^\mu \alpha_k^\nu (n-m) \\
&= i(m-n)L_{m+n}.
\end{aligned}$$

Similarly, we get all other Poisson brackets. □

3.3.2 Cannonical Commutation Relations

Following the usual way, promote the scalar fields X^μ to operator valued fields and impose the cannonical commutation relation following the rule:

$$\{\cdot,\cdot\}_{P.B.} = \frac{1}{i}[\cdot,\cdot].$$

Using the Poisson brackets for X^μ, Π^μ, we get the following commutation relations:

$$[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$$
$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = 0 = [X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)].$$

For the Fourier modes, using (3.3.1), we get

$$\begin{aligned}
[x^\mu, p^\nu] &= i\eta^{\mu\nu} \\
[\alpha_n^\mu, \alpha_m^\nu] &= m\eta^{\mu\nu}\delta_{m+n,0} = [\widetilde{\alpha}_m^\mu, \widetilde{\alpha}_n^\nu]
\end{aligned} \tag{3.3.3}$$

and all other combinations are zero. These commutation relations are similar to those of creation and annihilation operators. Indeed if we define

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu \quad , \quad (a_n^\mu)^\dagger = \frac{1}{\sqrt{n}}\alpha_{-n}^\mu \quad , \quad n>0,$$

then we will get the usual commutation relations:

$$\left[a_n^\mu, (a_m^\mu)^\dagger\right] = i\delta_{nm}.$$

Similarly we can put

$$\widetilde{a}_n^\mu = \frac{1}{\sqrt{n}}\widetilde{\alpha}_n^\mu \quad , \quad (\widetilde{a}_n^\mu)^\dagger = \frac{1}{\sqrt{n}}\widetilde{\alpha}_{-n}^\mu \quad , \quad n>0,$$

then we get the commutation relations:

$$\left[\widetilde{a}_n^\mu, (\widetilde{a}_m^\mu)^\dagger\right] = i\delta_{nm}.$$

So for every scalar field X^μ, $\mu = 0,1,\ldots,D-1$ we have two family of creation and annihilation operators corresponding to the left movers and the right movers.

Remark 3.3.2. We cannot directly get the commutation relations satisfied by the Virasoro generators from the Witt algebra. In subsequent sections, we will calculate the quantum algebra of Virasoro generators from the commutation relations of the Fourier modes. As we will discover later, the quantum algebra of Virasoro generators, called the *Virasoro algebra*, has an extra *central charge* term and hence the quantum algebra is the central extension of the Witt algebra. We will see that this is related related to the fact the Weyl symmetry which is a symmetry of the classical action does not survive quantisation.

3.3.3 Constructing the Fock Space

We will now construct the Fock space of the theory. We begin by constructing the ground state. We now have the creation and annilation operators to define the vacuum of the theory Denote it by $|0\rangle$. Then we demand:

$$\alpha_n^\mu|0\rangle = 0 = \widetilde{\alpha}_n^\mu|0\rangle \qquad \text{for } \mu = 0, 1, \ldots, D-1;\ n > 0.$$

Note that this condition alone does not uniquely fix the ground state. This is because, the ground state here is quite different from the one in field theory in the sense that there is a string specified by the center of mass position x^μ and momentum p^μ. So we denote the ground state by $|0; p^\mu\rangle$ which now has the property that

$$\widehat{p}^\mu|0; p^\mu\rangle = p^\mu\,|0; p^\mu\rangle\,, \tag{3.3.4}$$

where p^μ is the momentum of the string. So the ground state of the theory is now defined by

$$\alpha_n^\mu|0; p^\mu\rangle = 0 = \widetilde{\alpha}_n^\mu|0; p^\mu\rangle \qquad \text{for } \mu = 0, 1, \ldots, D-1;\ n > 0 \tag{3.3.5}$$

and (3.3.4). A general excitation of the string is

$$\left(\alpha_{-1}^{\mu_1}\right)^{n_{\mu_1}}\left(\alpha_{-2}^{\mu_2}\right)^{n_{\mu_2}}\cdots\left(\widetilde{\alpha}_{-1}^{\nu_1}\right)^{n_{\nu_1}}\left(\widetilde{\alpha}_{-2}^{\nu_2}\right)^{n_{\nu_2}}\cdots|0; p^\mu\rangle.$$

The norm of states is defined via the Hermiticity property $\left(\alpha_m^\mu\right)^\dagger = \alpha_{-m}^\mu, \left(\widetilde{\alpha}_m^\mu\right)^\dagger = \widetilde{\alpha}_{-m}^\mu$ and the normalisation

$$\langle 0; p|0; p'\rangle = (2\pi)^D \delta^{(D)}(p - p'). \tag{3.3.6}$$

This Fock space is not physical since we have not yet imposed the constraints. After imposing the constraint, each excited state will be interpreted as a particle. Hence we have infinitely many species of particles in this theory.

3.3.4 Ghosts

We immediately come across a problem. The theory has negative norm states – the so called ghost states[2]. Since $\eta^{00} = -1 < 0$ we have

$$\begin{aligned} \left[\alpha_n^0, \alpha_{-n}^0\right] &= \left[\alpha_n^0, \left(\alpha_{-n}^0\right)^\dagger\right] = -n \qquad \text{and} \\ \left[\widetilde{\alpha}_n, \tilde{\alpha}_n^0\right] &= \left[\widetilde{\alpha}_n^0, \left(\tilde{\alpha}_{-n}^0\right)^\dagger\right] = -n \end{aligned}$$

[2]these are different from Fadeev-Popov ghosts

Consider states of the form $|\psi\rangle = \alpha^0_{-m}|0;p^\mu\rangle$ for $m>0$. For these states we have

$$\begin{aligned}
\langle\psi \mid \psi\rangle &= \left\langle p^\mu;0\left|\left(\alpha^0_{-m}\right)^\dagger\alpha^0_{-m}\right|0;p^\mu\right\rangle \\
&= \left\langle p^\mu;0\left|\alpha^0_m\alpha^0_{-m}\right|0;p^\mu\right\rangle \\
&= \left\langle p^\mu;0\left|-m+\alpha^0_{-m}\alpha^0_m\right|0;p^\mu\right\rangle \\
&= -m\left\langle p^\mu;0 \mid 0;p^\mu\right\rangle + \left\langle p^\mu;0\left|\left(\alpha^0_m\right)^\dagger\alpha^0_m\right|0;p^\mu\right\rangle \\
&\propto -m<0.
\end{aligned}$$

Ghosts are problematic because these are in contradiction to the probabilistic interpretation of norm in Quantum mechanics. Our only hope is to apply the constraints and hope that these ghosts decouple from our theory. That is indeed the case when we fix the dimension of spacetime to be 26.

3.3.5 Normal Ordering and the Quantum Virasoro Algebra

As discussed in the previous section, the constraints in terms of Fourier modes is given by the vanishing of the Virasoro generators. But now, the Fourier modes are no more scalar valued functions but are operators on the Hilbert space. From the commutation relations (3.3.3), we see that the Virasoro generators

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{n-k} \quad \text{and} \quad \widetilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\widetilde{\boldsymbol{\alpha}}_k\cdot\widetilde{\boldsymbol{\alpha}}_{n-k}$$

can be defined unambiguously for $n\neq 0$ as $\alpha^\mu_k, \alpha^\nu_{n-k}$ and the respective tildes commute for $n\neq 0$. For $n=0$, we have

$$L_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{-k}, \quad \widetilde{L}_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}}\widetilde{\boldsymbol{\alpha}}_k\cdot\widetilde{\boldsymbol{\alpha}}_{-k},$$

but since $\alpha^\mu_k, \alpha^\nu_{-k}$ and the respective tildes do not commute, the definition of L_0 and $\widetilde{L}_0$ is ambiguous in the quantum theory. We need to pick an ordering convention to define L_0 and $\widetilde{L}_0$. The natural choice is the *normal ordering* – we put annihilation operators $\alpha^\mu_n, \quad n>0$ to the right of creation operator $\alpha^\mu_n, \quad n<0$. With this choice of normal ordering, we put

$$:L_0: = \frac{1}{2}\sum_{k\in\mathbb{Z}}:\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{-k}: = \sum_{k=1}^{\infty}\boldsymbol{\alpha}_{-k}\cdot\boldsymbol{\alpha}_k+\frac{1}{2}\boldsymbol{\alpha}_0^2, \quad :\widetilde{L}_0: = \frac{1}{2}\sum_{k\in\mathbb{Z}}:\widetilde{\boldsymbol{\alpha}}_k\cdot\widetilde{\boldsymbol{\alpha}}_{-k}: = \sum_{k=1}^{\infty}\widetilde{\boldsymbol{\alpha}}_{-k}\cdot\widetilde{\boldsymbol{\alpha}}_k+\frac{1}{2}\widetilde{\boldsymbol{\alpha}}_0^2.$$

Under the commutation relation on $\alpha^\mu_n, \widetilde{\alpha}^\mu_n$ and the choice of normal ordering, we can calculate the Virasoro algebra. We will show that

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.$$

where c is called the central charge. Recall that the classical Poisson bracket is

$$\{L_n, L_m\}_{P.B.} = i(m-n)L_{m+n}.$$

This extra term is due to *conformal anomaly* which is due to the breaking of Weyl symmetry in the quantum theory. As we will prove later, the expectation value of the trace of the energy momentum tensor $\langle T^\alpha_{\ \alpha}\rangle \propto R$ where R is the Ricci scalar. The nonvanishing of the trace implies that the Weyl symmetry is broken and hence we have the conformal anomaly. Hence the Virasoro algebra is the central extension of the Witt algebra. We will not define this term precisely here. From now on, we will omit the colons in the Virasoro generators but they are assumed to be normal ordered. We begin by proving a lemma.

Lemma 3.3.3. *For any $m, n \in \mathbb{Z}$, we have*

$$[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}, \quad \left[\widetilde{\alpha}^\mu_m, \widetilde{L}_n\right] = m\widetilde{\alpha}^\mu_{m+n}.$$

Proof. With the choice normal ordering we have

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_{\boldsymbol{k}}:.$$

So we have

$$[\alpha^\mu_m, L_n] = \frac{1}{2}\sum_{k\in\mathbb{Z}}[\alpha^\mu_m : \boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_k]$$

Now using $[A, BC] = [A,B]C + B[A,C]$ we get

$$\begin{aligned}
[\alpha^\mu_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left\{\eta_{\rho\sigma}{:}\alpha^\rho_{n-k}\left[\alpha^\mu_m, \alpha^\sigma_k\right]{:} + \eta_{\rho\sigma}{:}\left[\alpha^\mu_m, \alpha^\rho_{n-k}\right]\alpha^\rho_k{:}\right\} \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left\{\eta_{\rho\sigma}\left(\alpha^\rho_{n-k} m\eta^{\mu\sigma}\delta_{m+k,0} + \alpha^\sigma_k\eta^{\mu\rho}m\delta_{m+n-k,0}\right)\right\} \\
&= \frac{1}{2}\left\{\eta^\mu_\sigma\alpha^p_{n+m}m + n^\mu_\sigma\alpha^\sigma_{m+n} - m\right\} \\
&= m\alpha^\mu_{m+n}.
\end{aligned}$$

The proof for the tildes is identical. ∎

Theorem 3.3.4. *For any $m, n \in \mathbb{Z}$, we have*

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}, \\
\left[\widetilde{L}_n, \widetilde{L}_m\right] &= (n-m)\widetilde{L}_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}[:\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k:, L_n] \\
&= \frac{1}{2}\sum_{k\leq 0}[\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m-k}, L_n] + \frac{1}{2}\sum_{k=1}^{\infty}[\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k, L_n] \\
&= \frac{1}{2}\sum_{k\leq 0}\boldsymbol{\alpha}_k\cdot[\boldsymbol{\alpha}_{m-k}, L_n] + [\boldsymbol{\alpha}_k, L_n]\cdot\boldsymbol{\alpha}_{m-k} \\
&+ \frac{1}{2}\sum_{k\geq 1}\boldsymbol{\alpha}_{m-k}\cdot[\boldsymbol{\alpha}_k, L_n] + [\boldsymbol{\alpha}_{m-k}, L_n]\cdot\boldsymbol{\alpha}_k \\
&= \frac{1}{2}\sum_{k\leq 0}\left\{(m-k)\boldsymbol{\alpha}_\kappa\cdot\boldsymbol{\alpha}_{m+n-k} + k\boldsymbol{\alpha}_{n+k}\cdot\boldsymbol{\alpha}_{m-k}\right\} \\
&+ \frac{1}{2}\sum_{k\geq 1}\left\{k\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_{n+k} + (m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\right\},
\end{aligned}
$$

where in the second line we broke the normal ordering in the two sums and used Lemma 3.3.3 in last line. We now shift the second and third sum by n i,e. substitute $n+k$ by k. We get

$$
\begin{aligned}
[L_m, L_n] =& \frac{1}{2}\sum_{k\leq 0}(m-k)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \frac{1}{2}\sum_{k\leq n}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} \\
&+ \frac{1}{2}\sum_{k\geq n+1}(k-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \frac{1}{2}\sum_{k\geq 1}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k.
\end{aligned}
$$

Now we need to normal order the second and the third sum. If $n > 0$ then the second sum is not normal ordered and if $n \leq 0$ then the third sum is not normal ordered. We will assume $n > 0$ and proceed. One can get the result for $n \leq 0$ case using the same process. Breaking the second and fourth sum at 0 and $n+1$ respectively, we get

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-k)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\leq 0}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k=1}^{n}(k-n)d_k\cdot\boldsymbol{\alpha}_{m+n-k} \\
&+ \sum_{k\geq n+1}(k-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \sum_{k=1}^{n}(m-k)\delta_{n+n-k}\cdot\boldsymbol{\alpha}_k + \sum_{k\geq n+1}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg] \\
= \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq n+1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
&+ \sum_{k=1}^{n}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k=1}^{n}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg]
\end{aligned}
$$

We will now use $\left[\boldsymbol{\alpha}_k^\mu, \boldsymbol{\alpha}_{m+n-k}^\nu\right] = \eta^{\mu\nu} k \delta_{m+n,0}$ to normal order the third sum. We get

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq n+1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
&+ \sum_{k=1}^{n}(k-n)\left(\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \eta_\mu^\mu k\delta_{m+n,0}\right) + \sum_{k=1}^{n}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg] \\
= \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq 1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
&+ \sum_{k=1}^{n}(k-n)k\underbrace{n_\mu^\mu}_{D}\delta_{m+n,0}\Bigg] \\
&= (m-n)\frac{1}{2}\sum_{k\in\mathbb{Z}} {:}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k}{:} + \frac{D}{2}\delta_{m+n,0}\sum_{k=1}^{n}\left(k^2 - nk\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\left(\frac{n(n+1)(2n+1)}{6} - n\frac{n(n+1)}{2}\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\left(\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - n\right)\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\frac{n(n+1)}{2}\left(\frac{1-n}{3}\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\frac{n\left(1-n^2\right)}{6} \\
&= (m-n)L_{m+n} + \frac{D}{12}m\left(m^2-1\right)\delta_{m+n,0},
\end{aligned}
$$

where we replaced n by $-m$ in last step. The proof for the tildes is identical. □

Remark 3.3.5. We can also derive the structure of the central charge term by using Jacobi identity of the Lie bracket. We will rederive this algebra using the tools of conformal field theory.

Remark 3.3.6. In case of only free Bosonic fields, $c = \eta_\mu{}^\mu = D$ i,e. each scalar field contributes one unit to central charge. When we will rederive this algebra using conformal field theory and quantise the string using path integral, we will calculate the contribution of Fadeev-Popov ghosts to the central charge.

3.3.6 Imposing the Constraints

Recall that the constraints are $L_n = 0 = \widetilde{L}_n$ but this cannot be directly imposed on the Hilbert space of the theory. Indeed if $|\phi\rangle$ is any quantum mechanical state then for any

$n \in \mathbb{Z}$

$$0 = \langle \phi \,|[L_n, L_{-n}]|\, \phi \rangle = 2n \langle \phi \,|L_0|\, \phi \rangle + \frac{c}{12} n \left(n^2 - 1\right) \langle \phi \mid \phi \rangle$$

which does not hold if $n \neq 0, \pm 1$. So we cannot impose $L_n|\phi\rangle = 0$ for all n. So the alternative method of imposing the constraint would be to demand that the positive modes annihilate the physical states of the theory:

$$L_n|\text{phys}\rangle = 0, \quad \widetilde{L}_n|\text{phys}\rangle = 0, \quad n > 0, \tag{3.3.7}$$

where $|\text{phys}\rangle$ are the physical states of the theory. This way of imposing the constraints is equivalent to requiring that the matrix elements of all L_n (and the tildes) for $n \neq 0$ vanish. Indeed, we easily see that $L_n^\dagger = L_{-n}$ for $n \neq 0$, thus

$$\langle \text{phys}'|L_n|\text{phys}\rangle = 0, \quad \langle \text{phys}'|L_n|\text{phys}\rangle = 0, \quad \forall n.$$

We are left with imposing the constraint for L_0 and $\widetilde{L}_0$. Recall that we have an ordering ambiguity in defining L_0 and $\widetilde{L}_0$. We now define them using the normal ordering convention we have chosen and impose the constraints $L_0 = 0 = \widetilde{L}_0$ by shifting them by a constant which we will determine later:

$$(L_0 - a)|\text{phys}\rangle = 0 = \left(\widetilde{L}_0 - a\right)|\text{phys}\rangle \quad \text{(Mass-shell condition)}. \tag{3.3.8}$$

The constant a is called the *normal ordering constant*. In the classical theory, we saw that the constraints $L_0 = 0 = \widetilde{L}_0$ gave us the level matching condition. We want to understand its quantum version. Noting that

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu = \widetilde{\alpha}_0^\mu, \qquad \text{and} \quad p^\mu p_\mu = -M^2,$$

we see that (3.3.8) can be written as

$$\begin{aligned} \left(N - \frac{\alpha'}{4} M^2 - a\right)|\text{phys}\rangle &= 0 \\ \left(\widetilde{N} - \frac{\alpha'}{4} M^2 - a\right)|\text{phys}\rangle &= 0, \end{aligned}$$

where

$$N = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k \quad \text{and} \quad \widetilde{N} = \sum_{k=1}^{\infty} \widetilde{\boldsymbol{\alpha}}_{-k} \cdot \widetilde{\boldsymbol{\alpha}}_k \tag{3.3.9}$$

are the number operators. Thus the *quantum level matching condition* is

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\widetilde{N} - a). \tag{3.3.10}$$

Since the number operator gives the number of excitations of the string, we see that the number of left-moving and right moving excitations are equal. Thus quantum level matching condition imply equal number of left-moving and right-moving modes.

3.3.7 The Physical Hilbert Space

As discussed before, the physical Hilbert space of the theory consists of states satisfying the constraint (3.3.7) and (3.3.8). Let us now analyse the physical spectrum.

Definition 3.3.7. A state $|\psi\rangle$ is called *spurious* if it satisfies the mass-shell condition and is orthogonal to all physical states. That is

$$(L_0 - a)\,|\psi\rangle = 0 \quad \text{and} \quad \langle\phi \mid \psi\rangle = 0 \quad \forall \quad |\phi\rangle \text{ physical} \,.$$

Note that the spurious states are orthogonal to all physical states. Thus if we require the spurious states themselves to be physical we must have

$$\langle\psi \mid \psi\rangle = 0.$$

Definition 3.3.8. Spurious states which are also physical are called *null states*.

If $|\psi\rangle$ is a physical state and $|\chi\rangle$ is a null state then, it is easy to check that $|\psi\rangle + |\chi\rangle$ is also physical and that its inner product with any other physical state is same as that of $|\psi\rangle$. This means that these two states are physically indistinguishable and we have the identification

$$|\psi\rangle \cong |\psi\rangle + |\chi\rangle. \tag{3.3.11}$$

Thus the physical Hilbert space must be the quotient space

$$\mathscr{H}_{\text{CQ}} \cong \frac{\mathscr{H}_{\text{phys}}}{\mathscr{H}_{\text{null}}}, \tag{3.3.12}$$

where $\mathscr{H}_{\text{CQ}}$ is the physical Hilbert space of covariant quantisation and $\mathscr{H}_{\text{phys}}, \mathscr{H}_{\text{null}}$ denotes the space of physical states (states satisfying the constraint (3.3.7) and (3.3.8)) and null states respectively. Let us now look at the states at various levels. We will only determine the left moving sector since the right moving is analogous.

Level 0

At level $N = 0$, there is only one state $|0;p\rangle$. The constraint (3.3.8) implies that the mass of this state is

$$M^2 = -\frac{4a}{\alpha'}. \tag{3.3.13}$$

The norm of this state is

$$\langle 0;p|0;p'\rangle = (2\pi)^D \delta^{(D)}(p-p') \tag{3.3.14}$$

and hence we have no ghosts or null states at this level.

Level 1

At level $N = 1$, a general state is of the form

$$|e, p\rangle = \boldsymbol{e} \cdot \boldsymbol{\alpha}_{-1}|0; p\rangle \tag{3.3.15}$$

where $\boldsymbol{e} = e^{\mu}$ is a vector. The norm of this state is

$$\langle e, p'|e, p\rangle = e^{\mu\star} e_{\mu} (2\pi)^D \delta^{(D)}(p - p') \tag{3.3.16}$$

and hence is nonnegative given then $\boldsymbol{e}$ is not timelike: $e^2 \geq 0$. The constraint (3.3.8) gives the mass of this state

$$M^2 = \frac{4(1-a)}{\alpha'}. \tag{3.3.17}$$

The constraints $L_m|e, p\rangle = 0$ for $m \geq 2$ is automatically satisfied. We check the constraint $L_1|e, p\rangle = 0$. We have

$$L_1|e, p\rangle \propto (\widehat{\boldsymbol{p}} \cdot \boldsymbol{\alpha}_1)(\boldsymbol{e} \cdot \boldsymbol{\alpha}_{-1})|0; p\rangle = (\boldsymbol{e} \cdot \boldsymbol{p})|0; p\rangle = 0. \tag{3.3.18}$$

Thus we must have $\boldsymbol{e} \cdot \boldsymbol{p} = 0$ for this state to be physical. There are three cases to consider:

1. $a > 1$: In this case $M^2 < 0$ and is not allowed for a physical theory.

2. $a = 1$: In this case $M^2 = 0$. By doing a Lorentz transformation, we can take the momentum of the state to be

   $$p^{\mu} = (E, 0, 0, \cdots, E).$$

   The physicality condition then becomes

   $$p \cdot e = -Ee^0 + Ee^{D-1} = 0 \implies e^0 = e^{D-1}.$$

   Then
   $$e^2 = -\left(e^0\right)^2 + \left(e^1\right)^2 + \ldots + \left(e^{D-1}\right)^2 = \left(e^1\right)^2 + \ldots + \left(e^{D-2}\right)^2 \geq 0.$$
   This means the state has no negative norm state. There can be null states which we will determine later.

3. $a < 1$: In this case $M^2 > 0$ and we can go to the rest frame via a Lorentz transformation in which
   $$p^{\mu} = (M, 0, 0, \cdots, 0).$$
   The physicality condition $e \cdot p = 0$ implies $e^0 = 0$. Then a nontrivial state with $e \neq 0$ has positive norm and there are no null states.

The last two cases both physically make sense but the last case does not have a known way of introducing interactions. Later we will show that $a = 1$ along with $D = 26$ (to be argued below) gives rise to ghost free $\mathscr{H}_{\text{CQ}}$. We will take $a = 1$ from now on.

Level 2

At level $N=2$, a general state is of the form

$$|\zeta, e, p\rangle \equiv \left(e\cdot\alpha_{-2}+\zeta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}\right)|0;p\rangle.$$

The constraint implies

$$M^2=\frac{4(2-a)}{\alpha'}=\frac{4}{\alpha'}>0.$$

The physical constraint $L_m|\varphi, e, p\rangle=0$ for $m\geq 3$ is trivially satisfied. We have

$$L_1|\zeta, e, p\rangle=\left[e_\nu+\sqrt{\frac{\alpha'}{2}}\zeta_{\mu\nu}p^\mu\right]\alpha_{-1}^{\nu}|0,p\rangle=0,$$

which gives

$$e_\nu+\sqrt{\frac{\alpha'}{2}}\zeta_{\mu\nu}p^\mu=0, \quad \nu=0,1,\ldots,D-1. \tag{3.3.19}$$

Similarly $L_2|\varphi, e, p\rangle=0$ gives

$$2\left(\frac{\alpha'}{2}\right)^{1/2}p\cdot e+\zeta_{\mu\nu}\eta^{\mu\nu}=0. \tag{3.3.20}$$

Let us find a particular solution to these equations. Since this is a massive state, we can take $p^\mu=(M,0,0,\ldots,0)$ where $M=\left(\frac{4}{\alpha'}\right)^{1/2}$. Then from (3.3.19) we get

$$e_\mu=\sqrt{2}\zeta_{0\mu}. \tag{3.3.21}$$

From (3.3.20) we get

$$-2\sqrt{2}e_0+\zeta^{\mu}{}_{\mu}=0 \implies -4\zeta_{00}-\zeta_{00}+\zeta^{i}_{i}=0 \implies \zeta\equiv\zeta^{i}_{i}=5\zeta_{00} \tag{3.3.22}$$

Let us now take

$$\zeta_{ij}=\zeta\delta_{ij}, \quad \zeta_{00}=\frac{D-1}{5}\zeta, \quad e_0=\frac{\sqrt{2}(D-1)}{5}\zeta \tag{3.3.23}$$

and rest all components 0. This choice clearly satisfies (3.3.21) and (3.3.22). We then have

$$\begin{aligned}
\langle\zeta, e, p \mid \zeta, e, p'\rangle &= \left(2e^2+2\zeta_{\mu\nu}\zeta^{\mu\nu}\right)(2\pi)^D\delta^{(D)}\left(p-p'\right) \\
&= \left(-\frac{4}{25}(D-1)^2\zeta^2+2\frac{(D-1)^2}{25}\zeta^2+(D-1)\zeta^2\right)(2\pi)^D\delta^{(D)}\left(p-p'\right) \\
&= \frac{2\zeta^2}{25}\left(25(D-1)-(D-1)^2\right)(2\pi)^D\delta^{(D)}\left(p-p'\right) \\
&= \frac{2\zeta^2}{25}(26-D)(D-1)(2\pi)^D\delta^{(D)}\left(p-p'\right).
\end{aligned}$$

So if $D > 26$, then this is a negative norm state. If $D \leq 26$ then this state is a non-negative norm state. We will quantise the string in the next section in a different way called *lightcone quantisation* which is manifestly ghost free but breaks Lorentz invariance. We will show in Section 7.4 that lightcone Hilbert space is same as $\mathscr{H}_{\text{CQ}}$ when $D = 26$ which will prove the following no ghost theorem:

Theorem 3.3.9. (No ghost theorem) *The ghosts decouple in the critical dimension $D = 26$ and with $a = 1$.*

The reason we want to match $\mathscr{H}_{\text{CQ}}$ with lightcone Hilbert space is twofold:

1. Once we fix $D \leq 26$ in this step we will have to show that $\mathscr{H}_{\text{CQ}}$ is ghost free at all levels which is a tedius task. Moreover we do not have any free parameter to fix and guarantee that $\mathscr{H}_{\text{CQ}}$ is ghost free.

2. The way we will prove the no-ghost theorem is by quantising the string using BRST formalism which is a consistent way of quantising gauge theories. It turns out that

$$\mathscr{H}_{\text{BRST}} \cong \mathscr{H}_{\text{lightcone}}$$

   and hence no-ghost theorem for $\mathscr{H}_{\text{CQ}}$ requires us to prove $\mathscr{H}_{\text{CQ}} \cong \mathscr{H}_{\text{lightcone}}$.

We now determine $\mathscr{H}_{\text{null}}$ to fully characterise $\mathscr{H}_{\text{CQ}}$. We will show that for $a = 1, D = 26$, we can explicitly construct all states in $\mathscr{H}_{\text{null}}$. The construction of null states is based on [1].

Lemma 3.3.10. *A general spurious state is of the form*

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n} |\chi_n\rangle$$

where $|\chi_n\rangle$ are some states satisfying the modified mass-shell condition

$$(L_0 - a + n) |\chi_n\rangle = 0, \quad \forall n \geq 1.$$

Proof. By definition, we have

$$\langle \phi \mid \psi \rangle = 0 \quad \forall \quad |\phi\rangle \quad \text{physical.}$$

We know that

$$L_n|\phi\rangle = 0 \quad \forall n > 0$$

Thus we can write

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n} |\chi_n\rangle \quad \Big(\text{since} \quad L_{-n}^{\dagger} = L_n\Big)$$

for some state $|\chi_n\rangle$. Mass-shell condition implies

$$\begin{aligned} (L_0 - a)\,|\psi\rangle &= 0 \\ \Rightarrow \quad \sum_{n=1}^{\infty} (L_0 L_{-n} - a L_n)\,|\chi_n\rangle &= 0. \end{aligned}$$

By quantum Virasoro algebra $L_0 L_{-n} = L_{-n} L_0 + n L_{-n}$. We get

$$\begin{aligned} &\sum_{n=1}^{\infty} (L_{-n} L_0 + n - a L_{-n})\,|\chi_n\rangle = 0 \\ \Rightarrow \quad &\sum_{n=1}^{\infty} L_{-n}\,(L_0 - a + n)\,|\chi_n\rangle = 0 \\ \Rightarrow \quad &(L_0 - a + n)\,|\chi_n\rangle = 0 \quad \forall \quad n > 0. \end{aligned}$$

□

Definition 3.3.11. The states $|\chi_n\rangle$ satisfying $(L_0 - a + n)\,|\chi_n\rangle = 0$ are called level n states.

Lemma 3.3.12. *Any spurious state $|\psi\rangle$ can be written as*

$$|\psi\rangle = L_{-1}\,|\chi_1\rangle + L_{-2}\,|\chi_2\rangle$$

where $|\chi_1\rangle$ and $|\chi_2\rangle$ are level 1 level 2 states i,e. they satisfy

$$(L_0 - a + 1)\,|\chi_1\rangle = 0 \quad \textit{and} \quad (L_0 - a + 2)\,|\chi_2\rangle = 0.$$

Proof. By Lemma 3.3.10, we have

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}\,|\chi_n\rangle\,,$$

where $(L_0 - a + n)\,|\chi_n\rangle = 0$. We will use induction to show that $L_{-n}|\chi_n\rangle$ can be written as $L_{-1}\,|\chi_1\rangle + L_{-2}\,|\chi_2\rangle$ for some level 1 and level 2 states $|\chi_1\rangle$ and $|\chi_2\rangle$ respectively for all $n \geq 3$. Let us begin with the base case. Note that by quantum Virasoro algebra, we have

$$[L_{-1}, L_{-2}] = (-1 + 2) L_{-2-1} + 0 = L_{-3}.$$

Thus

$$L_{-3}\,|\chi_3\rangle = [L_{-1}, L_2]\,|\chi_3\rangle = L_{-1}\,(L - 2\,|\chi_3\rangle) + L_2\,(-\chi_{-1}\,(\chi_3))\,.$$

Take $|\chi_1\rangle = L_2 |\chi_3\rangle$ and $|\chi_2\rangle = -\chi_1 |\chi_3\rangle$. It remains to show that $|\chi_1\rangle$ and $|\chi_2\rangle$ are Level 1 and level 2 states respectively. Indeed since $(L_0 - a + 3) |\chi_3\rangle = 0$, we have

$$\begin{aligned}(L_0 - a + 1) L_{-2} |\chi_3\rangle &= (L_0 L - 2 + L - 2(-a + 1)) |\chi_3\rangle \\ &= (L_{-2} L_0 + 2L_{-2} + L_{-2}(-a + 1)) (\chi_3\rangle \\ &= L_2 (L_0 - a + 3) |\chi_3\rangle \\ &= 0,\end{aligned}$$

where we used the quantum Virasoro algebra: $L_0 L_{-2} = L_{-2} L_0 + 2L_{-2}$. Similarly we have

$$\begin{aligned}(L_0 - a + 2) (-L_{-1} |\chi_3\rangle) &= -(L_{-1} L_0 + L_{-1}(-a + 2)) |\chi_3\rangle \\ &= -(L_0 L_{-1} + L_{-1} + L_{-1}(-a + 2)) |\chi_3\rangle \\ &= -L_{-L} (L_0 - a + 3) |\chi_3\rangle \\ &= 0.\end{aligned}$$

For any n, we assume that $L_{-n+1} |\chi_{n-1}\rangle$ can be written as $L_{-1} |\chi_1\rangle + L_{-2} |\chi_2\rangle$. Then since

$$L_{-n} = \frac{1}{n} [L_{-1}, L_{-n+1}],$$

so that

$$L_n |\chi_n\rangle = \frac{1}{n} L_{-1} (L_{-n+1} |\chi_n\rangle) + \frac{1}{n} L_{-n+1} (-L_{-1} |\chi_n\rangle).$$

Following similar method as in the base case, we can show that $-\frac{1}{n} L_{-1} |\chi_n\rangle$ is a level $n - 1$ state. Indeed observe that

$$\begin{aligned}-\frac{1}{n} (L_0 - a + n - 1) L_{-1} |\chi_n\rangle &= -\frac{1}{n} (L_{-1} L_0 + L_{-1} + L_1(-a + n - 1)) |\chi_n\rangle \\ &= -\frac{1}{n} L_{-1} (L_0 - a + n) |\chi_n\rangle \\ &= 0\end{aligned}$$

So using induction hypothesis, we get

$$L_{-n} |\chi_n\rangle = L_{-1} \left(\frac{1}{n} L_{n+1} |\chi_n\rangle \right) + L_{-1} |\widetilde{\chi}_1\rangle + L_{-2} |\widetilde{\chi}_2\rangle$$

for some level 1 state $|\widetilde{\chi}_1\rangle$ a level 2 state $|\widetilde{\chi}_2\rangle$. It is also clear that $\frac{1}{n} L_{n+1} |\chi_n\rangle$ is a level 1 state. Thus define

$$|\widehat{\chi}_1\rangle = |\widetilde{\chi}_1\rangle + \frac{1}{n} L_{n+1} |\chi_n\rangle \quad |\widehat{\chi}_2\rangle = |\widetilde{\chi}_2\rangle$$

so that

$$L_{-n} |\chi_n\rangle = L_{-1} |\widehat{\chi}_1\rangle + L_{-2} |\widehat{\chi}_2\rangle$$

where $\widehat{\chi}_1$ and $\widehat{\chi}_2$ are level 1 and level 2 states respectively. $\square$

Physical Spurious States

In view of Lemma 3.3.12, it is sufficient to find the values of a and D such that the spurious states $L_{-1}\left|\chi_1\right\rangle$ and $L_{-2}\left|\chi_2\right\rangle$ become physical where $\left|\chi_1\right\rangle$ and $\left|\chi_2\right\rangle$ are level 1 and level 2 states respectively.

Theorem 3.3.13. *Let $\left|\chi_1\right\rangle$ be a level 1 state satisfying $L_m\left|\chi_1\right\rangle = 0$ for all $m > 0$. Then the spurious state $\left|\psi\right\rangle = L_{-1}\left|\chi_1\right\rangle$ is physical if and only if $a = 1$.*

Proof. ($\Longrightarrow$) Suppose $L_{-1}\left|\chi_1\right\rangle$ is physical. Then $L_1 L_{-1}\left|\chi_1\right\rangle = 0$ as physical states $\left|\phi\right\rangle$ satisfy $L_m|\phi\rangle = 0$ for all $m > 0$. We get

$$\begin{aligned} L_1 L_{-1}\left|\chi_1\right\rangle &= \left(L_1 L_1 + 2L_0\right)\left|\chi_1\right\rangle \\ &= 2L_0\left|\chi_1\right\rangle \\ &= 2(a-1)\left|\chi_1\right\rangle, \end{aligned}$$

since $\left|\chi_1\right\rangle$ is a level 1 state satisfying $\left(L_0 - a + 1\right)\left|\chi_1\right\rangle = 0$. Thus $L_1 L_{-1}\left|\chi_1\right\rangle = 0 \Rightarrow a = 1$.
($\Longleftarrow$) If $a = 1$ then backtracking above steps we get $L_1 L_{-1}\left|\chi_1\right\rangle = 0$. To check that $L_m L_{-1}\left|\chi_1\right\rangle = 0$, we proceed inductively. We have the base case. Next

$$L_m L_{-1}\left|x_1\right\rangle = L_{-1} L_m\left|\chi_1\right\rangle + (m+1) L_{m-1}\left|\chi_1\right\rangle = 0,$$

since $L_m\left|\chi_1\right\rangle = 0$ by assumption and $L_{m-1}\left|\chi_1\right\rangle = 0$ by induction hypothesis. Next thing to check is

$$\left(L_0 - a\right) L_{-1}\left|\chi_1\right\rangle = 0, \quad (a = 1).$$

Indeed

$$\begin{aligned} L_0 L_{-1}\left|\chi_1\right\rangle &= L_{-1} L_0\left|\chi_1\right\rangle + L_{-1}\left|\chi_1\right\rangle \\ &= 0 + L_{-1}\left|\chi_1\right\rangle, \end{aligned}$$

since $\left(L_0 - a + 1\right)\left|\chi_1\right\rangle = L_0\left|\chi_1\right\rangle = 0$. ∎

Next we look at level 2 spurious states. A general level 2 spurious state is

$$\left|\psi\right\rangle = \left(L_{-2} + \gamma L_{-1} L_{-1}\right)\left|\chi_2\right\rangle.$$

We will show that $\left|\psi\right\rangle$ is physical if and only if $\gamma = \frac{3}{2}$ and $D = 26$.

Theorem 3.3.14. *Let $\left|\chi_2\right\rangle$ be a level 2 state satisfying $L_m\left|\chi_2\right\rangle = 0$ for all $m > 0$. Then the spurious state $\left|\psi\right\rangle = \left(L_{-2} + \gamma L_{-1} L_{-1}\right)\left|\chi_2\right\rangle$ is physical if and only if $\gamma = \frac{3}{2}$ and $D = 26$.*

Proof. ($\Longrightarrow$) Suppose $\left|\psi\right\rangle$ is physical, then we must have $\langle\psi|\psi\rangle = 0$ since $\left|\psi\right\rangle$ is spurious. Next we demand $L_m|\psi\rangle = 0$ for all $m > 0$. In particular $L_1|\psi\rangle = 0$. We have

$$\begin{aligned} &\left(L_1 L_{-2} + \gamma L_1 L_{-1} L_{-1}\right)\left|\chi_2\right\rangle = 0 \\ \Longrightarrow\ &\left(L_{-2} L_1 + 3L_{-1} + \gamma L_{-1} L_1 L_{-1} + 2\gamma L_0 L_{-1}\right)\left|\chi_2\right\rangle = 0 \\ \Longrightarrow\ &\left(L_1 + 3L_{-1} + \gamma L_{-1} L_{-1} L_1 + 2\gamma L_{-1} L_0 + 2\gamma L_{-1} L_0 + 2\gamma L_{-1}\right)\left|\chi_2\right\rangle = 0 \\ \Longrightarrow\ &L_{-1}\left(3 + 4\gamma L_0 + 2\gamma\right)\left|\chi_2\right\rangle = 0, \end{aligned}$$

where we used $L_1|\chi_2\rangle = 0$. Now since $L_0|\chi_2\rangle = -|\chi_2\rangle$, we get

$$\begin{aligned}
&L_{-1}(3 + y(-1)\gamma + 2\gamma)\,|\chi_2\rangle = 0\\
&\implies (3-2\gamma)L_{-1}\,|\chi_2\rangle = 0\\
&\implies \gamma = \frac{3}{2}
\end{aligned}$$

So

$$|\psi\rangle = \left(L_{-2} + \frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle\,.$$

Next we impose $L_2|\psi\rangle = 0$. We have

$$\begin{aligned}
&L_2\left(L_{-2} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle = 0\\
&\implies \left[L_2, L_{-2} + \frac{3}{2}L_{-1}L_{-1}\right]|\chi_2\rangle + \left(L_{-2} + \frac{3}{2}L_{-1}L_{-L}\right)L_2\,|\chi_2\rangle = 0\\
&\implies \left(4L_0 + \frac{c}{12}2(3)\delta_{0,0} + \frac{3}{2}\left[L_2, L_1L_{-1}\right]\right)|\chi_2\rangle = 0,
\end{aligned}$$

where we used our assumption that $L_2\,|\chi_2\rangle = 0$. Now since

$$\begin{aligned}
[L_2,\ L_{-1}L_{-1}] &= [L_2, L_1]\,L_1 + L_{-1}\,[L_2, L_1]\\
&= 3L_1L_{-1} + 3L_{-1}L_1\\
&= 3\left(L_{-1}L_1 + 2L_0\right) + 3L_{-1}L_1\\
&= 6L_{-1}L_1 + 6L_0.
\end{aligned}$$

So we have

$$\begin{aligned}
&\left(4L_0 + \frac{c}{2} + \frac{3}{2}\left(6L_{-1}L_1 + 6L_0\right)\right)|\chi_2\rangle = 0\\
&\implies \left(13L_0 + 9L_{-1}L_1 + \frac{c}{2}\right)|\chi_2\rangle = 0\\
&\implies c = 26,
\end{aligned}$$

where we used $L_1|\chi_2\rangle = 0$ and $L_0|\chi_2\rangle = -|\chi_2\rangle$. In free Bosonic string theory, we know that $c = \eta_\mu{}^\mu = D$, so $D = 26$.
($\Longleftarrow$) Assuming $D = 26$, $\gamma = \frac{3}{2}$, we can show that $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$ back tracking the steps. For $m \geq 3$, it is easily proved using induction as in the proof of Theorem 3.3.13.

Finally we need to show that $(L_0 - 1)|\psi\rangle = 0$. To see that this is true, observe that

$$
\begin{aligned}
(L_0-1)\left(L_{-2}+\frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle &= \left(L_0L_{-2}+\frac{3}{2}L_0L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_{-2}L_0+2L_2+\frac{3}{2}L_{-1}L_0L_1+\frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_2(-1)+2L_{-2}+\frac{3}{2}L_{-1}L_{-1}L_0+\frac{3}{2}L_{-1}L_{-1}+\frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_{-2}-\frac{3}{2}L_{-1}L_{-1}+\frac{3}{2}L_{-1}L_{-1}+\frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= 0,
\end{aligned}
$$

where we used $L_0|\chi_2\rangle = -|\chi_2\rangle$. □

Thus we have shown that infinite classes of spurious states of zero norm appear in our theory when $D = 26$ and $a = 1$. Thus we have determined the boundary where positive norm states turn into negative norm states. Thus for these values of a and D, the ghosts decouple from the theory as infinitely many zero norm states appear in our theory. There are *non-critical string theories* free of ghosts for $a \le 1$ and $D \le 25$ but we will not pursue it here. We conclude the covariant quantisation of closed strings with the result that the spectrum is well defined and ghost free in the critical dimension. We will arrive at the same result in the next section using another quantisation scheme.

3.4 Lightcone Quantisation

In lightcone quantisation, we begin by solving the constraints first and separating the physical degrees of freedom. Before we begin, let us discuss about reparametrizations, conformal transformations and Weyl rescaling.

Given any reparametrization of the worldsheet, it corresponds to choosing a different coordinate chart for the manifold. This has no physical consequence as all points, curves remain same on the manifold (worldsheet). Thus any diffeomorphism automatically preserves circular and hyperbolic angles. On the other hand coordinate transformations which transform the metric as

$$g_{\mu\nu} \longrightarrow \Omega^2(\sigma)g_{\mu\nu}(\sigma)$$

are called conformal transformations. These transformations preserve angles (circular as well as hyperbolic). Another version of Conformal transformations are maps between manifolds. Let (M, g) and $(N, \widetilde{g})$ be Riemannian manifolds and $\varphi : M \longrightarrow N$ be a smooth map. Then φ is said to be a conformal map if the pullback $\varphi^*\widetilde{g} = \Omega^2 g$ for some smooth function Ω. Writing $x' = \varphi(x)$ we see that

$$\widetilde{g}_{\mu\nu}(x')\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)g_{\rho\sigma}.$$

Thus angles are preserved. In particular if $M = N$ and $\widetilde{g} = g$ then

$$g_{\mu\nu}\left(x'\right)\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x) g_{\rho\sigma}$$

which are usual conformal transformations. Weyl rescalings on the other hand, are completely different. They are not coordinate transformations. These do not act on the parametrizations but act on the metric. Since the metric is only scaled thus angles are preserved.

3.4.1 Residual Gauge Freedom: Lightcone Gauge

We have already fixed a gauge i,e. chosen two reparametrizations and used Weyl rescaling to fix the metric to $\eta_{\mu\nu}$. But we have some residual gauge symmetry. Indeed consider a reparametrization $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha = \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$ such that the metric changes by $\eta_{\alpha\beta} \longrightarrow \widetilde{\eta}_{\mu\beta} = \Omega^2(\boldsymbol{\sigma})\eta_{\mu\nu}$. If at the same time, we perform a Weyl transformation $\eta_{\alpha\beta} \longrightarrow \widetilde{\eta}_{\mu\beta} = \Omega^{-2}(\boldsymbol{\sigma})\eta_{\mu\nu}$, then we see that the action is invariant. This is the residual guage symmetry and these are exactly the conformal transformations. Thus we see that

$$\text{“conformal} = \text{diffeomorphisms} \times \text{Weyl”}.$$

This also shows that the guage fixed Polyakov action has *worldsheet conformal symmetry*. We will now quantise this system using lightcone coordinates. Introduce

$$\sigma^{\pm} = \tau \pm \sigma \implies \tau = \frac{\sigma^+ + \sigma^-}{2}, \quad \sigma = \frac{\sigma^+ - \sigma^-}{2}.$$

The metric is given by

$$\begin{aligned} ds^2 = -d\tau^2 + d\sigma^2 &= -\frac{1}{4}\left(d\sigma^+ + d\sigma^-\right)^2 + \frac{1}{4}\left(d\sigma^+ - d\sigma^-\right)^2 \\ &= -\frac{1}{4}d\sigma^{+2} - \frac{1}{4}d\sigma^{-2} - \frac{1}{2}d\sigma^+ d\sigma^- + \frac{1}{4}d\sigma^{+2} + \frac{1}{4}d\sigma^{-2} - \frac{1}{2}d\sigma^+ d\sigma^- \\ &= -d\sigma^+ d\sigma^-. \end{aligned}$$

So a reparametrization $\sigma^+ \longrightarrow \widetilde{\sigma}^+\left(\sigma^+\right)$ and $\sigma^- \longrightarrow \widetilde{\sigma}^-\left(\sigma^-\right)$, ds^2 simply changes by scaling. Indeed

$$ds^2 = -\frac{\partial\sigma^+}{\partial\widetilde{\sigma}^+}d\widetilde{\sigma}^+\frac{\partial\sigma^-}{\partial\widetilde{\sigma}^-}d\sigma^- = -\frac{\partial\sigma^+}{\partial\tilde{\sigma}^+}\frac{\partial\sigma^-}{\partial\widetilde{\sigma}}d\tilde{\sigma}^+ d\widetilde{\sigma}.$$

Note that the reparametrizations are single variable. We would like to fix the remnant gauge. The choice that we will make here is called *lightcone gauge*. Introduce

$$X = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right).$$

Such a choice breaks Lorentz invariance in classical as well as quantum theory as we have picked a special time and space part while Lorentz transformations mixes space and time coordinates. So when we quantise our system, we will look for conditions that restores the Lorentz invariance. It is now easy to see that

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} \left(dX^i\right)^2.$$

So the metric $\eta_{++} = 0 = \eta_{--}$ and $\eta_{+-} = \eta_{-+} = -1$ and $\quad \eta_{ii} = 1 \quad \forall \quad \dot{i} = 1, 2, \ldots, D-2$ and all other elements vanish. So any vector $A^\mu = (A^+, A^-, A^i)$ is lowered as

$$A_\mu = (-A_-, -A_+, \quad A_i)$$

and the dot product is

$$A^\mu B_\mu = -A^+B_- - A^-B_+ + A^iB^i.$$

Solution of the equation of motion is

$$X^+ = X_L^+\left(\sigma^+\right) + X_R^+\left(\sigma^-\right).$$

To see this, note that

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

so that

$$\begin{aligned} X^+ &= \frac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right) = \frac{1}{\sqrt{2}}\left[X_L^0\left(\sigma^+\right) + X_L^{D-1}\left(\sigma^+\right) + X_R^0\left(\sigma^-\right) + X_R^{D-L}\left(\sigma^-\right)\right] \\ &= X_L^+\left(\sigma^+\right) + X_R^+\left(\sigma^-\right). \end{aligned}$$

We now fix our gauge. Note that X^+ satisfies the wave equation $\partial_+\partial_-X^+ = 0$. Now note that a reparametrization $\widetilde{\sigma}^+ = \widetilde{\sigma}^+\left(\sigma^+\right)$ and $\widetilde{\sigma}^- = \widetilde{\sigma}^-\left(\sigma^-\right)$ corresponds to

$$\widetilde{\tau} = \frac{\sigma^+ + \sigma^-}{2}, \quad \widetilde{\sigma} = \frac{\widetilde{\sigma}^+ - \widetilde{\sigma}^-}{2}.$$

But $\widetilde{\tau}$ has to satisfy $\partial_+\partial_-\widetilde{\tau} = 0$. So we can choose

$$\widetilde{\tau} = \frac{X^+}{\alpha' p^+} - x^+.$$

This is called *lightcone gauge*. The coordinate X^- still satisfies the wave equation

$$\partial_+\partial_-X^- = 0$$

The usual solution is

$$X^- = X_L^-\left(\sigma^+\right) + X_R^-\left(\sigma^-\right).$$

Let us look at the constraints in lightcone gauge. We had the constraint $(\partial_+ X)^2 = 0 = (\partial_- X)^2$ with $X = (X^+, X^-, X^0)$. So we get

$$
\begin{aligned}
(\partial_+ X)^2 &= -2\partial_+ X^- \partial_+ X^+ + \sum_{i=1}^{D-2} \left(\partial_+ X^i\right)^2 \\
(\partial_- X)^2 &= -2\partial_- X^- \partial_- X^+ + \sum_{i=1}^{D-2} \left(\partial_- X^i\right)^2 .
\end{aligned}
$$

Since

$$
\partial_+ X^+ = \frac{\alpha' p^+}{2} = \partial_- X^+ \quad \left(\text{as} \quad \tau = \frac{\sigma^+ + \sigma^-}{2}\right),
$$

the constraints $(\partial_+ X)^2 = 0 = (\partial_- X)^2$ gives

$$
\begin{aligned}
\partial_+ X^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \left(\partial_+ X^i\right)^2 \\
\partial_- X^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \left(\partial_- X^2\right)^2 .
\end{aligned}
\tag{3.4.1}
$$

Thus we see that in lightcone gauge the $D-2$ scalar fields determine X^- upto an additive constant coming from integration. Indeed we see that if we write the mode expansion of $X^-_{L/R}$

$$
\begin{aligned}
X_L^- \left(\sigma^+\right) &= \frac{1}{2} x^- + \frac{\alpha'}{2} p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^- e^{-in\sigma^+} \\
X_R^- \left(\sigma^-\right) &= \frac{1}{2} x^- + \frac{\alpha'}{2} p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^- e^{-in\sigma^-}
\end{aligned}
$$

then x^- is coming as the integration constant while all other terms p^- and $\widetilde{\alpha}_n^-, \alpha_n^-$ is determined in terms of $\widetilde{\alpha}_n^i, \alpha_n^i$ and p^+. Indeed if we write

$$
\begin{aligned}
\partial_+ X_L^- &= \sqrt{\frac{\alpha}{2}} \sum_{n \in \mathbb{Z}} \widetilde{\alpha}_n^- e^{in\sigma^+} \quad \text{with} \quad \widetilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}} p^- \\
\partial_- X_R^- &= \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}}^{n} \alpha_n e^{-in\sigma^-} \quad \text{with} \quad \alpha_0^- = \sqrt{\frac{\alpha'}{2}} p^- .
\end{aligned}
$$

Then substituting $(\partial_+ X)^2$ using Fourier modes of X^i in (3.4.1), we get by comparing coefficients of $e^{-in\sigma^\pm}$ that

$$
\begin{aligned}
\alpha_n^- &= \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{m \in \mathbb{Z}} \sum_{i=1}^{D-1} \alpha_{n-m}^i \alpha_m^i \\
\widetilde{\alpha}_n^- &= \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{m \in \mathbb{Z}} \sum_{i=1}^{D-2} \widetilde{\alpha}_{n-m}^i \widetilde{\alpha}_m^i .
\end{aligned}
$$

For $n = 0$, we get two expressions for p^-:

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{\alpha'}{2}p^i p^i + \sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n\right) = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{\alpha'}{2}p^i p^i + \sum_{n\neq 0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n\right).$$

Using $p^\mu = (p^+, p^-, p^i)$ we see that

$$M^2 = -p^\mu p_\mu = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i.$$

Using the above equality for $\frac{\alpha' p^-}{2}$ above we get

$$M^2 = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n,$$

where we used

$$\sum_{n\neq 0}{}^{'}\alpha^i_{-n}\alpha^i_n = 2\sum_{n>0}{}^{'}\alpha^i_{-n}\alpha^i_n.$$

The oscillators $\alpha^i_n, \widetilde{\alpha}^i_n$ are called *transverse oscillators*. These are physical excitations in the sense that knowing α^i_n and $\widetilde{\alpha}'_n$ determines all other modes. Thus the most general classical solution can be determined in terms of $2(D-2)$ oscillator modes $\alpha^i_n, \widetilde{\alpha}^i_n$ and a bunch of zero modes $p^\pm, p^i, x^\pm$.

3.4.2 Quantisation

The usual way of quantisation is to compute the classical Poisson brackets and use Dirac prescription. As we did in covariant quantisation, using the Poisson brackets, the following commutation relations are obvious:

$$\begin{aligned}\left[x^i, p^j\right] = i\delta^{ij}, \quad \left[x^-, p^+\right] = -i, \quad \left[x^+, p^-\right] = -i \\ \left[\alpha^i_n, \alpha^j_m\right] = n\delta^{ij}\delta_{m+n,0} = \left[\widetilde{\alpha}^i_n, \widetilde{\alpha}^j_m\right].\end{aligned} \tag{3.4.2}$$

The ground state is again $|0; p^\mu\rangle$ with $|0\rangle$ being the string. To build the Fock space, we impose

$$\widehat{p}^\mu\,|0;p^\mu\rangle = p^\mu\,|0;p^\mu\rangle, \quad \widetilde{\alpha}^i_n\,|0;p^\mu\rangle = 0 = \alpha^i_n\,|0;p^\mu\rangle, \quad \forall \quad n>0 \quad \mu = 1, 2, \cdots, D-1.$$

We act with $\alpha^i_{-n}, \widetilde{\alpha}^i_{-n}, n > 0$ to build the Fock space. Notice that i runs only over spatial index $i = 1, 2\cdots, D-1$, so the theory does not have ghosts by construction. Its time to impose the constraints. As we had in covariant quantisation, level matching with normal ordering implies

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\widetilde{N}-a),$$

where now the number operators are

$$N = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_n^i\alpha_n^i \quad \text{and} \quad \widetilde{N} = \frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\widetilde{\alpha}_n^i\widetilde{\alpha}_n^i$$

and a is again the normal ordering constant which we again fix by requiring that the spectrum be Lorentz invariant. Note that

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i &= \frac{1}{2}\sum_i\left[\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{n<0}\alpha_{-n}^i\alpha_n^i\right] \\
&= \frac{1}{2}\sum_i\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{1}{2}\sum_i\left[\sum_{n<0}\alpha_n^i\alpha_{-n}^i - n\right] \\
&= \sum_i\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0}n,
\end{aligned}$$

where we used the commutator $\left[\alpha_n^i, \alpha_{-n}^i\right] = n$. The last sum is divergent but we need to extract physics out of this divergence. The result is the appearance of *Casimir force*. We will do this in two ways.

UV Cut-off $\varepsilon \ll 1$

Write

$$\sum_{n>0}n \longrightarrow \sum_{n>0}ne^{-\varepsilon n} = -\frac{\partial}{\partial\varepsilon}\sum_{n>0}e^{-\varepsilon n} = -\frac{\partial}{\partial\varepsilon}\left[\left(L - e^{-\varepsilon}\right)^{-1}\right].$$

Now

$$\begin{aligned}
-\frac{\partial}{\partial\varepsilon}\left[\frac{1}{1-e^{-\varepsilon}}\right] = \frac{e^{-\varepsilon}}{\left(1-e^{-\varepsilon}\right)^2} &= \frac{\left(1 - \varepsilon + \frac{\varepsilon^2}{2} + O\left(\varepsilon^3\right)\right)}{\left(1 - 1 + \varepsilon - \frac{\varepsilon^2}{2} + \cdots\right)^2} \\
&= \frac{\left(1 - \varepsilon + \frac{\varepsilon^2}{2} + O\left(\varepsilon^3\right)\right)}{\varepsilon^2\left(1 - \frac{\varepsilon}{2} + \ldots\right)^2} \\
&= \frac{1}{\varepsilon^2}\left(t - \varepsilon + \frac{\varepsilon^2}{2} + O\left(\varepsilon^3\right)\right)\left(1 + 2\frac{\varepsilon}{2} - 2\frac{\varepsilon^2}{3!} + \frac{3}{4}\varepsilon^2 + O\left(\varepsilon^3\right)\right) \\
&= \frac{1}{\varepsilon^2}\left(1 + \frac{\varepsilon^2}{2} - \varepsilon^2 - \frac{2}{6}\varepsilon^2 + \frac{3}{4}\varepsilon^2 + O\left(\varepsilon^3\right)\right) \\
&= \frac{1}{\varepsilon^2} - \frac{1}{12} + O(\varepsilon).
\end{aligned}$$

The $\frac{1}{\varepsilon^2}$ must be renormalised away. After renormalising and taking $\varepsilon \to 0$, we get the odd result

$$\sum_{n=1}^{\infty}n = -\frac{1}{12}. \tag{3.4.3}$$

Zeta Function Regularisation

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

The series defining $\zeta(s)$ converges absolutely and uniformly on compact subsets of the half plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and hence $\zeta(s)$ is holomorphic on this half plane. Moreover, the Riemann zeta function admits a unique analytic continuation to the whole $s-$plane. To be precise, Riemann in 1859 proved the following integral representation of the Riemann zeta function:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} W(x)\left(x^{s/2} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}, \tag{3.4.4}$$

where

$$W(x) = \sum_{n=1}^{\infty} e^{n^2\pi x}$$

and $\Gamma(s)$ is the gamma function. The integral on the right hand side of (3.4.5) converges for all $\mathbb{C}$. So this integral gives an analytic continuation of $\zeta(s)$. Indeed putting

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

we see that $\xi(s)$ is an entire function and satisfies

$$\xi(1-s) = \xi(s).$$

From the fact that $\xi(s)$ is entire, we see that $\zeta(s)$ (analytically continued) has simple zeros at $s = -2n,\ n \in \mathbb{N}$ corresponding to poles of $\Gamma\left(\frac{s}{2}\right)$.[3] Now at $s = -1$, we have that $\xi(-1) = \xi(2)$. This implies

$$\begin{aligned}
&2\pi^{1/2}\Gamma\left(-\frac{1}{2}\right)\zeta(-1) = 2\pi^{-1}\Gamma(1)\zeta(2)\\
\Rightarrow \quad &\pi^{1/2}\left(-\frac{1}{2}\right)\sqrt{\pi}\zeta(-1) = \pi^{-1}\frac{\pi^2}{6}\\
&\Rightarrow \zeta(-1) = -\frac{1}{12}.
\end{aligned}$$

So we see that both of the computation gives same result.

[3]These are called the *trivial zeros* of $\zeta(s)$. The *Riemann hypothesis* says that all other *non trivial zeros* of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This is still an open problem and a million dollar problem announced by the Clay Mathematical Institute.

3.4.3 String Spectrum

With the above regularisation, the level matching condition becomes

$$
\begin{aligned}
M^2 &= \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\alpha^2_{-n}\alpha^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:N: - \frac{D-2}{24}\right] \\
&= \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:\widetilde{N}: - \frac{D-2}{24}\right].
\end{aligned}
$$

We also identify the normal ordering constant as

$$
a = \frac{D-2}{24}.
$$

Let us look at the ground state $|0;p^\mu\rangle$. By our definition of vacuum $|0\rangle$, we have

$$
:N:|0;p^\mu\rangle = 0 = :\widetilde{N}:|0;p^\mu\rangle .
$$

So Level matching gives

$$
M^2 = -\frac{D-2}{6\alpha'} < 0.
$$

These are particles weth negative mass-squared. These are called *Tachyons*. These are a problem in Bosonic string theory. But when we study superstring theory where we include Fermionic fields on the worldsheet, then these states automatically vanish. Now let us look at excited states. First excited state is obtained by acting α^i_{-1} and $\widetilde{\alpha}^i_{-1}$. To see this observe that for $n>0$,

$$
\begin{aligned}
N\alpha^j_{-n}|0;p^\mu\rangle &= \sum_{i=1}^{D-1}\sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^i_k\alpha^j_{-n}|0;p^\mu\rangle \\
&= \left[\sum_i\sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^j_{-n}\alpha^i_k + k\delta^{ij}\delta_{k-n,0}\alpha^i_{-k}\right]|0;p^\mu\rangle \\
&= n\alpha_{-n}|0;p^\mu\rangle .
\end{aligned}
$$

So α^i_{-1} and $\widetilde{\alpha}^i_{-1}$ give first excited states. Thus level matching requires us to act α^i_{-1} and $\widetilde{\alpha}_{-1}$ together. So the first excited states are $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0;p^\mu\rangle$. Mass of each of these states is

$$
M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right). \tag{3.4.5}
$$

3.4.4 Fixing Lorentz Invariance

Our states are labelled by indices $i,j = 1,2,\ldots,D-1$ and hence these transform as vectors with respect to the group $SO(D-2)\hookrightarrow SO(1,D-1)$ where $SO(1,D-1)$ is the full Lorentz group. But finally we want our states to fit into some representation of the Lorentz group $S0(1, D-1)$. Here we invoke Wigner's classification of representations of Poincaré group. From the discussion in Appendix A[4], we see that if we want our states $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}\left|0;p^\mu\right\rangle$ to transform as some representation of the Lorentz group, then these states must be massless as these states fit into the representation of the little group $SO(D-2)$ which is the little group corresponding to massless representation. Thus (3.4.5) implies that $D = 26$ which also gives $a = 1$. Thus we have recovered the critical dimension by requiring that the first excited state be representations of the Lorentz group. We still need to make sure that the higher excited states also transform as some representations of the Lorentz invariant and we now have no choice other than to hope that with the values of a and D that we have chosen, we somehow manage to embed the higher excited states into the representation of Lorentz group. This is indeed the case. We will show this for the second excited state but one can check that the all higher excited states fit into some massive representation of the Lorentz group. We first note from (3.4.5) that all higher excited states are massive with the values of D that we have chosen. So by Wigner's classification, all these states must fit into some representation of $SO(D-1)$ as the little group for massive representations is $SO(D-1)$. For $N = \widetilde{N} = 2$, the states are

$$\begin{aligned}
&\alpha^i_{-1}\alpha^j_{-1}\left|0;p^\mu\right\rangle, \alpha^i_{-2}\left|0;p^\mu\right\rangle \quad &&\text{– Right moving} \\
&\widetilde{\alpha}^i_{-1}\widetilde{\alpha}^j_{-1}\left|0;p^\mu\right\rangle, \widetilde{\alpha}^i_{-2}\left|0;p^\mu\right\rangle \quad &&\text{– Left moving.}
\end{aligned}$$

Since $\alpha^i_{-1}, \alpha^j_{-1}$ commute, in the right moving sector there are a total of

$$\begin{aligned}
\frac{1}{2}(D-2)(D-1) + (D-2) &= (D-2)\left(\frac{D-1+2}{2}\right) \\
&= \frac{(D-2)(D+1)}{2} \\
&= \frac{1}{2}D(D-1) - 1
\end{aligned}$$

states. These easily fit into the symmetric traceless representation of $SO(D-1)$. Infact one can prove that all higher excited states fit into some representation of $SO(D-1)$. Hence we have recovered Lorentz invariance by fixing the dimension of spacetime.

There is one other way to explicitly check that we have recovered Lorentz invariance: We compute the conserved charges and currents corresponding to the global Poincaré symmetry $X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu + C^\mu$ of the action and require that they satisfy Poincaré algebra. Let us begin with translations $X^\mu \longrightarrow X^\mu + C^\mu$. One can compute the Noether current. It turns out to be:

$$P^\alpha_\mu = \frac{1}{2\pi\alpha'}\partial^\alpha X_\mu.$$

[4] I recommend going through Appendix A to understand the uses of Wigner's classification in string theory context.

It is easy to see that $\partial_\alpha P^\alpha_\mu = 0$ as $\partial_\alpha \partial^\alpha X_\mu = 0$ on-shell. Next the Noether charge corresponding to Lorentz transformation $X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu$ is

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu.$$

We can again check that $\partial_\alpha J^\alpha_{\mu\nu} = 0$. Indeed we have

$$\begin{aligned}
\partial_\alpha J^\alpha_{\mu\nu} &= \left(\partial_\alpha P^\alpha_\mu\right) X_\nu + P^\alpha_\mu \partial_\alpha X_\nu - \left(\partial_\alpha P^\alpha_\nu\right) X_\mu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= P^\alpha_\mu \partial_\alpha X_\nu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= \frac{1}{2\pi\alpha'} \left(\partial^\alpha X_\mu \partial_\alpha X_\nu - \partial^\alpha X_\nu \partial_\alpha X_\mu\right) \\
&= 0.
\end{aligned}$$

The conserved charges corresponding to J^τ_μ is

$$M_{\mu\nu} = \int_0^\pi d\sigma J^\tau_{\mu\nu}.$$

Now using the mode expansion for X^μ we get

$$\begin{aligned}
M^{\mu\nu} = \int_0^\pi d\sigma \left(X^\mu \Pi^\nu - X^\nu \Pi^\mu\right) &= \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \\
&= l^{\mu\nu} + E^{\mu\nu} + \widetilde{E}^{\mu\nu},
\end{aligned}$$

where

$$\begin{aligned}
l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, \\
E^{\mu\nu} &= -i \sum_{n=1}^\infty \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_n \alpha^\mu_n\right), \\
\widetilde{E}^{\mu\nu} &= -i \sum_{n=1}^\infty \frac{1}{n} \left(\widetilde{\alpha}^\mu_{-n} \widetilde{\alpha}^\nu_n - \widetilde{\alpha}^\nu_{-n} \widetilde{\alpha}^\mu_n\right).
\end{aligned}$$

The first piece $l^{\mu\nu}$ is the orbital angular momentum of the string while the other two pieces arise from excited states. Classically, one can check that the Poisson bracket for $M_{\mu\nu}$ satisfies Lorentz algebra. In covariant quantisation, it is easy to check that $M_{\mu\nu}$ satisfies the Lorentz algebra but in lightcone quantisation, things are not so clear. In lightcone gauge, we must be able to produce the Lorentz algebra i,e.

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}.$$

The only bracket which is non trivial is $[M^{i-}, M^{j-}] = 0$. This commutator involves p^- and α^-_n which has been fixed in lightcone gauge in terms of other transverse oscillators. A messy calculation gives

$$\begin{aligned}
\left[M^{i-}, M^{j-}\right] = &\frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{D-2}{24} - 1\right] n + \frac{1}{n}\left[a - \frac{D-2}{24}\right]\right) \left(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n\right) \\
&+ + \frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{D-2}{24} - 1\right] n + \frac{1}{n}\left[a - \frac{D-2}{24}\right]\right) \left(\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^j_n - \widetilde{\alpha}^j_{-n}\widetilde{\alpha}^i_n\right),
\end{aligned}$$

which is 0 if and only if $a = 1$ and $D = 26$. This is consistent with our earlier derivation of the critical dimension.

3.4.5 First String Excitation

The first excited states are massless representations of the little group $SO(D-2)$. There are a total $(D-2)^2$ particles (tensor product of the left and right-moving sectors) in the first excitation. So we want to get the irreducible representations of $SO(D-2)$ of dimension $(D-2)^2$ so that each irreducible factor would correspond to an elementary particle by Wigner's proposal. Using the method of Young Tableau, we can prove that the tensorial representation of $SO(D-2)$ of dimension $(D-2)^2$ consists of three irreducible parts:

$$\begin{array}{cccc} & \textbf{Traceless symmetric} \oplus & \textbf{Antisymmetric} \oplus & \textbf{Trace (Scalar)} \\ \text{Dim:} & \dfrac{(D-2)(D-1)}{2} - 1 & \dfrac{(D-2)(D-3)}{2} & 1 \end{array}$$

Following the usual method of constructing field theory from representations, we attach a tensor field to each of these representation. We get three particles.

1. $G_{\mu\nu}(X)$: the traceless symmetric tensor field which we will identify with *graviton*.
2. $B_{\mu\nu}(X)$: the antisymmetric tensor field. This is sometimes called the *Kalb-Ramond field.*
3. $\Phi(X)$: the trace part of the tensor representations. This scalar field is called the *dilaton.*

To see that these fields arise in our theory, we decompose the first excited state as follows:

$$\alpha_1^i \widetilde{\alpha}_{-1}^j |0;p^\mu\rangle = \underbrace{\left(\alpha_{-1}^{(i} \widetilde{\alpha}_{-1}^{j)} - \frac{1}{D-2}\delta^{ij}\alpha_{-1}^k\widetilde{\alpha}_{-1}^k\right)|0;p^\mu\rangle}_{\text{symmetric traceless}} + \underbrace{\alpha_{-1}^{[i}\widetilde{\alpha}_{-1}^{j]}|0;p^\mu\rangle}_{\text{antisymmetric}} + \frac{1}{D-2}\underbrace{\delta^{ij}\alpha_{-1}^k\widetilde{\alpha}_{-1}^k|0;p^\mu\rangle}_{\text{trace}},$$

where $(,)$ and $[,]$ are the symmetrized and antisymmetrized indices. The traceless symmetric field $G_{\mu\nu}$ is particularly interesting as it represents massless symmetric, traceless rank two tensor field. We will identify this field with the metric of spacetime, the graviton because Weinberg in 1965 [15] showed that any interacting theory of massless symmetric, traceless rank two tensor field is Einstein's gravity. Later we will explicitly derive Einstein's field equations from this field.

Chapter 4

Open Strings and D-Branes

In the previous chapter, we quantised the closed string and found that the spectrum contains three particles including the graviton. In this chapter, we will quantise the open strings with different boundary conditions.

4.1 Solving the Equations of Motion

We have already found the equations of motion of the open string subject to three different boundary conditions in Subsection 2.2.2. As already mentioned in Subsection 2.2.2, we will normalise the length of the string so that $\sigma \in [0, \pi)$. We will now solve the equations of motion for the first two boundary conditions.

4.1.1 Neumann Boundary Condition at Both Ends (NN)

This means that

$$\partial_\sigma X^\mu = 0 \quad \text{for} \quad \sigma = 0, \pi.$$

Since the equation of motion is

$$\partial_\alpha \partial^\alpha X^\mu = 0,$$

we again have

$$X^\mu(\sigma, \tau) = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

with

$$X_L^\mu\left(\sigma^+\right) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu e^{-in\sigma^+}$$

and

$$X_R^\mu\left(\sigma^-\right) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}.$$

Now $\sigma = \frac{\sigma^+ - \sigma^-}{2}$, so $\sigma = 0 \Rightarrow \sigma^+ = \widetilde{\sigma} = \tau/2$. Since

$$\partial_\sigma X^\mu = \frac{1}{2}\left(\partial_+ X^\mu - \partial_- X^\mu\right),$$

the condition $\partial_\sigma X^\mu = 0$ implies $\partial_+ X^\mu = \partial_- X^\mu$. Using the Fourier expansion above, we get

$$\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \Bigg|_{\sigma=0,\pi} = \alpha\prime p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \Bigg|_{\sigma=0,\pi}.$$

At $\sigma = 0$, we get

$$\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\widetilde{\alpha}_n^\mu - \alpha_n^k\right) e^{-n\pi\tau_2} = 0 \quad \Rightarrow \quad \widetilde{\alpha}_n^\mu = \alpha_n^\mu \quad \forall\ n \neq 0.$$

So we have

$$X^\mu = x^\mu + 2p^\mu \alpha' \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \left(e^{-in\sigma} + e^{in\sigma}\right).$$

This gives

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{4.1.1}$$

We can check that the boundary condition at $\sigma = \pi$ is automatically satisfied. Again we can check that x^μ le p^μ are center of mass position and momentum of the string. Constraints are

$$(\partial_+ X)^2 = 0 = (\partial_- X)^2.$$

With the given Fourier expansion, we still have the same classical constraints

$$L_n = 0 \text{ where } L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \boldsymbol{\alpha}_{n-k} \cdot \boldsymbol{\alpha}_k \quad \forall n \in \mathbb{Z}$$

where now $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$. The Poisson bracket for α_n^μ are still the same.

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{P.B.} = -im\eta^{\mu\nu}\delta_{m+n,0}, \quad \{x^\mu, p^\nu\}_{P.B.} = \eta^{\mu\nu}. \tag{4.1.2}$$

Virasoro algebra is also the same

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}.$$

The Poisson bracket for the Fourier modes and the Virasoro generators remain the same.

4.1.2 Dirichlet Boundary Condition at Both Ends (DD)

We impose $\delta X^\mu = 0$ at $\sigma = 0, \pi$. This means that $\dot{X}^\mu = 0$ at $\sigma = 0, \pi \quad \forall\ \tau$. Suppose $X^\mu(0,\tau) = x_0^\mu$ and $X^\mu(\pi,\tau) = x_1^\mu$. The constraint is the same. We can still write

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right).$$

where

$$\begin{aligned}
X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+}\\
X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}.
\end{aligned}$$

But the boundary condition implies

$$\begin{aligned}
X^\mu(0,\tau) = x_0^\mu \Rightarrow \quad & x^\mu + 2p^\mu\alpha'\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu + \alpha_n^\mu\right)e^{-in\tau}\\
& \Rightarrow p^\mu = 0, \quad x^\mu = x_0^\mu \quad \text{and} \quad \widetilde{\alpha}_n = -\alpha_n^\mu
\end{aligned}$$

But the second condition $X^\mu(\pi,\tau) = x_1^\mu$ is not satisfied. Thus the general solution must have the form

$$X^\mu = x_0^\mu + \frac{x_1^\mu - x_0^\mu}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma). \tag{4.1.3}$$

This is gotten by assuming the forms of X_L and X_R as

$$\begin{aligned}
X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+}\\
X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu - p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-},
\end{aligned}$$

so that

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu e^{in\sigma^+} + \alpha_n^\mu e^{-in\sigma^-}\right).$$

Then $X^\mu(0,\tau) = x_0^\mu \Rightarrow x^\mu = x_0^\mu \quad$ and $\quad \widetilde{\alpha}_n^\mu = -\alpha_n^\mu$ and $X^\mu(\pi,\tau) = x_1^\mu$ implies

$$\begin{aligned}
& x_0^\mu + 2\alpha' p^\mu\pi + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu e^{-in\pi} - \widetilde{\alpha}_n^\mu e^{in\pi}\right)e^{-in\tau} = x_1^\mu\\
& \Rightarrow x_0^\mu + 2\alpha' p^\mu\pi = x_1^\mu\\
& \Rightarrow 2\alpha' p^\mu = \frac{x_1^\mu - x_0^\mu}{\pi}.
\end{aligned}$$

There is no center of mass momentum and the center of mass position is $\frac{x_0^\mu+x_1^\mu}{2}$ as is easily computed:

$$q^\mu = \frac{L}{\pi}\int_0^\pi d\sigma x^\mu(\sigma,\tau) = x_0^\mu + \frac{1}{\pi}\frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2}\pi^2 + 0 = \frac{x_0^\mu + x_1^\mu}{2}.$$

Next, we find the classical constraints in terms of Fourier modes. The constraints are

$$\left(\partial_+ X^\mu\right)^2 = 0 = \left(\partial_- X^\mu\right)^2.$$

We have

$$\begin{aligned}
\partial_+ X^\mu &= \frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2} + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu \frac{1}{2in}\partial_+\left(e^{-in(\tau-\sigma)} - e^{-in(\tau+\sigma)}\right)\\
&= \frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2} + \sqrt{\frac{\alpha}{2}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^+}\\
&= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\sigma^+},
\end{aligned}$$

where

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{x_1^\mu - x_0^\mu}{\pi}.$$

Similarly

$$\partial_- X^\mu = -\sqrt{\frac{\alpha\prime}{2}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^-},$$

with same $\boldsymbol{\alpha}_0$. Thus constraints are

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_k = 0 \quad \forall n\in\mathbb{Z}.$$

All Poisson brackets remain the same.

4.1.3 Neumann at $\sigma = 0$ and Dirichlet at $\sigma = \pi$ (ND)

This means

$$\partial_\sigma X^\mu = 0 \text{ at } \sigma = 0,\tau \text{ and } X^\mu = x^\mu \text{ at } \sigma = \pi,\tau.$$

As usual

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right).$$

where

$$\begin{aligned}
X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+}\\
X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}.
\end{aligned}$$

The condition $\partial_\sigma X^\mu = 0$ at $\sigma = 0 \Rightarrow \alpha_n^\mu = \widetilde{\alpha}_n^\mu$ as in previous case. Next

$$
\begin{aligned}
X^\mu = x^\mu \text{ at } \sigma = \pi \quad &\Rightarrow p^\mu = 0 \\
i\sqrt{\frac{\alpha'}{2}} 2 \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\pi) = 0 \quad &\forall \quad \tau.
\end{aligned}
$$

This is possible only if $\cos(n\pi) = 0 \quad \forall n \Rightarrow n \in \mathbb{Z} + \frac{1}{2}$. So the sum must actually run over half integers. So we get

$$
X^\mu(\sigma, \tau) = x^\mu + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{4.1.4}
$$

One can again show that the oscillators, which are now half integral, satisfy the same Poisson bracket. It is easy to check that

$$
\partial_\pm X^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \alpha_n^\mu e^{-in\sigma^\pm} \Rightarrow (\partial_\pm X^\mu)^2 = \alpha' \frac{1}{2} \sum_{n\in\frac{1}{2}\mathbb{Z}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \boldsymbol{\alpha}_{n-r} \cdot \boldsymbol{\alpha}_r e^{-in\sigma^\pm},
$$

so that the classical constraints are again the same with the same expression for the Virasoro generators:

$$
L_n = \frac{1}{2} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \boldsymbol{\alpha}_{n-r} \cdot \boldsymbol{\alpha}_r = 0 \quad \forall\ n \in \frac{1}{2}\mathbb{Z}.
$$

4.1.4 Dirichlet at $\sigma = 0$ and Neumann at $\sigma = \pi$ (DN)

Following similar process as in Subsection 4.1.3, we get that

$$
X^\mu(\sigma, \tau) = x^\mu + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \sin(n\sigma), \tag{4.1.5}
$$

and

$$
\partial_\pm X^\mu = \pm\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}+\frac{1}{2}} \alpha_n^\mu e^{-in\sigma^\pm}.
$$

This gives us the same classical constraints. The Poisson bracket also remains the same.

4.1.5 NN for $0 \le \mu \le p$ and DD for $p+1 \le \mu \le D-1$: D-Branes

This means that

$$
\begin{aligned}
\partial_\sigma X^a = 0 \text{ for } a = 0, \cdots, p \text{ at } \sigma = 0, \pi \\
X^I(0,\tau) = c^I, \quad X^I(\pi, \tau) = d^I \quad \text{ for } I = p+1, \cdots, D-1.
\end{aligned}
$$

This fixes the endpoints of the string in the $D-p-1$ directions and hence is constrained to move in the $(p+1)-$dimensional hypersurface. This hypersurface is usually called a *Dp-Brane*. So a $D0$-brane is a particle, a $D1$-brane is itself a string, a $D2$-brane is a membrane and so on. In particular if $p = D-1$ then we get to NN case which means all space is a D-brane, that is we get space filling D-brane. Combining Fourier modes of NN and DD conditions, we get

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu + 2p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma), \quad \mu = 0,1,\cdots,p \\ X^\mu(\sigma,\tau) &= c^\mu + \frac{d^\mu - c^\mu}{2}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma), \quad \mu = p+1,\cdots,D-1. \end{aligned} \tag{4.1.6}$$

One can also work out the Poisson bracket and show that they remain the same.

4.2 Quantisation

We can again quantise the open string in the cannonical way or using path integral. Here we will discuss the cannonical quantisation. As usual, it can be done in two ways. We will quickly discuss covariant quantisation but the lighcone quantisation will be discussed in some detail.

4.2.1 Covariant Quantisation

Using the classical Poisson brackets (4.1.2), we impose the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\delta_{m+n,0}\eta^{\mu\nu}, \tag{4.2.1}$$

with all others being zero. Construct the Fock space as usual from ground state $|0;p^\mu\rangle$. We will again encounter ghosts which we can again get rid of by choosing a and D as in closed string case by the same spurious state analysis. Infact in open string case we only have one set of Virasoro generators

$$L_n = \sum_r \boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r,$$

where the summation index and mode index run over integers or half-integers depending on boundary conditions whether NN, DD or DN, ND. The quantum Virasoro algebra is again the same i,e. the central extension of the Witt algebra. Thus the constraints are again imposed as

$$\begin{aligned} L_n|\text{phys}\rangle &= 0 \\ (L_0 - a)\,|\text{phys}\rangle &= 0 \end{aligned}$$

where a is the normal ordering constant. The number operator is

$$N = \sum_{n=1}^{\infty}\left(\alpha_{-n}^\mu\alpha_{\mu,n} + \alpha_{-n}^i\alpha_{i,n}\right) + \sum_{r\in\mathbb{N}_0+\frac{1}{2}}\alpha_{-r}^a\alpha_{a,r},$$

where μ denotes NN direction, i denotes DD direction and a denotes the DN and ND directions and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Again using the spurious state discussion we have $a = 1, D = 26$ for our spectrum to be ghost free. Lorentz invariance is manifest and the the normal ordering constant drops out of any expressions involving angular momentum.

4.2.2 Lightcone Quantisation

As usual, we go to lightcone gauge by introducing

$$X^{\pm} = \frac{1}{\sqrt{2}} \left(X^0 \pm X^p \right)$$

and choosing $X^+ = 2\alpha' p^+ \tau$. It is easy to see that $X^{\pm}$ has to satisfy Neumann boundary condition (due to τ in X^+). The X^+ oscillators are all zero except the zero mode

$$\alpha_0^+ = \sqrt{2\alpha'} p^+.$$

As in closed string case, the oscillators of X^- is determined by the transverse oscillators upto a constant x^-. Let us now impose the commutation relations

$$\begin{aligned} \left[q^-, p^+ \right] = -i, \left[q^i, p^j \right] = i\delta^{ij} \\ \left[\alpha_n^i, \alpha_m^j \right] = n\delta^{ij} \delta_{n+m,0}. \end{aligned} \tag{4.2.2}$$

We can now construct the Fock space from vacuum $|0; p^\mu\rangle$ by acting $\alpha_m^i, \ m < 0$ on $|0; p^\mu\rangle$. The spectrum is manifestly ghost free. Let us look at the ordering ambiguity. We have

$$\begin{aligned} L_0 - \alpha_0^2 = \sum_{n \neq 0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n &= \sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \sum_{n<0}^{0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n \\ &= \sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \sum_{n<0} \left(\boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n - n(D-2) \right) \\ &= 2 \left(\sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \frac{D-2}{2} \sum_{n>0} n \right). \end{aligned}$$

Now the sum above can go over integer or half-integer depending on NN, DD or ND, DN boundary conditions. In integral case, the last term is regularised using zeta function:

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}.$$

If the sum goes over half integers then the last term is regularised using Hurwitz zeta function. The last sum can be written as

$$\sum_{n \in \mathbb{N}_0 + \frac{1}{2}} n = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right).$$

The Hurwitz zeta function is defined as

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}$$

and is holomorphic for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(q) > 0$. It can be analytically continued to the whole s-plane for a given value of q in the domain of definition but we do not need the complete sophisticated machinery here rather a simple trick here will do the job. We note that

$$\begin{aligned}
\zeta\left(s, \frac{1}{2}\right) &= 2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \\
&= 2^s \left[\zeta(s) - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}\right] \\
&= 2^s \left(\zeta(s) - 2^{-s}\zeta(s)\right) \\
&= (2^s - 1)\zeta(s).
\end{aligned}$$

Thus, using the analytic continuation of the Riemann zeta function, we get

$$\zeta\left(-1, \frac{1}{2}\right) = -\frac{1}{2}\zeta(-1) = \frac{1}{24}.$$

We have five boundary conditions. In general we can have a mix of all those boundary conditions. Let i_1 be NN and DD directions and i_2 be ND and DN directions. Then we have

$$L_0 - \alpha_0^2 = 2\sum_{n=1}^{\infty} \alpha_{-n}^{i_1}\alpha_{i_1,n} + 2\sum_{n\in\mathbb{N}_0+\frac{1}{2}} \alpha_{-n}^{i_2}\alpha_{i_2,n} + D_1\left(-\frac{1}{12}\right) + D_2\left(\frac{1}{24}\right)$$

where $D_1 + D_2 = D - 2$ where D_1 denotes the total number of NN and DD directions and D_2 denotes the total number of DN and ND directions. We recognise the last two constant terms as the contribution to normal ordering constant. In terms of the number operator, we have

$$\begin{aligned}
L_0 - a &= \alpha_0^2 + 2\sum_{n=1}^{\infty} \alpha_{-n}^{i_1}\alpha_{i_1,n} + 2\sum_{n\in\mathbb{N}_0+\frac{1}{2}} \alpha_{-n}^{i_2}\alpha_{i_2,n} + D_1\left(-\frac{1}{12}\right) + D_2\left(\frac{1}{24}\right) \\
&= \alpha_0^2 + 2N + D_1\left(-\frac{1}{12}\right) + D_2\left(\frac{1}{24}\right).
\end{aligned}$$

If we consider the first string excitation $\alpha_{-n}^i|0;p^\mu\rangle$ where $n = 1$ if $i = i_1$ and $n = \frac{1}{2}$ if $i = i_2$ then

$$N\alpha_{-n}^i\left|0;p^\mu\right\rangle = n\alpha_{-n}^i\left|0;p^\mu\right\rangle.$$

Next the mass-spectrum is calculated using the constraint

$$(L_0 - a)\,|\text{phys}\rangle = 0.$$

We now need to find an expression for α_0^2. Indeed, note that in the DD directions,

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{x_1^\mu - x_0^\mu}{\pi} = \frac{\Delta X^\mu}{\sqrt{2\alpha'}\pi}, \quad \Delta X^\mu := x_1^\mu - x_0^\mu,$$

where ΔX is the string length in the DD direction. In the NN direction,

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu,$$

and there are no zero modes in ND and DN directions. This implies that

$$\alpha_0^2 = 2\alpha' p^2 + \left(\frac{\Delta X}{\sqrt{2\alpha'}\pi}\right)^2 = -2\alpha' M^2 + 2\alpha'\left(\frac{\Delta X}{2\alpha'\pi}\right)^2.$$

Using the expression for $(L_0 - a)$ from previous calculation, we get

$$\begin{aligned} 2N - \frac{D_1}{12} + \frac{D_2}{24} - 2\alpha' M^2 + 2\alpha'\left(\frac{\Delta X}{2\pi\alpha'}\right)^2 &= 0 \\ \implies \qquad \alpha' M^2 = N - \frac{D-2}{24} + \frac{D_2}{16} + \alpha'\left(\frac{\Delta X}{2\pi\alpha'}\right)^2&, \end{aligned} \tag{4.2.3}$$

where N is the number of states in the physical excitation. Thus we see that any physical excitation has to satisfy the above mass-shell condition. Let us explore the origin of the extra term ΔX. Note that we have

$$\alpha_0^2 = -2\alpha' M^2 + \alpha'\left(\frac{\Delta X}{2\alpha'\pi}\right)^2.$$

The extra term has natural physical interpretation: it is the mass of the string stretched between two branes.

4.2.3 String Spectrum

Let us start with NN boundary conditions. The ground state is $|0; p^\mu\rangle$ and the mass spectrum gives

$$M^2 = -\frac{D-2}{24\alpha'} < 0.$$

So the ground state is Tachyonic. The first excited state is

$$\alpha_{-1}^i\,|0; p^\mu\rangle$$

which transforms as a vector representation of $SO(D-2)$. Again Wigner's theorem implies that this state is a massless representation. Thus we get

$$1 - \frac{D-2}{24} = 0 \Rightarrow D = 26.$$

We can go on constructing the higher excited states and show that $D = 26$ forces all of them to be massive representations of the Lorentz group. At level n, the mass spectrum is

$$\alpha' M^2 = n - 1$$

and at level n, the representation includes a symmetric tensor of rank n (this comes from Young Tableau method which we shall not describe here). This state corresponds to the maximum spin n of this excitation. Let us pause and prove this. For each spin component, we will produce a level N state and show that its spin eigenvalue corresponding to the particular component is N. To make this explicit, we first recall the spin generators

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^{\mu}_{-n} \alpha^{\nu}_{n} - \alpha^{\nu}_{n} \alpha^{\mu}_{n} \right), \quad \widetilde{E}^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\widetilde{\alpha}^{\mu}_{-n} \widetilde{\alpha}^{\nu}_{n} - \widetilde{\alpha}^{\nu}_{n} \widetilde{\alpha}^{\mu}_{n} \right).$$

We will distinguish between open and closed strings. In lightcone quantisation, the relevant spin generators are E^{ij} and $\widetilde{E}_{ij}$ for $1 \leq i, j \leq D-2$. In closed string case, the state corresponding to the spin component E^{ij} and $\widetilde{E}^{ij}$ is given by

$$\Omega^{ij} = \left(\alpha^{i}_{-1} + i \alpha^{j}_{-1} \right)^{N} \left(\widetilde{\alpha}^{i}_{-1} + i \widetilde{\alpha}^{j}_{-1} \right)^{N} |0; p^{\mu}\rangle .$$

Now observe that

$$\begin{aligned} E^{ij}\Omega^{ij} &= \left(\widetilde{\alpha}^{i}_{-1} + i \widetilde{\alpha}^{j}_{-1} \right)^{N} (-i) \sum_{n=1} \frac{1}{n} \left(\alpha^{i}_{-n} \alpha^{j}_{n} - \alpha^{j}_{-n} \alpha^{i}_{n} \right) \left(\alpha^{i}_{-1} + i \alpha^{i}_{-1} \right)^{N} |0; p^{\mu}\rangle \\ &= \left(\widetilde{\alpha}^{i}_{-1} + i \widetilde{\alpha}^{j}_{-1} \right)^{N} (-i) \left(\alpha^{i}_{-1} \alpha^{j}_{1} - \alpha^{j}_{-1} \alpha^{i}_{1} \right) \left(\alpha^{i}_{-1} + i \alpha^{j}_{-1} \right)^{N} |0; p^{\mu}\rangle , \end{aligned}$$

where we used the fact that $\left(\alpha^{i}_{-n} \alpha^{j}_{n} - \alpha^{j}_{-n} \alpha^{i}_{n} \right)$ commutes with $\alpha^{i}_{-1} + i \alpha^{j}_{-1} \quad \forall n > 1$. Now we have

$$\begin{aligned} \left[\alpha^{i}_{-1} \alpha^{j}_{1} - \alpha^{j}_{-1} \alpha^{i}_{1}, \ \alpha^{i}_{-1} + i \alpha^{j}_{-1} \right] &= \alpha^{i}_{-1} \left[\alpha^{j}_{1}, \alpha^{i}_{-1} \right] + i \alpha^{i}_{-1} \left[\alpha^{j}_{1}, \alpha^{j}_{-1} \right] - \alpha^{j}_{-1} \left[\alpha^{i}_{1}, \ \alpha^{i}_{-1} \right] - i \alpha^{j}_{-1} \left[\alpha^{i}_{1}, \alpha^{j}_{-1} \right] \\ &= \alpha^{i}_{-1} \delta^{ji} + i \alpha^{i}_{-1} - \alpha^{j}_{-1} - i \alpha^{j}_{-1} \delta^{ij} \\ &= \begin{cases} i \left(\alpha^{i}_{-1} + i \alpha^{j}_{-1} \right) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases} \end{aligned}$$

So assuming $i \neq j$, we get

$$\begin{aligned} E^{ij}\Omega^{ij} &= \left(\widetilde{\alpha}^{i}_{-1} + i \widetilde{\alpha}^{j}_{-1} \right)^{N} (-i) \left(\alpha^{i}_{-1} \alpha^{j}_{1} - \alpha^{j}_{-1} \alpha^{i}_{1} \right) \left(\alpha^{i}_{-1} + i \alpha^{j}_{-1} \right)^{N} |0; p^{\mu}\rangle \\ &= \left(\widetilde{\alpha}^{i}_{-1} + i \widetilde{\alpha}^{j}_{-1} \right)^{N} (-i)(iN) \left(\alpha^{i}_{-1} + i \alpha^{j}_{-1} \right)^{N} |0; p^{\mu}\rangle \\ &= N\Omega^{ij}. \end{aligned}$$

Similarly $\widetilde{E}^{ij}\Omega^{ij} = N\Omega^{ij}$. In case of open strings with NN boundary conditions:

$$E^{ij}\left(\alpha_{-1}^{i} + i\alpha_{-1}^{j}\right)^{N}|0;p^{\mu}\rangle = N\left(\alpha_{-1}^{i} + 2\alpha_{-1}^{j}\right)|0;p^{\mu}\rangle\,.$$

Thus the maximum spin at each level is

$$J_{\max} = n.$$

Hence we have

$$J_{\max} = \alpha' M^2 + 1.$$

If we plot $J_{\max}$ verses M^2 at each level, we get a straight line with slope α'. This is why α' is called the *Regge slope*. All states at a given level satisfy

$$J_{\max} \le \alpha' M^2 + 1,$$

and since J and M^2 are quantised, all states lie on straight lines with the Tachyon lying on the leading trajectory. These lines are called *Regge trajectories*. Regge trajectories are observed in nature both for Mesons and baryons.

We now consider Dp branes i,e. NN boundary conditions in $p{+}1$ directions and DD boundary conditions in $D-p-1$ direction. There are two cases to distinguish.

One Dp Brane

In this case, we have

$$X^{\mu}(0,\tau) = c^{I} = X^{\mu}(\pi,\tau) \quad \mu = p+1,\ldots,D-1.$$

Thus the ends of the string are constrained to lie on one Dp brane. The ground state is now defined by

$$\alpha_{n}^{i}|0;p^{\mu}\rangle = 0, \quad n>0, \quad i = 1,2,\cdots p-1, p+1,\cdots, D-1.$$

Note that the string momentum p^{μ} is actually only in $p+1$ directions. The $SO(1,D-1)$ Lorentz symmetry is broken to the subgroup $SO(1,p)\times SO(D-p-1)$. This means that Lorentz symmetry on the brane still holds while in the spacetime, the brane is like a "defect" wall. Again Lorentz invariance requires $D=26$ and $a=1$ as we can readily see by looking at the mass spectrum of first excited state. To be explicit, the first excited states are $\alpha_{-1}^{i}|0;p^{\mu}\rangle$ for $i=1,2,\ldots,p-1$ which transforms as a vector representation of $SO(p-1)$ while the transverse modes $\alpha_{-1}^{I}|0;p^{\mu}\rangle$ transforms as a vector under $SO(D-p-1)$. But since $SO(p-1)\times SO(D-p-1)$ is the little group for massless representation of the "effective" Lorentz group $SO(1,p)\times SO(D-p-1$, the first excited state must be massless to preserve Lorentz symmetry. Now using the mass-shell condition (4.2.3), we get $D=26$ and $a=1$. As the first excited state has maximum spin 1, the states $\alpha_{-1}^{i}|0;p^{\mu}\rangle$ for $i=1,2,\ldots,p-1$ are gauge fields as is known from quantum field theory. We introduce a gauge field A_i, $i=0,\ldots,p$ and its quanta represents spin 1 photons. The transverse oscillators

$$\alpha_{-1}^{I}|0;p^{\mu}\rangle\,, \quad I = p+1,\ldots,D-1.$$

These transform as scalar representations of $SO(1,p)$ and hence we introduce $D-p-1$ scalar fields ϕ^I. These ϕ^I have physical interpretation of fluctuations of the Dp brane. This suggests that Dp branes are themselves dynamical as we will see later. Although ϕ^I transform as scalars under the $SO(1,p)$ Lorentz group of the Dp brane they transform as vectors as representations of the $SO(D-p-1)$ rotation group. This appears as a global symmetry of the brane world volume. One can also consider ϕ^I as the Goldstone Bosons associated to the spontaneously broken translational symmetry.

Two Dp Branes: String Stretched Between Two Branes

In this case $X^\mu(0,\sigma) = x_0^\mu \neq x_1^\mu = X^\mu(\pi,\sigma), \quad \mu = p+1,\cdots,D-1$. From (4.2.3) we see that there is a shift in mass spectrum:

$$\alpha' M^2 = N - \frac{D-2}{24} + \alpha' \left(\frac{x_1^\mu - x_0^\mu}{2\alpha'\pi}\right)^2 .$$

Thus the states $\alpha^i_{-1}\,|\Delta x^\pm, p^i\rangle$ are no longer massless. In general we can stack N such Dp branes on top of each other (that is the branes are coincident) and denote the massless vector excitation as

$$\alpha^i_{-1}|k,\ell,p^i\rangle$$

where k,ℓ are labels which encode the Dp branes on which the endpoints of the string end. These are called *Chan-Paton labels*. The resulting N^2 states can be embedded in an $N\times N$ matrix and expanded in a complete set of $N\times N$ matrices

$$\left|k,\ell;p^i\right\rangle = \lambda^a_{k\ell}|a;p^i\rangle, \quad a \in \left\{1,\cdots,N^2\right\},$$

where $\lambda^a_{k\ell}$ are called *Chan-Paton factors*. The resulting fields T^k_ℓ, $\left(\phi^I\right)^k_\ell$ and $(A^a)^k_\ell$ can be fit into Hermitian matrices. Here T is the open string Tachyon. The diagonal fields arise from strings ending on same brane. We will later see that $(A^a)^k_\ell$ are identified with $U(N)$ Yang-Mills gauge Bosons and $\left(\phi^I\right)^k_\ell$ transform in the adjoint representation of $U(N)$.

4.3 Discrete Diffeomorphisms: Oriented verses Nonoriented Strings

Until now, we dealt with oriented string theories, that is we have not considered reparametrizations of the form

$$\sigma \to \sigma' = \pi - \sigma$$
$$\tau \to \tau' = \tau.$$

Such a reparametrization respects the periodicity of closed strings and maps the two ends of an open string to each other and reverses the orientation $d\sigma\wedge d\tau$ of the worldsheet[1]. The

[1]orientation of a manifold can be defined in terms of a top form on the manifold.

above discrete diffeomorphism can be implemented by a unitary operator Ω:

$$\Omega X^\mu(\sigma,\tau)\Omega^{-1} = X^\mu(\pi-\sigma,\tau).$$

Since the same operation twice is trivial we demand $\Omega^2 = 1$. So that the only eigenvalues of Ω can be ± 1. These actions can be expressed in terms of the oscillators. For closed string we substitute

$$X^\mu(\sigma,\tau) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in\sigma^-} + \widetilde{\alpha}_n^\mu e^{-in\sigma^+}\right).$$

Also the length of the string was normalised to 2π, so that the condition now becomes

$$\Omega X^\mu(\sigma,\tau)\Omega^{-1} = X^\mu(2\pi-\sigma,\tau).$$

This gives

$$\Omega\alpha_n^\mu\Omega^{-1} = \widetilde{\alpha}_n^\mu, \quad \Omega\widetilde{\alpha}_n^\mu\Omega^{-1} = \alpha_n^\mu.$$

For open string, we need to differentiate between different boundary conditions. Using similar calculation as for closed strings, we get the following:

- NN boundary condition: $\Omega\alpha_n^\mu\Omega^{-1} = (-1)^n\alpha_n^\mu$.
- DD boundary condition: $\Omega\alpha_n^\mu\Omega^{-1} = (-1)^{n+1}\alpha_n^\mu \quad \Omega x_{0,1}^\mu\Omega^{-1} = x_{1,0}^\mu$.
- ND-DN boundary condition: $\Omega\alpha_{n+\frac{1}{2}}^{\mu,ND}\Omega^{-1} = i(-1)^n\alpha_{n+\frac{1}{2}}^{\mu,DN}$.

We need to fix the action of Ω on the ground state. It turns out that $\Omega|0;p^\mu\rangle$ is determined upto a sign which is fixed by the so called *Tadpole cancellation* (will be investigated later). For closed strings, the unoriented string spectrum must be invariant under left moving - right moving sector exchange. This means that of the three massless fields, only graviton and dilaton survives. This is called the *restricted Shapiro-Virasoro model* and the oriented one is called the *extended Shapiro-Virasoro model.*

Let us now turn to the open strings. If Ω acts on the ground state with plus sign, then the unoriented open string spectrum with NN (respectively DD) boundary condition must consist of even (respectively odd) level number. For $2N$ branes stacked on top of each other, one must also consider the action of Ω on the Chan-Paton factors. Since Ω changes orientations $\left(\Omega x_{1,0}^\mu\Omega^{-1} = x_{0,1}^\mu\right)$ we have

$$\Omega\,|k,\ell;p^\mu\rangle = |\ell,k,p^\mu\rangle$$

at massless vector level. This means we only have $N(2N-1)$ (symmetric) surviving Chan-Paton factors. Thus we get a massless vector of a $SO(2N)\subset U(2N)$ gauge theory. If Ω acts with negative sign, the Chan-Paton labels are antisymmetrized and we get a massless vector of a $Sp(2N)\subset U(2N)$ gauge theory where $Sp(2N)$ is the symplectic group of rank N defined as follows:

$$Sp(2N) = \left\{M\in GL(2N,\mathbb{R}) : MJM^T = J\right\}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

where $\mathbb{I}$ is $N \times N$ identity matrix. This is because the dimension of the antisymmetric representation is

$$(2N)^2 - \frac{2N(2N-1)}{2} = (2N)\left[\frac{4N-(2N-1)}{2}\right] = N(2N+1)$$

which is equal to the real dimension of $Sp(2N)$.

Chapter 5

Conformal Field Theory

In this chapter, we will review conformal transformations and conformal group in detail. We will describe the conformal transformations in N-dimensional case but later specify to two dimensions which is relevant to string theory.

5.1 Conformal Transformations

We have already looked at conformal transformations Let us recall the definition:

Definition 5.1.1. Let (M, g) and $(N, \widetilde{g})$ be Riemannian (or pseudo-Riemmanian) manifolds and $\varphi : U \longrightarrow V$ be a smooth map where $U \subset M, V \subset N$ are open sets. Then φ is called a conformal map if the pullback φ^* of φ satisfies

$$\varphi^* \widetilde{g} = \Omega^2 g, \tag{5.1.1}$$

where $\Omega \in C^\infty(M)$ is called the *scale factor*. Here $C^\infty(M)$ denotes the space of smooth functions $f : M \longrightarrow \mathbb{R}$.

Suppose M is m dimensional and M is n dimensional. Let $(x^0, \ldots, x^{m-1})$ and $(y^0, \ldots, y^{n-1})$ be chart maps on U and V respectively[1]. Let the components of the metric tensor be given by

$$g_{\mu\nu}(x) := g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right), \quad \widetilde{g}_{\mu\nu}(y) := \widetilde{g}\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right)$$

Then writing $x'^\mu = y^\mu \circ \varphi$ and using the definition of pullback, (5.1.1) becomes

$$\widetilde{g}_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Omega^2(x) g_{\mu\nu}(x), \quad x' = \varphi(x). \tag{5.1.2}$$

Conformal transformations between distinct Riemannian manifolds are important for many

[1] these charts may not cover all of U and V.

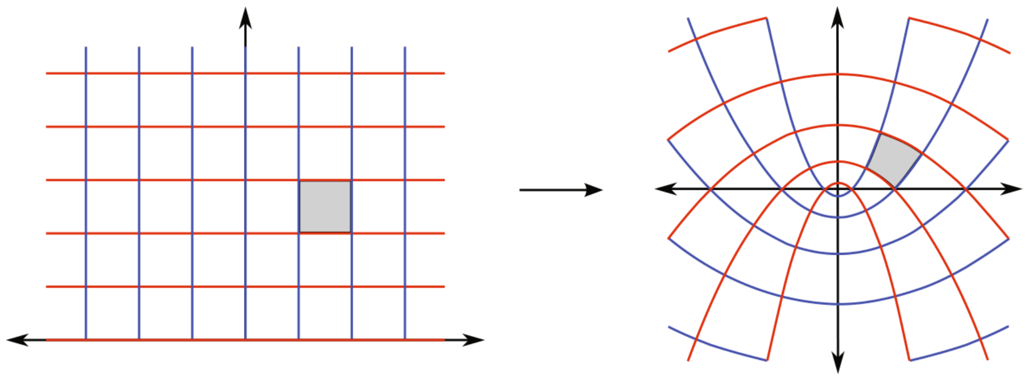

Figure 5.1: Conformal transformation in two dimensions. It is clearly visible that this transformation preserves angles.

applications but we will analyse the case when $M = N = \mathbb{R}^{1,D-1}$ and $g_{\mu\nu} = \widetilde{g}_{\mu\nu} = \eta_{\mu\nu}$. In this case, a conformal transformation is just a spacetime transformation which only scales the metric. It is also clear that the set of all conformal transformations forms a group under composition of maps. We denote this group by $\mathrm{Conf}(\mathbb{R}^{1,D-1})$. We would like to determine this group for various spacetime dimensions. It turns out to be the Lorentz group for dimensions greater than 3. In dimension 2, it is a bit more complicated and turns out to be infinite dimensional if we look at local conformal transformations which we will differentiate from the global conformal transformations in a precise sense. The global conformal transformations coincide with the local conformal transformations in $D \geq 3$. We begin by analysing infinitesimal conformal transformations which helps us get the Lie algebra of $\mathrm{Conf}(\mathbb{R}^{1,D-1})$ immediately.

5.1.1 Infinitesimal Conformal Transformations

We will now concentrate on infinitesimal spacetime transformations and find conditions on the infinitesimal parameter so that it is a conformal transformation. To this end, consider the local infinitesimal coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x) + O(\varepsilon^2). \tag{5.1.3}$$

Under this coordinate transformation, we have

$$\begin{aligned}
\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} &= \eta_{\rho\sigma}\left(\delta^\rho_\mu + \frac{\partial \varepsilon^\rho}{\partial x^\mu} + O\left(\varepsilon^2\right)\right)\left(\delta^\sigma_\nu + \frac{\partial \varepsilon^\sigma}{\partial x^\nu} + O\left(\varepsilon^2\right)\right) \\
&= \eta_{\mu\nu} + \eta_{\mu\sigma}\frac{\partial \varepsilon^\sigma}{\partial x^\nu} + \eta_{\rho\nu}\frac{\partial \varepsilon^\rho}{\partial x^\mu} + O\left(\varepsilon^2\right) \\
&= \eta_{\mu\nu} + \left(\frac{\partial \varepsilon_\mu}{\partial x^\nu} + \frac{\partial \varepsilon_\nu}{\partial x^\mu}\right) + O\left(\varepsilon^2\right),
\end{aligned}$$

where in the last step, we used

$$\frac{\partial \varepsilon_\nu}{\partial x^\mu} = \frac{\partial \eta_{\mu\nu}\varepsilon^\mu}{\partial x^\mu} = \eta_{\mu\nu}\frac{\partial \varepsilon^\nu}{\partial x^\mu}.$$

If we demand that this infinitesimal transformation be a conformal transformation, then we must have

$$\eta_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu + O(\varepsilon^2) = \Omega^2 \eta_{\mu\nu}\varepsilon_\mu$$

which implies that

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x)\eta_{\mu\nu} \tag{5.1.4}$$

for some function f. To determine the function f in terms of ε, we contract (5.1.4) with $\eta^{\mu\nu}$. We get

$$\begin{aligned}
&\eta^{\mu\nu}\left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) = f(x)\eta^{\mu\nu}\eta_{\mu\nu} \\
&\implies 2\partial^\mu \varepsilon_\mu = f(x)D \\
&\implies f(x) = \frac{2}{D}(\partial \cdot \varepsilon).
\end{aligned}$$

Plugging this expression in (5.1.4), we get

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{D}(\partial \cdot \varepsilon)\eta_{\mu\nu}. \tag{5.1.5}$$

The scale factor upto linear order in ε is given by

$$\Omega^2(x) = 1 + \frac{2}{D}(\partial \cdot \varepsilon) + O(\varepsilon^2).$$

We now derive several relations which will be useful in later computations. Taking partial derivative ∂^ν of (5.1.5), we obtain

$$\begin{aligned}
\partial^\nu\left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) &= \frac{2}{D}\partial^\nu(\partial \cdot \epsilon)\eta_{\mu v} \\
\implies \partial_\mu(\partial \cdot \varepsilon) + \Box \varepsilon_\mu &= \frac{2}{D}\partial_\mu(\partial \cdot \varepsilon),
\end{aligned}$$

where $\Box = \partial^\mu \partial_\mu$. Further taking partial derivative ∂_ν of above equation, we get

$$\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box \partial_\nu \varepsilon_\mu = \frac{2}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon). \tag{5.1.6}$$

Interchanging $\mu \leftrightarrow \nu$ in (5.1.6) and adding to the same equation, we obtain

$$\begin{aligned}
&(\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box(\partial_\nu \varepsilon_\mu)) + (\partial_\nu \partial_\mu(\partial \cdot \varepsilon) + \Box(\partial_\mu \varepsilon_\nu)) = \frac{4}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon) \\
&\implies 2\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box(\partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu) = \frac{4}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon) \\
&\implies 2\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box\left(\frac{2}{D}(\partial \cdot \varepsilon)\eta_{\mu\nu}\right) = \frac{4}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon) \\
&\implies (\eta_{\mu\nu}\Box + (D-2)\partial_\mu \partial_\nu)(\partial \cdot \varepsilon) = 0,
\end{aligned}$$

where we used (5.1.5) in the last step. Finally contracting this equation with $\eta^{\mu\nu}$, we get

$$(D-1)\Box(\partial\cdot\varepsilon)=0 \tag{5.1.7}$$

We now derive another equation for later use. Taking derivatives ∂_ρ of (5.1.5) and permuting indices we get

$$\begin{aligned}
\partial_\rho\partial_\mu\varepsilon_\nu+\partial_\rho\partial_\nu\varepsilon_\mu &= \frac{2}{D}\eta_{\mu\nu}\partial_\rho(\partial\cdot\varepsilon)\\
\partial_\nu\partial_\rho\varepsilon_\mu+\partial_\mu\partial_\rho\varepsilon_\nu &= \frac{2}{D}\eta_{\rho\mu}\partial_\nu(\partial\cdot\varepsilon)\\
\partial_\mu\partial_\nu\varepsilon_\rho+\partial_\nu\partial_\mu\varepsilon_\rho &= \frac{2}{D}\eta_{\nu\rho}\partial_\mu(\partial\cdot\varepsilon).
\end{aligned}$$

Adding the last two equations and subtracting the first gives

$$2\partial_\mu\partial_\nu\varepsilon_\rho=\frac{2}{D}\left(-\eta_{\mu\nu}\partial_\rho+\eta_{\rho\mu}\partial_\nu+\eta_{\nu\rho}\partial_\mu\right)(\partial\cdot\varepsilon). \tag{5.1.8}$$

5.2 Conformal Group in $D\geq 3$

Let us first define the *global* conformal group and its algebra.

Definition 5.2.1. The conformal group is the group consisting of *globally* defined, invertible and finite conformal transformations, that is conformal diffeomorphisms.

Definition 5.2.2. The conformal algebra is the Lie algebra corresponding to the conformal group.

To find the conformal group, we first work out the infinitesimal conformal transformation and then obtain the finite conformal transformation by exponentiating the infinitesimal ones.

5.2.1 Infinitesimal Conformal Transformations: $D\geq 3$

We begin by observing that (5.1.7) constraints $\varepsilon(x)$ to be atmost quadratic in x. Thus the most general form of the local infinitesimal parameter $\varepsilon(x)$ is

$$\varepsilon_\mu(x)=a_\mu+b_{\mu\nu}x^\nu+c_{\mu\nu\rho}x^\nu x^\rho, \tag{5.2.1}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho}\ll 1$ are constants and $c_{\mu\nu\rho}$ is symmetric in $\nu,\rho:\ c_{\mu\nu\rho}=c_{\mu\rho\nu}$. Now since the condition for conformal transformation is encoded only in the constants appearing in (5.1.7) and these conditions should not depend on the spacetime point, we can analyse the conditions on the constants order by order.

(i) The constant term a_μ (Translation): This term is not constrained by (5.1.5). This corresponds to spacetime translation, for which the generator[2] is $P_\mu=-i\partial_\mu$ as is well known.

[2] see Appendix B for details on symmetry generators and the explicit calculations of the generators of the conformal algebra.

(ii) The linear term $b_{\mu\nu}$ (Dilatation and Lorentz transformation): Plugging the expression in (5.2.1) upto linear term into (5.1.5), we obtain

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{D}\left(\eta^{\rho\sigma} b_{\sigma\rho}\right)\eta_{\mu\nu}.$$

Now if we split $b_{\mu\nu}$ into symmetric and antisymmetric part as

$$b_{\mu\nu} = \frac{b_{\mu\nu} + b_{\nu\mu}}{2} + \frac{b_{\mu\nu} - b_{\nu\mu}}{2}$$

then the above equation implies that the symmetric part is proportional to $\eta_{\mu\nu}$. Thus we see that $b_{\mu\nu}$ can be split in the following way

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$$

where $m_{\mu\nu} = -m_{\nu\mu}$. If we consider the symmetric term $\alpha\eta_{\mu\nu}$ alone, then we get the transformation $x'^{\mu} = (1+\alpha)x^{\mu}$ which describes infinitesimal scale transformations also called *dilatation*. The generator corresponding to this transformation is $D = -ix^{\mu}\partial_{\mu}$.

The antisymmetric part $m_{\mu\nu}$ corresponds to infinitesimal rotations $x'^{\mu} = \left(\delta^{\mu}_{\nu} + m^{\mu}{}_{\nu}\right)x^{\nu}$ with generator being the angular momentum operator $M_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)$.

(iii) The quadratic term $c_{\mu\nu\rho}$ (Special conformal transforamtion): Plugging the expression in (5.2.1) into (5.1.8), we get

$$\begin{aligned}
2\partial_{\mu}\partial_{\nu}c_{\rho\sigma\lambda}x^{\sigma}x^{\lambda} &= \frac{2}{D}\left(-\eta_{\mu\nu}\partial_{\rho} + \eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu}\right)\left(\eta^{\mu\nu}\partial_{\nu}(b_{\mu\rho}x^{\rho} + c_{\mu\sigma\lambda}x^{\sigma}x^{\lambda})\right)\\
\implies 2c_{\rho\mu\nu} &= \frac{1}{D}\left(-\eta_{\mu\nu}\partial_{\rho} + \eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu}\right)\left(b^{\mu}{}_{\mu} + 2c^{\mu}{}_{\mu\lambda}x^{\lambda}\right)\\
\implies 2c_{\rho\mu\nu} &= \frac{2}{D}\left(-\eta_{\mu\nu}c^{\sigma}{}_{\sigma\rho} + \eta_{\rho\mu}c^{\sigma}{}_{\sigma\nu} + \eta_{\nu\rho}c^{\sigma}{}_{\sigma\mu}\right)
\end{aligned}$$

Thus we have

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_{\nu} + \eta_{\mu\nu}b_{\rho} - \eta_{\nu\rho}b_{\mu} \quad \text{with} \quad b_{\mu} = \frac{1}{D}c^{\rho}{}_{\rho\mu}.$$

Thus the infinitesimal parameter is

$$\begin{aligned}
\varepsilon_{\mu}(x) &= (\eta_{\mu\rho}b_{\nu} + \eta_{\mu\nu}b_{\rho} - \eta_{\nu\rho}b_{\mu})x^{\nu}x^{\rho}\\
&= 2(b\cdot x)x_{\mu} + (x\cdot x)b_{\mu}.
\end{aligned}$$

The resulting transformations are called *Special Conformal Transformations* (SCT) which infinitesimally is given by:

$$x'^{\mu} = x^{\mu} + 2(x\cdot b)x^{\mu} - (x\cdot x)b^{\mu}. \tag{5.2.2}$$

The corresponding generator is written as

$$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu}\right).$$

So we have four infinitesimal transformations:

- Infinitesimal translation $x'^\mu = x^\mu + a^\mu,\ a^\mu \ll 1$ with generator $P_\mu = -i\partial_\mu$.
- Infinitesimal dilatation $x'^\mu = (1+\alpha)x^\mu,\ \alpha \ll 1$ with generator $D = ix^\mu\partial_\mu$.
- Lorentz transformation $x'^\mu = m^\mu{}_\nu x^\nu,\ m_{\mu\nu} = -m_{\nu\mu},\ m_{\mu\nu} \ll 1$ with generator $M_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$.
- Special conformal transformations $x'^\mu = x^\mu + 2(x\cdot b)x^\mu - (x\cdot x)b^\mu,\ b^\mu \ll 1$ with generator $K_\mu = -i\left(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu\right)$.

5.2.2 Finite Conformal Transformations: $D \geq 3$

To get finite conformal transformations, we need to exponentiate the generators with finite parameters.

(i) Translations: let a^μ be a finite translation. Then it is implemented on spacetime by the operator
$$T(a) := \exp\left(ia^\mu P_\mu\right).$$
Thus finite translations are given by
$$\begin{aligned}
T(a)x^\mu &= \exp\left(ia^\nu P_\nu\right)x^\mu \\
&= \left(1 + a^\nu\partial_\nu + \frac{1}{2!}a^\nu a^\rho\partial_\nu\partial_\rho + \dots\right)x^\mu \\
&= x^\mu + a^\nu\delta^\mu{}_\nu + 0 \\
&= x^\mu + a^\mu,
\end{aligned}$$
as expected.

(ii) Dilatation: let α be a finite dilatation parameter. It is implemented on spacetime by the operator
$$S(\alpha) := \exp\left(i\alpha D\right).$$
Thus finite dilatation is given by
$$\begin{aligned}
S(\alpha)x^\mu &= \exp\left(i\alpha D\right)x^\mu \\
&= \left(1 + \alpha x^\nu\partial_\nu + \frac{1}{2!}\alpha^2 x^\nu\partial_\nu x^\rho\partial_\rho + \dots\right)x^\mu \\
&= x^\mu + \alpha x^\nu\delta^\mu{}_\nu + \frac{1}{2!}\alpha^2 x^\nu\partial_\nu(x^\rho\delta^\mu{}_\rho) + \dots \\
&= x^\mu + \alpha x^\mu + \frac{1}{2!}\alpha^2 x^\mu + \dots \\
&= e^\alpha x^\mu.
\end{aligned}$$

(iii) Lorentz transformation: let $\omega_{\mu\nu}$ be finite rotation and boost parameters with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Then it is clear that $\exp(i\omega^{\mu\nu}M_{\mu\nu}) \in \mathrm{SO}(1, D-1)$. Then on spacetime, finite Lorentz transformation is implemented by the Lorentz transformation operator

$$\Lambda^{\mu}{}_{\nu} := \left(\exp(i\omega^{\rho\lambda}M_{\rho\lambda})\right)^{\mu}{}_{\nu},$$

and on spacetime it acts in the usual way.

(iv) Special conformal transformation: let b^μ be a finite SCT parameter. To get finite SCT transformation on spacetime, we need to compute $\exp(ib^\rho K_\rho)x^\mu$ by expanding the exponential. Observe that

$$ib^\rho K_\rho x^\mu = (2(b\cdot x)x^\rho - (x\cdot x)b^\rho)\,\partial_\rho x^\mu = 2(b\cdot x)x^\mu - (x\cdot x)b^\mu.$$

Next we have

$$\begin{aligned}(ib^\rho K_\rho)^2 x^\mu &= (2(b\cdot x)x^\rho - (x\cdot x)b^\rho)\,\partial_\rho(2(b\cdot x)x^\mu - (x\cdot x)b^\mu)\\ &= (2(b\cdot x)x^\rho - (x\cdot x)b^\rho)\,(2b_\rho x^\mu + 2(b\cdot x)\delta^\mu{}_\rho - 2x_\rho b^\mu)\\ &= 2(4(b\cdot x)^2x^\mu - 2(b\cdot x)(x\cdot x)b^\mu - (x\cdot x)(b\cdot b)x^\mu)\end{aligned}$$

We can go on computing higher powers of $ib^\rho K_\rho$. Adding up, we obtain

$$\begin{aligned}\exp(ib^\rho K_\rho)x^\mu &= x^\mu + 2(b\cdot x)x^\mu - (x\cdot x)b^\mu + \frac{1}{2!}2(4(b\cdot x)^2x^\mu - 2(b\cdot x)(x\cdot x)b^\mu\\ &\quad - (x\cdot x)(b\cdot b)x^\mu) + \dots\\ &= (x^\mu - (x\cdot x)b^\mu) + 2(b\cdot x)x^\mu - 2(b\cdot x)(x\cdot x)b^\mu - (x\cdot x)(b\cdot b)x^\mu\\ &\quad + (b\cdot b)(x\cdot x)^2b^\mu - (b\cdot b)(x\cdot x)^2b^\mu + 4(b\cdot x)^2x^\mu + \dots\\ &= (x^\mu - (x\cdot x)b^\mu) + (x^\mu - (x\cdot x)b^\mu)(2(b\cdot x) - (b\cdot b)(x\cdot x)) + \dots\\ &= (x^\mu - (x\cdot x)b^\mu)(1 + (2(b\cdot x) - (b\cdot b)(x\cdot x)) + \dots)\\ &= \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}\end{aligned}$$

We thus have the action of all finite conformal transformations on spacetime. We list them in the table below. It is clear that the metric remains invariant under translation and Lorentz

Transformations		Generators
translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
dilatation	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu\partial_\mu$
rotation	$x'^\mu = \Lambda^\mu{}_\nu x^\nu$	$M_{\mu\nu} = i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCT	$x'^\mu = \frac{x^\mu - (x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$	$K_\mu = -i\,(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu)$

Table 5.1: Global conformal transformations in $D \geq 3$

transformation. Under dilatation, the scale factor is $\Omega^2(x) = \alpha^2$. Under SCT, the metric scales non trivially. Indeed using SCT transformation given in Table 5.1 and (5.1.2), one can show that the scale factor is

$$\Omega^2(x) = (1 - 2(b \cdot x) + (b \cdot b)(x \cdot x))^2.$$

We know the geometrical meaning of three of the transformations that we have got above namely translation, dilatation and Lorentz transformation. Let us see what SCT means geometrically. First observe that

$$\frac{x'^\mu}{x' \cdot x'} = \frac{x^\mu}{x \cdot x} - b^\mu.$$

Indeed we have

$$\begin{aligned}
\frac{x'^\mu}{x' \cdot x'} &= \frac{x^\mu - (x \cdot x)b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} \frac{(1 - 2(b \cdot x) + (b \cdot b)(x \cdot x))^2}{(x^\mu - (x \cdot x)b^\mu)(x_\mu - (x \cdot x)b_\mu)} \\
&= \frac{x^\mu - (x \cdot x)b^\mu}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} \frac{(1 - 2(b \cdot x) + (b \cdot b)(x \cdot x))^2}{[(x \cdot x) - 2(x \cdot x)(b \cdot x) + (x \cdot x)(b \cdot b)]} \\
&= \frac{x^\mu}{x \cdot x} - b^\mu.
\end{aligned}$$

This suggests that SCT corresponds to inversion followed by translation followed by inversion. Moreover SCT is not defined globally. In particular, the transformation blows up at

$$x^\mu = \frac{b^\mu}{b \cdot b} \in \mathbb{R}^{1,D-1}$$

because the denominator $1 - 2(b \cdot x) + (b \cdot b)(x \cdot x) = 0$ at this point. Thus to define SCT globally, we need to compactify the Minkowski space by including the point at infinity by a construction in topology called one point compactification[3]. We will see this construction explicitly in two dimensional case where the one point compactification is explicitly known namely the Riemann sphere.

5.2.3 The Conformal Group and its Algebra

We begin by describing the algebra of conformal transformation generators.

Proposition 5.2.3. *The generators $P_\mu, D, L_{\mu\nu}, K_\mu$ of conformal transformations satisfy the following algebra:*

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK_\mu \\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu}D - M_{\mu\nu}\right) \\
[K_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right) \\
[P_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right)
\end{aligned} \tag{5.2.3}$$

[3]see Munkres topology for precise formulation

Proof. The last two Lie brackets are standard Lorentz algebra proved in preliminary quantum field theory course and hence we omit it here. We will prove the first Lie bracket relation and the rest is similar. Suppose f is a test function. Then we have

$$\begin{aligned}[D, P_\mu]f &= (-ix^\nu\partial_\nu)(-i\partial_\mu)f - (-i\partial_\mu)(-ix^\nu\partial_\nu)f \\ &= -x^\nu\partial_\nu\partial_\mu f + x^\nu\partial_\mu\partial_\nu f + \delta^\nu{}_\mu\partial_\nu f \\ &= i(-i\partial_\mu)f \\ &= iP_\mu f.\end{aligned}$$

□

We easily see that the Lorentz algebra is a subalgebra of the conformal algebra. Moreover we know that the Lorentz algebra is $D(D-1)/2$ dimensional. The dimension of the conformal algebra is thus

$$\begin{aligned}&1 \text{ dilatation } + D \text{ translations } + D \text{ special conformal} \\ &\qquad + \frac{D(D-1)}{2} \text{ Lorentz } = \frac{(D+2)(D+1)}{2} \text{ generators.}\end{aligned}$$

To identify this algebra with standard Lie algebra, let us consider certain linear combinations of the conformal generators. Define

$$\begin{aligned}&J_{\mu\nu} = M_{\mu\nu}, \quad \mu,\nu = 0,\dots,D-1 \\ &J_{-1\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right), \quad \mu = 0,\dots,D-1 \\ &J_{-1D} = D, \quad J_{D\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right), \quad \mu = 0,\dots,D-1.\end{aligned}$$

Moreover we define $J_{mn} = J_{nm}$ for $n, m = -1, 0, 1, \dots, D-1, D$.

Proposition 5.2.4. *The generators J_{ab} satisfy the following Lie bracket relation:*

$$[J_{mn}, J_{rs}] = i\left(\eta_{ms}J_{nr} + \eta_{nr}J_{ms} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}\right),$$

where $\eta_{mn} = \mathrm{diag}(-1, \underbrace{-1, 1, \dots, 1}_{m,n=0,\dots,D-1}, 1)$

Proof. For $n, m = 0, \dots, D-1$, the relation is immediate from the Lorentz algebra. We check the Lie bracket of $J_{-1\mu}$ and $J_{D\mu}$. We have

$$\begin{aligned}[J_{-1\mu}, J_{D\nu}] &= \frac{1}{4}[P_\mu - K_\mu, P_\nu + K_\nu] \\ &= \frac{1}{4}\left([P_\mu, K_\nu] - [K_\mu, P_\nu]\right) \\ &= \frac{1}{4}\left[-2i(\eta_{\nu\mu}D - M_{\nu\mu}) - 2i(\eta_{\mu\nu}D - M_{\mu\nu})\right] \\ &= -i\eta_{\mu\nu}D = -i\eta_{\mu\nu}J_{-1D},\end{aligned}$$

where we used (5.2.3) and the antisymmetry of $M_{\mu\nu}$. The right hand side of the algebra is

$$i\left(\eta_{-1\nu}J_{\mu D} + \eta_{\mu D}J_{-1\nu} - \eta_{-1D}J_{\mu\nu s} - \eta_{\mu\nu}J_{-1D}\right) = -i\eta_{\mu\nu}J_{-1D}.$$

Other brackets are similar. $\square$

To identify the conformal algebra with standard Lie algebra, we define the generalised orthogonal group.

Definition 5.2.5. (Generalised orthogonal group) Let n, k be two positive integers. Define a bilinear form $B : \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \longrightarrow \mathbb{R}$ by

$$B(\mathbf{x}, \mathbf{y}) := -\sum_{i=1}^{n} x_i y_i + \sum_{j=1}^{k} x_{n+j} y_{n+j},$$

where $\mathbf{x} = (x_1, \ldots, x_n, \ldots x_{n+k}), \mathbf{y} = (y_1, \ldots, y_n, \ldots y_{n+k}) \in \mathbb{R}^{n+k}$. Define the set $\mathrm{O}(n,k)$ as the set of matrices which preserve the bilinear form B:

$$\mathrm{O}(n,k) := \{A \in \mathrm{GL}(n+k, \mathbb{R}) \mid B(A\mathbf{x}, A\mathbf{y}) = B(\mathbf{x}, \mathbf{y}),\ \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+k}\},$$

where $\mathrm{GL}(n+k, \mathbb{R})$ denotes the set of all invertible real matrices of size $(n+k) \times (n+k)$. If we write

$$\mathbb{1}_{n,k} := \begin{pmatrix} -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_k \end{pmatrix}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix, then it is easy to see that

$$\mathrm{O}(n,k) = \left\{A \in \mathrm{GL}(n+k, \mathbb{R}) \mid A^T \mathbb{1}_{n,k} A = \mathbb{1}_{n,k}\right\}.$$

$\mathrm{O}(n,k)$ is called the generalised orthogonal group. We also define

$$\mathrm{SO}(n,k) := \{A \in \mathrm{O}(n,k) \mid \det A = 1\}.$$

Thus, we identify the conformal algebra with the Lie algebra $\mathfrak{so}(2, D-1)$ of $\mathrm{SO}(2, D-1)$. In general if $\mathbb{R}^{p,q}$ denotes the Minkowski space with metric $\mathbb{1}_{p,q}$, then following the same procedure, we can get the conformal algebra and the conformal group of $\mathbb{R}^{p,q}$. Thus we have the following theorem.

Theorem 5.2.6. *For Minkowski space $\mathbb{R}^{p,q}$ with dimension $D = p + q \geq 3$, the conformal group is* $\mathrm{SO}(p+1, q+1)$.

5.3 Conformal Group in $D = 2$

We work with Euclidean metric but everything can be formulated in Lorentzian signature equally well.

5.3.1 Local Conformal Transformations

In two dimensions, let (z^0, z^1) be the coordinates on the plane. Under a spacetime transformation $z^\mu \longrightarrow w^\mu(x)$ the metric tensor transforms as

$$g^{\mu\nu} \longrightarrow \widetilde{g}^{\mu\nu}(w) = \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}$$

Since $g'_{\mu\nu}(w) = \Omega^2(x) g_{\mu\nu}(z)$ under conformal transformations, thus for various μ, ν we get

$$\begin{aligned}
\Omega^2(x) &= \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2, \quad \mu,\nu = 0,0 \\
\Omega^2(x) &= \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2, \quad \mu,\nu = 1 \\
0 &= \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1}, \quad (\mu,\nu) = (1,0),(0,1).
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \\
\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} = 0.
\end{aligned} \tag{5.3.1}$$

Second equation of (5.3.1) gives

$$\frac{\frac{\partial w^0}{\partial z^0}}{\frac{\partial w^1}{\partial z^1}} = -\frac{\frac{\partial w^0}{\partial z^1}}{\frac{\partial w^1}{\partial z^0}} = \lambda \implies \frac{\partial w^0}{\partial z^0} = \lambda \frac{\partial w^1}{\partial z^1}, \quad \frac{\partial w^0}{\partial z^1} = -\lambda \frac{\partial w^1}{\partial z^0}.$$

Substituting this in first equation of (5.3.1), we get

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \lambda^2 \left(\frac{\partial w^0}{\partial z^1}\right)^2 + \left(\frac{\partial w^0}{\partial z^0}\right)^2 \implies \lambda^2 = 1.$$

Thus we obtain two other conditions which are independently equivalent to (5.3.1):

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}, \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{5.3.2}$$

or

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}, \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{5.3.3}$$

(5.3.2) resembles the Cauchy-Riemann equations for holomorphic functions. On the other hand, we define antiholomorphic functions using (5.3.3). To make this explicit, we make a transition to complex coordinates using the change of coordinates given below:

$$\begin{aligned}
z &= z^0 + iz^1, \quad \bar{z} = z^0 - iz^1, \\
z^0 &= \frac{1}{2}(z+\bar{z}), \quad z^1 = \frac{1}{2i}(z-\bar{z}). \\
\partial_z &= \frac{1}{2}\left(\partial_0 - i\partial_1\right), \quad \partial_0 = \partial_z + \partial_{\bar{z}}. \\
\partial_{\bar{z}} &= \frac{1}{2}\left(\partial_0 + i\partial_1\right), \quad \partial_1 = i\left(\partial_z - \partial_{\bar{z}}\right).
\end{aligned} \tag{5.3.4}$$

In terms of the coordinates z and $\bar{z}$, we have

$$ds^2 = (dz^0)^2 + (dz^1)^2 = \frac{1}{4}(dz + d\bar{z})^2 - \frac{1}{4}(dz - d\bar{z})^2 = dzd\bar{z}.$$

So in the metric tensor in complex coordinates is

$$g^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

where the index μ, ν run over $z, \bar{z}$. With this notation, if we define

$$w(z,\bar{z}) = w^0(z,\bar{z}) + iw^1(z,\bar{z}),$$

then (5.3.2) implies that $\partial_{\bar{z}} w(z,\bar{z}) = 0$ and (5.3.3) implies that $\partial_z \bar{w}(z,\bar{z}) = 0$. This means that that the function $w(z)$ and $\bar{w}(\bar{z})$ are holomorphic in some open set of the complex plane. Thus conformal transformation in two dimension amounts to a *holomorphic* change of coordinates:

$$z \longrightarrow z' = f(z), \quad \bar{z} \longrightarrow \bar{f}(\bar{z}),$$

where the two transformations result from the two equations (5.3.2) and (5.3.3) but in both cases the change of coordinates is holomorphic. Conversely, if we have a transformation $z \longrightarrow f(z)$ for a holomorphic function f in some open set of the complex plane, then the Euclidean metric $dzd\bar{z}$ on[4] $\mathbb{C}$ transforms as

$$dzd\bar{z} \longrightarrow \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z},$$

from which we see that the metric transforms conformally with scale factor $\left|\frac{\partial f}{\partial z}\right|^2$. An important point to note is that we require holomorphicity only in some open set, which means that the conformal transformations we have obtained are *local*. Thus we have proved the following theorem.

Theorem 5.3.1. *The group of local conformal transformations in dimension two is isomorphic to the group of all holomorphic functions*[5] *in some open set on the complex plane and hence is infinite dimensional.*

[4]induced from the Euclidean metric $dx^2 + dy^2$ on $\mathbb{R}^2$.

[5]the set of all holomorphic maps is a group under usual composition of maps.

Proof. The isomorphism is explicit from our discussion above. The dimensionality follows from the fact that the set of all holomorphic functions is infinite dimensional. To see this, note that any complex function f holomorphic in some open set admits a Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}.$$

Thus we need an infinite number of parameters, namely the coefficients in the Laurent expansion to specify a holomorphic function. □

5.3.2 Infinitesimal Generators: The Witt Algebra

Any infinitesimal conformal transformation can be written as

$$z \longrightarrow z' = z + \varepsilon(z), \quad \bar{z} \longrightarrow \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}),$$

where $|\varepsilon(z)| \ll 1$. Since $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are holomorphic in some open set, we can write its Laurent expansion around 0:

$$\begin{aligned} z' &= z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n \left(-z^{n+1}\right) \\ \bar{z}' &= \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right) \end{aligned}$$

where the infinitesimal parameters ε_n and $\bar{\varepsilon}_n$ are constants defining the Laurent expansion. Let l_n and $\bar{l}_n$ be the generators corresponding to the transformation $z \longrightarrow z - \varepsilon_n z^{n+1}$ and $\bar{z} \longrightarrow \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1}$ respectively. Then we have[6]

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{5.3.5}$$

Thus we have infinite number of generators for infinitesimal conformal transformations in two dimensions. Thus we conclude that local conformal transformation is infinite dimensional.

We now calculate the algebra of the infinitesimal generators. We have

$$\begin{aligned} [l_m, l_n] &= z^{m+1}\partial_z \left(z^{n+1}\partial_z\right) - z^{n+1}\partial_z \left(z^{m+1}\partial_z\right) \\ &= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\ &= -(m-n)z^{m+n+1}\partial_z \\ &= (m-n)l_{m+n}. \end{aligned}$$

Similarly we have

$$\left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n},$$

[6] see Appendix B for explicit calculations

and as expected

$$[l_m, \bar{l}_n] = 0.$$

Thus the algebra of infinitesimal generators is

$$[l_m, l_n] = (m-n)l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n}, \quad [l_m, \bar{l}_n] = 0.$$

The commutation relation satisfied by the generator l_n and $\bar{l}_n$ is called Witt algebra. Thus the algebra of infinitesimal generators of conformal transformations in two dimensions consists of two copies of the Witt algebra as subalgebras.

Remark 5.3.2. We may identify this algebra as the classical Virasoro generators that we obtained when we imposed the classical constraints on the string. We also saw that the quantum Virasoro algebra involved an additional term called the central charge. We will rederive the quantum Virasoro algebra in next section.

5.3.3 The Global Conformal Group

We now try to extract the subalgebra of the infinitesimal conformal algebra which is globally defined. We first analyse the generators l_n. Observe that these generators are not defined at $z = 0$. Thus we need to include the point at infinity to the complex plane and consider the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong S^2$. Even if we consider the Riemann sphere then also not all z=generators are well defined. For example the generators $l_n = -z^{n+1}\partial_z$ is non singular at $z = 0$ only for $n \geq -1$. The other problematic point is ∞. To understand the behaviour of l_n at ∞, we make a change of variable $z \longrightarrow -1/\omega$ and then study the limit $\omega \to 0$. The generators transform as

$$l_n = -z^{n+1}\partial_z \longrightarrow -\left(-\frac{1}{\omega}\right)^{n-1}\partial_\omega.$$

From this expression, we see that these generators are non singular at infinity only for $n \leq 1$. Thus we see that only three generators are globally defined namely $\{l_{-1}, l_0, l_1\}$. Thus we have proved the following theorem.

Theorem 5.3.3. *The global conformal group of the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong S^2$ is three dimensional and is generated by l_{-1}, l_0, l_1 which satisfies the Witt algebra.*

To identify the global conformal group, we will analyse the transformations generated by the generators l_{-1}, l_0, l_1.

It is clear that l_{-1} generates translations[7] $z \longrightarrow z + a$. It is also clear that l_0 generators dilatation[8] $z \longrightarrow \alpha z$. We are left with l_0. This corresponds to SCT. Let us work out the explicit transformation. We have

$$\exp(cl_1)z = \left(\sum_{n=0}^{\infty} \frac{(cl_1)^2}{n!}\right) z.$$

[7]compare the generator l_{-1} with the momentum operator P_μ.

[8]compare l_0 with D.

Observe that

$$l_1 z = -z^2 \partial_z z = -z^2.$$

By induction, we see that

$$l_1^n z = (-1)^n n! z^{n+1}.$$

Thus we have that

$$\exp(cl_1)z = \sum_{n=0}^{\infty} \frac{(-c)^n n! z^{n+1}}{n!} = z \sum_{n=0}^{\infty} (-cz)^n = \frac{z}{cz+1}.$$

In total, a combination of l_{-1}, l_0, l_1 produces the following transformation:

$$z \longrightarrow \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0,$$

where the last condition is required for invertibility of the map. We can rescale the complex numbers a, b, c, d such that $ad - bc = 1$. Now we can identify each such map with the matrix

$$\frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A straightforward calculation calculation shows that composition of two such maps corresponds to matrix multiplication of the corresponding matrices. Moreover observe that the matrices A and $-A$ produce the same conformal transformation. Hence we have proved the following theorem,

Theorem 5.3.4. *The global conformal group of the Riemann sphere* $\mathbb{C} \cup \{\infty\} \cong S^2$ *is isomorphic to* $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$ *where* $\mathrm{SL}(2,\mathbb{C})$ *denotes the group of* 2×2 *complex matrices with determinant 1.*

The Virasoro Algebra

Recall that the classical constraints we obtained when we quantised the Polyakov string action in cannonical formalism satisfies the Witt algebra. Whereas in the quantum theory, we got a nontrivial central term in the quantum Virasoro algebra. Here we rederive the quantum Virasoro algebra which is the so called central extension of the Witt algebra.

Roughly speaking, a central extension by $\mathbb{C}$ of a Lie algebra $\mathfrak{g}$ is $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ and is characterised by the Lie bracket

$$\begin{aligned}
&[\widetilde{X}, \widetilde{Y}]_{\widetilde{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} + cp(X, Y), \quad \widetilde{X}, \widetilde{Y} \in \widetilde{\mathfrak{g}}, \\
&[\widetilde{X}, c]_{\widetilde{\mathfrak{g}}} = 0, \\
&[c, c]_{\widetilde{\mathfrak{g}}} = 0, \quad c \in \mathbb{C},
\end{aligned}$$

where $p : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is a bilinear map.

Let $L_n,\ n \in \mathbb{Z}$ denote the elements of the central extension of the Witt algebra. Then by definition, we have

$$[L_m, L_n] = (m-n)L_{m+n} + cp(m,n)$$

We now determine $p(m,n)$ in three steps:

Step 1: $p(m,n) = -p(n,m)$ and we can assume $p(1,-1) = 0$ and $p(n,0) = 0$.

Proof. Since the Lie bracket is antisymmetric, we obtain the first assertion. Next, without the loss of generality, we can assume that $p(1,-1) = 0$ and $p(n,0) = 0$. If not then we can make the following redefinition:

$$\begin{aligned} \widehat{L}_n &= L_n + \frac{cp(n,0)}{n}, \quad n \neq 0 \\ \widehat{L}_0 &= L_0 + \frac{cp(1,-1)}{2}. \end{aligned}$$

We can check that with this redefinition, we have $p(1,-1) = 0$ and $p(n,0) = 0$. Indeed, for the modified generators we have

$$\begin{aligned} \left[\widehat{L}_n, \widehat{L}_0\right] &= nL_n + cp(n,0) = n\widehat{L}_n, \\ \left[\widehat{L}_1, \widehat{L}_{-1}\right] &= 2L_0 + cp(1,-1) = 2\widehat{L}_0 \end{aligned}$$

□

Step 2: $p(n,m) = 0$ for $n \neq -m$.

Proof. To prove this, we begin by observing that Jacobi identity gives

$$[[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = 0.$$

Using the characterisation of the central extension and the fact that the Lie bracket of the Witt algebra also satisfies Jacobi identity, we get

$$\begin{aligned} (m-n)cp(m+n,0) + ncp(n,m) - mcp(m,n) &= 0 \\ \implies (m+n)p(n,m) &= 0, \end{aligned}$$

where we used results of step 1. The result is now immediate. □

We are now left with the only non-vanishing central extensions $p(n,-n)$ for $|n| \geq 2$.

Step 3: $p(n,-n) = \frac{1}{12}(n^3 - n)$.

Proof. Again by Jacobi identity, we have

$$[[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] = 0.$$

Again proceeding as in Step 2, we obtain

$$(-2n+1)cp(1,-1) + (n+1)cp(n-1,-n+1) + (n-2)cp(-n,n) = 0.$$

Using $p(1,-1) = 0$, we obtain the recursion relation

$$p(n,-n) = \left(\frac{n+1}{n-2}\right) p(n-1,-n+1), \quad |n| \geq 3.$$

Thus we are free to choose $p(2,-2)$ to solve the recursion. We choose $p(2,-2)$ for later suitability. We get

$$\begin{aligned} p(n,-n) &= \frac{1}{2}\left(\frac{n+1}{n-2}\right)\left(\frac{n+1}{n-2}\right)\cdots\left(\frac{4}{3}\right) \\ &= \frac{1}{2}\binom{n+1}{3} \\ &= \frac{1}{12}(n+1)n(n-1) \\ &= \frac{1}{12}(n^3-n). \end{aligned}$$

□

Thus we see that the central extension of the Witt algebra satisfies the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.$$

Similar algebra is satisfies by the central extension of the generators $\bar{l}_n$.

Remark 5.3.5. For the Minkowski metric, we can perform a similar analysis. To do so, we define the lightcone coordinates $u = -t + x$ and $v = t + x$ where t denotes the time direction and x the space direction. The metric becomes

$$ds^2 = -dt^2 + dx^2 = dudv$$

and conformal transformations are given by $u \mapsto f(u)$ and $v \mapsto g(v)$ which gives

$$ds'^2 = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v}dudv.$$

Thus we see that again the Lie algebra of infinitesimal generators is infinite dimensional.

5.4 Primary Fields

We begin by discussing the transformation of fields under conformal transformation. This requires us to investigate representations of the conformal algebra.

5.4.1 Representation of the Conformal Group in D Dimensions

Let $\Phi(x)$ be a multicomponent classical field. We want to find representations of the conformal group and its action on the field Φ. We separately analyse $D \geq 3$ and $D = 2$ case.

Dimension $D \geq 3$

We use a cute little trick for this calculation. We begin by computing the generators which relate the transformed field to the original field at $x = 0$. We do this computation for the generators which keep the origin invariant. Then we use the translation generator to get the generator at any arbitrary spacetime point. Since Lorentz transformations, dilatations and special conformal transformations preserve the origin, we start by writing

$$\begin{aligned}
&M_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \\
&K_{\mu}\Phi(0) = \kappa_{\mu}\Phi(0) \\
&D\Phi(0) = \widetilde{\Delta}\Phi(0),
\end{aligned}$$

where $S_{\mu\nu}, \widetilde{\Delta}$ and κ_μ are the operators associated to the representation Φ corresponding to Lorentz transformation, dilatation and SCT respectively. Now recall that under translation $x \longrightarrow x + a$,

$$\Phi'(x') = \Phi(x) \implies \Phi'(x) = \Phi(x-a) \implies e^{ia^\mu P_\mu}\Phi(x) = \Phi\left(e^{-ia^\mu P_\mu}x\right).$$

This implies that

$$e^{-ix^\mu P_\mu}\Phi(0) = \Phi(x).$$

Now we have

$$M_{\mu\nu}\Phi(x) = M_{\mu\nu}e^{-ix^\lambda P_\lambda}\Phi(0) = e^{-ix^\lambda P_\lambda}\left[e^{ix^\lambda P_\lambda}M_{\mu\nu}e^{-ix^\lambda P_\lambda}\right]\Phi(0).$$

Now by first equation of (A.3.1), we have

$$e^{ix^\lambda P_\lambda}M_{\mu\nu}e^{-ix^\lambda P_\lambda} = M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu.$$

Thus we have

$$\begin{aligned}
M_{\mu\nu}\Phi(x) &= e^{-ix^\lambda P_\lambda}\left[M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu\right]\Phi(0) \\
&= e^{-ix^\lambda P_\lambda}M_{\mu\nu}\Phi(0) - (x_\mu P_\nu - x_\nu P_\mu)e^{-ix^\lambda P_\lambda}\Phi(0) \\
&= e^{-ix^\lambda P_\lambda}S_{\mu\nu}\Phi(0) - (x_\mu P_\nu - x_\nu P_\mu)\Phi(x)
\end{aligned}$$

Thus we conclude that

$$
\begin{aligned}
&P_\mu = -i\partial_\mu \Phi(x) \\
&M_{\mu\nu}\Phi(x) = S_{\mu\nu}\Phi(x) + i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi(x).
\end{aligned}
\tag{5.4.1}
$$

Instead of using (A.3.1) to evaluate $e^{ix^\lambda P_\lambda} M_{\mu\nu} e^{-ix^\lambda P_\lambda}$, one could have used the conformal algebra and the Hausdorff formula:

$$
e^{-A} B e^{A} = B + [B, A] + \frac{1}{2!}[[B, A], A] + \frac{1}{3!}[[[B, A], A], A] + \cdots
$$

for operators A and B. Indeed it will be useful for later computations.

We see that the operators $S_{\mu\nu}, \widetilde{\Delta}$ and κ_μ must satisfy the conformal algebra:

$$
\begin{aligned}
&[\widetilde{\Delta}, S_{\mu\nu}] = 0 \\
&[\widetilde{\Delta}, \kappa_\mu] = -i\kappa_\mu \\
&[\kappa_\nu, \kappa_\mu] = 0 \\
&[\kappa_\rho, S_{\mu\nu}] = i\left(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu\right) \\
&[S_{\mu\nu}, S_{\rho\sigma}] = i\left(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\right).
\end{aligned}
\tag{5.4.2}
$$

Using (5.2.3) and the Hausdorff formula, we have

$$
\begin{aligned}
&e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\nu P_\nu \\
&e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} = K_\mu + 2x_\mu D - 2x^\nu M_{\mu\nu} + 2x_\mu\left(x^\nu P_\nu\right) - x^2 P_\mu.
\end{aligned}
$$

This gives us the transformation of the field $\Phi(x)$ under dilatations and SCT:

$$
\begin{aligned}
&D\Phi(x) = \left(-ix^\nu \partial_\nu + \widetilde{\Delta}\right)\Phi(x) \\
&K_\mu \Phi(x) = \left(\kappa_\mu + 2x_\mu \widetilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\right)\Phi(x).
\end{aligned}
\tag{5.4.3}
$$

We now know how the field $\Phi(x)$ transforms under all generators of the conformal algebra. Let us assume that $(S_{\mu\nu}, \Phi)$ furnishes an *irreducible* representation of the Lorentz algebra. The following theorem will be crucial.

Theorem 5.4.1. (Schur's Lemma) *Let Π be an irreducible complex representation of a Lie group G. If A is in the center of G, then $\Pi(A) = \lambda I$, for some $\lambda \in \mathbb{C}$. Similarly, if π is an irreducible complex representation of a Lie algebra $\mathfrak{g}$ and if $[X, Y] = 0$ for every $Y \in \mathfrak{g}$, then $\pi(X) = \lambda I$.*

Since $\widetilde{\Delta}$ commutes with $S_{\mu\nu}$, thus it must act as a multiple of identity on Φ. From (B.2.3), it is clear that

$$
\widetilde{\Delta} = -i\Delta,
$$

where Δ is the scaling dimension of the field Φ. This obviates the fact that $\widetilde{\Delta}$ is not Hermitian.

Next, observe that since $\widetilde{\Delta}$ is a multiple of identity, it commutes with every other generator, in particular the generator of SCT. Thus the algebra of these generators in (5.4.2) implies that $\kappa_\mu = 0$. This is a crucial result:

the generators of SCT act trivially on fields Φ if they belong to irreducible representation of the Lorentz algebra.

Now we want to get the transformation of field Φ under finite conformal transformation. To do this we employ the fact that a Lie algebra representation gives rise to a Lie group representation via the exponential map. To make this precise, let $\Phi_\alpha(x)$ transform in the irreducible representation of the Lorentz algebra. Then under the conformal transformation $x \longrightarrow x'$ with parameter $a_\mu, \lambda, \omega_{\mu\nu}, b_\mu$ corresponding to translation, dilatation, rotation and SCT respectively, the field $\Phi_\alpha(x)$ transforms as

$$\Phi_\alpha(x) \longrightarrow \Phi'_\alpha(x) = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D} \left[\exp(i\omega^{\mu\nu}S_{\mu\nu})\right]_{\alpha\beta} \Phi_\beta(\Lambda^{-1}x),$$

where Λ is the Lorentz transformation acting on spacetime with parameters $\omega_{\mu\nu}$. To prove this, observe that the Jacobian for a general conformal transformation (excluding SCT as its generator acts trivially so that SCT acts as identity) is given by λ^D. Moreover by assumption $\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x)$. The transformation is now immediate.

In particular, for spinless field ϕ i.e. $S_{\mu\nu}\phi = 0$, the transformation is

$$\phi(x) \longrightarrow \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D} \phi(x). \tag{5.4.4}$$

Definition 5.4.2. A field $\phi(x)$ transforming as in (5.4.4) under global conformal transformations is called a *quasi primary* field with scaling dimension Δ. A field which is not quasi primary is called *secondary*.

Dimension $D = 2$

We have seen that the conformal algebra of the plane parametrized by (x^0, x^1) is most conveniently expressed in terms of Witt algebra generators which in turn are expressed in terms of the complexified coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$.[9] In what follows, we will consider z and $\bar{z}$ to be two independent complex variables but we also keep in mind that at the end of the calculation, we have to identify $\bar{z}$ with the complex conjugate of z. With this understanding, the fields ϕ on the plane transform to field on the four real dimensional $\mathbb{C}^2$ via the *complexification* $\mathbb{R}^2 \longrightarrow \mathbb{C}^2$:

$$\phi\left(x^0, x^1\right) \longrightarrow \phi(z, \bar{z}),$$

[9] note that the coordinates (x^1, x^2) is Euclidean. Recall that to get to Euclidean coordinates (x^1, x^2) from Minkowski coordinate (τ, σ), one has to perform a Wick rotation $\tau \to x^2 \equiv i\tau, x^1 = \sigma$. Thus the complex coordinates are really the Wick rotated lighcone coordinates.

where $(x^0, x^1) \in \mathbb{R}^2$ and $(z, \bar{z}) \in \mathbb{C}^2$. In two dimensions we already saw that the local conformal group is infinite dimensional, so we have two different definitions based on the transformation of the fields as we will see in a moment.

We know that the global conformal group of two dimensional Euclidean spacetime is generated by l_{-1}, l_0, l_1, so we work with a basis of eigenstates of the operators l_0 and $\bar{l}_0$. Let the corresponding eigenvalues be h and $\bar{h}$. These are known as the *conformal weights* of the state. Since $l_0 + \bar{l}_0$ and $i\left(l_0 - \bar{l}_0\right)$ are identified with the generators of dilatations and rotations (see Table 5.1 for the generators), the scaling dimension Δ and the spin s of the state are given by

$$\Delta = h + \bar{h}, \quad s = h - \bar{h}. \tag{5.4.5}$$

Definition 5.4.3. (i) Fields only depending on z, i.e. $\phi(z)$, are called *chiral fields* or *holomorphic* fields and fields $\phi(\bar{z})$ only depending on $\bar{z}$ are called *anti chiral* or *anti holomorphic* fields.

(ii) A field $\phi(z, \bar{z})$ which transforms under dilatations $z \longmapsto \lambda z$ according to

$$\phi(z, \bar{z}) \longmapsto \phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}),$$

is said to have *conformal dimensions* $(h, \bar{h})$.

(iii) A field which transforms under conformal transformations $z \longmapsto f(z)$ according to

$$\phi(z, \bar{z}) \longmapsto \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{5.4.6}$$

is called a *primary field of conformal dimension* $(h, \bar{h})$.

(iv) A field ϕ which transforms as a primary field only for global conformal transformations $f \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ is called a *quasi primary field*.

(v) A primary field is always quasi primary but the converse is not true. A field in a CFT which is neither primary nor quasi primary is called *secondary fields*.

We now find the infinitesimal version of transformation of primary fields. To this end, consider the infinitesimal conformal transformation $f(z) = z + \varepsilon(z)$ with $\varepsilon(z) \ll 1$. Up to first order in $\varepsilon(z)$, we have

$$\left(\frac{\partial f}{\partial z}\right)^h = 1 + h\partial_z \varepsilon(z) + O\left(\varepsilon^2\right),$$
$$\phi(z + \varepsilon(z), \bar{z}) = \phi(z, \bar{z}) + \varepsilon(z)\partial_z \phi(z, \bar{z}) + O\left(\varepsilon^2\right).$$

Using these expressions, we see that a primary field with conformal dimensions $h, \bar{h}$ transforms as

$$\phi(z, \bar{z}) \longmapsto \phi(z, \bar{z}) + \left(h\partial_z \varepsilon + \varepsilon \partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}}\right)\phi(z, \bar{z}).$$

Thus under infinitesimal conformal transformation, a primary field transforms as

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(z, \bar{z}) = \left(h\partial_z \varepsilon + \varepsilon \partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}}\right)\phi(z, \bar{z}). \tag{5.4.7}$$

5.5 Consequences of Conformal Invariance: Classical Aspects

A theory of fields invariant under conformal transformations is called a conformal field theory (CFT)[10]. We have seen in previous sections that the global conformal group in various dimensions includes translations, Lorentz transformations, SCT and dilatations. Invariance under conformal transformations has many consequences. We will analyse the classical aspects of a CFT.

5.5.1 Translation Invariance: Energy-Momentum Tensor

Recall that by Noether's theorem (see Appendix B), there is a classical conserved current corresponding to every classical continuous symmetry of the action. The current corresponding to translation invariance is called the energy momentum tensor. Suppose that a classical theory of fields Φ with Lagrangian $\mathcal{L}$ is invariant under infinitesimal translation $x \longrightarrow x+\varepsilon(x)$. Then by (B.2.4), the energy momentum tensor[11] is given by

$$T^{\mu}_{\ \nu} := j^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\partial_{\nu}\Phi - \eta^{\mu}_{\ \nu}\mathcal{L}. \tag{5.5.1}$$

An alternative way of deriving the energy momentum tensor is the following: consider the same theory but now let the background metric be dynamical $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$. Then translation invariance of the action may be thought of as a diffeomorphism. Under such a transformation, the metric transforms as

$$\begin{aligned}\widetilde{g}_{\mu\nu} &= \frac{\partial x^{\alpha}}{\partial x'}\frac{\partial x^{\beta}}{\partial x'}g_{\alpha\beta}\\ &= \left(\delta^{\alpha}_{\mu} - \partial_{\mu}\varepsilon^{\alpha}\right)\left(\delta^{\beta}_{\nu} - \partial_{\nu}\varepsilon^{\beta}\right)g_{\alpha\beta}\\ &= g_{\mu\nu} - \left(\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}\right).\end{aligned}$$

By (B.2.5), the action varies as

$$\begin{aligned}\delta S &= \int d^{D}x T^{\mu\nu}\partial_{\mu}\varepsilon_{\nu}\\ &= \frac{1}{2}\int d^{D}x T^{\mu\nu}\left(\partial_{\mu}\varepsilon_{\nu} + \partial_{\nu}\varepsilon_{\mu}\right),\end{aligned}$$

where we used the fact that the energy momentum tensor is symmetric[12]. Thus we have

$$\delta S = -\frac{1}{2}\int d^{D}x T^{\mu\nu}\delta g_{\mu\nu}.$$

[10] see Appendix B for precise definitions of symmetries

[11] the counting index a in the current as in (B.2.4) is now a spacetime index because of the spacetime index in the transformation parameter $\varepsilon(x)$.

[12] in general, the energy momentum tensor may not be symmetric. But one can show that it can always be made symmetric by adding the divergence of an antisymmetric tensor which neither affects the conservation of current nor the Ward identities. The new energy momentum tensor is called the Belinfante tensor. See Subsection 5.5.2 for the details of the construction.

This shows that the energy momentum tensor is given by

$$T^{\mu\nu} = -2\frac{\delta S}{\delta g_{\mu\nu}}.$$

In string theory, we usually choose a different normalisation and define the energy momentum to be

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{5.5.2}$$

where g denotes the determinant of the metric. If the space is flat, we evaluate $T_{\mu\nu}$ on $g_{\mu\nu} = \eta_{\mu\nu}$ and the resulting expression obeys $\partial^\alpha T_{\alpha\beta} = 0$. In general, the energy momentum tensor is covariantly conserved,

$$\nabla^\mu T_{\mu\nu} = 0.$$

Remark 5.5.1. The energy momentum tensor in string theory differs from that in usual QFT by a factor of 2π when the background metric is flat. We will keep this in mind and when we apply CFT to string theory, we will drop any extra factor of 2π that appear. We will mention this whenever we do so.

We now prove a typical consequence of conformal invariance.

Theorem 5.5.2. *In a classical CFT, the trace of the energy momentum tensor vanishes.*

Proof. Since the theory is invariant under dilatation, this, let us vary the action with respect to an infinitesimal dilatation $x \longrightarrow x' = (1+\alpha)x$. We have

$$\delta g_{\mu\nu} = \alpha g_{\mu\nu}.$$

The action varies as

$$\delta S = \int d^D x \frac{\delta S}{\delta g_{\mu\nu}}\delta g_{\mu\nu} = -\frac{1}{4\pi}\int d^D x\sqrt{g}\alpha T^\mu{}_\mu.$$

Since dilatations are symmetry of the theory, $\delta S = 0$ which implies

$$T^\mu{}_\mu = 0.$$

□

Remark 5.5.3. In a conformal field theory, vanishing trace of the energy momentum tensor is a typical feature, but as it turns out, this does not hold at quantum level in general, for example, in Yang-Mills theory it does not hold. In 2 dimensional CFT, it holds at quantum level only when the metric is flat. For curved background, we get an anomaly called the *trace anomaly* which was mentioned in Subsection 3.3.5. See Subsection 5.7.4 for details.

Energy Momentum Tensor in Two Euclidean Dimensions

We make the change of variables from real to complex as given in (5.3.4). Then using the transformation of the energy momentum tensor

$$T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}T_{\alpha\beta}.$$

Using $x^0 = \frac{1}{2}(z+\bar{z})$ and $x^1 = \frac{1}{2i}(z-\bar{z})$, it is straightforward to work out the components (we have removed the primes)

$$\begin{aligned} T_{zz} &= \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right), \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}\left(T_{00} + T_{11}\right) = \frac{1}{4}T^\mu{}_\mu = 0, \end{aligned} \tag{5.5.3}$$

where we used the fact that $T^\mu{}_\mu = 0$. Indeed, tracelessness gives

$$T_{zz} = \frac{1}{2}\left(T_{00} - iT_{10}\right), \quad T_{\bar{z}\bar{z}} = \frac{1}{2}\left(T_{00} + iT_{10}\right).$$

Now using translation invariance $\partial_\mu T^{\mu\nu} = 0$, we get

$$\partial_0 T_{00} + \partial_1 T_{10} = 0, \quad \partial_0 T_{01} + \partial_1 T_{11} = 0, \tag{5.5.4}$$

from which it follows that

$$\partial_{\bar{z}} T_{zz} = \frac{1}{4}\left(\partial_0 + i\partial_1\right)\left(T_{00} - iT_{10}\right) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 \underbrace{T_{00}}_{=-T_{11}} - i\partial_0 \underbrace{T_{10}}_{=T_{01}}) = 0,$$

where we used (5.5.4) and $T^\mu{}_\mu = 0$. Similarly, one can show that $\partial_z T_{\bar{z}\bar{z}} = 0$. Thus we have the following result:

Theorem 5.5.4. *The two non-vanishing components of the energy momentum tensor in two dimensions are a chiral and an anti-chiral field $T_{zz}(z,\bar{z})$ and $T_{\bar{z}\bar{z}}(z,\bar{z})$.*

5.5.2 Other Noether Currents

In this subsection, we compute the Noether current associated to other conformal transformations namely dilatations and Lorentz transformations.

Lorentz Invariance Current

Consider infinitesimal Lorentz transformations with parameters $\omega_{\mu\nu}$. The spacetime and field variations are

$$\frac{\delta x^\rho}{\delta \omega_{\mu\nu}} = \frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\rho\nu}x^\mu\right), \quad \frac{\delta \mathcal{F}}{\delta \omega_{\mu\nu}} = \frac{-i}{2}S^{\mu\nu}\Phi.$$

By (B.2.4), the associated conserved current is

$$\begin{aligned} j^{\mu\nu\rho} &= \left\{ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta^\mu{}_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a} \\ &= T_C^{\mu\nu} x^\rho - T_C^{\mu\rho} x^\nu + i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} S^{\nu\rho} \Phi, \end{aligned}$$

where we write $T_C^{\mu\nu}$ for the *cannonical* energy momentum tensor

$$T_C^{\mu\nu} = \frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \eta^\mu{}_\nu \mathcal{L} \right),$$

which is same as the energy momentum tensor we calculated in (5.5.1) upto a factor of half. We also note that the cannonical energy momentum tensor may not be symmetric. Also note that the current corresponding to Lorentz invariance has a nice compact expression modulo the nasty spin generator term. There is a way around to bypass both these problems, that is to make the energy momentum tensor symmetric and obtain a compact expression for the current corresponding to Lorentz invariance. Recall that we are free to add the divergence of an antisymmetric tensor in the current without affecting the conservation law. We will use this freedom. We try looking for a tensor $B^{\rho\mu\nu}$ antisymmetric in first two indices such that with the modified energy momentum tensor

$$T_B^{\mu\nu} := T_C^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}, \tag{5.5.5}$$

the Lorentz invariance current is given by

$$j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu \tag{5.5.6}$$

Proposition 5.5.5. *Let*

$$B^{\mu\rho\nu} = \frac{i}{2}\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\rho \Phi\right)} S^{\mu\nu} \Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \Phi\right)} S^{\mu\rho} \Phi \right].$$

Then the modified energy momentum tensor $T_B^{\mu\nu}$ is symmetric and the Lorentz invariance current is given by (5.5.6). The modified energy momentum tensor $T_B^{\mu\nu}$ is called the Belinfante *energy momentum tensor.*

Proof. It is clear that $B^{\mu\rho\nu}$ is antisymmetric in the first twp indices since $S^{\mu\nu} = -S^{\nu\mu}$. Checking the form of Lorentz invariance current is straightforward computation. To see that $T_B^{\mu\nu}$ is symmetric, note that

$$\partial_\mu j^{\mu\nu\rho} = 0 \implies T_B^{\mu\nu} \delta^\rho{}_\mu - T_B^{\mu\rho} \delta^\nu{}_\mu + x^\rho \partial_\mu T^{\mu\nu} - x^\nu \partial_\mu T^{\mu\rho} = 0.$$

Now the symmetric property of the energy momentum tensor is immediate from the fact that it is conserved. □

Dilatation Invariance Current

Consider an infinitesimal infinitesimal dilatation with parameter α

$$x'^{\mu} = (1+\alpha)x^{\mu}, \quad \mathcal{F}(\Phi) = (1-\alpha\Delta)\Phi,$$

where Δ is the scaling dimension of Φ. The variations are

$$\frac{\delta x^{\mu}}{\delta\alpha} = x^{\mu}, \quad \frac{\delta\mathcal{F}}{\delta\alpha} = -\Delta\Phi.$$

Thus the conserved current is given by

$$\begin{aligned} j^{\mu}_{D} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)} x^{\nu}\partial_{\nu}\Phi - \mathcal{L}x^{\mu} + \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\Delta\Phi \\ &= T^{\mu}_{C\ \nu}x^{\nu} + \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\Delta\Phi, \end{aligned}$$

where $T^{\mu\nu}_{C}$ is again the cannonical energy momentum tensor, which now we may assume to be symmetric. Again we have a nasty term in the current and it can again be removed by an appropriate choice of an antisymmetric tensor as we did in the previous case. This time the modified energy momentum tensor becomes traceless which we already concluded based on scale invariance. We will not describe the exact procedure here. The interested reader can look up section 4.2.2 of the Yellow book. Thus we conclude that the dilatation invariance current is given by

$$j^{\mu}_{D} = T^{\mu}_{\ \nu}x^{\nu},$$

where the energy momentum tensor is now symmetric and traceless.

Remark 5.5.6. The form of the current for scale invariance that we have concluded here involves some steps which do not go through for two dimensions. But we will assume it anyway and prove certain results which support our hypothesis.

SCT Invariance Current

An infinitesimal SCT with parameter b^{μ} is given by

$$x'^{\mu} = x^{\mu} + 2(x\cdot b)x^{\mu} - (x\cdot x)b^{\mu}, \quad \mathcal{F}(\Phi) = (1 - ib^{\mu}K_{\mu})\Phi.$$

Following similar methods, we see that the current is given by

$$j^{\mu\nu}_{K} = T^{\mu}_{C\ \rho}(2x^{\rho}x^{\nu} - \eta^{\rho\nu}x^{2}) - \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\frac{\delta\mathcal{F}}{\delta b^{\nu}}.$$

We can again do some manipulations and get a cannonical form for the current which we mention without further details:

$$j^{\mu\nu}_{K} = T^{\mu}_{\ \rho}(2x^{\rho}x^{\nu} - \eta^{\rho\nu}x^{2}),$$

where $T^{\mu\nu}$ is the energy momentum tensor.

5.6 Consequences of Conformal Invariance: Quantum Aspects in Dimension $D \geq 3$

So far we only discussed the classical aspects of a CFT. We will now discuss the quantum consequences of a CFT. Conformal invariance puts strong restrictions on the quantum theory.

5.6.1 Correlation Functions

In quantum field theory with classical action $S[\Phi]$, we define the correlation function of n number of *fields* $\phi_1, \ldots, \phi_n$ at spacetime points $x_1, \ldots, x_n$ respectively is defined in terms of the path integral

$$\langle \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle = \frac{\int [\mathcal{D}\Phi]\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\exp(-S[\Phi])}{\int [\mathcal{D}\Phi]\exp(-S[\Phi])}.$$

We can also define the correlation function of *local operators* $\mathcal{O}_1, \ldots, \mathcal{O}_n$ in a similar way:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \frac{\int [\mathcal{D}\Phi]\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\exp(-S[\Phi])}{\int [\mathcal{D}\Phi]\exp(-S[\Phi])}.$$

Remark 5.6.1. An important point is the following: in CFT, every local object is called a field as opposed to QFT where we call only the objects Φ appearing in the action as field. Thus Φ, $\partial_\mu\Phi, T^{\mu\nu}$ are all fields and consequently the functional integral measure $[\mathcal{D}\Phi]$ involves all possible fields in the theory.

We will compute several correlation function involving energy momentum tensor and primary fields later.

We can determine the two point correlation function of quasi primary fields exactly upto a normalisation constant using the constraints of conformal invariance. We will assume that the functional integral measure is invariant under conformal transformation[13]. Let us proceed.

Dimension $D \geq 3$

We begin by computing the two point correlation function of two quasi primary spinless fields ϕ_1, ϕ_2. By transformation rule (5.4.4) and invariance of functional integral measure, we obtain the following transformation rule for correlation function:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/D}\left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/D}\langle \phi_1(x_1')\phi_2(x_2')\rangle. \tag{5.6.1}$$

In particular for dilatation $x \to \lambda x$, we get

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle. \tag{5.6.2}$$

[13]this is a heavy assumption and may not hold in general.

Under Lorentz transformation and translation, we can easily check that the Jacobian factor in (5.6.1) is 1 and hence the correlation function remains invariant. This invariance under Lorentz transformation and translation and transformation in(5.6.2) requires that

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = f(|x_1 - x_2|)$$

where $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$. Only such function is given by $|x_1 - x_2|^{-\Delta_1-\Delta_2}$. Thus we have

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{5.6.3}$$

where C_{12} is some function. We are left to impose transformation under SCT. The Jacobian factor for SCT with parameter b^μ can easily be calculated to be

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 - 2b \cdot x + b^2 x^2)^D} \tag{5.6.4}$$

We need to compute the transformation of the term $|x_1 - x_2|$ under SCT to impose the covariance of the correlation function under SCT. Indeed one can easily check that

$$|x_i' - x_j'| = \frac{|x_i - x_j|}{(1 - 2b \cdot x_i + b^2 x_i^2)^{1/2} \left(1 - 2b \cdot x_j + b^2 x_j^2\right)^{1/2}},$$

for any two spacetime variables x_i, x_j. The correlation function transforms as

$$\begin{aligned} \langle \phi_1 (x_1') \phi_2 (x_2') \rangle &= \frac{C_{12}}{|x_1' - x_2'|^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12} (\gamma_1 \gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \end{aligned} \tag{5.6.5}$$

where

$$\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2.$$

Thus covariance of the correlation function and using (5.6.4), (5.6.5) and (5.6.1), we get

$$\begin{aligned} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} &= \frac{1}{\gamma_1^{D\Delta_1/D} \gamma_2^{D\Delta_2/D}} \frac{C_{12} (\gamma_1 \gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \frac{(\gamma_1 \gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \end{aligned}$$

This constraint is satisfied only if $\Delta_1 = \Delta_2$. Thus we conclude that

two quasi-primary fields are correlated only if they have the same scaling dimension.

The corresponding correlation function is given by

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2. \end{cases} \tag{5.6.6}$$

Thus we have determined the correlation function upto a normalisation constant. Similarly, we can determine the three point correlation function. We will not go through the complete details but mention the result:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \phi_3 (x_3) \rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{5.6.7}$$

where

$$x_{ij} = |x_i - x_j|,$$

and C_{123} is again some constant. This impressive feat stops at this stage. For four point correlation function, the result has a lot of freedom and cannot be fixed only using conformal invariance.

5.6.2 Ward Identities

In this section, we will write the Ward identities associated to conformal invariance. Ward identities is reviewed in Appendix B.3. We recall the general form of Ward identity. For a classical continuous symmetry with generator G_a and conserved current j_a^μ, the ward identity is given by

$$\frac{\partial}{\partial x^\mu} \langle j_a^\mu(x) \Phi_1 (x_1) \cdots \Phi_n (x_n) \rangle = -i \sum_{i=1}^{n} \delta (x - x_i) \langle \Phi_1 (x_1) \cdots G_a \Phi_i (x_i) \cdots \Phi_n (x_n) \rangle .$$

We will write $X \equiv \Phi_1 (x_1) \cdots \Phi_n (x_n)$ to simplify notations.

Translation invariance

The current associated to translation invariance is the energy momentum tensor $T^\mu{}_\nu$. So the Ward identity takes the form

$$\partial_\mu \langle T^\mu{}_\nu X \rangle = - \sum_i \delta (x - x_i) \frac{\partial}{\partial x_i^\nu} \langle X \rangle. \tag{5.6.8}$$

Lorentz invariance

The current associated to Lorentz invariance is

$$j^{\mu\nu\rho} = T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu .$$

Using the generator of Lorentz transformation given in (5.4.1), the Ward identity is given by

$$\partial_\mu \langle (T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu) X \rangle = \sum_i \delta (x - x_i) \left[(x_i^\nu \partial_i^\rho - x_i^\rho \partial_i^\nu) \langle X \rangle - i S_i^{\nu\rho} \langle X \rangle \right]$$

where $S_i^{\nu\rho}$ is the spin generator appropriate for the ith field of the set X. We can simplify this Ward identity by using the Ward identity (5.6.8). The left hand side becomes

$$\left\langle(\partial_\mu T^{\mu\nu})x^\rho X - (\partial_\mu T^{\mu\rho})x^\nu X + (T^{\mu\nu}\delta^\rho{}_\mu - T^{\mu\rho}\delta^\nu{}_\mu)X\right\rangle$$

Using the Ward identity (5.6.8), we get

$$\begin{aligned}\partial_\mu \langle (T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\, X\rangle &= \langle (T^{\rho\nu} - T^{\nu\rho})\, X\rangle - \sum_i \delta(x-x_i)\,[\partial_i^\nu\langle x^\rho X\rangle - \partial_i^\rho\langle x^\nu X\rangle]\\ &= \langle (T^{\rho\nu} - T^{\nu\rho})\, X\rangle - \sum_i \delta(x-x_i)\,[x_i^\rho\partial_i^\nu\langle X\rangle - x_i^\nu\partial_i^\rho\langle X\rangle\\ &\qquad\qquad\qquad\qquad\qquad\qquad + \delta^{\nu\rho}\langle X\rangle - \delta^{\rho\nu}\langle X\rangle]\\ &= \langle (T^{\rho\nu} - T^{\nu\rho})\, X\rangle - \sum_i \delta(x-x_i)\,[x_i^\rho\partial_i^\nu\langle X\rangle - x_i^\nu\partial_i^\rho\langle X\rangle]\end{aligned}$$

Thus the Ward identity for Lorentz invariance reduces to

$$\langle (T^{\rho\nu} - T^{\nu\rho})\, X\rangle = -i\sum_i \delta\,(x-x_i)\, S_i^{\nu\rho}\langle X\rangle. \tag{5.6.9}$$

It states that the energy momentum tensor is symmetric within correlation functions, except at the position of the other fields of the correlator.

Dilatation invariance

Using the generator $D = -ix^\nu\partial_\nu - i\Delta$ and conserved charge $j^\mu = T^\mu{}_\nu x^\nu$ of dilatation invariance, the Ward identity is given by

$$\partial_\mu \langle T^\mu{}_\nu x^\nu X\rangle = -\sum_i \delta\,(x-x_i)\left[x_i^\nu\frac{\partial}{\partial x_i^\nu}\langle X\rangle + \Delta_i\langle X\rangle\right]$$

Here again the derivative ∂_μ may act on $T^\mu{}_\nu$ and on the coordinate. By similar manipulations as above, we get the Ward identity corresponding to dilatation invariance

$$\left\langle T^\mu{}_\mu X\right\rangle = -\sum_i \delta\,(x-x_i)\,\Delta_i\langle X\rangle, \tag{5.6.10}$$

where Δ_i is the scaling dimension of Φ_i. This Ward identity says that the energy momentum tensor is traceless within the correlator, except at the position of the other fields of the correlator.

5.7 Consequences of Conformal Invariance: Quantum Aspects in Dimension $D = 2$

5.7.1 Correlation Functions

In two dimensions, we can consider the more general primary fields which transform in a nice way under local conformal transformation. Suppose $\phi_1, \ldots, \phi_n$ are primary fields with conformal dimensions $h_i, \bar{h}_i$. Under a local conformal transformation $z \longrightarrow w(z,\bar{z}),\ \bar{z} \longrightarrow \bar{w}(z,\bar{z})$, the correlation function transforms[14] as

$$\left\langle \phi_1\left(w_1,\bar{w}_1\right)\ldots\phi_n\left(w_n,\bar{w}_n\right)\right\rangle = \prod_{i=1}^{n}\left(\frac{dw}{dz}\right)_{w=w_i}^{-h_i}\left(\frac{d\bar{w}}{d\bar{z}}\right)_{\bar{w}=\bar{w}_i}^{-\bar{h}_i}\left\langle \phi_1\left(z_1,\bar{z}_1\right)\ldots\phi_n\left(z_n,\bar{z}_n\right)\right\rangle. \quad (5.7.1)$$

We can use (5.6.6) to write the two point function in two dimensions since the steps are the similar. In complex coordinates,

$$z_{ij} = |z_i - z_j| = \sqrt{z_{ij}\bar{z}_{ij}}.$$

So we have

$$\left\langle \phi_1\left(z_1,\bar{z}_1\right)\phi_2\left(z_2,\bar{z}_2\right)\right\rangle = \frac{C_{12}}{\left(z_1-z_2\right)^{2h}\left(\bar{z}_1-\bar{z}_2\right)^{2\bar{h}}} \quad \text{if} \quad \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases}$$

The two point function vanishes if the conformal dimensions of the two fields are different. Similarly, we can write down the three point function. It is given by

$$\left\langle \phi_1\left(x_1\right)\phi_2\left(x_2\right)\phi_3\left(x_3\right)\right\rangle = C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}}\frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}.$$

Again the four point function has freedom and cannot be fixed by conformal invariance alone.

5.7.2 Ward Identity

We wish to write the Ward identities corresponding to conformal invariance in complex coordinates. We observe that the spin generator $S_{\mu\nu}$ in two dimensions acts on a primary field ϕ as a multiple of the antisymmetric tensor $\varepsilon_{\mu\nu}$

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix}, \quad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}.$$

Thus the Ward identities take the form

$$\begin{aligned} \frac{\partial}{\partial x^\mu}\left\langle T^\mu{}_\nu(x)X\right\rangle &= -\sum_{i=1}^{n}\delta\left(x-x_i\right)\frac{\partial}{\partial x_i^\nu}\langle X\rangle \\ \epsilon_{\mu\nu}\left\langle T^{\mu\nu}(x)X\right\rangle &= -i\sum_{i=1}^{n}s_i\delta\left(x-x_i\right)\langle X\rangle \\ \left\langle T^\mu{}_\mu(x)X\right\rangle &= -\sum_{i=1}^{n}\delta\left(x-x_i\right)\Delta_i\langle X\rangle, \end{aligned} \quad (5.7.2)$$

where X denotes a collection of n primary fields and s_i is the spin of the ith field. To convert these into complex coordinates, we need the following lemma.

[14]assuming that the functional integral measure is invariant.

Lemma 5.7.1. *In two dimensions, we have for $x = (z, \bar{z})$*

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}}. \tag{5.7.3}$$

Proof. We will prove the first relation and the second is similar. We will show that for any holomorphic function $f(z)$ in a neighbourhood M of 0,

$$\frac{1}{\pi}\int_M d^2x f(z)\partial_{\bar{z}}\frac{1}{z} = f(0).$$

To do this, we need a version of Gauss's theorem in complex coordinates. For a vector field F^μ, Gauss theorem gives

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu,$$

where $d\xi_\mu$ is an outward directed differential of circumference, orthogonal to the boundary ∂M of the domain of integration. We can raise the index of the differential $d\xi_\mu$ using $\epsilon_{\mu\nu}$ which amounts to using a counterclockwise orientation on ∂M. We have $d\xi_\mu = \epsilon_{\mu\rho}ds^\rho$ where ds^ρ is $(dz, d\bar{z})$. Thus Gauss's theorem becomes

$$\begin{aligned}\int_M d^2x\partial_\mu F^\mu &= \int_{\partial M}\{dz\epsilon_{\bar{z}z}F^{\bar{z}} + d\bar{z}\epsilon_{z\bar{z}}F^z\} \\ &= \frac{1}{2}i\int_{\partial M}\{-dzF^{\bar{z}} + d\bar{z}F^z\}\end{aligned} \tag{5.7.4}$$

Here the contour ∂M circles counterclockwise. Taking $F^\mu = (0, f(z)/z)$, we get

$$\begin{aligned}\frac{1}{\pi}\int_M d^2x f(z)\partial_{\bar{z}}\frac{1}{z} &= \frac{1}{\pi}\int_M d^2x\partial_{\bar{z}}\left(\frac{f(z)}{z}\right) \\ &= \frac{1}{2\pi i}\int_{\partial M} dz\frac{f(z)}{z} \\ &= f(0),\end{aligned}$$

where we used the fact that $\partial_{\bar{z}} f(z) = 0$. $\square$

Substituting the delta function in (5.7.2) using (5.7.3), we easily see that the Ward identities take the form

$$\begin{aligned}2\pi\partial_z\left\langle T_{\bar{z}z}X\right\rangle+2\pi\partial_{\bar{z}}\left\langle T_{zz}X\right\rangle&=-\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle\\2\pi\partial_z\left\langle T_{\bar{z}\bar{z}}X\right\rangle+2\pi\partial_{\bar{z}}\left\langle T_{z\bar{z}}X\right\rangle&=-\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle\\2\left\langle T_{z\bar{z}}X\right\rangle+2\left\langle T_{\bar{z}z}X\right\rangle&=-\sum_{i=1}^{n}\delta\left(x-x_i\right)\Delta_i\langle X\rangle\\-2\left\langle T_{z\bar{z}}X\right\rangle+2\left\langle T_{\bar{z}z}X\right\rangle&=-\sum_{i=1}^{n}\delta\left(x-x_i\right)s_i\langle X\rangle\end{aligned}\tag{5.7.5}$$

Adding and subtracting the last two equations of (5.7.5) and using (5.7.3) gives two new equations

$$\begin{aligned}2\pi\left\langle T_{\bar{z}z}X\right\rangle&=-\sum_{i=1}^{n}\partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle\\2\pi\left\langle T_{z\bar{z}}X\right\rangle&=-\sum_{i=1}^{n}\partial_{z}\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle X\rangle\end{aligned}\tag{5.7.6}$$

where we used the fact that for primary fields $h_i=(\Delta_i+s_i)/2$ and $\bar{h}_i=(\Delta_i-s_i)/2$. Inserting these relations into the first two equations of (5.7.5), we get

$$\begin{aligned}\partial_{\bar{z}}\left[\langle T(z,\bar{z})X\rangle-\sum_{i=1}^{n}\left(\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle+\frac{h_i}{\left(z-w_i\right)^2}\langle X\rangle\right)\right]&=0\\\partial_{z}\left[\langle \bar{T}(z,\bar{z})X\rangle-\sum_{i=1}^{n}\left(\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle+\frac{\bar{h}_i}{\left(\bar{z}-\bar{w}_i\right)^2}\langle X\rangle\right)\right]&=0\end{aligned}$$

where we have defined

$$T(z,\bar{z})=-2\pi T_{zz}(z,\bar{z}),\quad \bar{T}(z,\bar{z})=-2\pi T_{\bar{z}\bar{z}}(z,\bar{z}).\tag{5.7.7}$$

This says that the expression in the square bracket in the above equation is holomorphic and antiholomorphic respectively. In particular T and $\bar{T}$ are functions of z and $\bar{z}$ respectively. Thus we may write

$$\langle T(z)X\rangle=\sum_{i=1}^{n}\left\{\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle+\frac{h_i}{\left(z-w_i\right)^2}\langle X\rangle\right\}+\text{reg.},\tag{5.7.8}$$

where "reg." stands for a holomorphic function of z, regular at $z=w_i$. A similar expression holds for the antiholomorphic part.

Conformal Ward Identity

We want to write the Ward identity as variation of the path integral due to infinitesimal conformal transformation, so that all the three Ward identity can be combined in a single Ward identity called the conformal Ward identity.

Theorem 5.7.2. *Let $x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x)$ be an infinitesimal conformal transformation. Then we have*

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\varepsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle,$$

where C is a closed curve in the complex plane containing the positions of all the fields in X.

Proof. We have

$$\begin{aligned}\partial_\mu\left(\varepsilon_\nu T^{\mu\nu}\right) &= \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right)T^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu\right)T^{\mu\nu}\\ &= \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}\left(\partial_\rho\varepsilon^\rho\right)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\epsilon_{\mu\nu}T^{\mu\nu}\end{aligned} \tag{5.7.9}$$

where we used

$$\begin{aligned}\frac{1}{2}\left(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right) &= \frac{1}{2}\left(\partial_\rho\varepsilon^\rho\right)\eta_{\mu\nu}\\ \frac{1}{2}\left(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu\right) &= \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\epsilon_{\mu\nu}\end{aligned} \tag{5.7.10}$$

First equation is same as (5.1.5) for $D = 2$, the second equation can be verified component wise. Now under a general infinitesimal transformation with parameters ω_a and generator $G_a^{(i)}$ in the representation Φ_i,

$$\begin{aligned}x'^\mu &= x^\mu + \omega_a\frac{\delta x^\mu}{\delta\omega_a}\\ \Phi_i'(x) &= \Phi_i(x) - i\omega_a G_a^{(i)}\Phi_i(x).\end{aligned}$$

Thus we have

$$\delta_\omega X = -i\sum_{i=1}^{n}\left(\Phi_1\left(x_1\right)\cdots G_a^{(i)}\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right)\omega_a\left(x_i\right).$$

This implies that

$$\delta_\omega\langle X\rangle = -i\omega_a\sum_{i=1}^{n}\left\langle\Phi_1\left(x_1\right)\cdots G_a^{(i)}\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle. \tag{5.7.11}$$

We now apply this formula to infinitesimal conformal transformation. We have seen that a general infinitesimal conformal transformation (without SCT) has the form given by (5.2.1)

$$\varepsilon_\mu = a_\mu + b_{\mu\nu}x^\nu,$$

where the symmetric and antisymmetric part of $b_{\mu\nu}$ parametrises dilatation and Lorentz transformation. The generators of translation, dilatation and Lorentz transformation in representation Φ_i is $-i\partial_\mu, -i\Delta_i$ and $s_i\epsilon_{\mu\nu}$ where Δ_i and s_i is the scaling dimension and spin of the field Φ_i. Using (5.7.11) and noting that the symmetric and antisymmetric part of $b_{\mu\nu}$ is precisely the left hand side of (5.7.10), we have

$$\begin{aligned}\delta_\varepsilon\langle X\rangle = -\sum_{i=1}^{n}&\left[\varepsilon^\nu(x_i)\left\langle\Phi_1\left(x_1\right)\cdots\partial_\nu\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle\right.\\ &+\alpha\left\langle\Phi_1\left(x_1\right)\cdots\Delta_i\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle\\ &\left.+i\omega_{\mu\nu}\left\langle\Phi_1\left(x_1\right)\cdots s_i\epsilon^{\mu\nu}\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle\right],\end{aligned}\tag{5.7.12}$$

where

$$\alpha = \frac{1}{2}\partial\cdot\varepsilon, \quad \omega_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\epsilon_{\mu\nu}.$$

Next, using (5.7.9) we have

$$\int_M d^2x\partial_\mu\left\langle T^{\mu\nu}(x)\varepsilon_\nu(x)X\right\rangle = \int_M d^2x\varepsilon_\nu(x)\partial_\mu\langle T^{\mu\nu}X\rangle + \alpha(x)\langle T^\mu{}_\mu X\rangle + \omega_{\mu\nu}(x)\langle T^{\mu\nu}X\rangle,$$

where M is a domain containing the positions of all the fields in the string X. We now use the Ward identities (5.7.2), we get

$$\begin{aligned}\int_M d^2x\partial_\mu\left\langle T^{\mu\nu}(x)\varepsilon_\nu(x)X\right\rangle = -\sum_{i=1}^{n}\int_M d^2x\delta(x-x_i)&\left[\varepsilon_\nu(x)\left\langle\Phi_1\left(x_1\right)\cdots\partial^\nu\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle\right.\\ &+\alpha\left\langle\Phi_1\left(x_1\right)\cdots\Delta_i\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle\\ &\left.+i\omega_{\mu\nu}\left\langle\Phi_1\left(x_1\right)\cdots s_i\epsilon^{\mu\nu}\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\right\rangle\right].\end{aligned}$$

Using (5.7.12), we conclude that

$$\delta_\varepsilon\langle X\rangle = \int_M d^2x\partial_\mu\left\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\right\rangle.\tag{5.7.13}$$

Using (5.7.4) for $F^\mu = \langle T^{\mu\nu}(x)\varepsilon_\nu(x)X\rangle$, we obtain

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = \frac{1}{2}i\int_C\left[-dz\left\langle(T^{z\bar{z}}+T^{\bar{z}\bar{z}})\varepsilon_{\bar{z}}X\right\rangle + d\bar{z}\left\langle(T^{zz}+T^{\bar{z}z})\varepsilon_z X\right\rangle\right],$$

where $\varepsilon = \epsilon^z$ and $\bar{\varepsilon} = \varepsilon^{\bar{z}}$ and the contour C is the boundary curve of M. Now by (5.5.3), we see that

$$\langle T^{z\bar{z}}X\rangle = \frac{1}{4}\left\langle T^\mu{}_\mu(x)X\right\rangle = \langle T^{\bar{z}z}X\rangle = \frac{1}{4}\left\langle T^\mu{}_\mu(x)X\right\rangle,$$

which by the Ward identity (5.7.2) vanished if x is different from all of the positions of the fields in X. Since C goes around those positions, we obtain

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = \frac{1}{2}i\int_C \left[-dz\,\langle T^{\bar{z}\bar{z}}\varepsilon_{\bar{z}}X\rangle + d\bar{z}\,\langle T^{zz}\varepsilon_z X\rangle\right].$$

Substituting the definition (5.7.7), we obtain the conformal Ward identity:

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\varepsilon}(\bar{z})\langle\bar{T}(\bar{z})X\rangle.$$

□

5.7.3 Operator Product Expansion and Primary Operators

When the position of two local operators in a correlator approaches each other, the correlation function diverges. This divergence is typical in quantum field theories and reflects the infinite fluctuations of quantum fields when "measured" at a precise position. An operator product expansion (OPE) exactly captures this feature. We define OPE precisely now.

Definition 5.7.3. Suppose $\mathcal{O}_k$ be the local operators in a CFT. For any two local operators $\mathcal{O}_i(z,\bar{z})$ and $\mathcal{O}_j(w,\bar{w})$, an OPE of $\mathcal{O}$ and $\mathcal{O}'$ is a relation of the form:

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}),$$

where $C_{ij}^k(z-w,\bar{z}-\bar{w})$ are functions of $z-w,\bar{z}-\bar{w}$ which diverge as $z\to w$.

Some remarks are in order.

Remark 5.7.4. (i) OPEs are always understood to be operator to be substituted in a time ordered correlation function:

$$\langle\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})X\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\left\langle\mathcal{O}_k(w,\bar{w})X\right\rangle,$$

where X is an string of local operators.

(ii) The string of operators X above is arbitrary their position must be distinct from the positions of $\mathcal{O}_i,\mathcal{O}_j$.

(iii) OPEs have singular behaviour as $z\to w$, which is all we care about. So in many case we write an OPE of operators $A(z)$ and $B(w)$ as

$$A(z)B(w) \sim \sum_{n=1}^{N}\frac{(AB)(w)}{(z-w)^n},$$

where (AB), called the composite field of A and B, are non singular at $z=w$ and $\sim$ indicates that the above relation is true modulo nonsingular terms. Thus every OPE has infinite number of nonsingular extra terms which we don't bother writing.

As an example, observe that in (5.7.8), we proved that for a primary field $\phi(w, \bar{w})$ with conformal dimensions $(h, \bar{h})$ we have

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w})$$
$$\bar{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}).$$

This OPE is characteristic of primary fields. Thus we may define primary operators alternatively by their OPE with energy momentum tensor.

Definition 5.7.5. A field $\phi(z, \bar{z})$ is called primary with conformal dimensions $(h, \bar{h})$, if the operator product expansion between the energy momentum tensors and $\phi(z, \bar{z})$ takes the following form:

$$T(z)\phi(w,\bar{w}) \sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}),$$
$$\bar{T}(\bar{z})\phi(w,\bar{w}) \sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}).$$

We will now discuss OPEs and primary operators with the help of an example.

Example: Free Scalar Field

Consider a massless scalar field $X(\sigma)$ where σ covers a 2 dimensional manifold. The action is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \ \partial_\alpha X \partial^\alpha X.$$

The classical equation of motion for this action can easily be computed. It is given by

$$\partial^2 X = 0.$$

To find the quantum consequences, we can use Ehrenfest's theorem which states that the expectation value of operators satisfy the classical equations of motion. We will derive this explicitly. To do this we need the following lemma.

Lemma 5.7.6. *Let $F[\phi]$ be a functional of field $\phi^\alpha(x)$ which vanishes on the boundary. Then the following holds*

$$\int [\mathcal{D}\phi] \frac{\delta F[\phi]}{\delta\phi^\alpha(x)} = 0.$$

Proof. Suppose x varies over a manifold M and $\alpha \in J$ where J is an index set. The variation of $F[\phi]$ is given as

$$\delta F[\phi] := F[\phi + \delta\phi] - F[\phi] = \int_M dx \sum_{\alpha \in J} \frac{\delta F[\phi]}{\delta\phi^\alpha(x)} \delta\phi^\alpha(x).$$

We now have to construct the functional integral measure appropriately. We do this by discretising space and hen taking the continuum limit. We now use DeWitt's notation to discretise the spacetime M. Put

$$i = (\alpha, x) \in I := J \times M, \quad \phi^i := \phi^\alpha(x), \quad i \in I.$$

To discretize spacetime M, we run i over a finite set I so that we now only have finitely many variables $\phi^i,\ i \in I$, in the theory. The functional derivative becomes a partial derivative

$$\frac{\partial F[\phi]}{\partial \phi^i}.$$

We thus have

$$\delta F := F[\phi + \delta\phi] - F[\phi] = \sum_{i \in I} \frac{\partial F[\phi]}{\partial \phi^i} \delta\phi^i,$$

and the functional integral measure is simply the product of finitely many measures:

$$\left[\prod_{j \in I} \int d\phi^j\right] \xrightarrow{\text{continuum limit}} \int [\mathcal{D}\phi].$$

This gives

$$\left[\prod_{j \in I} \int d\phi^j\right] \frac{\partial F[\phi]}{\partial \phi^i} \xrightarrow{\text{continuum limit}} \int [\mathcal{D}\phi] \frac{\delta F[\phi]}{\delta \phi^\alpha(x)}.$$

The left hand side in above equation is zero on account of the integral of a total derivative and the boundary condition satisfied by F and the proof is complete. $\square$

Using Lemma 5.7.6, we get

$$0 = \int [\mathcal{D}X] \frac{\delta e^{-S[X]}}{\delta X(\sigma)} = \int [\mathcal{D}X] e^{-S[X]} \left[\frac{1}{2\pi\alpha'} \partial^2 X(\sigma)\right].$$

Thus we get

$$\left\langle \partial^2 X(\sigma) \right\rangle = 0,$$

which is Ehrenfest's theorem.

The Propagator. We now want to compute the propagator for X. We again use path integral for this. Recall that the propagator in position space is the correlation function $\langle X(\sigma)X(\sigma')\rangle$ which is given by path integral

$$\langle X(\sigma)X(\sigma')\rangle = \frac{1}{Z} \int [\mathcal{D}X] X(\sigma) X(\sigma') e^{-S[X]},$$

where Z is the partition function of the theory given by

$$Z = \int [\mathcal{D}X] e^{-S[X]}.$$

Proposition 5.7.7. *The two point correlation function $\langle X(\sigma)X(\sigma')\rangle$ of the massless scalar field is*

$$\langle X(\sigma)X(\sigma')\rangle = -\frac{\alpha'}{2}\ln(\sigma-\sigma')^2 + const. \tag{5.7.14}$$

Proof. Using Lemma 5.7.6, we see that

$$0=\int[\mathcal{D}X]\frac{\delta e^{-S[X]}X(\sigma')}{\delta X(\sigma)} = \int[\mathcal{D}X]e^{-S[X]}\left[\frac{1}{2\pi\alpha'}\partial_\sigma^2 X(\sigma)X(\sigma')+\delta(\sigma-\sigma')\right]$$

Dividing throughout by the partition function, we get

$$\left\langle\partial_\sigma^2 X(\sigma)X(\sigma')\right\rangle = -2\pi\alpha'\delta(\sigma-\sigma')$$

So we find that the propagator satisfies the differential equation

$$\partial_\sigma^2\langle X(\sigma)X(\sigma')\rangle = -2\pi\alpha'\delta(\sigma-\sigma').$$

We now solve this differential equation. Since this correlator has to be translation and rotation invariant, thus it should only depend on the norm of separation i.e. $|\sigma-\sigma'|$. Put

$$r=|\sigma-\sigma'| \quad \text{and} \quad K(r)=\langle X(\sigma)X(\sigma')\rangle,$$

then the differential equation (5.7.15) in polar coordinates (r,θ) becomes

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial K(r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 K(r)}{\partial\theta^2} &= -2\pi\alpha'\delta(\sigma-\sigma')\\ \implies \frac{1}{r}\frac{d}{dr}\left(r\frac{dK(r)}{dr}\right) &= -2\pi\alpha'\delta(\sigma-\sigma').\end{aligned} \tag{5.7.15}$$

Let $\mathbb{D}_r$ be a disc of radius r centred at σ'. We now integrate both sides of (5.7.15) on $\mathbb{D}_r$ with respect to σ, we get

$$\begin{aligned}\int_0^{2\pi}d\theta\int_0^r \rho d\rho\frac{1}{r}\frac{d}{d\rho}\left(\rho\frac{dK(\rho)}{d\rho}\right) &= -2\pi\alpha'\int_{\mathbb{D}_r}d^2\sigma\delta(\sigma-\sigma')\\ \implies 2\pi r\frac{dK(r)}{dr} &= -2\pi\alpha'\\ \implies K(r) &= -\alpha'\ln r+\text{const}.\end{aligned}$$

Thus we conclude that

$$\langle X(\sigma)X(\sigma')\rangle = -\frac{\alpha'}{2}\ln(\sigma-\sigma')^2+\text{const}.$$

□

Note that the correlator has a divergence as $\sigma \to \sigma'$. This is a common feature of all quantum theories as explained in Subsection 5.7.3. In complex coordinates, the propagator looks as

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}[\ln(z-w)+\ln(\bar{z}-\bar{w})] + \text{ const.}$$

Operator Product Expansions. Taking partial derivatives of (5.7.14), we obtain the correlation function of the derivatives of X. Explicitly, we get

$$\begin{aligned}\langle \partial_z X(z,\bar{z})\partial_w X(w,\bar{w})\rangle &= -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + \text{reg.},\\ \langle \partial_{\bar{z}} X(z,\bar{z})\partial_{\bar{w}} X(w,\bar{w})\rangle &= -\frac{\alpha'}{2}\frac{1}{(\bar{z}-\bar{w})^2} + \text{reg.}\end{aligned} \tag{5.7.16}$$

Note that the classical equations of motion in complex coordinates is given by

$$\partial_z\partial_{\bar{z}}X(z,\bar{z}) = 0,$$

which enables us to write $X(z,\bar{z})$ as a sum of a holomorphic (left moving mode) and an antiholomorphic (right moving mode) function:

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}).$$

In the following we shall only consider the holomorphic field $X(z)$. We have already proved that the OPE of the field $\partial X \equiv \partial_z X(z)$ with itself is

$$\partial X(z)\partial X(w) \sim -\frac{\alpha'}{2}\frac{1}{(z-w)^2}.$$

Note that exchanging the two factors does not affect the correlator which is a characteristic of Bosonic fields. To know if these fields are primary or not, we need the energy momentum tensor. The energy momentum tensor associated with the free massless scalar field is

$$T_{\mu\nu} = \frac{1}{2\pi\alpha'}\left(\partial_\mu X\partial_\nu X - \frac{1}{2}\eta_{\mu\nu}\partial_\rho X\partial^\rho X\right).$$

In the notation of (5.7.7), in complex coordinates the energy momentum tensor is given by

$$T(z) = -\frac{1}{\alpha'}\partial X\partial X.$$

Similarly we can also calculate $\bar{T}(\bar{z})$.

Normal Ordering. Care must be taken when interpreting the energy momentum tensor as a quantum operator since it involves product of operators. In cannonical quantisation, we could have normal ordered the above expression by putting annihilation operator to the right of creation operators so that the vacuum expectation value vanishes. Here also we do the same but without referring to creation and annihilation operators. We denote this CFT normal ordering by $\mathop{\circ}\limits^{\circ}\mathop{\circ}\limits^{\circ}$. The exact meaning of the above expression is

$$T(z) = -\frac{1}{\alpha'} \,\mathop{\circ}\limits^{\circ} \partial X \partial X \mathop{\circ}\limits^{\circ}\, \equiv -\frac{1}{\alpha'} \lim_{w \to z} (\partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle). \tag{5.7.17}$$

Let us discuss the generalisation of this normal ordering which will be useful later. The essential idea is to subtract off the singular part from the operator which we get on taking the expectation value. We define

$$\begin{aligned} &\mathop{\circ}\limits^{\circ} X^\mu(z,\bar{z}) \mathop{\circ}\limits^{\circ} = X^\mu(z,\bar{z}) \\ &\mathop{\circ}\limits^{\circ} X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \mathop{\circ}\limits^{\circ} = X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) - \langle X^\mu(z,\bar{z}) X^\nu(w,\bar{w}) \rangle. \end{aligned} \tag{5.7.18}$$

The definition of normal ordering for arbitrary numbers of fields can be given recursively as

$$\begin{aligned} &\mathop{\circ}\limits^{\circ} X^{\mu_1}(z_1,\bar{z}_1) \dots X^{\mu_n}(z_n,\bar{z}_n) \mathop{\circ}\limits^{\circ} \\ &\quad = X^{\mu_1}(z_1,\bar{z}_1) \dots X^{\mu_n}(z_n,\bar{z}_n) - \sum \text{ subtractions }, \end{aligned} \tag{5.7.19}$$

where the sum runs over all ways of choosing one, two, or more pairs of fields from the product and replacing each pair with its expectation value. For example,

$$\begin{aligned} &\mathop{\circ}\limits^{\circ} X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) X^\lambda(z_3,\bar{z}_3) \mathop{\circ}\limits^{\circ} = X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) X^\lambda(z_3,\bar{z}_3) \\ &\quad - \langle X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) \rangle X^\lambda(z_3,\bar{z}_3) - 2 \text{ permutations.} \end{aligned}$$

This normal ordering prescription can be compactified as follows

$$\mathop{\circ}\limits^{\circ}\mathcal{O}\mathop{\circ}\limits^{\circ} = \exp\left(-\frac{1}{2} \int d^2z_1 d^2z_2 \langle X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) \rangle \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)} \frac{\delta}{\delta X^\nu(z_2,\bar{z}_2)} \right) \mathcal{O},$$

where $\mathcal{O}$ is any operator, that is a functional of the field X. One can easily check that this is equivalent to (5.7.19). Indeed the double derivative in the exponent contracts each pair of fields, and the exponential sums over any number of pairs with the factorial canceling the number of ways the derivatives can act. This gives us a useful relation when we use the inverse exponential:

$$\begin{aligned} \mathcal{O} &= \exp\left(-\frac{1}{2} \int d^2z_1 d^2z_2 \langle X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) \rangle \frac{\delta}{\delta X^\mu(z_1,\bar{z}_1)} \frac{\delta}{\delta X^\nu(z_2,\bar{z}_2)} \right) \mathop{\circ}\limits^{\circ}\mathcal{O}\mathop{\circ}\limits^{\circ} \\ &= \mathop{\circ}\limits^{\circ}\mathcal{O}\mathop{\circ}\limits^{\circ} + \sum \text{ contractions} \end{aligned}$$

where a contraction is the opposite of a subtraction: sum over all ways of choosing one, two, or more pairs of fields from $\mathop{\circ}\limits^{\circ}\mathcal{O}\mathop{\circ}\limits^{\circ}$ and replacing each pair with $\langle X^\mu(z_1,\bar{z}_1) X^\nu(z_2,\bar{z}_2) \rangle$. The OPE for any pair of operators can be obtained using

$$\mathop{\circ}\limits^{\circ}\mathcal{O}_1\mathop{\circ}\limits^{\circ}\,\mathop{\circ}\limits^{\circ}\mathcal{O}_2\mathop{\circ}\limits^{\circ} = \mathop{\circ}\limits^{\circ}\mathcal{O}_1\mathcal{O}_2\mathop{\circ}\limits^{\circ} + \sum \text{ cross-contractions}$$

for arbitrary operators $\mathcal{O}_1$ and $\mathcal{O}_2$. The sum now runs over all ways of contracting pairs with one field in $\mathcal{O}_1$ and one in $\mathcal{O}_2$. This can also be written

$$\overset{\circ}{\circ}\mathcal{O}_1\overset{\circ}{\circ}\,\overset{\circ}{\circ}\mathcal{O}_2\overset{\circ}{\circ} = \exp\left(\int d^2z_1 d^2z_2 \langle X^\mu(z_1,\bar z_1)\, X^\nu(z_2,\bar z_2)\rangle \frac{\delta}{\delta X^\mu_{\mathcal{O}_1}(z_1,\bar z_1)} \frac{\delta}{\delta X^\nu_{\mathcal{O}_2}(z_2,\bar z_2)}\right) \overset{\circ}{\circ}\mathcal{O}_1\mathcal{O}_2\overset{\circ}{\circ},$$

where the functional derivatives act only on the fields in $\mathcal{O}_1$ or $\mathcal{O}_2$ respectively.

We now compute the OPE of ∂X with $T(z)$.

Proposition 5.7.8. *The field ∂X is a primary field with conformal dimension $(h, \bar h) = (1, 0)$.*

Proof. We need to compute the correlation function

$$\langle T(z)\partial X(w)\rangle \equiv \langle 0|\mathcal{T}(T(z)\partial X(w))|0\rangle,$$

where $\mathcal{T}$ is the time ordering operator. We can use Wick's theorem to compute this time ordered product. Recall that by Wick's theorem, the time ordered product $\mathcal{T}(\phi_1 \dots \phi_n)$ of n fields is the normal ordered product $:\phi_1 \dots \phi_n:$ plus all possible contractions where a contraction of a pair of fields means that we replace the pair by the correlation function of the pair. So we obtain

$$\begin{aligned}\mathcal{T}(T(z)\partial X(w)) = -\frac{1}{\alpha'}\Big(&:\partial X(z)\partial X(z)\partial X(w): + :\partial X(z)\partial \overbracket{X(z) : \partial X}(w) \\ &+ :\partial \overbracket{X(z)\partial X(z) : \partial X}(w)\Big),\end{aligned}$$

where the square bracket indicates contraction. Thus we see that

$$\begin{aligned}\langle T(z)\partial X(w)\rangle &= -\frac{1}{\alpha'}\left(\langle 0| :\partial X(z)\partial X(z)\partial X(w): |0\rangle + 2\langle 0|\partial X(z)|0\rangle\langle \partial X(z)\partial X(w)\rangle\right) \\ &= \frac{\langle \partial X(z)\rangle}{(z-w)^2} + \text{reg.}\end{aligned}$$

where we used (5.7.16). Note that the reg. term contains the vacuum expectation value of the normal ordered product $:T(z)\partial X(w):$. Thus in standard form, the OPE has the form

$$T(z)\partial X(w) \sim \frac{\partial X(z)}{(z-w)^2}.$$

We can expand $\partial X(z)$ around $z = w$:

$$\partial X(z) = \partial X(w) + \partial_w^2 X(w)(z-w) + O((z-w)^2).$$

This gives

$$T(z)\partial X(w) \sim \frac{\partial X(w)}{(z-w)^2} + \frac{\partial_w^2 X(w)}{(z-w)}. \tag{5.7.20}$$

Similarly we can calculate the OPE of ∂X with $\bar T(\bar z)$. Thus according to Definition 5.7.5, ∂X is a primary operator of conformal weight $(h, \bar h) = (1, 0)$. $\square$

Corollary 5.7.9. *Higher order derivatives $\partial^n X$, $n > 1$ of the field X are not primary operators.*

Proof. Using (5.7.20), we see that

$$T(z)\partial^2 X(w) \sim \partial_w \left[\frac{\partial X(w)}{(z-w)^2} + \dots \right] \sim \frac{2\partial X(w)}{(z-w)^3} + \frac{2\partial_w^2 X(w)}{(z-w)^2}.$$

□

Corollary 5.7.10. *The field $: e^{ikX} :$ is a primary field with conformal dimensions $h = \bar{h} = \alpha' k^2/4$.*

Proof. We have

$$\begin{aligned}
:\partial X(z)\partial X(z) :: e^{ikX(w)} : &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} :\partial X(z)\partial X(z) :: \underbrace{X(w)X(w)\cdots X(w)}_{n \text{ terms}} : \\
&= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} :\partial X(z)\partial X(z) : X(w)\cdots X(w)\cdots X(w)\cdots X(w) : \\
&\qquad + :\partial X(z)\partial X(z) :: X(w)\cdots X(w)\cdots X(w) : \\
&\qquad + :\partial X(z)\partial X(z) :: X(w)\cdots X(w)\cdots X(w) : \\
&= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \left(-\frac{n(n-1)\alpha'^2}{4(z-w)^2} \right) : X^{n-2}(w) : + \frac{2n\alpha'}{z-w} : \partial X(z) X^{n-1}(w) : + \text{reg.} \\
&= -\frac{k^2\alpha'}{4} \frac{: e^{ikX(w)} :}{(z-w)^2} - \frac{ik\alpha'}{z-w} \sum_{n=1}^{\infty} \frac{(ik)^{n-1}}{(n-1)!} : \partial X(z) X^{n-1}(w) : + \text{reg.} \\
&= -\frac{k^2\alpha'^2}{4} \frac{: e^{ikX(w)} :}{(z-w)^2} - \frac{ik\alpha'^2 : \partial X(z) e^{ikX(w)} :}{z-w} + \text{reg.},
\end{aligned}$$

where we performed $n(n-1)$ contractions in first term and $2n$ contractions in the second and third term in the second step. Now observe that in $z \to w$

$$\frac{\partial_z X(z) : e^{ikX(w)} :}{z-w} - \frac{\partial_w X(w) : e^{ikX(w)} :}{z-w} \sim \text{regular}.$$

Thus we can add and subtract $\frac{\partial_w X(w) : e^{ikX(w)} :}{z-w}$ in the last step of the calculation of $T(z) : e^{ikX(w)} :$ to get

$$\begin{aligned}
T(z) : e^{ikX(w)} : &= -\frac{1}{\alpha'} : \partial X(z)\partial X(z) :: e^{ikX(w)} : \\
&= \frac{k^2\alpha'}{4} \frac{: e^{ikX(w)} :}{(z-w)^2} + \frac{ik : \partial_w X(w) e^{ikX(w)} :}{z-w} + \text{reg.} \\
&= \frac{k^2\alpha'}{4} \frac{: e^{ikX(w)} :}{(z-w)^2} + \frac{: \partial_w e^{ikX(w)} :}{z-w} + \text{reg.}
\end{aligned}$$

$\square$

Remark 5.7.11. The fields $V_k(z) =: e^{ikX(z)}:$ are called *vertex operators* in CFT and we will meet them again while discussing the application of CFT in string theory. Corollary 5.7.10 also shows that the conformal weights of the free boson CFT is continuous.

Lastly we check whether the energy momentum tensor is a primary operator or not.

Proposition 5.7.12. *The energy momentum tensor $T(z)$ is not a primary operator.*

Proof. Again using Wick's theorem, we have

$$\begin{aligned}
T(z)T(w) &= \frac{1}{\alpha'^2} : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \\
&= \frac{1}{\alpha'^2}\Big(: \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : + 2 : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \\
&\qquad\qquad + 4 : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \Big) \\
&= \frac{2}{\alpha'^2}\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2 - \frac{4}{\alpha^2}\frac{\alpha'}{2}\frac{:\partial X(z)\partial X(w):}{(z-w)^2} + \text{reg.}
\end{aligned}$$

where we used (5.7.16). Here again the reg. term includes the first normal ordered product. Again substituting $\partial X(z) = \partial X(w) + \partial_w^2 X(w)(z-w) + O((z-w)^2)$, we obtain

$$\begin{aligned}
T(z)T(w) &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} - \frac{2}{\alpha'}\frac{\partial_w^2 X(w)\partial X(w)}{z-w} + \text{reg.} \\
&= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}
\end{aligned} \tag{5.7.21}$$

Thus the OPE obviates the fact that $T(z)$ is not a primary operator. $\square$

Example: Free Fermionic System

Consider a free Majorana fermion in two dimensions with Euclidean metric. The action is given by

$$S = \frac{g}{2}\int d^2x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi, \tag{5.7.22}$$

where the Dirac matrices satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.$$

With $\eta^{\mu\nu} = \text{diag}(1,1)$, one choice of the Dirac matrices could be

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With this choice, the action can be simplified and we get

$$S = g \int d^2x(\overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi),$$

where we have written the two component Majorana fermion as $\Psi = (\psi, \overline{\psi})$. The equation of motions are

$$\partial\overline{\psi} = 0, \quad \bar{\partial}\psi = 0,$$

whose solutions are any holomorphic function $\psi(z)$ and any antiholomorphic function $\overline{\psi}(\bar{z})$.

The Propagator. We now calculate the propagator $\langle \Psi_i(\boldsymbol{x})\Psi_j(\boldsymbol{y})\rangle$ for $i, j = 1, 2$. As usual, the first step is to express the action in the form:

$$S = \frac{1}{2}\int d^2\boldsymbol{x}d^2\boldsymbol{y}\Psi_i(\boldsymbol{x})A_{ij}(\boldsymbol{x},\boldsymbol{y})\Psi_j(\boldsymbol{y}).$$

From the action in (5.7.22), we can identify A_{ij} with

$$A_{ij}(\boldsymbol{x},\boldsymbol{y}) = g\delta(\boldsymbol{x}-\boldsymbol{y})(\gamma^0\gamma^\mu)_{ij}\partial_\mu.$$

The propagator is then the inverse of A_{ij}:

$$K_{ij}(\boldsymbol{x},\boldsymbol{y}) \equiv \langle \Psi_i(\boldsymbol{x})\Psi_j(\boldsymbol{y})\rangle = (A^{-1})_{ij}(\boldsymbol{x},\boldsymbol{y}).$$

From the Gaussian integral of Grassmann variables, it is known that K_{ij} satisfies the differential equation:

$$g\delta(\boldsymbol{x}-\boldsymbol{y})(\gamma^0\gamma^\mu)_{i\ell}\partial_\mu K_{\ell j}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ij}. \tag{5.7.23}$$

We now return back to the notation $\boldsymbol{x} \to (z,\bar{z}), \boldsymbol{y} \to (w,\bar{w})$. Then (5.7.23) takes the form

$$2g\begin{pmatrix}\partial_{\bar{z}} & 0\\ 0 & \partial_z\end{pmatrix}\begin{pmatrix}\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle & \langle\psi(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle\\ \langle\bar{\psi}(z,\bar{z})\psi(w,\bar{w})\rangle & \langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle\end{pmatrix} = \frac{1}{\pi}\begin{pmatrix}\partial_{\bar{z}}\frac{1}{z-w} & 0\\ 0 & \partial_z\frac{1}{\bar{z}-\bar{w}}\end{pmatrix},$$

where we used the following representation of the delta function as in Lemma 5.7.1:

$$\delta((x)) = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z}. \tag{5.7.24}$$

The propagator is now easily seen to be

$$\begin{aligned}\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle &= \frac{1}{2\pi g}\frac{1}{z-w}\\ \langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle &= \frac{1}{2\pi g}\frac{1}{\bar{z}-\bar{w}}\\ \langle\psi(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle &= \langle\bar{\psi}(z,\bar{z})\psi(w,\bar{w})\rangle = 0.\end{aligned} \tag{5.7.25}$$

Operator Product Expansions. After differentiation, we get using (5.7.25)

$$\langle \partial_z \psi(z,\bar{z})\psi(w,\bar{w})\rangle = -\frac{1}{2\pi g}\frac{1}{(z-w)^2}$$
$$\langle \partial_z \psi(z,\bar{z})\partial_w\psi(w,\bar{w})\rangle = -\frac{1}{\pi g}\frac{1}{(z-w)^3}.$$

The OPE of two Fermions then reads

$$\psi(z)\psi(w) = \frac{1}{2\pi g}\frac{1}{z-w} + \text{reg.} \tag{5.7.26}$$

Note that the OPE reflects the anticommutativity of the Fermionic wavefunctions. The energy momentum tensor can be evaluated using the general expression in (5.5.2):

$$T^{\bar{z}\bar{z}} = 2\frac{\partial \mathcal{L}}{\partial \bar{\partial}\Phi}\partial\Phi = 2g\psi\partial\psi$$
$$T^{zz} = 2\frac{\partial \mathcal{L}}{\partial \partial\Phi}\bar{\partial}\Phi = 2g\bar{\psi}\bar{\partial}\bar{\psi}$$
$$T^{z\bar{z}} = 2\frac{\partial \mathcal{L}}{\partial \partial\Phi}\partial\Phi = -2g\psi\bar{\partial}\psi.$$

Note that the energy momentum tensor is not symmetric but it becomes symmetric onshell. We need not worry about this because we can always use Belinfante construction described above Proposition 5.5.5. In the notation of (5.7.7), we have

$$T(z) = -\pi g : \psi(z)\partial\psi(z) :,$$

where the normal ordering is again defined as

$$: \psi\partial\psi : (z) = \lim_{z\to w}\left[\psi(z)\partial\psi(w) - \langle\psi(z)\partial\psi(w)\rangle\right].$$

Proposition 5.7.13. *The Fermion wavefunction $\psi(z)$ is a holomorphic primary field of conformal dimension $h = \frac{1}{2}$.*

Proof. We calculate the OPE of $\psi(z)$ with $T(z)$. We have

$$\begin{aligned} T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w) \\ &= \frac{1}{2}\frac{\partial X(z)}{z-w} + \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \text{reg.} \\ &= \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{1}{2}\frac{\partial X(w)}{z-w} + \text{reg.}, \end{aligned}$$

where we carried over $\psi(z)$ over to $\partial X(z)$ resulting in a minus sign and we used the argument as in Corollary 5.7.10 to replace $\psi(z)$ and $\partial\psi(z)$ by $\psi(w)$ and $\partial\psi(w)$ respectively. □

Theorem 5.7.14. *The stress tensor $T(z)$ satisfies the OPE*

$$T(z)T(w) = \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

Proof. The proof is similar to the free Boson case but with larger number of contractions. $\square$

Example with Foresight: The Ghost System

Consider the following action:

$$S = \frac{g}{2}\int d^2x b_{\mu\nu}\partial^\mu c^\nu,$$

where both fields b, c are Fermionic fields and $b_{\mu\nu}$ is symmetric traceless tensor. These fields are called *ghosts* or *reparametrisation ghosts* and will appear in our discussion when we discuss path integral quantisation. The classical equations of motion are

$$\partial^\alpha b_{\alpha\mu} = 0, \quad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0.$$

In complex notation, we usually write $c = c^z, \bar{c} = c^{\bar{z}}$ and $b = b^{zz}, \bar{b} = b^{\bar{z}\bar{z}}$. The equations of motion then reads

$$\begin{gathered} \bar{\partial} b = 0, \quad \partial\bar{b} = 0; \\ \bar{\partial} c = 0, \quad \partial\bar{c} = 0, \quad \partial c = -\bar{\partial}\bar{c}. \end{gathered}$$

Propagator. It is calculated as usual by writing the action as

$$S = \frac{1}{2}\int d^2\boldsymbol{x} d^2\boldsymbol{y} b_{\mu\nu}((x))A^{\mu\nu}_\alpha(\boldsymbol{x},\boldsymbol{y})c^\alpha(\boldsymbol{y}),$$

so that

$$A^{\mu\nu}_\alpha(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}g\delta^\nu_\alpha\delta(\boldsymbol{x}-\boldsymbol{y})\partial^\mu.$$

The factor of $1/2$ takes care of the double counting of the terms in the sum since $b_{\mu\nu}$ and hence $A^{\mu\nu}$. As usual the propagator is given by $K = A^{-1}$ where K satisfies the differential equation

$$\frac{1}{2}g\delta^\mu_\alpha\partial^\nu K^\beta_{\mu\nu}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{\alpha\beta}.$$

Solving this, one gets

$$b(z)c(w) = \frac{1}{\pi g}\frac{1}{z-w} + \text{reg}.$$

This immediately gives us the OPE

$$\begin{aligned} \langle c(z)b(w)\rangle &= \frac{1}{\pi g}\frac{1}{z-w} \\ \langle b(z)\partial c(w)\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^2} \\ \langle \partial b(z)c(z)\rangle &= \frac{1}{\pi g}\frac{1}{(z-w)^2}. \end{aligned}$$

The usual energy momentum tensor is given by

$$T_C^{\mu\nu} = \frac{g}{2}\left(b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu} b^{\alpha\beta}\partial_\alpha c_\beta\right).$$

It turns out that the canonical energy momentum tensor as above is not symmetric even on-shell, so we need the full machinery of Belinfante construction to make the energy momentum tensor symmetric. We add $\partial_\rho B^{\rho\mu\nu}$ as in Proposition 5.5.5 where

$$B^{\rho\mu\nu} = -\frac{1}{2}g(b^{\nu\rho}c^\mu - b^{\nu\mu}c^\rho)$$

It can then be shown that the Belinfane energy momentum tensor given by

$$T_B^{\mu\nu} = \frac{g}{2}\left[b^{\mu\alpha}\partial^\nu c_\alpha + b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right]$$

is symmetric and traceless onshell. In complex coordinates, we have

$$T(z) = \pi g : (2\partial c\; b + c\partial b) :$$

The OPE of T with c can again be calculated using Wick's theorem:

$$\begin{aligned} T(z)c(w) &= \pi g : (2\partial c\; b + c\partial b) : c(w) \\ &= -\frac{c(z)}{(z-w)^2} + 2\frac{\partial_z c(z)}{z-w} + \text{reg.} \\ &= -\frac{c(z)}{(z-w)^2} + 2\frac{\partial_w c(w)}{z-w} + \text{reg.} \end{aligned}$$

Therefore c is a primary field with conformal weight $h = -1$. Similarly, we have

$$\begin{aligned} T(z)b(w) &= \pi g : (2\partial c\; b + c\partial b) : b(w) \\ & 2\frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w}. \end{aligned}$$

This implies that b is a primary field with conformal weight $h = 2$. Note that in both the OPEs we used the anticommutativity of b and c. Finally we have

$$\begin{aligned} T(z)T(w) &= \pi g^2 : (2\partial c(z)b(z) + c(z)\partial b(z)) :: (2\partial c(w)b(w) + c(w)\partial b(w)) : \\ &= \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \end{aligned}$$

Thus we see that this OPE has same form as the previous two examples except for the coefficient of the quartic term. One can tacitly modify this coefficient by modifying the action in such a way that the OPE of b and c remains the same but the energy momentum tensor changes. To be precise, we subtract a total derivative : $\partial(cb)$ from the original action. This gives the new energy momentum tensor to be

$$T(z) = \pi g : \partial c\; b : .$$

This new theory is called the *simple ghost system*. New OPEs are

$$T(z)c(w) = \frac{\partial c(w)}{z-w} + \text{reg.}$$
$$T(z)b(w) = \frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \text{reg.}$$

These OPE imply that in simple ghost theory, c is a primary field with conformal dimension $h = 0$ and b is a primary field of dimension $h = 1$. The OPE of T with T is

$$T(z)T(w) = \frac{-1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

Thus we see that the new coefficient is simply -1.

Remark 5.7.15. Note that the OPE of $T(z)$ with itself in all the three examples above fails to be that of a primary operator only on account of the $(z-w)^{-4}$ term, without which $T(z)$ would be a primary operator of conformal weight $(h, \bar{h}) = (2, 0)$. The coefficient of the $(z-w)^{-4}$ term thus decides this fact. This is a general feature of CFTs. The coefficient of this inverse quartic term is called the *central charge* which we now explore in the next subsection.

5.7.4 Central Charge

We begin by proving the OPE structure of the energy momentum tensor in general CFTs.

Theorem 5.7.16. *The energy momentum tensor in a 2d unitary CFT*[15] *satisfies the OPE:*

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$
$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + reg.$$

The constant $c, \bar{c}$ are called the central charges of the CFT.

Proof. We begin by observing that since the stress tensor is a symmetric tensor of rank 2, thus it must represent a spin $s = 2$ representation of the Lorentz group. Next, the stress tensor has scaling dimension 2. Thus the general form of the OPE of $T(z)$ with itself will be of the form:

$$T(z)T(w) = \cdots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

where the dots in front indicate higher order singularity. Other singular terms in the OPE have the form:

$$\frac{\mathcal{O}_n}{(z-w)^n},$$

[15] we will define and discuss this notion in next section.

which implies that the scaling dimension of $\mathcal{O}_n$ will be

$$\Delta[\mathcal{O}_n] = 4 - n$$

since the left hand side has scaling dimension 4 and $(z-w)^{-n}$ has scaling dimension $-n$. We will shortly prove that any unitary CFT cannot have conformal weights $h, \bar{h} < 0$. Thus the scaling dimension of $\mathcal{O}_n$ cannot be negative which implies that the most singular term that can appear in the OPE is $(z-w)^{-4}$. So the OPE reduces to

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{\mathcal{O}(w)}{(z-w)^3} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

where $\mathcal{O}(w)$ is some field of scaling dimension 1 and c is a complex number. We now only need to rule out the $(z-w)^{-3}$ term. Note that this term violates the $T(z)T(w) = T(w)T(z)$ which is required since we interpret OPEs as operators inside a correlation function. So now it suffices to show that $(z-w)^{-1}$ term does indeed satisfy this upto regular terms. To see this, note that we can expand Taylor $T(z)$ around w:

$$T(z) = T(w) + \partial_w T(w)(z-w) + O((z-w)^2) \implies \partial_z T(z) = 0 + \partial_w T(w) + O(z-w).$$

Thus

$$\begin{aligned}
T(w)T(z) &= \frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \text{reg.} \\
&= \frac{c/2}{(w-z)^4} + \frac{2(T(w) + \partial_w T(w)(z-w) + O((z-w)^2))}{(w-z)^2} + \frac{\partial_w T(w) + O(z-w)}{w-z} + \text{reg.} \\
&= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \\
&= T(z)T(w),
\end{aligned}$$

where the higher order terms in the expansion have been included in the regular terms. This trick does not work for $(z-w)^{-3}$ term. Similar analysis gives the OPE of $\bar{T}(\bar{z})$. □

Transformation of $T(z)$ and the Schwartzian derivative

The anomalous OPE of $T(z)$ with itself due to nontrivial central charge results in a nontrivial transformation of the stress tensor under conformal transformations. This naturally leads to the notion of Schwartzian derivative. To this end we use the conformal Ward identity of Theorem 5.7.2 to obtain the transforamtion of T under infinitesimal conformal transformations $z \to z + \varepsilon(z)$. We have

$$\begin{aligned}
\delta_\varepsilon T(w) &= -\frac{1}{2\pi i}\int_C dz \varepsilon(z) T(z)T(w) \\
&= -\frac{1}{2\pi i}\int_C dz \varepsilon(z)\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.}\right] \\
&= -\frac{c}{12}\partial_w^3 \varepsilon(w) - 2T(w)\partial_w \varepsilon(w) - \varepsilon(w)\partial_w T(w),
\end{aligned} \tag{5.7.27}$$

where we used the Cauchy residue theorem and the fact that $\varepsilon(z)$ and reg. are holomorphic function. Thus we have found the infinitesimal transformation of the stress tensor under infinitesimal conformal transformation. It is not obvious what the finite transformations are corresponding to this infinitesimal transformation. Here is the answer: under a finite conformal transformation $z \to w(z)$, the energy momentum tensor transforms as follows[16]

$$T'(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w; z\}\right], \tag{5.7.28}$$

where $\{w; z\}$ is the *Schwartzian derivative* of w defined by

$$\{w; z\} := \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\left(\frac{d^2w/dz^2}{dw/dz}\right)^2.$$

We will just verify that the infinitesimal version of this transformation is as in (5.7.27). Indeed for infinitesimal conformal transformation, since $w = z + \varepsilon(z)$, upto first order in ε, we have

$$\{z + \varepsilon; z\} = \frac{\partial_z^3 \varepsilon}{1 + \partial_z \varepsilon} - \frac{3}{2}\left(\frac{\partial_z^2 \varepsilon}{1 + \partial_z \varepsilon}\right)^2 \approx \partial_z^3 \varepsilon.$$

At first order in ε then, we have

$$\begin{aligned} T'(z + \varepsilon) &= T'(z) + \varepsilon(z)\partial_z T(z) \\ &= (1 + \partial_z \varepsilon)^{-2} \left[T(z) - \frac{c}{12}\partial_z^3 \varepsilon\right] \\ &= (1 - 2\partial_z \varepsilon) \left[T(z) - \frac{c}{12}\partial_z^3 \varepsilon\right], \end{aligned}$$

where we used (5.7.28). This implies that

$$\delta_\varepsilon T(w) = T'(w) - T(w) = -\frac{c}{12}\partial_w^3 \varepsilon(w) - 2T(w)\partial_w \varepsilon(w) - \varepsilon(w)\partial_w T(w),$$

which coincides with (5.7.27) exactly. We now show that (5.7.28) satisfies the group composition law of consecutive conformal transformation: two consecutive transformations $z \to w(z) \to u(w)$ is equivalent to one transformation $z \to u$. Indeed we have

$$\begin{aligned} T''(u) &= \left(\frac{du}{dw}\right)^{-2} \left[T'(w) - \frac{c}{12}\{u; w\}\right] \\ &= \left(\frac{du}{dw}\right)^{-2} \left[\left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w; z\}\right] - \frac{c}{12}\{u; w\}\right]. \end{aligned}$$

Once we convince ourselves that the Schwartzian derivative satisfies

$$\{u; z\} = \{w; z\} + \left(\frac{dw}{dz}\right)^2 \{u; w\}, \tag{5.7.29}$$

[16]primes are not derivatives here, they just denote the transformed tensor.

then we can easily see that

$$T''(u) = \left(\frac{du}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{u; z\}\right].$$

This is what we wanted to prove. The required identity (5.7.29) of Schwartzian derivative is pretty straightforward and we omit its proof here. Moreover, we can confirm something we already know. If we put $u = z$ in (5.7.29), we get

$$\{w; z\} = -\left(\frac{dw}{dz}\right)^{2} \{u; w\},$$

which gives another version of (5.7.28):

$$T'(w) = \left(\frac{dw}{dz}\right)^{2} T(z) + \frac{c}{12}\{z, w\}. \tag{5.7.30}$$

This equation says that $T(z)$ is a primary field of conformal dimension 2 modulo the central charge (compare with (5.4.6)). Also for global conformal transformations

$$z \to w = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}),$$

it is easy to see that $\{z, w\} = 0$ and we conclude that $T(z)$ is a quasi-primary field (see Definition 5.4.3 (iv)).

Casimir Energy

When we introduce an intrinsic scale into our theory, it breaks conformal symmetry, since conformal invariance includes scale invariance which means that the theory has no scale. This introduction of scale automatically leads to the central charge. We understand this using a toy example. Consider a conformal field theory on the whole complex plane. We map the theory to a cylinder of circumference L using the transformation

$$z \to w = \frac{L}{2\pi} \ln z.$$

Thus we have introduced a periodic boundary condition which inturn is equivalent to introducing a scale in the theory. Using the transfomation of energy momentum tensor as in (5.7.28), we get

$$T_{\mathrm{cyl}}(w) = \left(\frac{2\pi}{L}\right)^{2} \left[T_{\mathrm{pl}}(z) z^2 - \frac{c}{24}\right],$$

where we used the fact that

$$\frac{dw}{dz} = \frac{L}{2\pi z}, \quad \{w; z\} = \frac{1}{2z^2}.$$

Here T_{cyl} and T_{pl} denotes the stress tensor on the cylinder and the plane respectively. If we assume that the vacuum energy on the plane $\langle T_{\text{pl}}\rangle = 0$, then the vacuum energy on the cylinder is given by

$$\langle T_{\text{cyl}}\rangle = -\frac{\pi^2 c}{6L^2}. \tag{5.7.31}$$

Thus the central charge can be interpreted as the Casimir energy of a CFT on the cylinder.

The Trace Anomaly

In Subsection 3.3.5, we proved that the Virasoro generators satisfy the Virasoro algebra which is the central extension of the Witt algebra. We will now show that this is related to the fact that the stress tensor fails to be traceless in the quantum theory in the sense that the expectation value $\langle T^{\alpha}_{\ \alpha}\rangle$ in general is non-zero. This means that the conformal symmetry is broken at the quantum level. To be precise, we will show that for a conformal field theory on a manifold with Ricci scalar R, the expectation value of the stress tensor satisfies:

$$\langle T^{\alpha}_{\ \alpha}\rangle = -\frac{c}{12}R. \tag{5.7.32}$$

Note that conformal anomaly is not present for conformal field theories on flat manifolds since the Ricci is identically zero for flat manifolds. Before we prove this important result, we make some comments:

- (5.7.32) is true with expectation value is taken for an arbitrary state and not just the vacuum. This implies that the expectation value must not depend on any physical state except for the background metric with an added condition that it must have conformal dimension 2. This leaves us only with R being our only choice. The question is what is the coefficient which is the content of the derivation here.
- In 2d, we can always transform the metric to $g_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}$ by a suitable choice of charts[17] on the manifold for some function ω. The Ricci scalar then takes the form:

  $$R = -2e^{-2\omega}\partial^2\omega. \tag{5.7.33}$$

  Thus we see that the Ricci scalar depends explicitly on the parameter ω. (5.7.32) implies that the expectation value $\langle T^{\alpha}_{\ \alpha}\rangle$ is dependent on the parameter of transformation, which means that the expectation value is not invariant under Weyl transformations. This explicitly shows that the quantum theory is not conformally invariant.
- Note that (5.7.32) involves the central charge of only the left-moving sector. There is nothing special about this. If we calculated the expectation of the trace in the right-moving sector, we would have gotten the same result with the right-moving central charge $\tilde{c}$. If we want both sectors to be consistent with each other with fixed curved background, then we require $c = \tilde{c}$. This is an example of something called the *gravitational anomaly*.

[17] see Subsection 3.1.1

Theorem 5.7.17. *In a 2d CFT with central charge c on a curved background with Ricci scalar R, the expectation value of the energy momentum tensor satisfies*

$$\langle T^{\alpha}_{\ \alpha}\rangle = -\frac{c}{12}R.$$

Proof. We begin by calculating the OPE[18] of $T_{z\bar{z}}$ with $T_{w\bar{w}}$. The energy conservation equation $\partial_\mu T^{\mu\nu}$ gives

$$\partial_z T_{z\bar{z}} = -\partial_{\bar{z}} T_{zz}. \tag{5.7.34}$$

We thus have

$$\partial_z T_{z\bar{z}} \partial_w T_{w\bar{w}} = \partial_{\bar{z}}\partial_{\bar{w}} T_{zz}T_{ww} = \partial_{\bar{z}}\partial_{\bar{w}}\left[\frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{reg.}\right] \tag{5.7.35}$$

Note that the derivative of the regular term vanishes but the derivative of singular terms gives rise to delta functions. Indeed using Lemma 5.7.1, we have

$$\partial_{\bar{z}}\partial_{\bar{w}}\left(\frac{1}{(z-w)^4}\right) = \frac{1}{3!}\partial_{\bar{z}}\partial_{\bar{w}}\left(\partial_z^2\partial_w\frac{1}{z-w}\right) = \frac{\pi}{3}\partial_z^2\partial_w\partial_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}). \tag{5.7.36}$$

Integrating the ∂_z, ∂_w in (5.7.35), we get

$$T_{z\bar{z}}T_{w\bar{w}} = \frac{c\pi}{6}\partial_z\partial_{\bar{w}}\delta(z-w,\bar{z}-\bar{w}). \tag{5.7.37}$$

Note that the other terms in the OPE (5.7.35) do not contribute since the integral gives regular terms. The expression in (5.7.37) is called the *contact term*. Usng the definition of stress-nergy tensor in terms of action, we now calculate the variation of the expectation value of trace with respect to ariation in the metric $\delta g^{\alpha\beta}$:

$$\delta\left\langle T_{\alpha}^{\ \alpha}(\sigma)\right\rangle = \delta\int[\mathcal{D}\phi]e^{-S}T^{\alpha}_{\ \alpha}(\sigma) = \frac{1}{4\pi}\int[\mathcal{D}\phi]e^{-S}\left(T^{\alpha}_{\ \alpha}(\sigma)\int d^2\sigma'\sqrt{g}\delta g^{\beta\gamma}T_{\beta\gamma}(\sigma')\right).$$

In particular, for Weyl transformation, $\delta g_{\alpha\beta} = 2\omega\delta_{\alpha\beta}$, which implies that $\delta g^{\alpha\beta} = -2\omega\delta^{\alpha\beta}$. Thus we get

$$\delta\left\langle T^{\alpha}_{\ \alpha}(\sigma)\right\rangle = -\frac{1}{2\pi}\int[\mathcal{D}\phi]e^{-S}\left(T^{\alpha}_{\ \alpha}(\sigma)\int d^2\sigma'\omega(\sigma')T^{\beta}_{\ \beta}(\sigma')\right). \tag{5.7.38}$$

We can now use the OPE (5.7.37), but first we need to change the coordinates from σ^α to $(z,\bar{z})$. Using (5.5.3) we see that

$$T^{\alpha}_{\alpha}(\sigma)T^{\beta}_{\beta}(\sigma') = 16T_{z\bar{z}}(z,\bar{z})T_{w\bar{w}}(w,\bar{w}). \tag{5.7.39}$$

[18] note that in the classical theory the mixed components $T_{z\bar{z}}, T_{\bar{z}z}$ is identically zero. See Subsection 5.5.1 for details.

Moreover we see that

$$8\partial_z \bar{\partial}_{\bar{w}} \delta(z-w, \bar{z}-\bar{w}) = -\partial^2 \delta\left(\sigma - \sigma'\right).$$

Thus we get

$$T^{\alpha}_{\ \alpha}(\sigma) T^{\beta}_{\ \beta}\left(\sigma'\right) = -\frac{c\pi}{3} \partial^2 \delta\left(\sigma - \sigma'\right). \tag{5.7.40}$$

We can now plug this into (5.7.38) and integrate by parts to get

$$\delta \left\langle T^{\alpha}_{\alpha} \right\rangle = \frac{c}{6} \partial^2 \omega \qquad \Longrightarrow \qquad \left\langle T^{\alpha}_{\alpha} \right\rangle = -\frac{c}{12} R \tag{5.7.41}$$

where in the last step we used (5.7.33) and replaced $e^{-2\omega} \approx 1$ since ω is infinitesimal. Note that our derivation above only applies to spaces whose metrics are close to the flat metric. But as discussed before, once we determine the coefficient in front of the Ricci scalar for the expectation value of trace, we can write the trace anomaly for any general 2d surface. $\square$

Chapter 6

2d Conformal Field Theory and its Application to Bosonic String Theory

In the previous chapter, we explored some general features of classical and quantum conformal field theory in various dimensions. We saw that the conformal symmetry gets broken at the quantum level and results in a nontrivial central charge. We discussed operators in 2d CFT and the general notion of operator product expansion. In this chapter we explore the quantum Hilbert space of 2d CFTs and the famous state-operator correspondence. We will also discuss various other mathematical notions stemming from 2d CFTs. We begin by discussing the quantisation of 2d CFT.

6.1 Quantisation of a 2d CFT

In this section, we explore the general procedure of quantising a 2d CFT. In usual QFT, we quantise a classical system on time slices which are time ordered in the usual way and construct the Hilbert space at those time slices. The Weyl invariance in a CFT enables us to formulate an equivalent ordering on the plane called radial ordering. This enables us to discover many amazing aspects of a 2d CFT and its quantum theory. We will describe its various aspects beginning with the radial quantisation.

6.1.1 Radial Quantisation

Consider a quantum field theory on the plane $\mathbb{R}^2$. We usually parametrize the plane by coordinates (x, t) and in the quantum theory, the states live on time slices and the Hamiltonian $H = \partial_t$ evolves the states in time (in the Schrödinger picture). Thus time ordering is defined by the vertical stacking of the time slices. In a CFT, we reformulate time ordering in a way which helps us in uncovering new results, we call this ordering the *radial ordering.* To understand the radial ordering, we first map a quantum field theory on the plane to a quantum field theory on the cylinder. To do this, we first parametrize the plane with the origin removed by a complex variable $z = x + it$ and then map it to the cylinder via

$$z \mapsto \ln z =: \sigma + i\tau, \quad \sigma \in [0, 1), \quad \tau \in \mathbb{R}.$$

This gives a one to one mapping of the plane with the origin removed to the cylinder. If we now consider a CFT on the plane with flat metric[1]

$$(ds^2)_{\text{pl}} = dt^2 + dx^2.$$

On the cylinder, the metric is

$$(ds^2)_{\text{cyl}} = d\sigma^2 + d\tau^2.$$

Since

$$x + it = z = \exp(\sigma + i\tau) = \exp(\sigma)(\cos\tau + i\sin\tau).$$

This means that

$$\begin{aligned} dx &= \exp(\sigma)\cos\tau d\sigma - \exp(\sigma)\sin\tau d\tau, \\ dy &= \exp(\sigma)\sin\tau d\sigma + \exp(\sigma)\cos\tau d\tau. \end{aligned}$$

This implies that

$$(ds^2)_{\text{pl}} = \exp(2\sigma)\left(ds^2\right)_{\text{cyl}}.$$

Thus the punctured plane and the cylinder are conformally equivalent. Thus any CFT on the plane is same as a CFT on the cylinder. Thinking of a CFT on the plane as a CFT on the cylinder, we get a new interpretation of the time ordering on the plane induced by the time ordering on the cylinder. To be precise, in a CFT on a cylinder, the time ordering is given by the vertical stacking of the circles. When this time ordering is translated to the plane, we end up with radial ordering. To make this quantitative, consider the inverse mapping of cylinder to the plane as[2]

$$\omega = \sigma + i\omega \mapsto z = e^{-i\omega}.$$

Then $\tau_1 > \tau_2$ implies that $|z_1| > |z_2|$ where $z = e^{\tau}e^{i\sigma}$. Since $|z|$ on plane represents the radial distance, larger τ on the cylinder translates to larger radial distance. Thus time ordering on cylinder translates to radial ordering on plane. Thus the Hilbert space now live on radial slices and the Hamiltonian evolves the states in the radial position on the plane. Thus the Hamiltonian takes the form of the dilatation operator

$$H = z\partial_z + \bar{z}\partial_{\bar{z}}. \tag{6.1.1}$$

[1] we consider the Euclidean metric. The analysis for Minkowski metric is similar.

[2] the extra minus sign is introduced so that greater τ on cylinder gives larger radial distance on the plane.

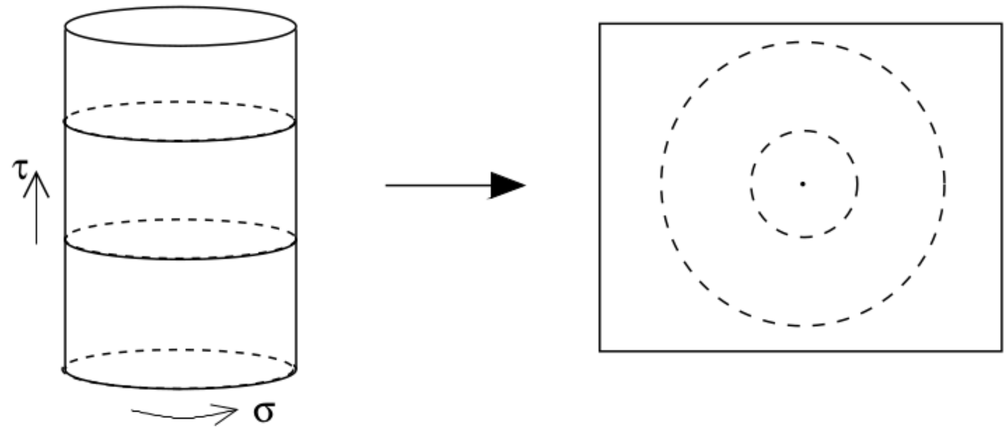

Figure 6.1: Mapping the cylinder to the complex plane

Hermitian structure

To construct the quantum states of the theory, we need to assume the existence of a vacuum state $|0;0\rangle$ on which we can act by creation operators to construct the Hilbert state of the theory. We will discuss more about it in the next section. In usual QFT, a field $\phi(x,t)$ gives rise to an asymptotic state $\phi_{\text{in}} \propto \lim_{t\to-\infty}\phi(x,t)$. On radial quantisation, this translates to

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0;0\rangle \tag{6.1.2}$$

An important point to note is that there are two asymptotic states associated to a field in a QFT with the usual time ordering: namely the "in" and "out" state corresponding to $t \to \mp\infty$ respectively. But in radial quantisation, we get a single asymptotic state corresponding to $z,\bar{z}\to 0$. This is remeniscent of the "operator-state correspondence" in conformal field theory which we will discuss in detail in coming sections.

We now endow a Hermitian structure to the states of the theory as follows: for a conformal field $\phi(z,\bar{z})$ with conformal weights $(h,\bar{h})$, define the Hermitian conjugate on the real surface $\bar{z}=z^*$:

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi(1/\bar{z},1/z). \tag{6.1.3}$$

This definition can be motivated by the fact the on the plane, the Wick-rotated Euclidean time $\tau = it$ must map to $\tau \to -\tau$ inside an operator under Hermitian conjugate if t is to be interpreted as time. This is precisely done by $z\to 1/\bar{z}$ and $\bar{z}\to 1/z$. The prefactor will make sense in a moment. The Hermitian conjuate of the asymptotic in state is then defined to be the asymptotic out state:

$$|\phi_{\text{out}}\rangle = |\phi_{\text{in}}\rangle^\dagger.$$

This is called BPZ (Belavin-Polyakov-Zamalodchikov) conjugation. The inner product between states can now be calculated to be

$$\begin{aligned}\langle\phi_{\text{out}}|\phi_{\text{in}}\rangle &= \lim_{z,\bar{z},w,\bar{w}\to 0}\left\langle 0;0\left|\phi(z,\bar{z})^\dagger\phi(w,\bar{w})\right|0;0\right\rangle \\ &= \lim_{z,\bar{z},w,\bar{w}\to 0}\bar{z}^{-2h}z^{-2\bar{h}}\langle 0;0|\phi(1/\bar{z},1/z)\phi(w,\bar{w})|0;0\rangle \\ &= \lim_{\xi,\bar{\xi}\to\infty}\bar{\xi}^{2h}\xi^{2\bar{h}}\langle 0;0|\phi(\bar{\xi},\xi)\phi(0,0)|0;0\rangle.\end{aligned}$$

Note that the last line is a correlator while the first line was a vacuum expectation value, since the operators are clearly radial (time) ordered. Also by the usual transformation of correlator under conformal transformation (5.7.1), the last line is independent of ξ and hence the inner product is well-defined.

Mode expansions

A quasi-primary operator $\phi(z,\bar{z})$ of conformal dimension $(h,\bar{h})$ can be expanded as a Laurent series in z and $\bar{z}$ as follows:

$$\phi(z,\bar{z}) = \sum_{m,n\in\mathbb{Z}}\phi_{m,n}z^{-m-h}\bar{z}^{-n-\bar{h}}. \tag{6.1.4}$$

By residue theorem, the modes $\phi_{m,n}$ can be expressed as a contour integral

$$\phi_{m,n} = \frac{1}{2\pi i}\oint dz\ z^{m+h-1}\oint d\bar{z}\ \bar{z}^{n+\bar{h}-1}\phi(z,\bar{z}). \tag{6.1.5}$$

Hermitian conjugate of the operator gives

$$\phi(z,\bar{z})^\dagger = \sum_{m,n\in\mathbb{Z}}\phi^\dagger_{m,n}\bar{z}^{-m-h}z^{-n-\bar{h}}. \tag{6.1.6}$$

Using the definition in (6.1.3), we easily see that

$$\phi^\dagger_{m,n} = \phi_{-m,-n}. \tag{6.1.7}$$

We introduce the notation

$$|h,\bar{h}\rangle := \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0;0\rangle = \phi_{-h,-\bar{h}}|0;0\rangle \tag{6.1.8}$$

for the asymptotic states. Note that for the asymptotic "in" and "out" states to be well defined, we must have:

$$\phi_{m,n}|0;0\rangle = 0; \quad m > -h, n > -\bar{h}. \tag{6.1.9}$$

For a chiral field $\phi(z)$ we write the expansion as

$$\phi(z) = \sum_{n\in\mathbb{Z}}\phi_n z^{-n-h}, \quad \phi^\dagger_n = \phi_{-n}, \tag{6.1.10}$$

and similarly for anti-chiral field. In particular, for the stress tensor, we have

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad \overline{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}, \tag{6.1.11}$$

so that

$$L_n = \frac{1}{2\pi i}\oint dz\, T(z) z^{n+1}, \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \overline{T}(\bar{z})\bar{z}^{n+1}. \tag{6.1.12}$$

The asymptotic state corresponding to the stress tensor is called the *conformal vector*. Note that the condition (6.1.9) for the stress tensor gives

$$L_n|0;0\rangle = 0, \quad \bar{L}_n|0;0\rangle = 0, \quad n \geq 1. \tag{6.1.13}$$

In particular, the vacuum vector is invariant under the global conformal transformations generated by $L_0, L_{\pm 1}$.

6.1.2 The state-operator correspondence: vertex operators

In conformal field theory, there is a distinguished property: the states and local operators are in one to one correspondence. In a generic quantum field theory, this is never true; one local operator corresponds to many different states. To understand this correspondence, recall that a state in quantum mechanics can be represented by a square integrable function, called the wavefunction, satisfying the Schrödinger equation. More concretely, suppose we have a particle moving in on dimension under some potential. Suppose the particle is in quantum state[3] $|\Psi(t)\rangle$ at time t. In the position basis $|x\rangle$ the state $|\Psi(t)\rangle$ can be represented by the wavefunction $\Psi(x,t)$:

$$\Psi(x,t) := \langle x|\Psi(x)\rangle. \tag{6.1.14}$$

One can understand states in QFT in a similar way. Instead of position basis, which describes the trajectory of particle in space, we construct a field configuration basis $|\phi(\boldsymbol{\sigma})\rangle$. A state is then represented by a Schrödinger wavefunctional $\Psi[\phi(\boldsymbol{\sigma}),t]$ - the corresponding state being $|\Psi(t)\rangle$ and

$$\Psi[\phi(\boldsymbol{\sigma}),t] = \langle\phi(\boldsymbol{\sigma})|\Psi(t)\rangle. \tag{6.1.15}$$

Consider a CFT on the plane which is radially quantised. Suppose we have a local operator $\mathcal{O}(z)$. Let us now construct a state at radius $|z| = r$ on the plane[4]. Let us consider fields in a unit disc in the z-plane with fixed boundary condition ϕ_b at the boundary circle. We consider the path integral with the operator insertion $\mathcal{O}(z=0)$ in the path integral[5]. Then the functional

$$\Psi_{\mathcal{O}}\left[\phi_b(\boldsymbol{\sigma})\right] := \int \left[\mathcal{D}\phi_i\right]_{\phi_b} \exp\left(-S\left[\phi_i\right]\right)\mathcal{O}(0) \tag{6.1.16}$$

[3]Recall that in a QFT the Hilbert space of states is defined on time slices of spacetime so that for every $t \in \mathbb{R}$ there is a Hilbert space of states. Similarly in quantum mechanics, where we do not have the notion of fields or spacetime, we have a Hilbert space of states for each time determined by the wavefunctions obtained by solving the Schrödinger equation.

[4]Recall that in radial quantisation, time ordering changes to radial ordering. Thus Hilbert space of states exists at various radii.

[5]We consider Euclidean path integral.

defines a wavefunctional and hence a state of the QFT. To construct a local operator from a wavefunctional, we recall how wavefunction is propagated in time. This is done using the propagator

$$G\left(x_f, x_i\right)=\int_{x(t_i)=x_i}^{x\left(t_f\right)=x_f}[\mathcal{D} x] \exp (-S) \tag{6.1.17}$$

where the path integral is over all classical paths with position x_i at time t_i and position x_f at time t_f. Now a wavefunction $\psi_i\left(x_i, t_i\right)$ can be evolved to $\psi_f\left(x_f, t_f\right)$ using the above propagator:

$$\begin{aligned}
\psi_f\left(x_f, t_f\right) & =\int d x_i G\left(x_f, x_i\right) \psi_i\left(x_i, t_i\right) \\
& =\int d x_i \int_{x(t_i)=x_i}^{x\left(t_f\right)=x_f}[\mathcal{D} x] \exp (-S) .
\end{aligned} \tag{6.1.18}$$

Let us now do the same with states in QFT. We have

$$\Psi_f\left[\phi_f(\boldsymbol{\sigma}), t_f\right]=\int\left[D \phi_i\right] \int_{\phi(t_i)=\phi_i}^{\phi\left(t_f\right)=\phi_f}[D \phi] \exp (-s[\phi]) \Psi_i\left[\phi_i(\boldsymbol{\sigma}), t_i\right] \tag{6.1.19}$$

where the path integral is over classical solutions with bounder y conditions as given above. In the context of radial quantisation the time coordinate (the vertical direction) on the cylinder turns into radial coordinate on the plane. The path integral becomes

$$\Psi_f\left[\phi_f(\boldsymbol{\sigma}), r_f\right]=\int\left[\mathcal{D} \phi_i\right] \int_{\phi(r_i)=\phi_i}^{\phi\left(r_f\right)=\phi_f}[\mathcal{D} \phi] \exp (-S[\phi]) \Psi_i\left[\phi_i(\boldsymbol{\sigma}), r_i\right] . \tag{6.1.20}$$

Thus the path integral is over fields on the annulus $r_i \leq |z| \leq r_f$ with boundary condition $\phi\left(r_i\right)=\phi_i$ and $\phi\left(r_f\right)=\phi_f$ along with state insertion $\Psi\left[\phi_i(\boldsymbol{\sigma}), r_i\right]$ on the inner circle. What happens when we take $r_i \rightarrow 0$. We would obtain a state as in (6.1.16). The insertion of the state turns into an operator insertion. We get

$$\Psi_{\mathcal{O}}\left[\phi_b(\boldsymbol{\sigma})\right]:=\int\left[\mathcal{D} \phi_i\right]_{\phi_b} \exp \left(-S\left[\phi_i\right]\right) \mathcal{O}(z=0). \tag{6.1.21}$$

This defines a local operator from a state. We call these operators *vertex operators*.

Remark 6.1.1. (i) From above considerations, it is clear that the state-operator correspondence is true for any CFT defined on a cylinder $\mathbb{R} \times S^{D-1}$ which gets mapped to $\mathbb{R}^D$ under radial quantisation.

(ii) If we consider the state corresponding to the identity operator, the above map (6.1.16) gives us the vacuum state. We usually denote this state by $|0;0\rangle$ as above. Clearly the conformal dimensions of the identity operator is (0,0).

Example: the closed string

6.1.3 Operator product expansion and Mode Algebra

As we have seen, in radial quantisation, time ordering is replaced by radial ordering on the cylinder. So whenever we talk about correlation function of operators, they are assumed to be radially ordered in the following sense: for any two operators $A(z)$ and $B(z)$ we define their radially ordered product to be the operator

$$\mathcal{R}(A(z)B(w)) := \begin{cases} A(z)B(w) & \text{if } |z| > |w|, \\ B(w)A(z) & \text{if } |w| > |z| \end{cases}. \tag{6.1.22}$$

If both the operators A and B are fermionic then a minus sign must be added to the second expression owing to spin-statistics theorem. Let $C^z_r(w)$ denote a circle of radius $r > 0$ in variable z centered around w and oriented counterclockwise and put $C^z_r := C^z_r(0)$. Then we see that for $r_1 > |w|$ and $r_2 < |w|$ we have

$$\oint_{C^z_{r_1}} dz A(z)B(w) - \oint_{C^z_{r_2}} dz B(w)A(z) = \oint_{C^z_{r_3}(w)} dz \mathcal{R}(A(z)B(w)), \tag{6.1.23}$$

for some $r_3 > 0$, see Figure below for the contour deformations. In particular for $r_1 > r_2 > r_3 > 0$ we have

$$\oint_{C^w_{r_2}} dw \oint_{C^z_{r_1}} dz A(z)B(w) - \oint_{C^w_{r_2}} dw \oint_{C^z_{r_3}} dz B(w)A(z) = \oint_{C^w_{r_2}} dw \oint_{C^z_{\delta}(w)} dz \mathcal{R}(A(z)B(w)) \tag{6.1.24}$$

for some $\delta > 0$. We can now use this to derive the algebra of modes of operators from their OPE. Let us derive the Virasoro algebra from the OPE (**??**) of stress tensor with itself.

Using (6.1.24), we have

$$
\begin{aligned}
[L_m, L_n] &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} z^{m+1} w^{n+1} [T(z), T(w)] \\
&= \oint_{C^w_{r_2}} \frac{dw}{2\pi i} w^{n+1} \oint_{C^z_\delta(w)} \frac{dz}{2\pi i} z^{m+1} \mathcal{R}(T(z)T(w)) \\
&= \oint_{C^w_{r_2}} \frac{dw}{2\pi i} w^{n+1} \oint_{C^z_\delta(w)} \frac{dz}{2\pi i} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \right) \\
&= \oint_{C^w_{r_2}} \frac{dw}{2\pi i} w^{n+1} \Big((m+1)m(m-1) w^{m-2} \frac{c}{2 \cdot 3!} \\
&\quad + 2(m+1) w^m T(w) + w^{m+1} \partial_w T(w) \Big) \\
&= \oint \frac{dw}{2\pi i} \Big(\frac{c}{12} \left(m^3 - m\right) w^{m+n-1} \\
&\quad + 2(m+1) w^{m+n+1} T(w) + w^{m+n+2} \partial_w T(w) \Big) \\
&= \frac{c}{12} \left(m^3 - m\right) \delta_{m,-n} + 2(m+1) L_{m+n} \\
&\quad + 0 - \underbrace{\oint \frac{dw}{2\pi i} (m+n+2) T(w) w^{m+n+1}}_{=(m+n+2)L_{m+n}} \\
&= (m-n) L_{m+n} + \frac{c}{12} \left(m^3 - m\right) \delta_{m,-n},
\end{aligned}
\tag{6.1.25}
$$

where we used integration by parts.

6.1.4 Hilbert space of the theory

Let us now describe the Hilbert of a conformal field theory. The state operator correspondence dictates that corresponding to every operator there must be a state in the Hilbert space of the theory. Suppose $\phi(z, \bar{z})$ is a primary field with conformal dimensions $(h, \bar{h})$. Then there is a state $|h, \bar{h}\rangle$ constructed as in (6.1.2). In particular, the vacuum vector $|0; 0\rangle$ is obtained from the identity operator. Clearly, the OPE (**??**) with the stress tensor, it is clear that

$$
\begin{aligned}
&L_0 |h, \bar{h}\rangle = h |h, \bar{h}\rangle, \quad \bar{L}_0 |h, \bar{h}\rangle = \bar{h} |h, \bar{h}\rangle \\
&L_n |h, \bar{h}\rangle = \bar{L}_n |h, \bar{h}\rangle = 0, \quad n > 0
\end{aligned}
\tag{6.1.26}
$$

We then construct the Fock space over $|h, \bar{h}\rangle$ by applying $L_{-n}, \bar{L}_{-m}$ with $n, m > 0$. The Fock space thus obtained is called the *Verma module* over the primary $|h, \bar{h}\rangle$. More precisely, the Verma module has the structure of a tensor product

$$
V(h, c) \otimes \overline{V}(\bar{h}, \bar{c}) \tag{6.1.27}
$$

where

$$
\begin{aligned}
V(h,c) &:= \{L_{-n_1}L_{-n_2}\dots L_{-n_k}|h,\bar{h}\rangle : k\in\mathbb{N},\ n_1,n_2,\dots,n_k\in\mathbb{N}\} \\
\overline{V}(\bar{h},\bar{c}) &:= \{\bar{L}_{-m_1}\bar{L}_{-m_2}\dots \bar{L}_{-m_\ell}|h,\bar{h}\rangle : \ell\in\mathbb{N},\ m_1,m_2,\dots,m_\ell\in\mathbb{N}\}.
\end{aligned}
\tag{6.1.28}
$$

The states $L_{-n_1}L_{-n_2}\dots L_{-n_k}|h,\bar{h}\rangle, \bar{L}_{-m_1}\bar{L}_{-m_2}\dots \bar{L}_{-m_\ell}|h,\bar{h}\rangle$ are called *descendents*. The operator corresponding to them is given by

$$
\begin{aligned}
L_{-n_1}L_{-n_2}\dots L_{-n_k}|h,\bar{h}\rangle &\leftrightarrow \prod_{i=1}^{k}\frac{1}{(n_i-1)!}\partial^{n_1}\partial^{n_2}\dots\partial^{n_k}\phi(z,\bar{z}) \\
\bar{L}_{-m_1}\bar{L}_{-m_2}\dots \bar{L}_{-m_\ell}|h,\bar{h}\rangle &\leftrightarrow \prod_{j=1}^{\ell}\frac{1}{(m_j-1)!}\bar{\partial}^{m_1}\bar{\partial}^{m_2}\dots\bar{\partial}^{m_\ell}\phi(z,\bar{z}).
\end{aligned}
\tag{6.1.29}
$$

The full Hilbert space is the direct sum

$$
\mathscr{H}(c,\bar{c}) := \bigoplus_{h,\bar{h}} V(h,c)\otimes\overline{V}(\bar{h},\bar{c}) \tag{6.1.30}
$$

where the sum is over all primaries of the theory.

6.2 Torus partition function and modular invariance

Up until now, we have studied conformal field theory on the plane (and hence the Riemann sphere by one point compactification) and the cylinder. We will now study conformal field theories on the torus which is a genus one 2 dimensional Riemannian manifold. Infact one can (and has to for string theory applications[6]) study conformal field theories on any higher genus Riemann surfaces. But physically in context of critical phenomenas described by conformal field theories, it is somewhat unnatural to study CFTs on genus higher than 1. On the torus, it is equivalent to the plane with periodic boundary condition in both space and time directions and hence has some significance.

Another motivation to study CFTs on higher genus Riemann surfaces is as follows: on the plane we saw that the holomorphic and antiholomorphic sector completely decouples and we may study the two sectors independently, infact we can treat them as different theories. But this is very unphysical, such a decoupling is a feature of a conformally invariant point on the infinite plane in the parameter space. The Hilbert space of the theory must continuously deform as we move away from the conformally invariant point. This should lead to constraints on the Hilbert space at the conformal point itself. Such constraints are obtained from the consitency of the CFT on higher genus Riemann surface. In particular, the requirement of *modular invariance* of the partition function of the theory on the torus will give us one such constraint.

[6] We will see in later chapters that CFTs on higher genus Riemann surfaces are required to compute higher loop string amplitudes.

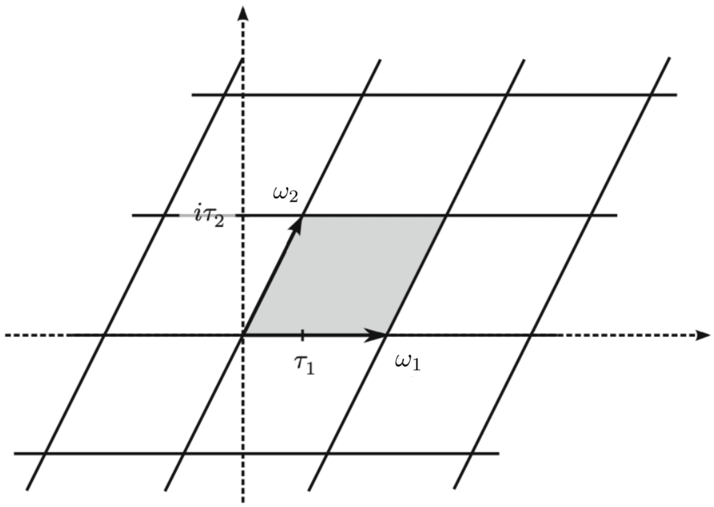

Figure 6.2: The fundamental region of the torus parametrized by (ω_1, ω_2). In the figure, we have chosen the basis of the lattice to be $(1, \tau)$ where $\tau = \tau_1 + i\tau_2 = \omega_2/\omega_1$.

6.2.1 Geometry of the torus

The torus can be obtained from the cylinder by cutting off the infinite ends of the cylinder at some finite length and identifying the boundary circles. Alternatively, it can be obtained as a quotient of the complex plane by some discrete lattice. It is useful to describe the torus in terms of the quotient of the complex plane. For a pair of complex numbers $\omega_1, \omega_2 \neq 0$, consider the quotient space $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ obtained by identifying points on the complex plane as follows:

$$z \sim z + m\omega_1 + n\omega_2, \quad m, n \in \mathbb{Z}. \tag{6.2.1}$$

It is convenient to rescale and rotate the basis (ω_1, ω_2) of the lattice to the basis $(1, \tau)$ where

$$\tau := \frac{\omega_2}{\omega_1}. \tag{6.2.2}$$

The smallest fundamental domain of this equivalence relation is called the *fundamental region* of the torus, see Figure 6.2. The torus itself is obtained by identifying the boundaries of the fundamental region. The complex number τ is called the *complex structure* or the *modular parameter* of the torus. Form the equivalence relation (6.2.1) we see that several choices of the basis (ω_1, ω_2) determine the same lattice and hence the same torus. Indeed any integer linear combination of ω_1, ω_2 determines the same lattice given that the linear combination is invertible by integer linear combinations again. More precisely

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}. \tag{6.2.3}$$

The inverse of this transforamtion is given by

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}. \tag{6.2.4}$$

The matrix i integral if and only if its determinant $ad - bc = \pm 1$.

6.3 Conformal Field Theory with Boundary

Recall that the worldsheet of a closed string has the topology of a cylinder while that of an open string is an infinite plane bounded by the lines traced by the ends of the open string. Since a cylinder is conformally equivalent to th punctured complex plane, we can apply the results of CFT on the plane to the worldsheet CFT of a closed string, but fro open string this does not work since the worldsheet in this case has a boundary. In this section we develop the theory of boundary conformal field theory (BCFT) and analyze some essential features of the theory.

6.3.1 Stress tensor and boundary conditions

A prototype of a space with boundary is the complex upper-half plane

$$\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}. \tag{6.3.1}$$

The real axis is the boundary of this space. We will study conformal field theory on $\mathbb{H}$. First note that a global conformal transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}$ that maps $\mathbb{H}$ to $\mathbb{H}$ has to have real matrix elements since it must map the real axis to the real axis. Thus the group of global conformal transformations of $\mathbb{H}$ is $\mathrm{SL}(2, \mathbb{R})$. Let us now look at the local conformal transformation. An infinitesimal conformal transformation is given by

$$z \to z + \varepsilon(z).$$

This transformation maps $\mathbb{H}$ to $\mathbb{H}$ only if $\varepsilon(z) = \bar{\varepsilon}(\bar{z})$ when $z = \bar{z}$. This means that ε has to be real on the real axis. This imposes strong constriants on the conformal algebra and the two sets of Virasoro algebra for the holomorphic and antiholomorphic sector breaks down to one set of Virasoro algebra as we will now see. Under conformal transformations from $\mathbb{H}$ to $\mathbb{H}$, the correlators must transform covariantly. The fact the correlators transform covariantly is captured by the following theorem:

Theorem 6.3.1. *Let $T^{\mu\nu}$ be the stress tensor for a CFT on $\mathbb{H}$. Then a given boundary condition on a set of conformal primary fields ϕ_i is conformally invariant if and only if $T^{01}(x, 0) = 0$ or equivalently in complex coordinates*

$$T(z) = \overline{T}(\bar{z}) \quad on \quad z = \bar{z}.$$

Proof. Let $X = \prod_{i=1}^{n} \phi_i(x_i, y_i)$ be a set of fields. Let

$$x^\mu \to x^\mu + \varepsilon^\mu(x),$$

be a transformation on $\mathbb{H}$ viewed as a subset of $\mathbb{R}^2$ satisfying the following conditions:

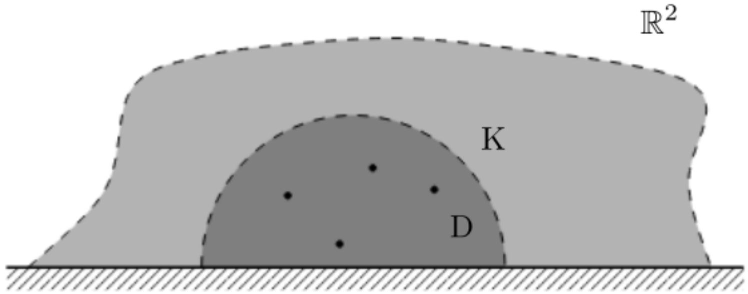

Figure 6.3: Infinitesimal transformation on $\mathbb{H}$

1. $\varepsilon : \mathbb{H} \longrightarrow \mathbb{R}^2$ is a continuous function.
2. The transformation $(x, y) \mapsto (x, y) + (\varepsilon^0(x, y), \varepsilon^1(x, y))$ maps $\mathbb{H}$ to itself, i.e. for all x we have $\varepsilon^1(x, 0) = 0$.
3. $\varepsilon(x, y)$ is conformal in a semi-disc D (see Figure 6.3) containing the point of insertions of all the fields X.
4. Let K be a compact set such that $\mathrm{D} \subset \mathrm{K}$. $\varepsilon(x, y)$ is arbitrary in $\mathrm{K} - \mathrm{D}$.
5. $\varepsilon(x, y)$ is zero on $\mathbb{R}^2 - \mathrm{K}$.

Then using (5.7.13), the variation of the correlation function $\langle X \rangle$ in terms of the path integral can be written as

$$\int [\mathcal{D}\Phi] \delta X e^{-S[\Phi]} = -\int_{\partial(\mathrm{K}-\mathrm{D})} n_\nu(x) \varepsilon_\mu(x) \langle T^{\mu\nu}(x) X \rangle + \int_{\mathrm{K}-\mathrm{D}} \varepsilon_\mu(x) \partial_\nu \langle T^{\mu\nu}(x) X \rangle \tag{6.3.2}$$

where n_ν is the normal to the boundary $\partial(\mathrm{K} - \mathrm{D})$ of $\mathrm{K} - \mathrm{D}$. The second term is zero on account of the stress tensor being conserved due to conformal invariance of the theory. THe first term gives us something physical. If suppose that the semi-disk D covers the interval (a, b) on the real axis, then using $n_\nu = (0, 1)$ and the fact that $\varepsilon^1(x, 0) = 0$, the first integral is

$$\int_a^b \varepsilon_0(x, 0) \langle T^{01}(x, 0) X \rangle. \tag{6.3.3}$$

Demanding that this be independent of $\varepsilon_0(x, 0)$, we get the boundary condition for stress tensor

$$T^{01}(x, 0) = 0.$$

The secons relation follows from (5.5.3). □

The condition $T^{01}(x, 0) = 0$ implies that no energy flows across the boundary. This constraint is not enough to fix the boundary condition uniquely, but this gives us a class of boundary conditions allowed by conformal invariance. Another important consequence of this is that now we only have one set of Virasoro generators. We will describe this by the *doubling trick*. Define $T(z)$ in the lower half-plane by

$$T(z) := \overline{T}(z), \quad \mathrm{Im} z < 0. \tag{6.3.4}$$

Thus we have an analytic continuation of the holomorphic stress tensor to the whole complex plane $\mathbb{C}$. The Virasoro mode can then be given by the integral

$$L_n = \frac{1}{2\pi i} \int_{\mathcal{C}} dz T(z) z^{n+1} + \frac{1}{2\pi i} \int_{-\mathcal{C}} d\bar{z} \overline{T}(\bar{z}) \bar{z}^{n+1} \tag{6.3.5}$$

where $\mathcal{C}$ is a semicircular contour on the origin and $-\mathcal{C}$ is the same contour with opposite orientation. The two integral can be combined and we can write

$$L_n = \frac{1}{2\pi i} \oint dz T(z) z^{n+1} \tag{6.3.6}$$

where now the contour is a circle around the origin and T is the analytically continued stress tensor. These modes satisfy the Virasoro algebra. Similar doubling trick can be applied to all other operators of the theory.

6.4 Applying Conformal Field Theory to String Theory

Chapter 7

The Polyakov Path Integral and BRST Quantisation

In the modern formulation of quantum field theory, path integral is the most important object since we can get all the observables of the theory from the path integral by simple functional differentiation. In this chapter, we develop the string path integral and define the observables of string theory – the string S-matrix. We then discuss the string spectrum of the theory via the BRST quantisation procedure and prove the no-ghost theorem mentioned in Chapter 3.

7.1 Polyakov path integral

Given a QFT of fields $\{\phi\}$ with action $S(\{\phi\})$, the partition function is defined by the path integral

$$Z := \int [\mathcal{D}\phi] \exp(iS[\{\phi\}]). \tag{7.1.1}$$

Since the action is real number, the exponential factor is just a phase and the path integral diverges. To overcome this difficulty, we define the *Euclidean path integral* by

$$Z := \int [\mathcal{D}\phi] \exp(-S_E[\{\phi\})]), \tag{7.1.2}$$

where S_E is the Euclidean action obtained by analytically continuing the time coordinate t to Euclidean time $t \to it_E$. Under this change $S \to iS_E$. The Euclidean path integral is thus analytically continued version of the origin path integral. From now on we will omit the subscript E but all path integrals will be Euclidean.

We will start with the Polyakov action[1]

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma d\sigma d\tau \sqrt{g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{7.1.3}$$

where Σ is the string worldsheet. One can also add the topological term:

$$\chi = \frac{1}{4\pi}\int_\Sigma d\sigma d\tau \sqrt{g} R, \tag{7.1.4}$$

where R is the Ricci scalar of the worldsheet. This is called the *Euler characteristic* and is independent of the choice of metric on Σ – hence the term is topological. This term is clearly Diff invariant since R is a scalar. To see that it is also Weyl invariant, recall that under $g_{\alpha\beta} \to g'_{\alpha\beta} = e^{2\omega(\boldsymbol{\sigma})} g_{\alpha\beta}$, we have

$$\sqrt{g'}R' = \sqrt{g}\left(R - 2\nabla^2\omega\right). \tag{7.1.5}$$

Thus the Weyl variation is

$$\delta_W\left(\sqrt{g}R\right) = -2\sqrt{g}\nabla^2\omega. \tag{7.1.6}$$

But note that for any vector v^α,

$$\sqrt{g}\nabla_\alpha v^\alpha = \partial_\alpha\left(\sqrt{g}v^\alpha\right). \tag{7.1.7}$$

This implies that

$$\sqrt{g}\nabla^2\omega = \sqrt{g}\nabla_\alpha\left(\nabla^\alpha\omega\right) = \partial_\alpha\left(\sqrt{g}\nabla^\alpha\omega\right) \tag{7.1.8}$$

which means that the variation is a total derivative and its integral over Σ vanishes by Stoke's theorem if Σ has no boundary. But this is true only for manifolds without boundary, *i.e.* for closed string worldsheet. For open string worldsheet we have to add a boundary term to make it Weyl invariant:

$$\chi = \frac{1}{4\pi}\int_\Sigma d\sigma d\tau \sqrt{g}R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\ k \tag{7.1.9}$$

where $\partial\Sigma$ denotes the boundary of Σ, ds is the proper time in metric $g_{\alpha\beta}$ along the boundary and k is the geodesic curvature of the boundary

$$k = \pm t^a n_a \nabla_b t^b, \tag{7.1.10}$$

where n^a, t^b are the normal and tangent vectors to the boundary respectively, see Figure ?? below. The upper sign is for Lorentzian signature and lower sign is for Euclidean signature. The extra term is required to make the Euler characteristic term Weyl invariant. We will now consider the path integral

$$Z = \int [\mathcal{D}X\mathcal{D}g]\exp(-S) \tag{7.1.11}$$

where

$$S = S_{\mathrm{P}} + \lambda\chi, \tag{7.1.12}$$

and the path integral is over every embedding $X^\mu : \Sigma \longrightarrow \mathbb{R}^D$ and every metric $g_{\alpha\beta}$ on the worldsheet. As already discussed, we take the metric to be Euclidean with $(+,+)$ signature so that the path integral is well-defined.

[1]Note that now we are in Euclidean signature and hence the determinant of the metric is positive.

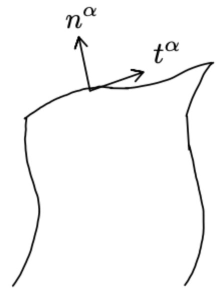

Figure 7.1: The normal and tangent vectors on the boundary.

7.1.1 Fadeev-Popov gauge fixing

Path integrals as in (7.1.2) for gauge theories necessarily diverge unless we remove the unphysical gauge degrees of freedom. Removing the gauge degrees of freedom schematically results in the following expression for the path integral:

$$Z := \int \frac{[\mathcal{D}\phi]}{\mathrm{V}_{\text{gauge}}} \exp(-S[\{\phi\}]), \tag{7.1.13}$$

where $\mathrm{V}_{\text{gauge}}$ is the volume of the gauge group[2]. In string theory, the gauge group is $\mathcal{G} :=$ Diff $\ltimes$ Weyl. The string partition function thus takes the form

$$Z \equiv \int \frac{[\mathcal{D}X\mathcal{D}g]}{\mathrm{V}_{\mathcal{G}}} e^{-S[X,g]} \tag{7.1.14}$$

Fadeev-Popov methods tells us how to divide by the volume of the gauge group. More precisely, it gives us a method to integrate against a measure on the field space $\mathcal{X} = \{(X, g)\}$ which cuts through each gauge equivalence class exactly once as shown in Figure 7.2 below. This is equivalent to saying that we want a measure on $\mathcal{X}/\mathcal{G}$ where $\mathcal{X}$ is the space of fields and $\mathcal{G}$ is the group of gauge transformations.

The idea is to separate the integral over the space of fields into an integral over a gauge slice times an integral over the gauge group. To do this, we change variables carefully. The method is called the *Fadeev-Popov method* and the Jacobian of the change of variables is called the *Fadeev-Popov determinant*. Under a gauge transformation $\zeta \in \mathcal{G}$:

$$\begin{aligned} &X^\mu(\sigma,\tau) \longrightarrow X^{\mu\zeta}(\sigma',\tau')\,; \\ &g_{\alpha\beta}(\sigma,\tau) \longrightarrow g^{\zeta}_{\alpha\beta}(\sigma',\tau') = e^{2\omega(\sigma)} \frac{\partial\sigma^\gamma}{\partial\sigma'^a}\frac{\partial\sigma^\delta}{\partial\sigma'^b} g_{\gamma\delta}(\sigma,\tau). \end{aligned} \tag{7.1.15}$$

We denote gauge equivalent field to (X, g) by $\left(X^\zeta, g^\zeta\right)$. Note that the embedding does not change under ζ. To choose a gauge slice is thus equivalent to choosing one metric $\hat{g}_{\alpha\beta}$ on each

[2] one can prove this explicitly. See [16, Chapter] for derivation.

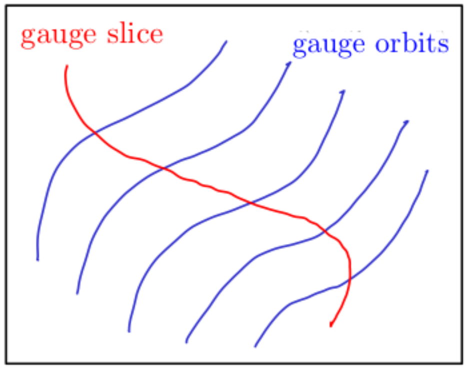

Figure 7.2: The box represents the space of all field configurations $\mathcal{X}$. The blue curves represent gauge equivalent field configuration while the red line represents gauge inequivalent field configurations. It is called a *gauge slice* since it cuts through each gauge orbit exactly once.

slice and removing the $[\mathcal{D}g]$ with $[\mathcal{D}\zeta]$ which is a measure on the gauge group. The choice of metric $\hat{g}_{\alpha\beta}$ on each gauge slice is called the *fiducial metric*. As discussed in Subsection 3.1.1, we can always choose $\hat{g}_{\alpha\beta} = \delta_{\alpha\beta}$ locally on each slice. For now we keep $\hat{g}_{\alpha\beta}$ explicit. This change of variable is only possible if there exists a measure on $\mathcal{G}$. The variable change incurs a Jacobian $\Delta_{\mathrm{FP}}^{-1}[g]$ defined

$$\Delta_{\mathrm{FP}}^{-1}[g] = \int [\mathcal{D}\zeta]\delta\left(g - \hat{g}^{\zeta}\right) \tag{7.1.16}$$

where $\delta(g - \hat{g}^{\zeta})$ is a delta functional which sets $g = \hat{g}^{\zeta}$ at every worldsheet point (σ, τ). The measure $[\mathcal{D}\zeta]$ is assumed to be gauge invariant, we will demonstrate this in Subection **??** below. Now to change coordinates in the path integral, we insert

$$1 = \Delta_{\mathrm{FP}}^{-1}[g] \int [\mathcal{D}\zeta]\delta\left(g - \hat{g}^{\zeta}\right). \tag{7.1.17}$$

The path integral is

$$\begin{aligned} Z[\hat{g}] &= \int \frac{[\mathcal{D}\zeta\mathcal{D}X\mathcal{D}g]}{V_{\mathcal{G}}} \Delta_{\mathrm{FP}}[g]\delta\left(g - \hat{g}^{\zeta}\right) e^{-S[X,g]} \\ &= \int \frac{\left[\mathcal{D}\zeta\mathcal{D}X^{\zeta}\right]}{V_{\mathcal{G}}} \Delta_{\mathrm{FP}}^{-1}\left[\hat{g}^{\zeta}\right] e^{-S\left[X^{\zeta},\hat{g}^{\zeta}\right]} \end{aligned} \tag{7.1.18}$$

where we performed the $\mathcal{D}g$ integral and renamed the dummy variable $X \longrightarrow X^{\zeta}$. To proceed, we need to prove the gauge invariance of $\Delta_{\mathrm{FP}}\left[g^{\zeta}\right]$.

Lemma 7.1.1. *The Fadeev-Popov determinant* $\Delta_{\mathrm{FP}}[g]$ *is gauge invariant.*

Proof. We have

$$\begin{aligned}
\Delta_{\mathrm{FP}}^{-1}\left[g^{\zeta}\right] &= \int [\mathcal{D}\zeta']\,\delta\left(g\zeta - \hat{g}^{\zeta'}\right) \\
&= \int [\mathcal{D}\zeta']\,\delta\left(g - \hat{g}^{\zeta^{-1}\cdot\zeta'}\right) \\
&= \int [\mathcal{D}\zeta'']\,\delta\left(g - \hat{g}^{\zeta''}\right) \\
&= \Delta_{\mathrm{FP}}^{-1}[g]
\end{aligned} \tag{7.1.19}$$

where we changed variable from $\zeta' \longrightarrow \zeta^{-1}\cdot\zeta' =: \zeta''$ and used the gauge invariance of the measure $[\mathcal{D}\zeta]$. □

Now since $[\mathcal{D}X]$ is a gauge invariant measure and the action is gauge invariant, the path integral becomes

$$\begin{aligned}
Z[\hat{g}] &= \int \frac{[\mathcal{D}\zeta][\mathcal{D}X]}{V_{\mathcal{G}}}\Delta_{\mathrm{FP}}[\hat{g}]e^{-S[X,\hat{g}]} \\
&= \int [\mathcal{D}X]\Delta_{\mathrm{FP}}[\hat{g}]e^{-S[X,\hat{g}]}
\end{aligned} \tag{7.1.20}$$

where the volume $V_{\mathcal{G}}$ of the gauge group is canceled by the $[\mathcal{D}\zeta]$ integral. Thus now it is clear that $\Delta_{\mathrm{FP}}[\hat{g}]$ is the Jacobian for variable change.

Evaluating the Fadeev-Popov determinant: Ghosts

We want to evaluate the integral

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g}] = \int [\mathcal{D}\zeta]\delta\left(\hat{g} - \hat{g}^{\zeta}\right). \tag{7.1.21}$$

Since the gauge slice cuts the gauge orbits at exactly one point, $\delta\left(\hat{g} - \hat{g}^{\zeta}\right)$ is nonzero only for $\hat{g}^{\zeta} = \hat{g}$. In particular, the gauge transformations ζ near identity are the ones which contribute. So let us take an infinitesimal gauge transformation with Weyl transformation parametrized by $\omega(\boldsymbol{\sigma})$ and reparametrization $\delta\sigma^{\alpha} = v^{\alpha}(\boldsymbol{\sigma})$. The metric changes as

$$\delta\hat{g}_{\alpha\beta} = 2\omega(\boldsymbol{\sigma})\hat{g}_{\alpha\beta} + \nabla_{\alpha}v_{\beta} + \nabla_{\beta}v_{\alpha} \tag{7.1.22}$$

where we used the fact that under reparametrization

$$\begin{aligned}
&\delta\sigma^{\alpha} = v^{\alpha}(\boldsymbol{\sigma}) \\
&\delta\hat{g}_{\alpha\beta} = \mathcal{L}_{v}\hat{g}_{\alpha\beta} = \nabla_{\alpha}v_{\beta} + \nabla_{\beta}v_{\alpha},
\end{aligned} \tag{7.1.23}$$

where $\mathcal{L}_{\sqsubseteq}$ denotes the Lie derivative in the direction of the vector field v^{α}. Thus locally, we have

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g}] = \int [\mathcal{D}\omega\mathcal{D}v]\delta\left(2\omega(\boldsymbol{\sigma})\hat{g}_{\alpha\beta} + \nabla_{\alpha}v_{\beta} + \nabla_{\beta}v_{\alpha}\right) \tag{7.1.24}$$

The delta functional can be written as

$$\begin{aligned}\delta\left(2\omega(\boldsymbol{\sigma})\hat{g}_{\alpha\beta}\right. &\left.+\nabla_\alpha v_\beta + \nabla_\beta v_\alpha\right)\\ &= \int[\mathcal{D}\beta]\exp\left(2\pi i\int d^2\sigma\sqrt{g}\beta^{\alpha\beta}\left[2\omega(\boldsymbol{\sigma})\hat{g}_{\alpha\beta}+\nabla_\alpha v_\beta+\nabla_\beta v_\alpha\right]\right).\end{aligned}\tag{7.1.25}$$

where $\beta^{\alpha\beta}$ is a symmetric rank 2 tensor. Compare with

$$\delta(x)\sim\int dp\; e^{2\pi i p x}.\tag{7.1.26}$$

So the FP determinant becomes

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g}]=\int[\mathcal{D}\omega\mathcal{D}v\mathcal{D}\beta]\exp\left(2\pi i\int d^2\sigma\sqrt{\hat{g}}\beta^{\alpha\beta}\left[2\omega(\boldsymbol{\sigma})\hat{g}_{\alpha\beta}+\nabla_\alpha v_\beta+\nabla_\beta v_\alpha\right]\right).\tag{7.1.27}$$

Performing the ω integral is simple. It gives

$$\begin{aligned}\int[\mathcal{D}\omega]\exp\left(4\pi i\int d^2\sigma\sqrt{\hat{g}}\beta^{\alpha\beta}\hat{g}_{\alpha\beta}\omega(\boldsymbol{\sigma})\right)&=\delta\left(2\int d^2\sigma\sqrt{\hat{g}}\beta^{\alpha\beta}\hat{g}_{\alpha\beta}\right)\\ &=\delta\left(\hat{g}_{\alpha\beta}\beta^{\alpha\beta}\right)\end{aligned}\tag{7.1.28}$$

Thus the $[\mathcal{D}\omega]$ integral forces $\beta^{\alpha\beta}$ to be traceless:

$$\beta^{\alpha\beta}\hat{g}_{\alpha\beta}=0.\tag{7.1.29}$$

We take this as the definition of $\beta^{\alpha\beta}$ and then we have

$$\begin{aligned}\Delta_{\mathrm{FP}}^{-1}[\hat{g}]&=\int[\mathcal{D}v\mathcal{D}\beta]\exp\left(2\pi i\int d^2\sigma\sqrt{\hat{g}}\beta^{\alpha\beta}\left(\nabla_\alpha v_\beta+\nabla_\beta v_\alpha\right)\right).\\ &=\int[\mathcal{D}v\mathcal{D}\beta]\exp\left(4\pi i\int d^2\sigma\sqrt{\hat{g}}\beta^{\alpha\beta}\nabla_\alpha v_\beta\right)\end{aligned}\tag{7.1.30}$$

where we used the fact that $\beta^{\alpha\beta}$ is symmetric. We now want to invert this to obtain $\Delta_{\mathrm{FP}}[\hat{g}]$. The usual Fadeev-Popov procedure is to realise that the path integral gives the inverse determinant of the differential operator ∇_α which takes vectors to symmetric tensors. But there is a problem now. The operator ∇_α is now not between vector spaces of same dimension and hence determinant of ∇_α is not defined. To remedy this, we do the following. We write

$$\begin{aligned}\delta g_{\alpha\beta}&=2\omega(\boldsymbol{\sigma})g_{\alpha\beta}+\nabla_\alpha v_\beta+\nabla_\beta v_\alpha-\nabla_\gamma v^\gamma g_{\alpha\beta}+\nabla_\gamma v^\gamma g_{\alpha\beta}\\ &=2(P_1 v)_{\alpha\beta}+\left(2\omega(\boldsymbol{\sigma})+\nabla_\gamma v^\gamma\right)g_{\alpha\beta}\end{aligned}\tag{7.1.31}$$

where

$$(P_1 v)_{\alpha\beta}=\frac{1}{2}\left(\nabla_\alpha v_\beta+\nabla_\beta v_\alpha-\nabla_\gamma v^\gamma g_{\alpha\beta}\right).\tag{7.1.32}$$

Thus P_1 now projects to the symmetric traceless part of $\nabla_\alpha v_\beta$. Repeating the above argument

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g}] = \int [\mathcal{D}v\mathcal{D}\beta] \exp\left(4\pi i \int d^2\sigma \sqrt{\hat{g}} \beta^{\alpha\beta}(Pv)_{\alpha\beta}\right) \tag{7.1.33}$$

Now since v_α is 2 dimensional and $(P_1 v)_{\alpha\beta}$ is traceless, symmetric which means the dimension of the $(P_1 v)_{\alpha\beta}$ space is $4-2=2$. Now recall the formula for the determinant of an operator in terms of bosonic path integral:

$$\int [\mathcal{D}\phi_1\mathcal{D}\phi_2] \exp\left(i \int d^d x\ \phi_1(x)\Delta\phi_2(x)\right) = (\det \Delta)^{-1}. \tag{7.1.34}$$

In general if the fields ϕ_i are defined on a Riemannian manifold with metric $g_{\mu\nu}$ then we replace $d^d x$ by $d^d x\sqrt{g}$. Another generalisation is replacing scalar fields by vector and tensor field. That is precisely what we have. Thus we see that

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g}] = (4\pi)^{-2}(\det P_1)^{-1}. \tag{7.1.35}$$

Thus

$$\Delta_{\mathrm{FP}}[\hat{g}] = (4\pi)^2 \det P_1. \tag{7.1.36}$$

Finally recall the formula for the determinant of an operator in terms of grassmannian path integral:

$$\det \Delta = \int [\mathcal{D}\psi_1\mathcal{D}\psi_2] \exp\left(\int d^d x \psi_2(x)\Delta\psi_1(x)\right) \tag{7.1.37}$$

for grassmannian variables ψ_1, ψ_2. Thus we can replace $\beta^{\alpha\beta}$ and v^α by fermionic fields $b^{\alpha\beta}$ and c^α to obtain the FP determinant:

$$\begin{aligned}\Delta_{\mathrm{FP}}[\hat{g}] &= \int [\mathcal{D}b\mathcal{D}c] \exp\left(-\frac{1}{2\pi}\int d^2\sigma \sqrt{\hat{g}}\ b^{\alpha\beta}(P_1 c)_{\alpha\beta}\right) \\ &= \int [\mathcal{D}b\mathcal{D}c] \exp\left(-\frac{1}{2\pi}\int d^2\sigma \sqrt{\hat{g}}\ b^{\alpha\beta}\hat{\nabla}_\alpha c_\beta\right)\end{aligned} \tag{7.1.38}$$

where we used the tracelessness of $b^{\alpha\beta}$ to remove the last term from $(P_1 c)_{\alpha\beta}$. A hat over ∇ indicates that it is a covariant derivative in the metric $\hat{g}$. Writing

$$\Delta_{\mathrm{FP}}[\hat{g}] = \int [\mathcal{D}b\mathcal{D}c] \exp\left(-S_g\right) \tag{7.1.39}$$

where

$$S_g[b, c, \hat{g}] = \frac{1}{2\pi}\int d^2\sigma \sqrt{\hat{g}}\ b^{\alpha\beta}\hat{\nabla}_\alpha v_\beta \tag{7.1.40}$$

is the *ghost action* and $b^{\alpha\beta}$ and c_α are called *Fadeev-Popov ghosts* the partition becomes

$$Z[\hat{g}] = \int [\mathcal{D}X\mathcal{D}b\mathcal{D}c] \exp\left(-S[X,\hat{g}] - S_g[b, c, \hat{g}]\right) \tag{7.1.41}$$

where $S[X, \hat{g}]$ is the total string action (7.1.12). We now choose the fiducial metric $\hat{g}_{\alpha\beta}$ to be in the *conformal gauge*:

$$\hat{g}_{\alpha\beta} = e^{2\omega(\boldsymbol{\sigma})}\delta_{\alpha\beta} \tag{7.1.42}$$

and the complex coordinates $z = \sigma + i\tau, \bar{z} = \sigma - i\tau$. We see that

$$\sqrt{\hat{g}} = e^{2\omega}, d^2\sigma = \frac{1}{2}d^2z, \nabla^z = g^{z\bar{z}}\nabla_{\bar{z}} = 2e^{-2\omega}\nabla_{\bar{z}}. \tag{7.1.43}$$

The particular Christoffel connection we need is

$$\Gamma^z_{\bar{z}\alpha} = 0 \text{ for } \alpha = z, \bar{z}. \tag{7.1.44}$$

The ghost action is

$$S_g = \frac{1}{2\pi}\int d^2z\frac{1}{2}e^{2\omega}\left(b_{zz}\nabla^z c^z + b_{\bar{z}\bar{z}}\nabla^{\bar{z}}c^{\bar{z}}\right), \tag{7.1.45}$$

where we used the fact that $b_{\alpha\beta}$ is symmetric so that there is no $b_{z\bar{z}}$ or $b_{\bar{z}z}$ components. Note that we lowered the index on b and raised it on c. The reason will become apparent in a moment. We can rewrite S_g as

$$\begin{aligned} S_g &= \frac{1}{2\pi}\int d^2z\left(b_{zz}\nabla_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\nabla_z c^{\bar{z}}\right) \\ &= \frac{1}{2\pi}\int d^2z\left(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right) \end{aligned} \tag{7.1.46}$$

where we used $\Gamma^z_{\bar{z}\alpha} = 0$. So, we have ended up with an action which is Weyl invariant. Note that b_{zz} and c^z (and the antiholomorphic counterparts) are neutral under Weyl transfomation. But b^{zz} and c_z (and antiholomorphic counterparts) are not. The ghost theory is thus a CFT because we can cancel the factor due to a coordinate transformation by a Weyl transformation. We have already met this CFT earlier. It is a CFT with central charge $c = -26$.

Remark 7.1.2. For open strings, the worldsheet has a boundary and so the embeddings X^μ are embeddings with boundary. We can take the worldsheet coordinates (σ, τ) to range over a fixed subset of $\mathbb{R}^2$. In this case coordinate transformations must map the boundary to itself. This means that v^α satisfies

$$n_\alpha v^\alpha = 0 \tag{7.1.47}$$

where n_α is the normal to the boundary. This translates to the condition $n_\alpha c^\alpha = 0$ on the ghost field. This also means that c^α is proportional to the tangent vector t^α. In the ghost CFT, we get a boundary term

$$\int_{\partial\Sigma} ds\; n^\alpha b_{\alpha\beta}\delta c^\beta. \tag{7.1.48}$$

If we impose the boundary condition

$$n^\alpha b_{\alpha\beta}t^\beta = 0, \tag{7.1.49}$$

the equations of motion remain unchanged since c^α is proportional to t^α. This is the boundary condition we used previously.

7.1.2 The Weyl anomaly

We fixed the gauge to get an expression for path integeral. But it still depends on the fiducial metric $\hat{g}_{\alpha\beta}$. Also we had residual conformal symmetry after gauge fixing which we didn't bother about. The path integral must also be invariant under the residual gauge symmetry for a physical theory. That is

$$Z[\hat{g}^{\zeta}] = Z[\hat{g}] \tag{7.1.50}$$

for ζ a conformal transformation. We also want the correlation functions to be gauge invariant:

$$\begin{aligned}\langle \cdots \rangle_{\hat{g}} &\equiv \int [\mathcal{D}X\mathcal{D}b\mathcal{D}c] \exp(-S[X,\hat{g}] - S_g[b,c,\hat{g}]) \cdots \\ \langle \cdots \rangle_{\hat{g}^{\zeta}} &= \langle \cdots \rangle_{\hat{g}}.\end{aligned} \tag{7.1.51}$$

We have already seen the there is an anomaly in the trace ${T_\alpha}^\alpha$:

$$\langle {T_\alpha}^\alpha \rangle_g = \frac{c}{12} R, \tag{7.1.52}$$

where R is the Ricci scalar for g and c is the central charge. Let us derive it by a different route here. Under variation of metric

$$\delta \langle \cdots \rangle_g = -\frac{1}{4\pi} \int d^2\sigma \sqrt{g(\boldsymbol{\sigma})} \delta g_{\alpha\beta} \left\langle T^{\alpha\beta}(\sigma) \ldots \right\rangle, \tag{7.1.53}$$

where

$$T_{\alpha\beta}(\boldsymbol{\sigma}) = \frac{4\pi}{\sqrt{g(\boldsymbol{\sigma})}} \frac{\delta S}{\delta g^{\alpha\beta}}. \tag{7.1.54}$$

Under a Weyl transformation, $\delta g_{\alpha\beta} = 2\omega(\boldsymbol{\sigma}) g_{\alpha\beta}$, so that

$$\delta_W \langle \cdots \rangle_g = -\frac{1}{4\pi} \int d^2\sigma \sqrt{g(\sigma)} 2\omega(\boldsymbol{\sigma}) \left\langle {T^\alpha}_\alpha(\boldsymbol{\sigma}) \cdots \right\rangle_g \tag{7.1.55}$$

Since we have preserved diffeomorphism and Poincaré, the trace must be invariant under these. Also in flat background, the theory is conformally invariant. Thus we can only have

$$\langle {T^\alpha}_\alpha \rangle = a_1 R + a_2, \tag{7.1.56}$$

where a_2 is a constant. This is an example of a *counterterm*. But we can add a term in the action, analogous to a cosmological constant, of the form

$$S_{\text{ct}} = -\frac{a_2}{4\pi} \int d^2\sigma \; \sqrt{g}. \tag{7.1.57}$$

This term is not Weyl invariant and adds a constant $-a_2$ in the trace of the stress tensor. Thus we can assume that a_2 is zero. More than two derivatives in the metric is not allowed in the trace because T has conformal dimension 2 and taking derivative increases conformal dimension by 1. We now calculate a_1. We have

$$\begin{aligned} &\langle T_z^z + T_{\bar{z}}^{\bar{z}} \rangle = a_1 R \\ \Longrightarrow\ &\langle g^{z\bar{z}} T_{\bar{z}z} + g^{\bar{z}z} T_{z\bar{z}} \rangle = a_1 R \\ \Longrightarrow\ &\langle T_{z\bar{z}} \rangle = \frac{a_1}{2} g_{z\bar{z}} R. \end{aligned} \tag{7.1.58}$$

Taking covariant derivative gives (dropping $\langle \cdots \rangle$)

$$\nabla^{\bar{z}} T_{\bar{z}z} = \frac{a_1}{2} \nabla^{\bar{z}} \left(g_{z\bar{z}} R \right) = \frac{a_1}{2} \partial_z R \tag{7.1.59}$$

where we used the fact that $\nabla^{\bar{z}} g_{z\bar{z}} = 0$. By conservation of T

$$\nabla^z T_{zz} = -\nabla^{\bar{z}} T_{\bar{z}z} = \frac{a_1}{2} \partial_z R. \tag{7.1.60}$$

We now compare the Weyl transformation of LHS and RHS. We know that under Weyl transformation

$$g_{\alpha\beta} \longrightarrow e^{2\omega} g_{\alpha\beta} =: g'_{\alpha\beta}, \tag{7.1.61}$$

we have

$$\sqrt{g'} R' = \sqrt{g} \left(R - 2\nabla^2 \omega \right). \tag{7.1.62}$$

This implies

$$\begin{aligned} \left(1 + e^{2\omega}\right) R' = R - 2\nabla^2 \omega \\ \Longrightarrow\ \delta_W R = -2\nabla^2 \omega. \end{aligned} \tag{7.1.63}$$

We now expand around the flat metric, so that

$$\nabla^2 \omega \approx 4 \partial_z \partial_{\bar{z}} \omega. \tag{7.1.64}$$

Thus RHS of (7.1.60) transforms as

$$4 a_1 \partial_z^2 \partial_{\bar{z}} \omega. \tag{7.1.65}$$

Now, under a general conformal transformation (infinitesimal)

$$\begin{aligned} \delta x^\alpha = \varepsilon^\alpha(z) \quad &\text{(coordinate transformation)} \\ 2\omega = \partial \varepsilon^z + \left(\bar{\partial} \varepsilon^{\bar{z}} \right) \quad &\text{(Weyl counteraction)} \end{aligned} \tag{7.1.66}$$

the conformal Ward identity gives

$$\delta T_{zz}(z) = -\frac{c}{12} \partial_z^3 \varepsilon^z(z) - 2\partial_z \varepsilon^z T_{zz} - \varepsilon^z \partial_z T_{zz}. \tag{7.1.67}$$

The last two terms are the variation due to coordinate transformation. Thus

$$\delta_W T_{zz} = -\frac{c}{6} \partial_z^2 \omega \tag{7.1.68}$$

where we used $2\omega = \partial\varepsilon^z$. Thus upto linear order in ω

$$\begin{aligned}\delta_W \nabla^2 T_{zz} &\approx \delta_W \partial^z T_{zz}\\ &= 2\delta_W \partial_z T_{zz}\\ &= -\frac{c}{3}\partial_z^2\partial_z\omega\end{aligned}$$

Thus we get

$$a_1 = -\frac{c}{12}.$$

Critical dimension: In the worldsheet CFT of D bosonic fields X^μ, the central charge is D. Addition of ghosts adds central charge $-3(2\cdot 2-1)^2+1=-26$. So the Weyl anomaly cancels only when

$$c = D - 26 = 0$$

which gives the critical dimension $D = 26$.

Weyl anomaly in boundary CFT

In presence of boundary, there are extra terms in the variation of $\langle\cdots\rangle_g$. Let us start with

$$\begin{aligned}\delta_W \ln\langle\cdots\rangle_g &= \frac{1}{\langle\cdots\rangle_g}\delta_W\langle\cdots\rangle_g\\ &= \frac{\delta S}{\delta g^{\alpha\beta}}\delta g^{\alpha\beta}\\ &= -\frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{g}\left[\omega(\boldsymbol{\sigma})T^\alpha_\alpha(\sigma) + \partial_\alpha\left(\frac{\delta\mathcal{L}}{\partial\left(\partial_\alpha g_{\beta\gamma}\right)}\delta g_{\beta\gamma}\right)\right]\\ &= -\frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{g}\omega(\boldsymbol{\sigma})\left(a_1 R + a_2\right) - \frac{1}{2\pi}\int_{\partial\Sigma} ds\frac{\delta\mathcal{L}}{\partial\left(\partial_\alpha g_{\beta r}\right)}\omega(s)g_{\beta\gamma}\end{aligned}$$

where $\mathcal{L}$ is the Lagrangian given as

$$\mathcal{L} = \partial_\alpha X^\mu\partial^\alpha X_\mu + R + \partial_\alpha k$$

where $k = -t^a n_b \nabla_a t^b$ is the geodesic curvature on boundary. One can then show that

$$\delta_W \ln\langle\cdots\rangle_g = -\frac{1}{2\pi}\int_\Sigma d^2\sigma\sqrt{g}\omega(\boldsymbol{\sigma})\left(a_1 R + a_2\right) - \frac{1}{2\pi}\int_{\partial\Sigma} ds\left(a_3 + a_4 k + a_5 n^a\partial_a\right)\omega(s).$$

We can add a counterterm of the form

$$S_{\text{ct}} = \int_\Sigma d^2\sigma\sqrt{g}b_1 + \int_{\partial\Sigma} ds\left(b_2 + b_3 k\right).$$

One can check that

$$\delta_W S_{\text{ct}} = 2\int_\Sigma d^2\sigma\sqrt{g}b_1\omega(\boldsymbol{\sigma}) + \int_{\partial\Sigma} ds\left(b_2 + b_3 n^a\nabla_a\right)\omega(\boldsymbol{\sigma}).$$

Thus choosing b_1, b_2, b_3 appropriately, we can set $a_2, a_3, a_5 = 0$. To fix a_4, we use the fact that consecutive Weyl variations must commute. This is called the *Wess-Zumino consistency condition*. We have

$$\begin{aligned}\delta_{W_1}\left(\delta_{W_2}\ln\langle\cdots\rangle_g\right) &= \frac{a_1}{\pi}\int_\Sigma d^2\sigma\sqrt{g}\omega_2(\boldsymbol{\sigma})\nabla^2\omega_1(\boldsymbol{\sigma}) - \frac{a_4}{2\pi}\int_{\partial\Sigma} ds\ \omega_2 n^a\partial_a\omega_1 \\ &= -\frac{a_1}{\pi}\int_\Sigma d^2\sigma\sqrt{g}\partial_a\omega_2(\boldsymbol{\sigma})\partial^a\omega_1 - \frac{2a_1 - a_4}{2\pi}\int_{\partial\Sigma} ds\ \omega_2 n^a\partial_a\omega_1\end{aligned}$$

The first term is symmetric under $\omega_1 \leftrightarrow \omega_2$ but the boundary term is not. Thus we must have

$$a_4 = 2a_1.$$

So even in open string CFT, we do not get new Weyl anomalies.

The Wess-Zumino consistency condition also shows that the central charge has to be a constant, which we have already seen in Theorem 5.7.16.

7.2 The String S-matrix

In the previous sections, we described how to consistently define the gauge fixed string path integral. We now want to formulate the scattering amplitude of string states. We will not be bothered about calculating path integrals in this section, so we will use the original string path integral (**??**) to describe the notion of string amplitudes. Let us start with the string action: where

$$S[X,g] = S_{\mathrm{P}}[X,g] + \lambda\chi \tag{7.2.1}$$

where

$$\chi = \frac{1}{4\pi}\int_\Sigma d\sigma d\tau\sqrt{g}R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\ k \tag{7.2.2}$$

and λ is some coupling constant. To start talking about scattering amplitude, we must first find out how strings interact.

7.2.1 String interactions and sum over woldsheet topologies

What are the ways strings can interact? A noninteracting open/closed string can be represented by a worldsheet with topology of a sheet/cylinder as shown in Figure **??** below. Two strings might join to form one string and so on. See Figure **??** for some possibilities. But is it possible to add interactions terms in the Polyakov action to introduce these interactions? It turns out that it is not possible to do it so that the symmetries are preserved. What is the way out?

Notice that the worldsheets in the interactions in Figure **??** have different topology. So a possible way to introduce interactions in the theory is to introduce a new data in the path integral: *sum over worldsheet topologies*. So we define

$$Z_{\text{string}} \equiv \sum_{\substack{\text{topologies} \\ \text{on } \Sigma}} \int \frac{[\mathcal{D}X^{\Sigma}\mathcal{D}g^{\Sigma}]}{V_{\mathcal{G}}} e^{-S_{\Sigma}[X^{\Sigma},g^{\Sigma}]}, \tag{7.2.3}$$

where we have emphasized that $X^{\Sigma} : \Sigma \longrightarrow \mathbb{R}^D$ is an embedding of the worldsheet, g^{Σ} is the metric on the worldsheet and the action depends on the worldsheet in the sense that it is an integral over the worldsheet. Sum over topologies also gives us the perturbative expansion of the string S-matrix as we now describe. The term $\lambda\chi$ in the action is topological and only depends on the topology of the worldsheet. Thus we can write

$$Z_{\text{string}} = \sum_{\substack{\text{topologies} \\ \text{on } \Sigma}} e^{-\lambda\chi(\Sigma)} \int \frac{[\mathcal{D}X\mathcal{D}g]}{V_{\mathcal{G}}} e^{-S_{\text{P}}}. \tag{7.2.4}$$

Thus different topologies are weighted by e^{λ}. Now in going from Figure **??** to Figure **??**, we have added an extra strip. The Euler characteristic of the strip is 1 using the famous formula $\chi = V - E + F$ where V, E, F denote the number of vertices, edges and faces respectively. Adding an extra strip thus decreases the Euler characteristic by 1 since there is an extra circular edge. This corresponds to an extra factor of e^{λ} and hence the corresponding amplitude is $e^{\lambda/2}$. Similarly for closed string adding a handle increases the genus by 1 and decreases $\chi = 2 - 2g$, where g is the genus – the number of holes in the worldsheet, by 2, so that the amplitude for emitting and reabsorbing a virtual closed string is e^{λ}. Thus perturbative expansion is defined by expansion in

$$g_o^2 \sim g_c \sim e^{\lambda}. \tag{7.2.5}$$

We will show later that λ is not a free parameter.

Clasification of string theories based on interactions

Another important information that we get from the possible string interactions is a classification of possible string theories: based on the topologies of worldsheet we include in the sum there are four different string theories:

1. Closed oriented:

String amplitudes

Now that we know the interactions of strings, let us discuss how to define the amplitude for a given process. Consider two closed strings interaction as shown in Figure **??** below. Before we attempt to do make any definition, let us recall how this is defined in field theory. In field theory we calculate correlation functions $\langle\phi_1(x_1)\ldots\phi_n(x_n)\rangle$. It is easier to calculate the Fourier transform of this - the momentum space correlation functions. The special thing about momentum space correlation functions is that the external legs can have arbitrary momenta *i.e.* they can be *offshell*. The LSZ reduction formula then gives the scattering amplitude.

Can we compute offshell string amplitudes? The current answer is - it does not make sense. To see this recall that the correlators of only gauge invariant operators make sense in a gauge theory. Now if we have gravity in the theory, diffeomorphism is a gauge symmetry and hence the coordinates of operators are not well defined. There are no local offshell gauge invariant observable in a theory of quantum gravity. Now in string theory, we will see that Weyl invariance of the amplitude will set the external string states on-shell which in position space turns into a statement that we cannot fix the position of external fields. To see this note that if we want to fix the position of external field to X_0, then we need to insert a delta function

$$\delta^D\left(X(\boldsymbol{\sigma})-X_0\right)=\int\frac{d^Dk}{(2\pi)^D}e^{ik\cdot(X(\boldsymbol{\sigma})-X_0)} \tag{7.2.6}$$

which involves all momentas inconsistent with Weyl invariance, since Weyl invariance as noted above and shown below fixes the external string state onshell. This also means that we may not be able to compute string scattering amplitudes at finite times. So we, for now, focus on *string S-matrix* defined to be string amplitude where the external string states are taken to $X^0=\pm\infty$. The external legs are represented by cylinders which can be described by a complex coordinate w with $\mathrm{Re}(w)$ going around the cylinder and $-2\pi t\le\mathrm{Im}(w)\le 0$. $\mathrm{Re}(w)$ is periodic and we have have a string state and the scattering process would mean $t\to\pm\infty$. It will turn out that this limit is equivalent to $X^0\to\pm\infty$. The end $\mathrm{Im}(w)=0$ is the end on the world sheet. Now we can map the cylinder to a disk via the conformal map

$$z=e^{-iw},\quad e^{-2\pi t}\le|z|\le 1 \tag{7.2.7}$$

and the circular end at $\mathrm{Im}(w)=-2\pi t$ maps to a circle of radius $e^{-2\pi t}$. Thus under this map the worldsheet maps to a sphere with small circular holes at external states. When we take $t\to\pm\infty$, these holes shrink to points. Thus the worldsheet reduces to sphere with punctures at external states. Similarly for open string, the long strips...

To write the amplitude as path integral we need to specify the external string states. Here the state operator correspondence comes to our rescue: the states at the punctures/dents can be represented by local operators, that is, the vertex operators $V_j(k)$ where j defines the internal state and k^μ is the spacetime momenta. Incoming and outgoing states are distinguished by sign k^0; incoming state has $k^\mu=(E,\vec{k})$ and outgoing states have $k^\mu=-(E,\vec{k})$. We can restrict our attention to compact topologies indicating localised string interactions. Thus we see that an n-particle string S-matrix element must be defined as

$$S_{j_1}\dots j_n\left(k_1,\cdots,k_n\right)=\sum_{\substack{\text{compact}\\ \text{topologies}}}\int\frac{[\mathcal{D}X\mathcal{D}g]}{V_{\mathcal{G}}}e^{-S_{\mathrm{P}}-\lambda\chi}\prod_{i=1}^n\int d^2\sigma_i\sqrt{g\left(\boldsymbol{\sigma}_i\right)}V_{j_i}\left(k_i,\boldsymbol{\sigma}_i\right) \tag{7.2.8}$$

A few comments are in order about this definition:

Remark 7.2.1.

1. In a typical term in the sum over topologies, the path integral over X results in the correlation function of the product of vertex operators in the CFT on the given topology. For example, at tree level, the closed string S-matrix would involve the correlation function of the vertex operators representing external string states on the sphere S^2. We will develop methods to calculate these correlation functions in subsequent sections.

2. The vertex operators have been integrated over the whole worldsheet to make the S-matrix Diff-invariant.

3. Depending on the type of string theory we are doing calculations in, the sum over topologies can include unoriented worldsheets and/or worldsheets with boundary.

4. In general, one may also include disconnected topologies which would physically mean two or more widely separated sets of particles scattering independent. But we will focus on connected topologies.

Let us now discuss how to sum over topologies. We have to sum over all topologies. The classification of all 2d topologies is well known: oriented 2d manifolds without boundary is classified by the Euler characteristic

$$\chi = 2 - 2g, \tag{7.2.9}$$

where g is the genus of the surface *i.e.* the number of holes or handles. If we include unoriented surfaces with boundaries, then we also have to specify the number of boundary components b and number of crosscaps c to classify 2d topological spaces. The Euler characteristics is then given by

$$\chi = 2 - 2g - b - c. \tag{7.2.10}$$

Some examples are shown in Figure **??**. A boundary component is introduced in the surface by cutting out a disk. For example, a sphere with one boundary is a disk, with two boundaries is an annulus and with three boundaries is a pair of pants. Crosscap is a protype of an unoriented surface...

7.2.2 Vertex Operators

The most important ingredients in the string S-matrix are the vertex operator insertions which represent asymptotic string states. To make the string S-matrix well-defined, these vertex operators must be Diff $\ltimes$ Weyl invariant. We want to find out the constraints for the vertex operators to be Diff $\ltimes$ Weyl invariant for general topology of the worldsheet. As already announced before, the constraint turns out to be that the asymptotic states be onshell.

Vertex operators on flat worldsheet

Let us start with the tachyon vertex operator:

$$V_0 = 2g_c \int d^2\sigma \sqrt{g}\; e^{ik\cdot X}.$$

where we introduced a coupling g_c which will turn out to be related to the dilaton. On flat worldsheet, this becomes

$$V_0 = g_c \int d^2z :e^{ik\cdot X}: .$$

This must be Diff $\ltimes$ Weyl invariant and in particular conformally invariant, d^2z has conformal dimensions $(h, \bar{h}) \equiv (-1, -1)$. Thus $e^{ik\cdot X}$ must have conformal dimensions $(h, \bar{h}) = (1, 1)$. By our calculation in Corollary **??**, we get

$$\frac{\alpha' k^2}{4} = 1. \tag{7.2.11}$$

Onshell, this gives

$$m^2 = -k^2 = -\frac{4}{\alpha'}.$$

This is exactly what we got in lightcone quantisation. The first excited state vertex operator on flat worldsheet is given by

$$V_1(k) = \frac{2g_c}{\alpha'} \int d^2z :\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}:$$

The conformal weights of $:\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}:$ is

$$h = \bar{h} = 1 + \frac{\alpha' k^2}{4}.$$

Conformal invariance onshell then implies

$$m^2 = 0,$$

again agreeing with the lightcone quantisation result. The goal is now to do the same on curved worlolsheets. As we will see, this will give us more constraints.

Vertex operators on curved worldsheet

The expression for vertex operator in for curved background is only schematic since we did not explicitly derive it from the state-operator correspondence. Note that the tachyon vertex operator on flat background is just the integral of normal ordered $e^{ik\cdot X}$. So if we can generalize the normal ordering to curved background, the we can define the vertex operator.

The appropriate generalisation is to generalize the normal ordering defined in. To this, define the renormalised operator

$$[\mathcal{F}]_{\mathrm{r}} := \exp\left(\frac{1}{2}\int d^2\sigma d^2\sigma' \Delta\left(\sigma,\sigma'\right)\frac{\delta}{\delta X^\mu(\sigma)}\frac{\delta}{\delta X_\mu\left(\sigma'\right)}\right)\mathcal{F}. \tag{7.2.12}$$

where

$$\Delta\left(\sigma,\sigma'\right) = \frac{\alpha'}{2}\ln d^2\left(\sigma,\sigma'\right) \tag{7.2.13}$$

and $d\left(\sigma,\sigma'\right)$ is the geodesic distance [3] between σ,σ'. For the flat worldsheet,

$$d^2\left(\sigma,\sigma'\right) = \left|z-z'\right|^2 \tag{7.2.14}$$

and the renormalised operator is simply the normal ordered operator This is consistent with the tachyon vertex operator on flat background. As in normal ordering, in renormalised operator, we sum over all ways of contracting pairs of fields in $\mathcal{F}$ i.e. replace the pair by $\Delta\left(\sigma,\sigma'\right)$. Note that $\Delta\left(\sigma,\sigma'\right)$ has singularity as $\sigma\to\sigma'$. This singularity cancels singularity from self-contractions. We can now use this renormalised operator to defined renormalised vertex operator. For example, the tachyon vertex operator on curved worldsheet is defined as

$$V_0 = 2g_c\int d^2\sigma\sqrt{g}\left[e^{ik\cdot X}\right]_{\mathrm{r}}. \tag{7.2.15}$$

The next vertex operator that can be constructed using derivatives of X in a Diff-invariant way is

$$V_1 = \frac{g_c}{\alpha}\int d^2\sigma\sqrt{g}\left\{\left(g^{\alpha\beta}s_{\mu\nu}+i\epsilon^{\alpha\beta}a_{\mu\nu}\right)\left[\partial_\alpha X^\mu\partial_\beta X^\nu e^{ik\cdot X}\right]_{\mathrm{r}}+\phi R\left[e^{ik\cdot X}\right]_{\mathrm{r}}\right\} \tag{7.2.16}$$

where R is the Ricci scalar of the worlsheet, $s_{\mu\nu},a_{\mu\nu},\phi$ are a constant symmetric matrix, a constant antisymmetric matrix and a constant scalar, $\epsilon^{\alpha\beta}$ is an antisymmetric tensor satisfying $\sqrt{g}\epsilon^{12}=1$. The factor of i in front of $\epsilon^{\alpha\beta}$ is because $\partial_\alpha X^\mu\partial_\beta X^\nu$ contains exactly one time derivative when contracted with $\epsilon^{\alpha\beta}$ in the Minkowski signature. So when we continue to Euclidean signature, we put a factor of i. We now want to check if renormalised vertex operator defined using above rule is Diff $\times$ Weyl invariant. Diff-invariance of obvious since the vertex operators are integrated over the worldsheet. To check Weyl variation we use the identity

$$\delta_{\mathrm{W}}[\mathcal{F}]_{\mathrm{r}} = [\delta_{\mathrm{W}}\mathcal{F}]_{\mathrm{r}}+\frac{1}{2}\int d^2\sigma d^2\sigma'\delta_{\mathrm{W}}\Delta\left(\sigma'\sigma'\right)\frac{\delta}{\delta X^\mu(\sigma)}\frac{\delta}{\delta X_\mu\left(\sigma'\right)}[\mathcal{F}]_{\mathrm{r}}. \tag{7.2.17}$$

This is easily derivable using product rule. Let us calculate the Weyl variation of V_0. We have where we used $\delta_{\mathrm{W}}g_{\alpha\beta}=2\delta\omega g_{\alpha\beta}$. and $\delta_{\mathrm{W}}e^{ik\cdot X}=0$. Now, when σ and σ' are "close" we have

$$d^2\left(\sigma,\sigma'\right)\approx\left(\sigma-\sigma'\right)^2\exp(2\omega(\sigma)). \tag{7.2.18}$$

[3]Recall that the geodesic distance between two point is the length of the geodesic joining the two points

Thus

$$\Delta(\sigma,\sigma') \approx \alpha'\omega(\sigma) + \frac{\alpha'}{2}\ln(\sigma-\sigma')^2 \tag{7.2.19}$$

so that

$$\delta_{\mathrm{W}}\Delta(\sigma,\sigma) \approx \alpha'\delta\omega(\sigma). \tag{7.2.20}$$

We then have

$$\begin{aligned}
\delta_{\mathrm{W}}V_0 =& 2g_c\int d^2\sigma\delta_{\mathrm{W}}\sqrt{g}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\\
=& 2g_c\int d^2\sigma\left\{\frac{1}{2}g^{-1/2}gg^{ab}2\delta\omega g_{ab}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} + \sqrt{g}\left[\delta_{\mathrm{W}}e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right.\\
&\left.+\frac{1}{2}\sqrt{g}\int d^2\sigma' d^2\sigma''\delta_{\mathrm{W}}\Delta(\sigma',\sigma'')\frac{\delta}{\delta X^\mu(\sigma')}\frac{\delta}{\delta X_\mu(\sigma'')}\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right\}\\
=& 2g_c\int d^2\sigma\left\{\sqrt{g}2\delta\omega\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right.\\
&\left.+\frac{1}{2}\sqrt{g}\int d^2\sigma' d^2\sigma''\delta_{\mathrm{W}}\Delta(\sigma',\sigma")\left(-k^2\right)\delta^2(\sigma-\sigma')\delta^2(\sigma-\sigma'')\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right\}\\
=& 2g_c\int d^2\sigma\left\{\sqrt{g}2\delta\omega\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}} - \frac{k^2}{2}\sqrt{g}\delta_{\mathrm{W}}\Delta(\sigma,\sigma)\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}\right\}\\
=& 2g_c\int d^2\sigma\sqrt{g}\left(2\delta\omega - \frac{k^2}{2}\delta_{\mathrm{W}}\Delta(\sigma,\sigma)\right)\left[e^{ik\cdot X(\sigma)}\right]_{\mathrm{r}}
\end{aligned} \tag{7.2.21}$$

Thus $\delta_W V_0 = 0$ if and only if

$$k^2 = \frac{4}{\alpha'} \tag{7.2.22}$$

which is same as the constraint that we obtained for the flat worldsheet. We will work out the conditions that Weyl invariance imposes on the vertex operator V_1.

Theorem 7.2.2. *The Weyl variation of V_1 is given by*

$$\delta_{\mathrm{W}}V_1 = \frac{g_c}{2}\int d^2\sigma\sqrt{g}\delta\omega\left\{\left(g^{\alpha\beta}S_{\mu\nu} + i\epsilon^{\alpha\beta}A_{\mu\nu}\right)\left[\partial_\alpha X^\mu\partial_\beta X^\nu e^{ik\cdot X}\right]_{\mathrm{r}} + \phi RF\left[e^{ik\cdot X}\right]_{\mathrm{r}}\right\} \tag{7.2.23}$$

where

$$\begin{aligned}
S_{\mu\nu} &= -k^2 s_{\mu\nu} + k_\nu k^\rho s_{\mu\rho} + k_\mu k^\rho s_{\nu\rho} - (1+\gamma)k_\mu k_\nu s^\rho_\rho + 4k_\mu k_\nu\phi,\\
A_{\mu\nu} &= -k^2 a_{\mu\nu} + k_\nu k^\rho a_{\mu\rho} - k_\mu k^\rho a_{\nu\rho},\\
F &= (\gamma-1)k^2\phi + \frac{1}{2}\gamma k^\mu k^\nu s_{\mu\nu} - \frac{1}{4}\gamma(1+\gamma)k^2 s^\rho{}_\rho,
\end{aligned} \tag{7.2.24}$$

where $\gamma = -\frac{2}{3}$.

Proof. □

Thus $\delta_{\mathrm{W}} V_1 = 0$ implies

$$S_{\mu\nu} = A_{\mu\nu} = F = 0 \tag{7.2.25}$$

since these constants appear as coefficients of linearly independent operators. We should note that this vertex operators are not all independent for general $s_{\mu\nu}, a_{\mu\nu}$ and ϕ relation

$$\left[\nabla^2 X^\mu e^{ik\cdot X}\right]_{\mathrm{r}} = i\frac{\alpha'\gamma}{4} k^\mu R \left[e^{ik\cdot X}\right]_{\mathrm{r}}, \quad \gamma = -\frac{2}{3} \tag{7.2.26}$$

implies equivalences between these constants. The relation is not easy to derive - we do not include proof of this but refer to [6, Section 2] for some details used in the proof.

Proposition 7.2.3. *The vertex operator V_1 is invariant under*

$$\begin{aligned} s_{\mu\nu} &\longrightarrow s_{\mu\nu} + \xi_\mu k_\nu + k_\mu \xi_\nu, \\ a_{\mu\nu} &\longrightarrow a_{\mu\nu} + \xi_\mu k_\nu - k_\mu \xi_\nu, \\ \phi &\longrightarrow \phi + \frac{\gamma}{2} k \cdot \xi, \end{aligned} \tag{7.2.27}$$

where $\gamma = -\frac{2}{3}$.

Proof. □

To see what conditions Weyl invariance puts, we need to consider independent vertex operators by taking into account the equivalences in Proposition 7.2.3. To remove the extra degrees of freedom from $s_{\mu\nu}, a_{\mu\nu}$, we fix a ξ and ζ as follows: for each k choose a null vector n^μ satisfying $n \cdot k = 1$, then restrict to $s_{\mu\nu}, a_{\mu\nu}$ satisfying

$$n^\mu s_{\mu\nu} = 0, \quad \eta^\mu a_{\mu\nu} = 0. \tag{7.2.28}$$

This fixes ξ_μ and ξ_μ freedom since

$$\begin{aligned} n^\mu s'_{\mu\nu} = n^\mu \left(s_{\mu\nu} + k_\mu \xi_\nu + k_\nu \xi_\mu\right) = 0 \\ \Longrightarrow \xi_\nu = -(n \cdot \xi) k_\nu \end{aligned} \tag{7.2.29}$$

and similarly $\zeta_\mu = (n \cdot \zeta) k_\mu$. We now solve (7.2.25). Since $S_{\mu\nu} = 0$ we must have $S_{\mu\nu} n^\mu n^\nu = 0$. This implies, using the expression (7.2.23) that

$$\phi = \frac{1+\gamma}{4} s^\mu{}_\mu. \tag{7.2.30}$$

This fixes the constant ϕ. Next $S_{\mu\nu} n^\mu = 0$ in

$$k^\mu S_{\mu\nu} = 0. \tag{7.2.31}$$

Next $A_{\mu\nu} \eta^\mu = 0$ (holds because $A_{\mu\nu} = 0$) implies

$$k^\mu a_{\mu\nu} = 0. \tag{7.2.32}$$

Finally $S_{\mu\nu} = 0$ implies

$$k^2 = 0. \tag{7.2.33}$$

In total, the Weyl invariance of the vertex operator implies

$$\begin{aligned} &k^2 = 0 \\ &k^\mu s_{\mu\nu} = 0, \quad k^\mu a_{\mu\nu} = 0 \\ &\phi = \frac{1+\gamma}{4} s^\mu{}_\mu . \end{aligned} \tag{7.2.34}$$

The first condition is the mass-shell condition – this vertex operator corresponds to the massless state. The second condition says that the polarization of the fields $s_{\mu\nu}, a_{\mu\nu}$ be transverse to the momenta – this is the expected condition for a physical massless gauge symmetry emerging from worlsheet gauge symmetry. The vectors ξ_μ, ζ_μ parametrize the gauge transformations of the tensor fields $s_{\mu\nu}, a_{\mu\nu}$ respectively.

Open string vertex operators

7.2.3 Calculating the path integral

In this section, we start with a systematic discussion of how to calculate the string S-matrix. Recall that the string S-matrix with n asymptotic states was defined to be

$$\begin{aligned} S_{j_1,j_2\ldots j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int \frac{[D\phi Dg]}{V_{\mathcal{G}}} \exp\left(-S_{\text{m}} - \lambda\chi\right) \\ \times \prod_{i=1}^{n} \int d^2\sigma_i \sqrt{g(\sigma_i)} V_{j_i}(k_i,\sigma_i) \end{aligned} \tag{7.2.35}$$

where S_{m} is a general $c = \bar{c} = 26$ matter CFT with fields ϕ, χ is the Euler characteristic of the worldsheet, V_{j_i} are the vertex operators corresponding to external asymptotic states j_i. To calculate this path integral we would like to identify a gauge slice. Locally, we did this by fixing the metric in our discussion of Fadeev-Popov ghosts but globally, this does not account for the complete gauge fixing as we will see. Let us start with the point particle example.

Point particle:

The point particle partition function is

$$Z = \int [DeDX] \exp\left[-\frac{1}{2}\int d\tau \left(e^{-1}\dot{X}^\mu \dot{X}_\mu + em^2\right)\right]. \tag{7.2.36}$$

Consider the topology of a circle so that x maps onto a closed loop in spacetime. The parameter τ can be taken to be $0 \le \tau \le 1$ with ends identified. $X^\mu(\tau)$ and the einbein $e(\tau)$

are both periodic on $0 \leq \tau \leq 1$. Now suppose we fix the gauge to be $e' = 1$. The gauge transformation corresponding to this gauge choice satisfies

$$e'(\tau') = e(\tau)\frac{\partial \tau'}{\partial \tau} \tag{7.2.37}$$

which implies

$$\tau'(\tau) = \int_0^\tau d\widetilde{\tau}\; e(\widetilde{\tau}) \tag{7.2.38}$$

where we assumed that $\tau'(0) = 0$. But now $\tau'(1) \neq 1$. Infact

$$\tau'(1) = \int_0^t e(t) d\tau = \ell \tag{7.2.39}$$

where ℓ is the length of the circle. So the coordinate region of τ is not preserved. This means that gauge fixing can be done by a one parameter family of diffeomorphisms parametrized by ℓ. Suppose we want to preserve the coordinate region $0 \leq \tau \leq 1$. Then we have

$$\int_0^1 e'(\tau')\, d\tau' = \int_0^1 e(\tau) d\tau. \tag{7.2.40}$$

Thus if we want to fix e' to a constant value, then it is constrained to be $e' = \ell$. This means that not all einbeins on the circle are diff-equivalent. This is intuitively clear – diffeomorphism transformation cannot change the length of the circle. They are parametrized by ℓ. So we see that there are two choices for gauge fixing:

- $e' = 1, \quad 0 \leq \tau' \leq \ell, \quad \ell \in \mathbb{R}_+$
- $e' = \ell, \quad 0 \leq \tau' \leq 1, \quad \ell \in \mathbb{R}_+$.

Thus the path integral turns into an ordinary integral over ℓ.

Definition 7.2.4. The space of gauge-inequivalent field configurations for a fixed topology is called the moduli space of fields.

Now note that the gauge choice $e = \ell$ is preserved under translation

$$\tau \longrightarrow \tau + v (\text{mod } 1). \tag{7.2.41}$$

The corresponds to choosing a point on the circle with $\tau = 0$. Thus in either of the gauge fixing, there is some *residual gauge symmetry* which is not fixed by this gauge choice. This residual gauge group is called the *conformal Killing group* (CKG) and the generators of this group are called *conformal Killing vectors* (CKV). Thus to calculate amplitudes, we need to identify the moduli space and the CKG.

Let us now turn to the string S-matrix. As we saw, string S-matrix involves sum over topologies of the worldsheets which provides perturbative expansion of the scattering amplitude. Let us start discussing the moduli space and CKG for some initial topologies of string wolrdsheet.

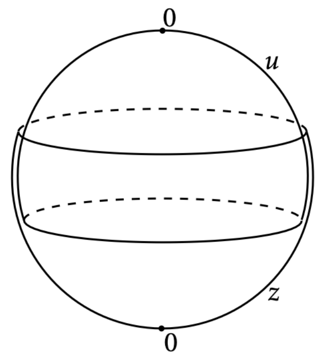

Figure 7.3: The sphere constructed from two disks. The north and south poles are the centers of the two disks.

Positive Euler characteristics

There are only three Riemann surfaces with positive Euler characteristics – the sphere S^2, the disk $\mathbb{D}$ and the real projective plane $\mathbb{RP}^2$.

The sphere: This topology corresponds to the tree level closed oriented string. The path integral for the amplitude over metrics and spacetime embeddings on the Riemann sphere is then given by

$$S_{j_1 j_2 \cdots j_n}(k_1, \ldots, k_n) = e^{-2\lambda} \int \frac{[\mathcal{D}\phi \mathcal{D}g]}{V_{\mathcal{G}}} \exp(-S_{\mathrm{m}}) \prod_{i=1}^{n} \int_{S^2} d^2\sigma_i \sqrt{g(\sigma_i)} V_{j_i}(k_i, \sigma_i) \qquad (7.2.42)$$

since $\chi = 2$ for S^2. To perform the inner integral over S^2, we need to construct coordinate patches on the sphere. The sphere can be obtained from the complex plane $\mathbb{C}$ by one-point compactification. That is by adding a point at infinity to the complex plane: $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. It can be covered by two patches. To describe it choose a real number $\rho > 1$ and consider disks $|z| < \rho, |u| < \rho$. Join them by identifying points such that

$$u = 1/z \qquad (7.2.43)$$

The two coordinates u, z covers the upper and lower part of the sphere as shown is Figure ?? below. The overlapping region satisfies (7.2.43). Thus the transition maps are $u^{-1} \circ z = z^{-1} \circ u = 1$ which is trivially holomorphic. Any metric on S^2 can be mapped to a metric on the plane using the stereographic projection and as we have already seen, we can always make a metric on the plane flat using a gauge transformation from $\mathcal{G}$. Thus the general conformal metric on S^2 is

$$ds^2 = \exp(2\omega(z, \bar{z})) dz d\bar{z}, \qquad (7.2.44)$$

for some function ω. In the u patch, the metric transforms as

$$\begin{aligned} g_{u\bar{u}} &= \frac{\partial z}{\partial u}\frac{\partial \bar{z}}{\partial \bar{u}} g_{z\bar{z}} \\ &= \frac{1}{|u|^4}\exp(2\omega(z,\bar{z})) \\ &= |z|^4 \exp(2\omega(z,\bar{z})). \end{aligned} \tag{7.2.45}$$

Thus $g_{u\bar{u}}$ is nonsingular at $u = 0$ (corresponds to $z \to \infty$) if

$$\lim_{z\to\infty} |z|^4 |\exp(\omega(z,\bar{z}))| < \infty. \tag{7.2.46}$$

Thus any $\omega(z,\bar{z})$ satisfying (7.2.46) defines a metric on S^2. The round metric on S^2 of radius r and Ricci scalar curvature $R = 2/r^2$ is

$$ds^2 = \frac{4r^2}{(1+z\bar{z})^2} dz d\bar{z} = \frac{4r^2}{(1+u\bar{u})^2} du d\bar{u}. \tag{7.2.47}$$

We now look at the CKG. As we already discussed in Subsection 5.3.3, the conformal transformations on the sphere generated by the Witt algebra

$$\begin{aligned} l_n &= z^{n+1}\frac{\partial}{\partial z} \\ \{l_n, l_m\} &= (m+n) l_{m+n}. \end{aligned} \tag{7.2.48}$$

We also saw that not all of these generators are globally defined. Indeed only $l_0, l_{\pm 1}$ is globally defined which generates the global conformal killing group

$$\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2. \tag{7.2.49}$$

Thus the for CKG is $\mathrm{PSL}(2,\mathbb{C})$ and to calculate the tree level amplitude we need to integrate over PSL $(2,\mathbb{C})$.

Real projective plane: This topology corresponds to the unoriented closed string tree level amplitude. To see this, note that the real projective plane $\mathbb{RP}^2$ can be obtained from the sphere by identifying antipodal points: identify points z, z' related by

$$z' = -1/\bar{z}. \tag{7.2.50}$$

Since there are no fixed points under this identification, there is no boundary of this manifold but it is unoriented. There is again no moduli for this space and the CKG is the subgroup of PSL $(2,\mathbb{C})$ which respects (7.2.50). Indeed, that subgroup consists of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ such that

$$\begin{aligned} \frac{az'+b}{cz'+d} &= -\frac{1}{\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}} \\ \implies \frac{-a+b\bar{z}}{-c+d\bar{z}} &= -\frac{\bar{c}\bar{z}+\bar{d}}{\bar{a}\bar{z}+\bar{b}}. \end{aligned} \tag{7.2.51}$$

One can show that the only solution to this is $a = \bar{d}, c = -\bar{b}$ with $|a|^2 + |b|^2 = 1$. This subgroup is precisely SU(2). Modding out by $\mathbb{Z}_2$, we obtain $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2$.

The disk: This topology corresponds to the open string tree level amplitude. The open disk $\mathbb{D}$ is conformally equivalent to the upper half plane under the map

$$\begin{aligned} q : \mathbb{H} &\longrightarrow \mathbb{D} \\ \tau &\longmapsto e^{2\pi i \tau}. \end{aligned} \tag{7.2.52}$$

Moreover the CKG for $\mathbb{H}$ is $\mathrm{PSL}(2,\mathbb{R})$ since one needs to preserve the boundary of $\mathbb{H}$. Thus we need to integrate over PSL(2,$\mathbb{R}$) to get open string n-point tree level amplitude.

Zero Euler characteristic

Let us start with coordinates σ^1, σ^2 in the region

$$0 \leq \sigma^1 \leq 2\pi, \quad 0 \leqslant \sigma^2 \leq 2\pi. \tag{7.2.53}$$

The fields $X^\mu(\sigma^1, \sigma^2)$ and $g_{\alpha\beta}(\sigma^1, \sigma^2)$ are periodic in both directions. The torus is described by the identification

$$\left(\sigma^1, \sigma^2\right) \equiv \left(\sigma^1, \sigma^2\right) + 2\pi(m,n); \quad m,n \in \mathbb{Z}. \tag{7.2.54}$$

Theorem 7.2.5. *Let $g_{\alpha\beta}$ be a metric on the torus described by the coordinates $\sigma^1, \sigma^2 \in [0, 2\pi]$ with $(\sigma^1, \sigma^2) \equiv (\sigma^1, \sigma^2) + 2\pi\mathbb{Z}^2$. Then there exist coordinates $\widetilde{\sigma}^1, \widetilde{\sigma}^2 \in [0, 2\pi]$ satisfying $(\widetilde{\sigma}^1, \widetilde{\sigma}^2) \equiv (\widetilde{\sigma}^1, \widetilde{\sigma}^2) + 2\pi\mathbb{Z}^2$ such that*

$$ds^2 = \left|d\widetilde{\sigma}^1 + \tau d\widetilde{\sigma}^2\right|^2 \tag{7.2.55}$$

with a complex constant τ.

Proof. We first make the torus flat using a Weyl transformation as described in Remark 3.1.1. The flat torus can be described in complex coordinates as a parallelogram spanned by 1 and some $\tau \in \mathbb{C}\backslash\mathbb{R}$ with the identification

$$\begin{aligned} &z \cong z + 2\pi m, \\ &z \cong z + 2\pi n\tau; \quad z = a + b\tau \in \mathbb{R} + \mathbb{R}\tau, m, n \in \mathbb{Z}. \end{aligned} \tag{7.2.56}$$

Clearly on the parallelogram, the metric can be written as $dzd\bar{z}$. Taking coordinates on the torus to be $\widetilde{\sigma}^1, \widetilde{\sigma}^2$ such that $z = \widetilde{\sigma}^1 + \tau\widetilde{\sigma}^2$ we see that $dzd\bar{z}$ takes the desired form. ☐

This theorem implies that the moduli space of gauge inequivalent metrics is parametrized by a complex parameter τ. Observe that the metric (7.2.55) is invariant under $\tau \to \bar{\tau}$. Thus we can restrict to $\mathrm{Im}(\tau) > 0$. Hence the moduli space is parametrized by $\tau \in \mathbb{H} := \{\tau = x + iy \in \mathbb{C} : y > 0\}$. The parameter τ is called the *Teichmüler parameter* or *modulus.* There are further identifications. Replacing τ by $\tau+1$ and τ by $-\frac{1}{\tau}$ gives the same torus since this only modifies the identifications of points on $\mathbb{R}+\mathbb{R}\tau$ by changing $(m,n)\to(m-n,n)$ and $(m,n)\to(n,-m)$ respectively. These two transformations

$$\begin{aligned} T &: \tau \longrightarrow \tau+1 \\ S &: \tau \longrightarrow -\frac{1}{\tau} \end{aligned} \tag{7.2.57}$$

generate the modular group $\mathrm{PSL}(2,\mathbb{Z}) := \mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2$. Thus the moduli space of metrics on the torus can be identified with $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$ which is the fundamental domain of $\mathbb{H}$ defined below:

Definition 7.2.6. If Γ is a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and $\mathcal{F}\subset\mathbb{H}$ is a closed set with connected interior, we say that $\mathcal{F}$ is a fundamental domain for Γ (or $\Gamma\backslash\mathbb{H}$) if

(i) any $z\in\mathbb{H}$ is Γ-equivalent to a point in $\mathcal{F}$;

(ii) no two interior points of $\mathcal{F}$ are Γ-equivalent;

(iii) the boundary of $\mathcal{F}$ is a finite union of smooth curves.

As an example let Γ_∞ be the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ given by

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n\in\mathbb{Z} \right\}. \tag{7.2.58}$$

A fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$ is given by

$$\mathcal{F} = \left\{ z\in\mathbb{H} : -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \right\} \tag{7.2.59}$$

Theorem 7.2.7. *Let*

$$\mathcal{F} = \left\{ z\in\mathbb{H} : |\,\mathrm{Re}(z)| \leq \frac{1}{2}, |z|\geq 1 \right\} \tag{7.2.60}$$

Then $\mathcal{F}$ is a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ *on* $\mathbb{H}$.

Proof. See Theorem 3.2.2 of [11] $\square$

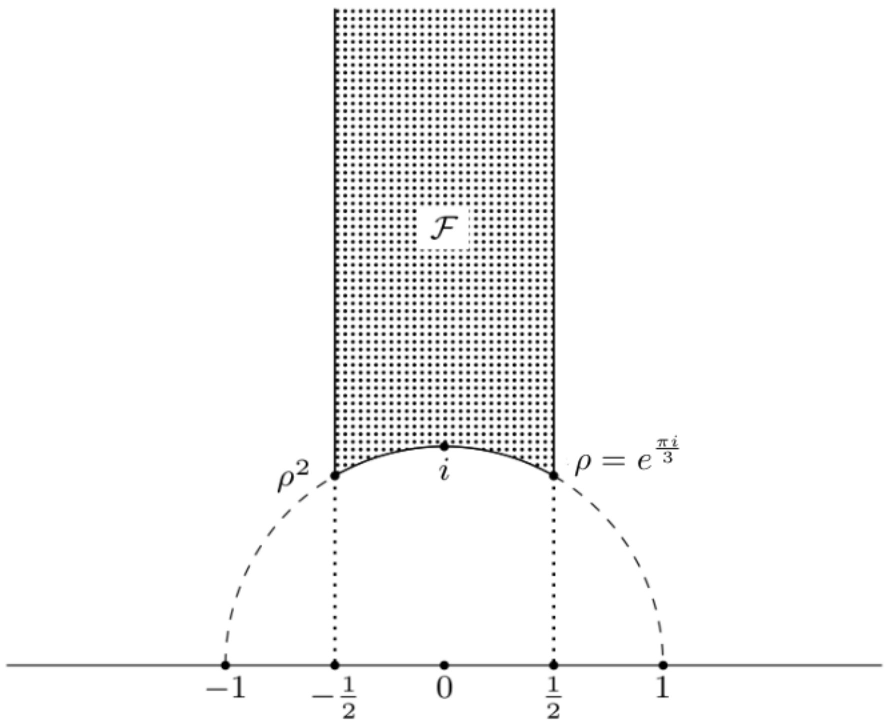

Figure 7.4: Fundamental domain for $\mathrm{SL}(2,\mathbb{Z})$

The modular group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathbb{H}$ as

$$\tau \longmapsto \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{7.2.61}$$

To integrate over $\mathcal{F}$, we need an $\mathrm{SL}(2,\mathbb{Z})$ invariant measure.

Theorem 7.2.8. *The measure*

$$d\mu = \frac{d^2\tau}{\mathrm{Im}(\tau)^2}$$

is $\mathrm{SL}(2,\mathbb{R})$ *invariant.*

Proof. Under $\mathrm{SL}(2,\mathbb{R})$,

$$\tau \to \tau' = \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}).$$

Then

$$d\tau' = \frac{d\tau'}{d\tau} d\tau = \frac{1}{(c\tau+d)^2} d\tau.$$

Thus

$$d\tau' d\bar{\tau}' = \frac{1}{|c\tau + d|^4} d\tau d\bar{\tau}.$$

It is easy to check that

$$\mathrm{Im}\,(\tau') = \frac{\mathrm{Im}\,\tau}{|c\tau + d|^2}$$

Thus

$$\frac{d^2\tau'}{(\mathrm{Im}\,\tau')^2} = \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}.$$

□

In general, it is not possible to bring this metric to unit form such that the new coordinates also satisfy the periodicity condition (7.2.54). Suppose under a coordinate transformation to $\widetilde{\sigma}^\alpha$, the metric reduces to $\delta_{\alpha\beta}$. Then in general

$$\widetilde{\sigma}^\alpha \cong \widetilde{\sigma}^\alpha + 2\pi\,(mu^\alpha + nv^\alpha) \tag{7.2.62}$$

where u^α and v^α are general translations. By rotating and rescaling the coordinate system, we can take $u = (1, 0)$ which leaves us with two real parameters v^1, v^2. If we redefine $z = \widetilde{\sigma}^1 + i\widetilde{\sigma}^2$, the metric is $dzd\bar{z}$ and the periodicity is

$$z \cong z + 2\pi(m + n\tau)$$

where $\tau = v^1 + iv^2$. Thus the torus is a parallelogram in the z-plane with periodic boundary condition. The moduli space is again parametrized by two real parameters v^1, v^2.

We now look at CKG. The rigid translations

$$\sigma^\alpha \to \sigma^\alpha + v^\alpha$$

where $v^\alpha \in \mathbb{R}^2$ leaves the metric and periodicity invariant. Thus this 2 parameter subgroup isomorphic to $\mathbb{R}^2$ is not fixed. Moreover the discrete transformations $\sigma^\alpha \longrightarrow -\sigma^\alpha$ and $(\sigma^1, \sigma^2) \longrightarrow (-\sigma^1, \sigma^2)$ with $\tau \to -\bar{\tau}$ in the unoriented case are also part of CKG.

7.2.4 Moduli space of higher genus surface

As already discussed, the topology of a 2d surface is classified by the Euler characteristic. In case the surface is oriented and closed, the genus classifies the surfaces. Let $\mathcal{G}_g$ be the space of all metrics on a 2d surface with topology g. The moduli space is then

$$\mathcal{M}_g = \frac{\mathcal{G}_g}{(\mathrm{Diff} \ltimes \mathrm{Weyl})_g}. \tag{7.2.63}$$

Then there are the residual symmetries, the CKG. If there are vertex operators in the path integeral, then one way to fix the CKG is to specify the coordinate of the vertex operators.

For example, for torus, the $\mathbb{R}^2$ CKG can be fixed by fixing the position of one of the vertex operators. Additionally the $\mathbb{Z}_2$ from $\sigma^\alpha \longrightarrow -\sigma^\alpha$ can be fixed by restricting another vertex operator to half the torus. In general if there are n vertex operator positions over the worldsheet Σ then the moduli space at topology g is

$$\mathcal{M}_{g,n} = \frac{\mathcal{G}_g \times \Sigma^n}{(\text{Diff} \ltimes \text{Weyl})_g}. \tag{7.2.64}$$

This space is called the *moduli space of metrics*. We now want to find the dimension of $\mathcal{M}_{g,n}$ so we construct a local model of $\mathcal{M}_{g,n}$, that is, its tangent space at some $g_{\alpha\beta}$. To do this we look at the infinitesimal variations $\delta' g_{\alpha\beta}$ of the metric orhtogonal to the variation $\delta g_{\alpha\beta}$ along the gauge orbit (see Figure 7.5 below).

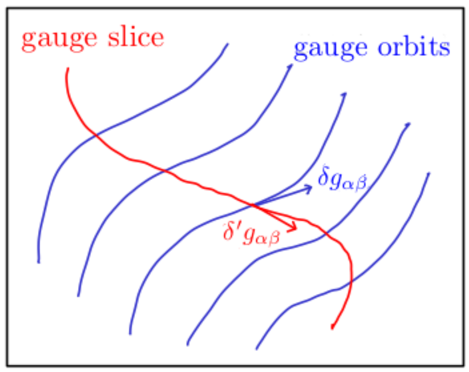

Figure 7.5: Gauge inequivalent infinitesimal variations of the metric

Recall that an infinitesimal Diff$\ltimes$Weyl variation $\delta g_{\alpha\beta}$ of the metric is given by

$$\delta g_{ab} = -2\left(P_1 \delta\sigma\right)_{\alpha\beta} + \left(2\delta\omega - \nabla \cdot \delta\sigma\right) g_{\alpha\beta} \tag{7.2.65}$$

so that we have

$$\int d^2\sigma \sqrt{g} \delta' g^{\alpha\beta} \left[-2\left(P_1 \delta\sigma\right)_{\alpha\beta} + (2\delta\omega - \nabla \cdot \delta\sigma) g_{\alpha\beta}\right] = 0 \tag{7.2.66}$$

Note that we are using the inner product on tensors given by

$$\langle T^{\alpha_1 \cdots \alpha_n}, R_{\beta_1 \cdots \beta_n} \rangle = \int d^2\sigma \sqrt{g} T^{\alpha_1 \cdots \alpha_n} R_{\alpha_1 \cdots \alpha_n}. \tag{7.2.67}$$

To proceed further, we will need some results about the operator P_1. Recall that P_1 acts on vectors and gives traceless symmetric rank 2 tensors:

$$\left(P_1 v\right)_{\alpha\beta} = \frac{1}{2}\left(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha - g_{\alpha\beta} \nabla_\gamma v^\gamma\right). \tag{7.2.68}$$

Indeed, we can define a more general operator P_n which maps traceless symmetric n-tensor to traceless symmetric $(n+1)$-tensor: let $v_{\alpha_1\cdots\alpha_n}$ be a traceless symmetric tensor. Define P_n by

$$(P_n v)_{\alpha_1\cdots\alpha_{n+1}} \equiv \nabla_{(\alpha_1} v_{\alpha_2\cdots\alpha_{n+1})} - \frac{n}{n+1} g_{(\alpha_1\alpha_2} \nabla_{|\beta|} v^{\beta}{}_{\alpha_3\cdots\alpha_{n+1})}, \tag{7.2.69}$$

where () indicates the symmetrization of the indices and $|\beta|$ indicates that the index β is not included in the symmetrization. This tensor is symmetric by construction. Contracting with $g^{\alpha_1\alpha_2}$, the first term becomes

$$\begin{aligned} g^{\alpha_1\alpha_2}\nabla_{(\alpha_1} v_{\alpha_2\cdots\alpha_{n+1})} &= g^{\alpha_1\alpha_2}\frac{1}{n+1}\left[\nabla_{\alpha_1} v_{(\alpha_2\cdots\alpha_{n+1})} + \nabla_{\alpha_2} v_{(\alpha_1\alpha_3\cdots\alpha_{n+1})} + \sum_{i\neq 1,2} \nabla_{\alpha_i} v_{(\alpha_1\cdots\alpha_{n+1})}\right] \\ &= \frac{2}{n+1}\nabla_\beta v^{\beta}{}_{\alpha_3\cdots\alpha_{n+1}}, \end{aligned}$$

where we have used the symmetry and tracelessness of v. The second similarly becomes

$$\begin{aligned} g^{\alpha_1\alpha_2} g_{(\alpha_1\alpha_2}\nabla_{|\beta|} v^{\beta}{}_{\alpha_3\cdots\alpha_{n+1})} &= \frac{2}{n(n+1)} g^{\alpha_1\alpha_2} g_{\alpha_1\alpha_2}\nabla_\beta v^{\beta}{}_{\alpha_3\cdots\alpha_{n+1}} + \frac{2(n-1)}{n(n+1)} g^{\alpha_1\alpha_2} g_{\alpha_1\alpha_3}\nabla_\beta v^{\beta}{}_{\alpha_2\alpha_4\cdots\alpha_{n+1}} \\ &= \frac{2}{n}\nabla_\beta v^{\beta}{}_{\alpha_3\cdots\alpha_{n+1}}. \end{aligned}$$

Formally, we can define the transpose of P_n using the inner product (7.2.67):

$$\langle u, P_n v\rangle = \langle P_n^T u, v\rangle. \tag{7.2.70}$$

We claim that for $u_{\alpha_1\cdots\alpha_{n+1}}$ a traceless symmetric tensor, define P_n^T by

$$\left(P_n^T u\right)_{\alpha_1\cdots\alpha_n} \equiv -\nabla_\beta u^{\beta}{}_{\alpha_1\cdots\alpha_n}. \tag{7.2.71}$$

This inherits the symmetry and tracelessness of u.

$$\begin{aligned} \langle u, P_n v\rangle &= \int d^2\sigma\sqrt{g}\, u^{\alpha_1\cdots\alpha_{n+1}} \left(P_n v\right)_{\alpha_1\cdots\alpha_{n+1}} \\ &= \int d^2\sigma\sqrt{g}\, u^{\alpha_1\cdots\alpha_{n+1}} \left(\nabla_{\alpha_1} v_{\alpha_2\cdots\alpha_{n+1}} - \frac{n}{n+1} g_{\alpha_1\alpha_2}\nabla_\beta v^{\beta}{}_{\alpha_3\cdots\alpha_{n+1}}\right) \\ &= -\int d^2\sigma\sqrt{g}\,\nabla_{\alpha_1} u^{\alpha_1\cdots\alpha_{n+1}} v_{\alpha_2\cdots\alpha_{n+1}} \\ &= -\int d^2\sigma\sqrt{g}\,\nabla_{\alpha_1} u^{\alpha_1\cdots\alpha_{n+1}} v_{\alpha_2\cdots\alpha_{n+1}} \\ &= \int d^2\sigma\sqrt{g}\left(P_n^T u\right)^{\alpha_2\cdots\alpha_{n+1}} v_{\alpha_2\cdots\alpha_{n+1}} \\ &= \langle P_n^T u, v\rangle. \end{aligned}$$

Returing back to (7.2.66), we have from orthogonality

$$\int d^2\sigma\sqrt{g}\left(-2\left(P_1^T\delta' g\right)_\alpha \delta\sigma^\alpha + \delta' g_{\alpha\beta} g^{\alpha\beta}(2\delta\omega - \nabla\cdot\delta\sigma)\right) = 0 \tag{7.2.72}$$

For arbitrary $\delta\sigma$ and $\delta\omega$, this implies

$$\begin{aligned} g^{\alpha\beta}\delta' g_{\alpha\beta} &= 0 \\ \left(P_1^T \delta' g\right)_\alpha &= 0. \end{aligned} \tag{7.2.73}$$

The first condition means that $\delta' g_{\alpha\beta}$ is traceless so that the second condition is meaningful since P_1^T acts only on traceless symmetric tensors. CKG is determined by the condition

$$\delta g_{\alpha\beta} = 0. \tag{7.2.74}$$

This corresponds to those transformations which does not change the metric after a gauge has been chosen. From (7.2.65), this implies

$$-2\left(P_1\delta\sigma\right)_{\alpha\beta} + (2\delta\omega - \nabla\cdot\delta\sigma)g_{\alpha\beta} = 0. \tag{7.2.75}$$

Taking trace, we get

$$\begin{aligned} -2g^{\alpha\beta}(P_1\delta\sigma)_{\alpha\beta} + (2\delta\omega - \nabla\cdot\delta\sigma)g_\alpha^\alpha &= 0 \\ \implies \delta\omega &= \frac{\nabla\cdot\delta\sigma}{2} \end{aligned} \tag{7.2.76}$$

since $(P_1\delta\sigma)$ is traceless. Thus $\delta g_{\alpha\beta} = 0$ gives

$$\left(P_1\delta\sigma\right)_{\alpha\beta} = 0. \tag{7.2.77}$$

Thus the tangent space to moduli space is $\mathrm{Ker} P_1^T$ and CKG is $\mathrm{Ker}\, P_1$. Let

$$\begin{aligned} \mu &= \dim \mathrm{Ker}\, P_1^T \\ \kappa &= \dim \mathrm{Ker}\, P_1. \end{aligned} \tag{7.2.78}$$

Then by Riemann-Roch theorem

$$\mu - \kappa = -3\chi \tag{7.2.79}$$

where χ is the Euler characteristic of the 2d surface. We will prove this using path integral in the subsequent sections.

Theorem 7.2.9. *Let χ, μ, κ be as above. Then the following is true*

(i) If $\chi > 0$ then $\kappa = 3\chi, \mu = 0$

(ii) If $\chi < 0$ then $\mu = -3\chi, \kappa = 0$.

Proof. Without the loss of generality, we can assume that the Ricci scalar R of the surface is a constant. Indeed we can make a Weyl transformation to make R constant. Thus the sign of R is the sign of χ since

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}R.$$

Now we claim that

$$P_1^T P_1 = -\frac{1}{2}\nabla^2 - \frac{R}{4}.$$

Indeed acting on a test function v, we see that

$$\begin{aligned}
\left(P_1^T P_1 v\right)_\alpha &= -\nabla^\beta \left(P_1 v\right)_{\alpha\beta} \\
&= -\frac{1}{2}\left[\nabla^\beta \nabla_\alpha v_\beta + \nabla^\beta \nabla_\beta v_\alpha - \nabla^\beta \left(g_{\alpha\beta} \nabla_\gamma v^\gamma\right)\right] \\
&= -\frac{1}{2}\nabla^2 v_\alpha - \frac{1}{2}\left(\nabla^\beta \nabla_\alpha v_\beta - \nabla_\alpha \nabla_\gamma v^\gamma\right) \\
&= -\frac{1}{2}\nabla^2 v_\alpha - \frac{1}{2} g^{\beta\gamma}\left(\nabla_\gamma \nabla_\alpha v_\beta - \nabla_\alpha \nabla_\gamma v_\beta\right) \\
&= -\frac{1}{2}\nabla^2 v_\alpha - \frac{1}{2} g^{\beta\gamma} R_{\gamma\alpha} v_\beta
\end{aligned}$$

But in 2d, one can show that $R_{\gamma\alpha} = \frac{R}{2} g_{\gamma_\alpha}$. Thus

$$\begin{aligned}
\left(P_1^T P_1 v\right)_\alpha &= -\frac{1}{2}\nabla^2 v_\alpha - \frac{R}{2} g^{\beta\gamma} g_{\gamma\alpha} v_\beta \\
&= \left(-\frac{1}{2}\nabla^2 - \frac{R}{4}\right) v_\alpha.
\end{aligned}$$

Using this

$$\begin{aligned}
\int d^2\sigma \sqrt{g} \left(P_1 \delta\sigma\right)_{\alpha\beta} \left(P_1 \delta\sigma\right)^{\alpha\beta} &= \int d^2\sigma \sqrt{g} \delta\sigma_\alpha \left(P_1^T P_1 \delta\sigma\right)^\alpha \\
&= \int d^2\sigma \sqrt{g} \left(-\frac{1}{2}\delta\sigma_\alpha \nabla^2 \delta\sigma^\alpha - \frac{R}{4}\delta\sigma_\alpha \delta\sigma^\alpha\right) \\
&= \int d^2\sigma \sqrt{g} \left(\frac{1}{2}\nabla_\alpha \delta\sigma_\beta \nabla^\alpha \delta\sigma^\beta - \frac{R}{4}\delta\sigma_\alpha \delta\sigma^\alpha\right)
\end{aligned}$$

where we used integration by parts. Now if $\chi < 0$ then the RHS is strictly positive and hence $P_1 \delta\sigma \neq 0$ for any $\delta\sigma$. Thus $\kappa = 0$ and (7.2.79) implies $\mu = -3\chi$. Doing similar calculation, we can show that $P_1^T \delta' g \neq 0$ for $\chi > 0$ which gives $\mu = 0$ and we can show that $\kappa = 3\chi$ for $\chi > 0$. □

We see that this theorem is valid for the sphere for which $\chi = 2$ since as we saw, the moduli space of metrics on sphere is trivial and the CKG is $\mathrm{SL}(2, \mathbb{C})$ which has six real parameters in accordance with the above theorem. The theorem is not valid for $\chi = 0$ but there is only one oriented 2d surface without boundary – the torus and we saw that it has a nontrivial moduli space as well as non trivial CKG.

2d Reimannian Manifolds and Riemann surfaces

Recall that the string S-matrix has integrals over the worldsheet with vertex operator insertions. The path integral includes integral of space of all metrics modulo the diff $\times$ Weyl. As discussed in previous section, the gauge fixing procedure picks out a metric from each gauge orbit leading to the moduli space of metrics. Since we have freedom to choose the metric from each gauge orbit, we can choose it to be such that the integral over the worldsheet simplifies. In this section, we will show that 2d Riemannian manifolds modulo Weyl equivalence can be considered as a Riemann surface. This means that there exists a choice of metric, explicitly the locally flat metric, on the worldsheet such that the worldsheet admits a complex atlas and hence is a Riemann surface (see Definition 7.2.11 below). This will simplify the integral over worldsheet to integral over a Riemann surface. We start with a brief survey of Riemann surface.

Definition 7.2.10. A function $f : \mathbb{C}^n \to \mathbb{C}^m$ is called holomorphic if each component $f_i : \mathbb{C}^n \to \mathbb{C}, 1 \leq i \leq m$ is holomorphic in the sense that it satisfies the Cauchy-Riemann equations: write $f_i = f_i^1 + i f_i^2$ and $z^\mu \in \mathbb{C}^n$ as $z^\mu = x^\mu + i y^\mu$ then

$$\frac{\partial f_i^1}{\partial x^\mu} = \frac{\partial f_i^2}{\partial y^\mu}, \quad \frac{\partial f_i^2}{\partial x^\mu} = -\frac{\partial f_i^2}{\partial y^\mu}; \quad \mu = 1, \dots, n. \tag{7.2.80}$$

Definition 7.2.11. A $2n$ dimensional manifold X is called a *complex manifold* of dimension n if there is a covering of X by a family of open sets $\{U_\alpha\}$ and homeomorphisms $z_\alpha : U_\alpha \longrightarrow V_\alpha$ where $V_\alpha \subset \mathbb{C}^n$ is some open subset such that

$$f_{\alpha\beta} \equiv z_\alpha \circ z_\beta^{-1} : z_\beta \left(V_\alpha \cap V_\beta\right) \longrightarrow z_\alpha \left(V_\alpha \cap V_\beta\right)$$

is *biholomorphic* i,e $f_{\alpha\beta}$ as well as $f_{\alpha\beta}^{-1}$ is holomorphic. Such a collection of charts is called a *complex atlas*. We write $\dim_{\mathbb{R}} X = 2n$ and $\dim_{\mathbb{C}} X = n$ for the real and complex dimensions of the manifold respectively. A complex manifold of dimension 1 is called a *Riemann surface*.

Thus $\mathbb{C}^n$ is trivially a complex manifold of complex dimension $\dim_{\mathbb{C}} \mathbb{C}^n = n$ with a single chart $(\mathbb{C}^n, z^\mu)$ in the complex atlas. In particular $\mathbb{C}$ is a Riemann surface. Once we include the point at infinity, it turns into a sphere called the *Riemann sphere*.

Definition 7.2.12. A continuous map $f : X \to Y$ between Riemann surfaces is said to be *holomorphic* if it is holomorphic in charts, that is, for every $p \in X$, if (U_α, z_α) is a chart containing p on X and (V_β, w_β) is a chart containing $f(p)$ on Y then $w_\beta \circ f \circ z_\alpha^{-1}$ is holomorphic in the usual sense. Similarly, $f : X \to Y$ is *meromorphic* if it is meromorphic in charts.

Let g be a Riemannian metric on X. Define $2n$ vector fields as follows: on a chart (U, z^μ) from a complex atlas on X define

$$\frac{\partial}{\partial z^\mu} = \frac{1}{2}\left\{\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\right\}, \quad \frac{\partial}{\partial \bar{z}^\mu} = \frac{1}{2}\left\{\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\right\} \tag{7.2.81}$$

where $z^\mu : U \longrightarrow \mathbb{C},\ \mu = 1, \dots, n$ is written as $z^\mu = x^\mu + i y^\mu$. Note that these vector fields form a basis for the complexifield tangent space $T_p X^{\mathbb{C}}$ with $p \in U$. Using this basis, the Riemannian metric can be extended to $T_p X^{\mathbb{C}}$ as follows: write $Z = X + iY, W = U + iV \in T_p X^{\mathbb{C}}$ with $X, Y, U, V \in T_p X$, then

$$g_p(Z, W) \equiv g_p(X, U) - g_p(Y, V) + i \left[g_p(X, V) + g_p(Y, U)\right]. \tag{7.2.82}$$

In terms of components, if we use the basis

$$\begin{aligned} dz^{\mu} &= dx^{\mu} + idy^{\mu} \\ d\bar{z}^{\bar{\mu}} &= dx^{\bar{\mu}} + idy^{\bar{\mu}} \end{aligned} \tag{7.2.83}$$

where $1 \leq \mu, \bar{\mu} \leq m$ for [4] $T_p^* X^{\mathbb{C}}$ then one has

$$\begin{aligned} g_{\mu\nu}(p) &= g_p\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial z^{\nu}}\right) \\ g_{\mu\bar{\nu}}(p) &= g_p\left(\frac{\partial}{\partial z^{\mu}}, \frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right) \\ g_{\bar{\mu}\nu}(p) &= g_p\left(\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}, \frac{\partial}{\partial z^{\nu}}\right) \\ g_{\bar{\mu}\bar{\nu}}(p) &= g_p\left(\frac{\partial}{\partial \bar{z}^{\bar{\mu}}}, \frac{\partial}{\partial \bar{z}^{\bar{\nu}}}\right) \end{aligned} \tag{7.2.84}$$

and

$$g = g_{\mu\nu} dz^{\mu} \otimes dz^{\nu} + g_{\mu\bar{\nu}} dz^{\mu} \otimes d\bar{z}^{\bar{\nu}} + g_{\bar{\mu}\nu} d\bar{z}^{\bar{\mu}} \otimes dz^{\nu} + g_{\bar{\mu}\bar{\nu}} d\bar{z}^{\bar{\mu}} \otimes d\bar{z}^{\bar{\nu}}. \tag{7.2.85}$$

We need one more ingredient before we prove the main theorem.

Definition 7.2.13. Let $(U_\alpha, \varphi_\alpha)$ be an atlas on a manifold X. A *partition of unity subordinate to* $(U_\alpha, \varphi_\alpha)$ is a family of smooth functions $\rho_\alpha : X \longrightarrow \mathbb{R}$ such that

(i) ρ_α is smooth and $\operatorname{supp}(\rho_\alpha) \subset U_\alpha$.

(ii) for any $x \in X$, there a neighbourhood U of x such that

$$\#\{\alpha : \operatorname{supp}(\rho_\alpha) \cap U \neq \phi\} < \infty.$$

(iii) the functions ρ_α sum to 1:

$$\sum_\alpha \rho_\alpha = 1. \tag{7.2.86}$$

The standard result is:

Theorem 7.2.14. [14, Appendix C] *For any atlas on a manifold, there exists a partition of unity subordinate to it.*

Let us now prove the main theorem.

Theorem 7.2.15. *There is a one-to-one correspondence between Riemann surfaces and Riemannian manifolds Σ with $\dim_{\mathbb{R}}\Sigma = 2$ upto Weyl equivalence.*

Remark 7.2.16. We did not include modulo diffeomorphism in the theorem because it is implicit in the definition of a manifold.

[4] one can easily prove that $T_p^* X^{\mathbb{C}} \cong (T_p X^{\mathbb{C}})^*$.

Proof of Theorem 7.2.15. Let us first start with a $2d$ Riemannian manifold. Let $(U_\alpha, \varphi_\alpha)$ be an atlas on it with metric $g_{\alpha\beta}$ on chart $U_\alpha \cap U_\beta$. Without loss of generality we can assume that $g_{\alpha\beta} \propto \delta_{\alpha\beta}$. If not, we can use diffeomorphism, which is equivalent to changing chart, as described in Subsection 3.1.1 to bring the metric to this form. Now using Weyl transformation, we can take $g_{\alpha\beta} = \delta_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. Let $\varphi_\alpha : U_\alpha \longrightarrow \mathbb{R}^2$ be given by $\varphi_\alpha = (x_\alpha, y_\alpha)$. Define $z_\alpha = x_\alpha + iy_\alpha$. Then $g_{\alpha\beta} = \delta_{\alpha\beta}$ in coordinates z_α is $ds^2 = dz_\alpha \otimes d\bar{z}_\alpha$ on U_α. We now want to show that the transition function in terms of z_α is holomorphic. In terms of φ_α, the transition function on $U_\alpha \cap U_\beta$ is

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2.$$

In terms of z_α, z_β, the transition function is

$$z_\alpha \circ z_\beta^{-1} : \mathbb{C} \longrightarrow \mathbb{C}.$$

But on $U_\alpha \cap U_\beta$ the metric has form $dz_\alpha d\bar{z}_\alpha$ and $dz_\beta d\bar{z}_\beta$. Then denoting the components of the metric on U_α as $g^\alpha_{zz}, g^\alpha_{z\bar{z}}, g^\alpha_{\bar{z}z}, g^\alpha_{z\bar{z}}$ we see that

$$\begin{aligned} 0 = g^\alpha_{zz} &= 2\frac{\partial\left(z_\alpha \circ z_\beta^{-1}\right)}{\partial z}\frac{\partial\left(z_\alpha \circ z_\beta^{-1}\right)}{\partial \bar{z}} g^\beta_{z\bar{z}} \\ &= \frac{\partial\left(z_\alpha \circ z_\beta^{-1}\right)}{\partial z}\frac{\partial\left(z_\alpha \circ z_\beta^{-1}\right)}{\partial \bar{z}} \end{aligned} \tag{7.2.87}$$

where z is the standard coordinate on $\mathbb{C}$. This implies

$$\frac{\partial\left(z_\alpha \circ z_\beta^{-1}\right)}{\partial z} = 0 \quad \text{or} \quad \frac{\partial\left(z_\alpha \circ z_\beta^{-1}\right)}{\partial \bar{z}} = 0 \tag{7.2.88}$$

Thus $z_\alpha \circ z_\beta^{-1}$ is either holomorphic or antiholomorphic. Similary $z_\beta \circ z_\alpha^{-1}$ is either holomorphic or antiholomorphic. The present case of oriented closed string theory restricts to holomorphic transition functions since antiholomorphic transition functions destroy the orientability and hence we obtain a Riemann surface.

Conversely start with a Riemann surface with complex atlas (U_α, z_α). Take the metric on U_α to be $ds^2 = dz_\alpha \otimes d\bar{z}_\alpha$. We just have to patch these on the overlaps. We use partition of unity for that. Define the metric on the surface as follows

$$g = \sum_\alpha \rho_\alpha g^\alpha$$

where $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$ and $g^\alpha = dz_\alpha \otimes d\bar{z}_\alpha$. Then clearly g is a smooth 2-form field since the coefficients of $dz_\alpha \otimes d\bar{z}_\alpha$ are smooth. Moreover on U_α

$$\begin{aligned} g_p &= g^\alpha_p \sum_\beta \rho^\beta \\ &= g^\alpha, \quad p \in U_\alpha \end{aligned}$$

where we used the fact that g^α is fixed on U_α and also the property of partitions of unity. Thus we have defined a Riemannian metric on the surface in the complex coordinates which can be converted to the usual coordinates using the inverse construction of (7.2.84) and (7.2.85). □

Remark 7.2.17. To interpret the CKG in this picture, note that we can define the worldsheet as union of patches and use a $\mathcal{G}$-transformation to bring the metric to $dz_\alpha d\bar{z}_\alpha$ on patches U_α. The gauge choices on the overlaps can differ only by $\mathcal{G}$-invariances of this metric which are exactly the conformal transformations *i,e* CKG. Thus a Riemann surface is the natural background for a CFT.

Measure on moduli space of metrics

In this section, we want to derive a measure on the moduli space of metrics and prove various properties of it. In this section, we denote the gauge group as $\mathcal{G} \equiv \mathrm{Diff} \ltimes \mathrm{Weyl}$. Recall the string S-matrix of n external asymptotic states

$$S_{j_1,\dots j_n}(k_1,\dots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int \frac{[\mathcal{D}\phi\mathcal{D}g]}{\mathrm{Vol}(\mathcal{G})}\exp(-S_\mathrm{m}-\lambda\chi)\prod_{i=1}^{n}\int_{\Sigma_g} d^2\sigma_i\sqrt{g(\sigma_i)}V_{j_i}(\sigma_i,k_i)\,, \tag{7.2.89}$$

where Σ_g is the worldsheet of genus g and the sum is over the genus in case of oriented worldsheet without boundary. After gauge fixing, $\mathcal{D}g$ changes to a measure $\mathcal{D}\zeta$ on the gauge groups times a measure $d^\mu t$ on the moduli space which is μ-dimensional. Moreover the integral $d^{2n}\sigma$ on the worldsheet for the n vertex operators can be used to fix the CKG. Suppose CKG be κ dimensional. Then we can write

$$[\mathcal{D}g]d^{2n}\sigma \to [\mathcal{D}\zeta]d^\mu t d^{2n-\kappa}\sigma$$

To implement this in the path integral, we need to introduce a Jacobian for this variable change - the Fadeev-Popov determinant $\Delta_{\mathrm{FP}}(g,\sigma)$. This Jacobian is defined as follows: to fix gauge we choose a metric $\hat{g}(t)^\zeta$ from each orbit of $\mathcal{G}$ depending on the moduli t. Moreover k of the vertex operators are also fixed $\sigma_i^\alpha \to \hat{\sigma}_i^\alpha$. Let $(\alpha,i)\in f$ be the indices of fixed coordinates. Then

$$1 = \Delta_{\mathrm{FP}}(g,\sigma)\int_{\mathcal{G}_g/\mathcal{G}} d^\mu t\int_{\mathcal{G}}[\mathcal{D}\zeta]\delta\left(g-\hat{g}(t)\right)\prod_{(\alpha,i)\in f}\delta\left(\sigma_i^\alpha-\hat{\sigma}_i^{\zeta\alpha}\right) \tag{7.2.90}$$

where $\mathcal{G}_g$ denotes the space of all metrics on a genus g Riemann surface (worldsheet). Since every metric on worlsheet is $\mathcal{G}$-equivalent to $\hat{g}(t)^\zeta$ for some moduli t and gauge parameter ζ, the δ-function picks out a unique value of ζ upto some discrete elements of CKG. As for the discrete CKG, there are finitely many say n_R discrete symmetries which do not change $\hat{g}(t)^\zeta$. Thus the second delta function is nonzero at n_R points. To fix them, we just divide by n_R.

Plugging this into the path integral we obtain

$$S_{j_1,\ldots j_n}(k_1,\ldots,k_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \int_{\mathcal{M}_g} d^\mu t \Delta_{\mathrm{FP}}(\hat{g}(t),\hat{\sigma}) \int [\mathcal{D}\phi] \int \prod_{(\alpha,i)\notin f} d\sigma_i^\alpha \exp\left(-S_{\mathrm{m}}[\phi,\hat{g}(\sigma_i)] - \lambda\chi\right) \times \prod_{i=1}^{n} \sqrt{\hat{g}(\sigma_i)} V_{j_i}(\sigma_i,k_i), \tag{7.2.91}$$

where $\mathcal{M}_g$ denotes the moduli space $\mathcal{G}_g/\mathcal{G}$ of the worldsheet Σ_g. Now we evaluate $\Delta_{\mathrm{FP}}(\hat{g}(t),\hat{\sigma})$. Let $t^1,\ldots,t^\mu$ be components of coordinate t on $\mathcal{M}_g$. Then a generic infinitesimal variation of the metric $g_{\alpha\beta}$ is

$$\delta g_{\alpha\beta} = \sum_{k=1}^{\mu} \delta t^k \partial_{t^k} \hat{g}_{\alpha\beta} - 2\left(\hat{P}_1 \delta\sigma\right)_{\alpha\beta} + (2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g}_{\alpha\beta} \tag{7.2.92}$$

where hat on $\hat{P}_1$ and $\hat{\nabla}$ indicates that the metric in the definition of these operates is $\hat{g}_{\alpha\beta}$. Thus if $g_{\alpha\beta}$ is close to $\hat{g}_{\alpha\beta}(t)$ then we can write

$$\int_{\mathcal{G}} [\mathcal{D}\zeta] \delta\left(g - \hat{g}(t)^\zeta\right) = \int [\mathcal{D}\delta\omega \mathcal{D}\delta\sigma] \delta\left(\delta g_{\alpha\beta}\right). \tag{7.2.93}$$

Thus

$$\Delta_{\mathrm{FP}}^{-1}(\hat{g},\hat{\sigma})^{-1} = n_R \int d^\mu \delta t [\mathcal{D}\delta\omega \Delta\delta\sigma] \delta\left(\delta g_{\alpha\beta}\right) \prod_{(\alpha,i)\in f} \delta\left(\delta\sigma^\alpha(\hat{\sigma}_i)\right) \tag{7.2.94}$$

where the n_R factor takes care of the discrete symmetries. Now as in our calculation of Fadeev-Popov measure in Subsection 7.1.1 we again replace δ functions by integrals:

$$\begin{aligned} \delta\left(\delta g_{\alpha\beta}\right) &= \int [\mathcal{D}\beta] \exp\left(2\pi i \int d^2\sigma \sqrt{g(\sigma)} \beta^{\alpha\beta} \delta g_{\alpha\beta}\right) \\ &= \int [\mathcal{D}\beta] \exp[2\pi i \langle \beta, \delta g\rangle] \\ \delta\left(\delta\sigma_i^\alpha\right) &= \int d^\kappa x_{\alpha i} \exp\left(2\pi i x_{\alpha i} \delta\sigma^\alpha(\hat{\sigma}_i)\right). \end{aligned} \tag{7.2.95}$$

Plugging this in we get

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g},\hat{\sigma}] = n_R \int d^\mu \delta t [\mathcal{D}\delta\omega \mathcal{D}\delta\sigma \mathcal{D}\beta] \exp\left(2\pi i \int d^2\sigma \sqrt{g(\sigma)} \beta^{\alpha\beta} \delta g_{\alpha\beta}\right) \times \prod_{(\alpha,i)\in f} \int dx_{\alpha i} \exp\left(2\pi i x_{\alpha i} \delta\sigma^\alpha(\hat{\sigma}_i)\right). \tag{7.2.96}$$

The path integral over $\delta\omega$ can be easily performed now. In the integrand it only appears as

$$\int [\mathcal{D}\delta\omega] \exp\left(2\pi i \int d^2\sigma \sqrt{\hat{g}(\sigma)} \beta^{\alpha\beta} (2\delta\omega) \hat{g}_{\alpha\beta}\right) = \delta\left(2\beta^{\alpha\beta}\hat{g}_{\alpha\beta}\right). \tag{7.2.97}$$

Thus the path integral over $\delta\omega$ forces $\beta^{\alpha\beta}$ to be traceless:

$$\beta^{\alpha\beta}\hat{g}_{\alpha\beta} = 0. \tag{7.2.98}$$

So we remove the $\mathcal{D}\delta\omega$ from $\Delta_{\mathrm{FP}}^{-1}[\hat{g},\hat{\sigma}]$ and take $\beta \to \beta'$ where β' is traceless. This also forces the term in $\delta g_{\alpha\beta}$

$$\beta'^{\alpha\beta}(\hat{\nabla}\cdot\delta\sigma)\hat{g}_{\alpha\beta} = 0. \tag{7.2.99}$$

Thus we get

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g},\hat{\sigma}] = n_R \int d^\mu \delta t d^\kappa x \left[\mathcal{D}\beta'\mathcal{D}\delta\sigma\right] \exp\left[2\pi i \left\langle \beta', 2\hat{P}_1\delta\sigma - \delta t^k \partial_{t^k}\hat{g}\right\rangle + 2\pi i \sum_{(\alpha,i)\in f} x_{\alpha i}\delta\sigma^\alpha\left(\hat{\sigma}_i\right)\right] \tag{7.2.100}$$

where k is summed over. To invert $\Delta_{\mathrm{FP}}^{-1}[\hat{g},\hat{\sigma}]$ to get $\Delta_{\mathrm{FP}}[\hat{g},\hat{\sigma}]$, we just replace bosonic variables by Grassmann variables as before:

$$\begin{aligned} \delta\sigma^\alpha &\longrightarrow c^\alpha \\ \beta'_{\alpha\beta} &\longrightarrow b_{\alpha\beta} \\ x_{\alpha i} &\longrightarrow \eta_{\alpha i} \\ \delta t^k &\longrightarrow \xi^k. \end{aligned} \tag{7.2.101}$$

We can normalise the Grassmann variables such that the Fadeev-Popov measure takes the form

$$\Delta_{\mathrm{FP}}^{-1}[\hat{g},\hat{\sigma}] = \frac{1}{n_R}\int [\mathcal{D}b\mathcal{D}c] d^\mu\xi d^k\eta \exp\left[-\frac{1}{4\pi}\left\langle b, 2\hat{P}_1 c - \xi^k\partial_{t^k}\hat{g}\right\rangle + \sum_{(\alpha,i)\in f}\eta_{\alpha i}c^\alpha\left(\hat{\sigma}_i\right)\right]. \tag{7.2.102}$$

Integrating over ξ and η, we get

$$\Delta_{\mathrm{FP}}[\hat{g},\hat{\sigma}] = \frac{1}{n_R}\int [\mathcal{D}b\mathcal{D}c] \exp\left(-S_{gh}\right) \prod_{k=1}^{\mu} \frac{1}{4\pi}\left\langle b, \partial_k\hat{g}\right\rangle \prod_{(\alpha,i)\in f} c^\alpha\left(\hat{\sigma}_i\right) \tag{7.2.103}$$

where

$$S_{gh} = \frac{1}{2\pi}\left\langle b, \hat{P}_1 c\right\rangle \tag{7.2.104}$$

and we used the integral

$$\int \left[\mathcal{D}\xi^k\right] \exp\left(\frac{1}{4\pi}\xi^k\left\langle b, \partial_k\hat{g}\right\rangle\right) = \frac{1}{4\pi}\left\langle b, \partial_k\hat{g}\right\rangle. \tag{7.2.105}$$

Thus the gauge fixed amplitude takes the form

$$S_{j_1}\dots j_n\left(k_1,\dots,k_n\right)=\sum_{\substack{\text{compact}\\ \text{topologies}\\ \{g\}}}\int_{\mathcal{M}_g}\frac{d^\mu t}{n_R}\int\left[\mathcal{D}\phi\mathcal{D}b,\mathcal{D}c\right]\exp\left(-S_{\mathrm{m}}-S_{gh}-\lambda\chi\right)\times$$
$$\left.\prod_{(\alpha,i)\notin f}\int_{\Sigma_g}d\sigma_i^\alpha\prod_{k=1}^{\mu}\frac{1}{4\pi}\left\langle b,\partial_k\hat{g}\right\rangle\prod_{(\alpha,i)\in f}c^\alpha\left(\hat{\sigma}_i\right)\prod_{i=1}^{n}\sqrt{\hat{g}\left(\sigma_i\right.}\right)V_{j_i}\left(k_i,\sigma_i\right).\tag{7.2.106}$$

Simplifying the Fadeev-Popov determinant

We want to simplify the path integral expression for Δ_{FP} in (7.2.103). Here, we will express Δ_{FP} in terms of functional determinant. Recall that the operator P_1 maps vectors to symmetric, traceless rank 2 tensors and P_1^T is a map in opposite direction. In particular $P_1^TP_1$ and $P_1P_1^T$ are symmetric operators on the space of vectors on symmetric, traceless rank 2 tensors. By spectral theorem, they can be diagonalised and there exists complete orthonormal sets $\{C_J^\alpha\}$ and $\{B_{K\alpha\beta}\}$ for the two spaces with respect to the inner product defined in (7.2.67). More explicitly,

$$P_1^TP_1C_J^\alpha=\nu_J'^2C_J^\alpha,\quad P_1P_1^TB_{K\alpha\beta}=\nu_K^2B_{K\alpha\beta}\tag{7.2.107}$$

for real numbers ν_K',ν_K and

$$\begin{aligned}\left\langle C_J,C_{J'}\right\rangle&=\int d^2\sigma\sqrt{g}C_J^\alpha C_{J'\alpha}=\delta_{JJ'},\\ \left\langle B_K,B_{K'}\right\rangle&=\int d^2\sigma\sqrt{g}B_K^{\alpha\beta}B_{K'\alpha\beta}=\delta_{KK'}.\end{aligned}\tag{7.2.108}$$

We claim that there is a one-to-one correspondence between the eigenfunction of $P_1P_1^T$ and $P_1^TP_1$ with nonzero eigenvalue. Indeed if $P_1^TP_1C_J=\nu_J'^2C_J$ then

$$P_1P_1^T\left(P_1C_J\right)=\nu_J'^2P_1C_J\tag{7.2.109}$$

so $C_J\mapsto P_1C_J$ and similarly $B_K\longmapsto P_1^TB_K$. We will thus identify these functions as

$$B_{K\alpha\beta}=\frac{1}{\nu_K}\left(P_1C_K\right)_{\alpha\beta},\quad \nu_K=\nu_K'\neq0\tag{7.2.110}$$

Let B_{0K} and C_{0J} be the eigenvalue 0 eigenvectors of $P_1P_1^T$ and $P_1^TP_1$. Note that the C_{0j} are elements of CKG since it corresponds to solutions of $P_1C=0$ and B_{0K} are elements of moduli space of metrics since it corresponds to $P_1^TB=0$, see discussion around (7.2.77). Thus there are μ and κ number of C_{0j} and B_{0k} respectively. We now expand the ghost fields as

$$
\begin{aligned}
c^\alpha(\sigma) &= \sum_J c_J C_J^\alpha(\sigma) = \sum_{j=1}^{\mu} C_{0j} C_{0j}^\alpha(\sigma) + \sum_{J\neq 0} c_J C_J^\alpha(\sigma), \\
b_{\alpha\beta}(\sigma) &= \sum_K b_K B_{K\alpha\beta}(\sigma) = \sum_{k=1}^{\kappa} b_{0k} B_{0k\alpha\beta}(\sigma) + \sum_{K\neq 0} b_K B_{K\alpha\beta}(\sigma),
\end{aligned}
\tag{7.2.111}
$$

Here c_J and b_K are Grassmann odd variables. Plugging this in the ghost action, we get

$$
\begin{aligned}
S_{gh} &= \frac{1}{2\pi}\langle b, P_1 c\rangle \\
&= \frac{1}{2\pi}\left\langle \sum_{k=1}^{\kappa} b_{0k} B_{0k}(\sigma) + \sum_{K\neq 0} b_K B_K(\sigma), \sum_{j=1}^{\mu} c_{0j} P_1 C_{0j}(\sigma) + \sum_{J\neq 0} c_J P_1 C_J(\sigma)\right\rangle \\
&= \frac{1}{2\pi}\sum_{J,K} b_K c_J \langle B_K, P_1 C_J\rangle \\
&= \frac{1}{2\pi}\sum_{J,K} \frac{b_K c_J}{\nu_K}\langle P_1 C_K, P_1 C_J\rangle \\
&= \frac{1}{2\pi}\sum_{J,K} \frac{b_k c_J}{\nu_K}\left\langle C_K, P_1^T P_1 C_J\right\rangle \\
&= \frac{1}{2\pi}\sum_J \nu_J b_J c_J
\end{aligned}
\tag{7.2.112}
$$

where we used the fact that $P_1 C_{0j} = 0 = P_2^T B_{0k}$ in the third step. The ghost path integral for $\Delta_{\rm FP}$ becomes

$$
\Delta_{\rm FP} = \int \prod_{k=1}^{\kappa} db_{0k} \prod_{j=1}^{\mu} dc_{0j} \prod_{J\neq 0} db_J dc_J \exp\left(-\frac{\nu_J b_J c_J}{2\pi}\right) \prod_{k'=1}^{\mu} \frac{1}{4\pi}\langle b, \partial_{k'}\hat{g}\rangle \prod_{(\alpha,i)\in f} c^\alpha(\sigma_i)\,.
\tag{7.2.113}
$$

This path integral vanishes unless there are μ number of b_{0k} and κ number of c_{0j} in the integrand. These ghost zeromodes can come only from vertex operator insertions $c^\alpha(\sigma_i)$ and $\langle b, \partial_{k'}\hat{g}\rangle$. Note that $(\alpha, i) \in f$ runs over the conformal killing vectors which are κ in number.

Thus we see that we can write

$$
\begin{aligned}
\prod_{(\alpha,i)\in f} c^{\alpha}\left(\sigma_i\right) &= \prod_{(\alpha,i)\in f}\left[\sum_{j'=1}^{\kappa} c_{0j'}C^{\alpha}_{0j'}\left(\sigma_i\right)+\sum_{J\neq 0} c_J C^{\alpha}_{J}\left(\sigma_i\right)\right] \\
&\rightarrow \prod_{(\alpha,i)\in f}\left[\sum_{j'=1}^{k} C_{0j'}C^{\alpha}_{0j'}\left(\sigma_i\right)\right] \\
&= \prod_{j'=1}^{\kappa} c_{0j'} \prod_{(\alpha,i)\in f}\left[\sum_{P\in S_\kappa} \operatorname{sign}(P) C^{\alpha}_{0P(1)}\left(\sigma_i\right) C^{\alpha}_{0P(2)}\left(\sigma_i\right)\ldots C^{\alpha}_{0P(\kappa)}\left(\sigma_i\right)\right] \\
&= \prod_{j=1}^{\kappa} c_{0j'} \operatorname{Det}\left[C^{\alpha}_{0j}\left(\sigma_i\right)\right]
\end{aligned}
\tag{7.2.114}
$$

Other terms in the product give vanishing path integral due to lack of enough (exactly κ) number of ghost zero modes c_{0j}. Here S_κ is the permutation group on κ letters. Note that $\operatorname{Det}\left[C^{\alpha}_{0j}\left(\sigma_i\right)\right]$ makes sense since $(\alpha, i)\in f$ runs over κ values, so does j and hence $C^{\alpha}_{0j}\left(\sigma_i\right)$ is a square matrix. For the same reason

$$
\begin{aligned}
\prod_{k'=0}^{\mu} \frac{1}{4\pi}\left\langle b, \partial_{k'}\hat{g}\right\rangle &= \prod_{k'=1}^{\mu}\left[\sum_{k''=1}^{\mu}\frac{b_{0k''}}{4\pi}\left\langle B_{0k''}, \partial_{k'}\hat{g}\right\rangle\right]. \\
&= \prod_{k'=1}^{\mu} b_{0k} \operatorname{Det}\left[\frac{\left\langle B_{0k}, \partial_{k''}\hat{g}\right\rangle}{4\pi}\right]
\end{aligned}
\tag{7.2.115}
$$

The path integral over c_J, b_J is easily done using

$$
\int d\theta_1 d\theta_2 \exp\left(a\theta_1\theta_2\right) = a. \tag{7.2.116}
$$

Thus the path integral becomes

$$
\begin{aligned}
\Delta_{\mathrm{FP}} &= \prod_{J}\frac{\nu_J}{2\pi}\operatorname{Det}\left[C^{\alpha}_{0j}\left(\sigma_i\right)\right]\operatorname{Det}\left[\frac{\left\langle b_{0k}, \partial_{k''}\hat{g}\right\rangle}{4\pi}\right] \\
&= \operatorname{Det}\left[C^{\alpha}_{0j}\left(\sigma_i\right)\right]\operatorname{Det}\left[\frac{\left\langle b_{0k}, \partial_{k''}\hat{g}\right\rangle}{4\pi}\right]\operatorname{Det}'\left[\sqrt{\frac{P_1^T P_1}{4\pi^2}}\right]
\end{aligned}
\tag{7.2.117}
$$

where we used the fact that ν_J^2 are non-zero eigenvalues of $P_1^T P_1$. Also Det' indicates that zero eigenvalues are ignored.

7.3 Tree Level Amplitudes

7.4 BRST Quantisation and No-Ghost Theorem

In this section, we finally complete the picture sketched in Section 3.3.7. We start by discussing BRST quantisation of the string and then prove the no-ghost theorem.

7.4.1 Generalities on BRST quantisation

Let us take a QFT with fields $\{\phi_i\}$ and action $S[\phi_i]$. Also suppose the theory has gauge symmetry under which the fields transform as

$$\delta\phi_i = \epsilon^\alpha \delta_\alpha \phi_i, \tag{7.4.1}$$

where δ_α are the generators of gauge transformations and ϵ_α are the real parameters of the infinitesimal transformation. Suppose $f^\gamma_{\alpha\beta}$ be the structure constants of the gauge algebra:

$$[\delta_\alpha, \delta_\beta] = f^\gamma_{\alpha\beta}\delta_\gamma. \tag{7.4.2}$$

Consider now the path integral

$$Z = \int [D\phi_i] \exp\left(-S\left[\phi_i\right]\right) \tag{7.4.3}$$

of the theory. As already discussed, the path integral over-counts the physical degrees of freedom since gauge equivalent fields are physically equivalent. Thus we need to fix the gauge such that we choose only one representative from each gauge orbit. This is equivalent to dividing by the volume of the gauge group V_{gauge}. So the physical path integral is

$$Z = \int \frac{[D\phi_i]}{V_{\text{gauge}}} \exp\left(-S\left[\phi_i\right]\right). \tag{7.4.4}$$

In general gauge fixing can be represented by a function

$$F^A\left(\phi_i\right) = 0. \tag{7.4.5}$$

Then we perform the Faddeev-Popov gauge fixing by introducing ghosts and a gauge fixing term in the action:

$$\int \frac{[D\phi_i]}{V_{\text{gauge}}} \exp\left(-S\left[\phi_i\right]\right) \longrightarrow \int [D\phi_i DB_A db_A dc^\alpha] \exp\left(-S\left[\phi_i\right] - S_{\text{gf}}[B] - S_{\text{ghost}}[b,c]\right), \tag{7.4.6}$$

where

$$S_{\text{gf}}[B] := -iB_A F^A(\phi) \tag{7.4.7}$$

is the gauge fixing term and

$$S_{\text{ghost}} := b_A C^\alpha \delta_\alpha F^A(\phi). \tag{7.4.8}$$

is the Faddeev-Popov ghost action. The path integral on B_A produces a delta function $\delta\left(F^A\right)$ which fixes the gauge. The gauge fixed action $S[\phi_i] + S_{\text{gf}} + S_{\text{ghost}}$ is invariant under the BRST (Becchi-Rouet-Stora-Tyutin) transformation generated by an operator δ_B :

$$\begin{aligned}
\delta_B\phi_i &= -i\epsilon c^\alpha \delta_\alpha \phi_i \\
\delta_B B_A &= 0, \\
\delta_B b_A &= \epsilon B_A \\
\delta_B c^\alpha &= \frac{i}{2}\epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma
\end{aligned} \tag{7.4.9}$$

where ϵ is an anti-commuting parameter for the infinitesimal transformation. Let us check that the action is indeed invariant under BRST transformation. The fact that ϵ is grassmann odd follows from the fact that BRST transformation maps bosonic and fermionic fields into each other. We have

$$\delta_B S[\phi_i] = 0 \tag{7.4.10}$$

since $\delta_B \phi_i$ is just a gauge transformation with gauge parameter $i\epsilon c^\alpha$. Next

$$\begin{aligned} \delta_B S_{\text{gf}} &= -i\left(\delta_B B_A\right) F^A(\phi) - iB_A \delta_B F^A(\phi) \\ &= -iB_A\left(-i \in c^\alpha \delta_\alpha F^A(\phi)\right) \\ &= -\epsilon B_A c^\alpha \delta_\alpha F^A(\phi). \end{aligned} \tag{7.4.11}$$

Note that since F^A only depends on ϕ_i

$$\delta_B F^A = -i\epsilon c^\alpha \delta_\alpha F^A(\phi). \tag{7.4.12}$$

Finally

$$\begin{aligned} \delta_B S_{\text{ghost}} &= \left(\delta b_A\right) c^\alpha \delta_\alpha F^A(\phi) + b_A(\delta_B c^\alpha)\delta_\alpha F^A(\phi) + b_A c^\alpha \delta_\alpha(\delta_B F^A(\phi)) \\ &= \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + \frac{i}{2} b_A \epsilon f^\alpha_{\beta\gamma} c^\beta c^\gamma \delta_\alpha F^A(\phi) - i\epsilon b_A c^\alpha c^\beta \delta_\alpha \delta_\beta F^A(\phi) \\ &= \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + \frac{i\epsilon}{2} b_A c^\beta c^\gamma \left[\delta_\beta, \delta_\gamma\right] F^A(\phi) - i\epsilon b_A c^\alpha c^\beta \delta_\alpha \delta_\beta F^A(\phi) \\ &= \epsilon B_A c^\alpha \delta_\alpha F^A(\phi) + i\epsilon b_A c^\beta c^\gamma \delta_\beta \delta_\gamma F^A(\phi) - i\epsilon b_A c^\alpha c^\beta \delta_\alpha \delta_\beta F^A(\phi) \\ &= \epsilon B_A c^\alpha \delta_\alpha F^A(\phi), \end{aligned} \tag{7.4.13}$$

where we used the fact that $c^\alpha c^\beta = -c^\beta c^\alpha$. Combining (7.4.10), (7.4.12) and (7.4.13), we see that

$$\delta_B(S[\phi_i] + S_{\text{gf}} + S_{\text{ghost}}) = 0. \tag{7.4.14}$$

Using the Noether procedure, one can compute the conserved current and charge. Let the conserved charge be Q_B. There is a global $U(1)$ symmetry of the gauge-fixed action:

$$\begin{aligned} b_A &\longrightarrow e^{-i\theta} b_\alpha \\ c_\alpha &\longrightarrow e^{i\theta} b_\alpha \end{aligned} \tag{7.4.15}$$

and all other fields neutral. The corresponding conserved charge is called the *ghost number*. So b_A, c_α has ghost number -1 and 1 respectively and 0 for other fields. To match the ghost number on both sides of (7.4.9), we assign ghost number -1 to the BRST parameter ϵ.

This is the classical description of BRST symmetry. Now, in the quantum theory, Q_B becomes an operator and acts on the asymptotic initial and final states $|i\rangle, |f\rangle$ respectively. After quantisation

$$\delta_B \Phi = i\epsilon[Q_B, \Phi]_\pm \tag{7.4.16}$$

where $\pm$ indicates that the right hand side is a commutator or anitcommutator depending on whether Φ is bosonic or fermionic operator. This also implies that Q_B has ghost number 1.

The gauge fixing condition $F^A(\phi) = 0$ is unphysical. We can do the same procedure by changing the gauge fixing condition to $F^A + \delta F^A$. Then the inner product of asymptotic states changes by

$$\begin{aligned} \epsilon\delta\langle f \mid i\rangle &= i\left\langle f\left|\delta_B\left(b_A\delta F^A\right)\right| i\right\rangle \\ &= -\epsilon\left\langle f\left|\left\{Q_B, b_A\delta F^A\right\}\right| i\right\rangle . \end{aligned} \tag{7.4.17}$$

Here we used the path-integral representation of asymptotic state (give details). This unphysicality of gauge fixing condition implies that

$$\epsilon\delta\langle f \mid i\rangle = 0 \tag{7.4.18}$$

which implies that

$$\left\langle\psi\left|\left\{Q_B, b_A\delta F^A\right\}\right|\psi'\right\rangle = 0 \tag{7.4.19}$$

for physical states $|\psi\rangle$ and $|\psi'\rangle$. This condition for arbitrary δF^A implies

$$Q_B|\psi\rangle = Q_B\left|\psi'\right\rangle = 0 \tag{7.4.20}$$

Here we have assumed that $Q_B^\dagger = Q_B$. This gives us the aphorism: *physical states must be* BRST-*invariant*.

Proposition 7.4.1. *The* BRST *charge is nilpotent*

$$Q_B^2 = 0. \tag{7.4.21}$$

Proof. We must be able to freely change the gauge choices without changing anything physical. This means that Q_B must be conserved and hence must commute with the Hamiltonian (?)[5]. This implies

$$\left[Q_B, \left\{Q_B, b_A\delta F^A\right\}\right] = 0 \tag{7.4.22}$$

which implies

$$\begin{aligned} &Q_B^2 b_A\delta F^A + Q_B b_A\delta F^A Q_B - Q_B b_A\delta F^A Q_B - b_A\delta F^A Q_B^2 = 0 \\ \Longrightarrow\ &\left[Q_B^2, b_A\delta F^A\right] = 0. \end{aligned} \tag{7.4.23}$$

This requirement for arbitrary δF^A implies that $Q_B^2 =$ constant. But note that Q_B^2 has ghost number 2 while any nonzero constant has ghost number zero. □

[5]Recall that the Hamiltonian generates time translation and hence any conserved quantity must commute with the Hamiltonian

Alternatively, one can check the action of δ_β twice on all fields. For example

$$\begin{aligned}\delta_B\left(\delta'_B c^\alpha\right) &= \delta_B\left(\frac{i}{2}\epsilon f^\alpha_{\beta\gamma}c^\beta c^\gamma\right)\\ &= -\frac{\epsilon\epsilon'}{4}\left[f^\alpha_{\beta\gamma}f^\beta_{\sigma\tau}c^\sigma c^\tau c^\gamma + f^\alpha_{\beta\gamma}f^\gamma_{\sigma\tau}c^\beta c^\sigma c^\tau\right]\\ &= -\frac{\epsilon\epsilon'}{4}\left[f^\alpha_{\beta\gamma}f^\beta_{\sigma\tau}c^\gamma c^\sigma c^\tau - f^\alpha_{\gamma\beta}f^\gamma_{\sigma\tau}c^\beta c^\sigma c^\tau\right]\\ &= 0,\end{aligned} \tag{7.4.24}$$

where we used antisymmetry of structure constants and renamed the dummy indices in the second term to cancel it with first. Similarly one can check that $\delta_B\delta'_B = 0$ on all other fields.

Remark 7.4.2. In the gauge algebra, we assumed that the structure constants do not depend on the fields and that the RHS of does not contain terms proportional to the equations of motion and hence would vanish onshell. These assumptions break down in general and in such a case one needs the more general Batalin-Vilkoviski (BV) formalism. This formalism is particularly useful for string field theory which we will discuss later.

Since $Q_B^2 = 0$, one can look at the Q_B-cohomology called the BRST cohomology as we explain below. Let $\mathscr{H}$ be the string Hilbert space. Define the subspace $\mathscr{H}_{\text{closed}} \subseteq \mathscr{H}$ as follows:

$$\mathscr{H}_{\text{closed}} := \{|\psi\rangle \in \mathscr{H} : Q_B|\psi\rangle = 0\} \tag{7.4.25}$$

and the subspace $\mathscr{H}_{\text{exact}} \subseteq \mathscr{H}$ as

$$\mathscr{H}_{\text{exact}} := \{|\psi\rangle \in \mathscr{H} : |\psi\rangle = Q_B\,|\psi'\rangle \text{ for some } |\psi'\rangle \in \mathscr{H}\}\,. \tag{7.4.26}$$

These subspaces are called Q_B-closed and Q_B-exact subspaces respectively. Clearly $\mathscr{H}_{\text{exact}} \subseteq \mathscr{H}_{\text{closed}}$ because $Q_B^2 = 0$. Then we define the BRST cohomology as the quotient space

$$\mathscr{H}_{\text{BRST}} := \frac{\mathscr{H}_{\text{closed}}}{\mathscr{H}_{\text{exact}}}. \tag{7.4.27}$$

Theorem 7.4.3. *The physical string Hilbert space*

$$\mathscr{H}_{\text{phy}} \cong \mathscr{H}_{\text{BRST}}. \tag{7.4.28}$$

Proof. As already discussed above, physical states $|\psi\rangle \in \mathscr{H}_{\text{phy}}$ must satisfy $Q_B|\psi\rangle = 0$. This means that $|\psi\rangle \in \mathscr{H}_{\text{closed}}$. Next consider a state of the form $Q_B|\chi\rangle$. We claim that this is a null state. Clearly $Q_B|\chi\rangle \in \mathscr{H}_{\text{phy}}$ since $Q_B^2 = 0$ and for any $|\psi\rangle \in \mathscr{H}_{\text{phy}}$

$$\langle\psi\,|Q_B|\,\chi\rangle = \left(\langle\psi|Q_B^\dagger\right)|\chi\rangle = 0. \tag{7.4.29}$$

These states are modded out in $\mathscr{H}_{\text{phy}}$. Thus

$$\mathscr{H}_{\text{phy}} \cong \frac{\mathscr{H}_{\text{closed}}}{\mathscr{H}_{\text{exact}}} = \mathscr{H}_{\text{BRST}}. \tag{7.4.30}$$

Recall from Eq. 6.1.15 that

$$\langle f \mid i\rangle = \int [D\phi_i]_i^f \exp\left(-S - S_{gf} - S_{\text{ghost}}\right) \tag{7.4.31}$$

where $[D\phi_i]_i^f$ is short to denote that the fields have to satisfy boundary conditions corresponding to states $|i\rangle$ and $|f\rangle$. □

There is a key issue that we need to address. In many cases, as in strings, gauge fixing does not completely fix the gauge. There is residual symmetry after a gauge has been fixed. For example, in strings case, fixing the worldsheet metric to a fiducial metric leaves us with residual conformal symmetry. In such cases we are required to impose the constraints arising from the residual symmetry. More precisely, let G^I be the generators of the residual symmetry. These are called constraints. These satisfy an algebra

$$[G_I, G_J] = i f_{IJ}^K G_K. \tag{7.4.32}$$

One has to impose the constraint on the BRST Hilbert space that the matrix elements of these generator vanish:

$$\langle\psi\,|G_I|\,\psi'\rangle = 0, \quad |\psi\rangle, |\psi'\rangle \in \mathscr{H}_{\text{BRST}}\,. \tag{7.4.33}$$

For each generator G_I, there are a pair of ghosts c^I and b_J satisfying

$$\left\{c^I, b_J\right\} = \delta_J^I, \quad \left\{c^I, c^J\right\} = \{b_I, b_J\} = 0. \tag{7.4.34}$$

The BRST charge has the general form

$$\begin{aligned} Q_B &= c^I G_I^{\text{m}} - \frac{i}{2} f_{IJ}^K c^I c^J b_K \\ &\equiv c^I\left(G_I^{\text{m}} + \frac{1}{2} G_I^{\text{g}}\right) \end{aligned} \tag{7.4.35}$$

where G_I^{m} is the matter part of G_I and

$$G_I^{\text{g}} = -i f_{IJ}^K c^J b_K \tag{7.4.36}$$

is the ghost part and they satisfy the same algebra as (7.4.32). Using the GGG Jacobi identity, we have

$$Q_B^2 = \frac{1}{2}\{Q_B, Q_B\} = -\frac{1}{2} f_{IJ}^K f_{KL}^M c^I c^J c^K b_M = 0. \tag{7.4.37}$$

More generally, one might also have central terms in the constraint algebra, for example in the Virasoro algebra. In that case, we will need to check that Q_B squares to zero with the contribution of the central terms.

7.4.2 BRST quantisation of point particle

Recall that the point particle action is given by

$$S = \int dt \left(\frac{1}{2}e^{-2}\dot{x}^\mu \dot{x}_\mu + \frac{1}{2}em^2\right) \tag{7.4.38}$$

where e is an auxuliary field (einbein). The gauge symmetry of the theory is reparametrization invariance $\tau \to \widetilde{\tau}(\tau)$ under which X^μ is invariant $\widetilde{X}^\mu(\widetilde{\tau}) = X^\mu(\tau)$ while the einbein acts as a metric on the worldline so that

$$\widetilde{e}(\tau)d\widetilde{\tau} = e(\tau)d\tau \tag{7.4.39}$$

So the index α in (7.4.1) is the coordinate τ. Let us now find the infinitesimal transformations of the field. Taking $\widetilde{\tau}(\tau) = \tau + \varepsilon(\tau)$ for some infinitesimal parameter ε, we see that

$$\begin{aligned} \delta X^\mu(\tau) &= \widetilde{X}^\mu(\widetilde{\tau}) - X^\mu(\widetilde{\tau}) \\ &= X^\mu(\tau) - X^\mu(\tau + \varepsilon(\tau)) \\ &= -\varepsilon(\tau)\partial_\tau X^\mu(\tau) + O\left(\varepsilon^2\right). \end{aligned} \tag{7.4.40}$$

Note that in the calculation of δX^μ, it is important that we evaluate both $\widetilde{X}$ and X at τ or $\widetilde{\tau}$. To compare with (7.4.9), we perform some manipulations. We write

$$\begin{aligned} \delta X^\mu(\tau) &= -\int d\tau_1 \delta\left(\tau - \tau_1\right)\varepsilon\left(\tau_1\right)\partial_\tau X^\mu(\tau) \\ &= \int d\tau_1 \varepsilon^{\tau_1}\left(-\delta\left(\tau - \tau_1\right)\partial_\tau X^\mu(\tau)\right) \\ &\equiv \varepsilon^{\tau_1}\delta_{\tau_1} X^\mu(\tau) \end{aligned} \tag{7.4.41}$$

where we defined $\varepsilon^{\tau_1} \equiv \varepsilon\left(\tau_1\right)$ and the contraction of τ_1 is the integral over τ_1. Thus

$$\delta_{\tau_1} X^\mu(\tau) = -\delta\left(\tau - \tau_1\right)\partial_\tau X^\mu(\tau). \tag{7.4.42}$$

For the einbein we have

$$e(\tau) = \tilde{e}(\tilde{\tau})\frac{d\tilde{\tau}}{d\tau} = \tilde{e}(\tilde{\tau})\left[1 + \partial_\tau \varepsilon(\tau)\right] \tag{7.4.43}$$

Thus

$$\begin{aligned} \delta e(\tau) &= \tilde{e}(\tilde{\tau}) - e(\tilde{\tau}) \\ &= \tilde{e}(\tilde{\tau}) - e(\tau + \varepsilon(\tau)) \\ &= \tilde{e}(\tilde{\tau}) - e(\tau) - \varepsilon(\tau)\partial_\tau e(\tau) + O\left(\varepsilon^2\right) \\ &= e(\tau)\left[1 + \partial_\tau \varepsilon(\tau)\right]^{-1} - e(\tau) - \varepsilon(\tau)\partial_\tau e(\tau) + O\left(\varepsilon^2\right) \\ &= -e(\tau)\partial_\tau \varepsilon(\tau) - \varepsilon(\tau)\partial_\tau e(\tau) + O\left(\varepsilon^2\right) \\ &= -\int d\tau_1 \varepsilon^{\tau_1}\partial_\tau\left[\delta\left(\tau - \tau_1\right)e(\tau)\right] \\ &\equiv \varepsilon^{\tau_1}\delta_{\tau_1} e(\tau) \end{aligned} \tag{7.4.44}$$

so that

$$\delta_{\tau_1} e(\tau) = -\partial_\tau \left[\delta\left(\tau - \tau_1\right) e(\tau)\right]. \tag{7.4.45}$$

This shows that a basis of gauge algebra is δ_{τ_1} defined by

$$\delta_{\tau_1} \tau(\tau) = \delta\left(\tau - \tau_1\right). \tag{7.4.46}$$

We now calculate the structure constants of the gauge algebra. We have

$$\begin{aligned}
\left[\delta_{\tau_1}, \delta_{\tau_2}\right] X^\mu(\tau) &= -\delta_{\tau_1}\left(\delta\left(\tau - \tau_2\right) \partial_\tau X^\mu(\tau)\right) + \delta_{\tau_2}\left(\delta\left(\tau - \tau_1\right) \partial_\tau X^\mu(\tau)\right) \\
&= -\int d\tau_3 \delta\left(\tau - \tau_3\right)\left[\delta\left(\tau_3 - \tau_1\right) \partial_{\tau_3} \delta\left(\tau_3 - \tau_2\right) - \delta\left(\tau_3 - \tau_2\right) \partial_{\tau_3} \delta\left(\tau_3 - \tau_1\right)\right] \partial_\tau X^\mu(\tau) \\
&\equiv \int d\tau_3 f_{\tau_1 \tau_2}^{\tau_3} \delta_{\tau_3} X^\mu(\tau)
\end{aligned} \tag{7.4.47}$$

where we can easily recognize

$$f_{\tau_1 \tau_2}^{\tau_3} = \delta\left(\tau_3 - \tau_1\right) \partial_{\tau_3} \delta\left(\tau_3 - \tau_2\right) - \delta\left(\tau_3 - \tau_2\right) \partial_{\tau_3} \delta\left(\tau_3 - \tau_1\right). \tag{7.4.48}$$

Recall that we choose the gauge $e(\tau) = 1$. Thus the gauge fixing function is

$$F(\tau) = 1 - e(\tau). \tag{7.4.49}$$

The gauge fixed action is then

$$S_e = \int d\tau \left(\frac{1}{2} \frac{\dot{X}^\mu \dot{X}_\mu}{e} - \frac{1}{2} e m^2 + i B(e-1) - e \dot{b} c\right) \tag{7.4.50}$$

where equation of motion of the auxiliary field B fixes the gauge and b, c are reparametrization ghosts coming from Fadeev -Popov gauge fixing. From the general BRST transformation (7.4.9), we find the BRST transformation for the free particle to be

$$\begin{aligned}
\delta_B X^\mu &= -i \epsilon \int d\tau_1 c\left(\tau_1\right)\left(-\delta\left(\tau - \tau_1\right) \partial_\tau X^\mu(\tau)\right) \\
&= i \epsilon c \dot{X}^\mu. \\
\delta_B e(\tau) &= -i \epsilon \int d\tau_1 c\left(\tau_1\right)\left(-\partial_\tau \delta\left(\tau - \tau_1\right) e(\tau)\right) \\
&= i \epsilon \partial_\tau \int d\tau_1 c\left(\tau_1\right) e\left(\tau_1\right) \delta\left(\tau - \tau_1\right) \\
&= i \epsilon \partial_\tau(c e). \\
\delta_B c\left(\tau_1\right) &= \frac{i}{2} \epsilon \int d\tau_2 d\tau_3 f_{\tau_2 \tau_3}^{\tau_1} c\left(\tau_2\right) c\left(\tau_3\right) \\
&= i \epsilon c \dot{c}.
\end{aligned} \tag{7.4.51}$$

Thus BRST transformations are

$$\begin{aligned}
&\delta_B X^\mu = i\epsilon c\dot{X}^\mu \\
&\delta_B e = i\epsilon\partial_\tau(ce) \\
&\delta_B B = 0 \\
&\delta_B b = \epsilon B \\
&\delta_B c = i\epsilon c\dot{c}.
\end{aligned} \tag{7.4.52}$$

In the path integral, we can perform the path integral over B whose sole effect is to set $e = 1$ in the action. This is called *integrating out* B. The resultant action is

$$S = \int d\tau \left(\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right). \tag{7.4.53}$$

The BRST transformations for the remaining fields will change. To get the new BRST transformation for the remaining fields X^μ, b, c, we need to express B in terms of X^μ, b, c. This is readily obtained from equation of motion for e and then setting $e = 1$. Equation of motion for e is:

$$-\frac{1}{2}\frac{\dot{X}^\mu\dot{X}_\mu}{e^2} + \frac{1}{2}em^2 + iB - \dot{b}c = 0 \tag{7.4.54}$$

which after setting $e = 1$ gives

$$B = -\frac{i}{2}\dot{X}^\mu\dot{X}_\mu + \frac{i}{2}m^2 - i\dot{b}c. \tag{7.4.55}$$

The BRST transformation then is

$$\begin{aligned}
&\delta_B X^\mu = i\epsilon c\dot{X}^\mu \\
&\delta_B b = i\epsilon\left(-\frac{1}{2}\dot{X}^\mu\dot{X}_\mu + \frac{1}{2}m^2 - \dot{b}c\right) \\
&\delta_B c = i\epsilon c\dot{c}.
\end{aligned} \tag{7.4.56}$$

This BRST transformation is nilpotent only onshell. Indeed

$$\begin{aligned}
\delta'_B\delta_B X^\mu &= i\epsilon\delta'_B\left(c\dot{X}^\mu\right) \\
&= i\epsilon\left(i\epsilon' c\dot{c}\dot{X}^\mu + c\partial_\tau\left(i\epsilon' c\dot{X}^\mu\right)\right) \\
&= -\epsilon\epsilon' c\left(\dot{c}\dot{X}^\mu - \dot{c}\dot{X}^\mu - c\ddot{X}^\mu\right) \\
&= 0
\end{aligned} \tag{7.4.57}$$

where the $-$ sign in second term comes by commuting ϵ' past c and the last term vanishes since $c^2 = 0$ or $\ddot{X}^\mu = 0$ is the equation of motion of X^μ. For c, we have **fill in details**.

BRST Hilbert space of the point particle

To construct the BRST Hilbert space, we first construct the canonically quantized Hilbert space and then construct the BRST-cohomology over it.

The Hilbert space is the tensor of the ghost Hilbert space and X Hilbert space. Recall that the ghost Hilbert space is a two state system: there are two states in the system $|\uparrow\rangle, |\downarrow\rangle$ with the action of operators given by

$$\begin{aligned} b|\uparrow\rangle &= |\downarrow\rangle, \quad b|\downarrow\rangle = 0 \\ c|\uparrow\rangle &= 0, \qquad c|\downarrow\rangle = |\uparrow\rangle. \end{aligned} \tag{7.4.58}$$

Note that this is a quantum mechanical system, that is $0+1$d field theory. Thus we don't have oscillators. The the total Hilbert state is just the two ground states. The X Hilbert space is a momentum eigenstate $|k\rangle$ with $p^\mu|k\rangle = k^\mu|k\rangle$. The total Hilbert space is thus

$$\mathscr{H} = \left\{|k,\downarrow\rangle, |k,\uparrow\rangle : k \in \mathbb{R}^{1,D}\right\} \tag{7.4.59}$$

where we have defined $|k,\downarrow\rangle = |k\rangle \otimes |\downarrow\rangle$ and so on. The action of operators is

$$\begin{aligned} &p^\mu|k,\downarrow\rangle = k^\mu|k,\downarrow\rangle, \quad p^\mu|k,\uparrow\rangle = k^\mu|k,\uparrow\rangle, \\ &b|k,\downarrow\rangle = 0, \quad b|k,\uparrow\rangle = |k,\downarrow\rangle, \\ &c|k,\downarrow\rangle = |k,\uparrow\rangle, \quad c|k,\uparrow\rangle = 0. \end{aligned} \tag{7.4.60}$$

Moreover

$$\begin{aligned} &Q_B|k,\downarrow\rangle = cH|k,\downarrow\rangle = \frac{1}{2}\left(k^2+m^2\right)|k,\uparrow\rangle \\ &Q_B|k,\uparrow\rangle = 0. \end{aligned} \tag{7.4.61}$$

Thus

$$\mathscr{H}_{\text{closed}} = \left\{|k,\downarrow\rangle : k^2+m^2=0\right\} \cup \left\{|k,\uparrow\rangle : k \in \mathbb{R}^{1,D}\right\} \tag{7.4.62}$$

and

$$\mathscr{H}_{\text{exact}} = \left\{|k,\uparrow\rangle : k^2+m^2 \neq 0\right\}. \tag{7.4.63}$$

Thus the BRST Hilbert space is

$$\mathscr{H}_{\text{BRST}} = \frac{\mathscr{H}_{\text{closed}}}{\mathscr{H}_{\text{exact}}} \cong \left\{|k,\downarrow\rangle, |k,\uparrow\rangle : k^2+m^2=0\right\}. \tag{7.4.64}$$

We see that the physical states must satisfy mass-shell condition as expected. But note that we have two copies of expected states. To remedy this situation, we claim that states not satisfying

$$b|\psi\rangle = 0 \tag{7.4.65}$$

have vanishing amplitudes with any other state. This will restrict the physical Hilbert space to $\{|k,\downarrow\rangle : k^2+m^2=0\}$. To justify the claim, note that for $k^2+m^2 \neq 0$, the states $|k_2\uparrow\rangle$ are exact and hence they are orthogonal to all physical states (which are Q_B-closed) and their amplitudes vanish identically. So the amplitudes of the states $|k,\uparrow\rangle$ must be proportional to $\delta\left(k^2+m^2\right)$. But in field theory and string theory, amplitudes have poles and cuts but never delta function (except in $D+1=2$ case). So the amplitudes for states $|k,\uparrow\rangle$ must vanish identically.

7.4.3 BRST quantisation of the string

In string theory, the fields are X^μ. We want to derive the BRST transformation of the fields.

Theorem 7.4.4. *The* BRST *transformation of the fields are*

$$\begin{aligned}
\delta_B X^\mu &= i\epsilon(c\partial + \bar{c}\bar{\partial})X^\mu \\
\delta_B b &= i\epsilon\left(T^x + T^g\right) \\
\delta_B \bar{b} &= i\epsilon(\bar{T}x + \bar{T}g) \\
\delta_B c &= i\epsilon c\partial c \\
\delta_B \bar{c} &= i\epsilon\bar{c}\bar{\partial}\bar{c}.
\end{aligned} \tag{7.4.66}$$

Proof. The total action for the string is

$$S = S_P + S_{\text{ghost}} + S_{\text{gf}} \tag{7.4.67}$$

where S_P is the Polyakov action, S_{ghost} is the ghost action and

$$S_{\text{gf}} = \frac{i}{4\pi}\int d^2\sigma\sqrt{g}B^{\alpha\beta}\left(\delta_{\alpha\beta} - g_{\alpha\beta}\right). \tag{7.4.68}$$

The gauge symmetry is now $\mathcal{G} := \text{Diff}\times\text{Weyl}$ with c, b being the Diff, Weyl ghost respectively. We now want to find the gauge algebra. Under diffeomorphism $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$ we have

$$\begin{aligned}
\widetilde{X}^\mu(\widetilde{\boldsymbol{\sigma}}) &= X^\mu(\boldsymbol{\sigma}) \\
\widetilde{g}_{\alpha\beta}(\widetilde{\boldsymbol{\sigma}}) &= \frac{\partial\sigma^\gamma}{\partial\widetilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\widetilde{\sigma}^\delta}g_{\gamma\delta}(\boldsymbol{\sigma}).
\end{aligned} \tag{7.4.69}$$

Taking $\widetilde{\sigma}^\alpha(\boldsymbol{\sigma}) = \sigma^\alpha + \varepsilon^\alpha(\boldsymbol{\sigma})$ with ε an infinitesimal parameter, we find that

$$\begin{aligned}
\delta X^\mu(\boldsymbol{\sigma}) &= \widetilde{X}^\mu(\widetilde{\boldsymbol{\sigma}}) - X^\mu(\widetilde{\boldsymbol{\sigma}}) \\
&= X^\mu(\boldsymbol{\sigma}) - X^\mu(\boldsymbol{\sigma} + \boldsymbol{\varepsilon}) \\
&= -\varepsilon^\alpha\partial_\alpha X^\mu(\boldsymbol{\sigma}) \\
&\equiv -\int d^2\sigma_1\delta^{(2)}\left(\boldsymbol{\sigma} - \boldsymbol{\sigma}_1\right)\varepsilon^\alpha\left(\boldsymbol{\sigma}_1\right)\partial_\alpha X^\mu(\boldsymbol{\sigma})
\end{aligned} \tag{7.4.70}$$

Following the point particle example, this readily gives us the first BRST transformation

$$\delta_B X^\mu = i\epsilon c^\alpha\partial_\alpha X^\mu \tag{7.4.71}$$

or in complex coordinates

$$\delta_B X^\mu = i\epsilon(c\partial + \bar{c}\bar{\partial})X^\mu. \tag{7.4.72}$$

We also have

$$\begin{aligned}
\delta_B B_{\alpha\beta} &= 0 \\
\delta_B b_{\alpha\beta} &= \epsilon B_{\alpha\beta}.
\end{aligned} \tag{7.4.73}$$

To get $\delta_B c^\alpha$, we need the structure constant of the gauge algebra. From the above expression (7.4.70) we see that

$$\delta X^\mu = \varepsilon^{\alpha,\sigma_1}\delta_{\alpha,\sigma_1}X^\mu \tag{7.4.74}$$

where $\varepsilon^{\alpha,\sigma_1} = \varepsilon^\alpha\left(\boldsymbol{\sigma}_1\right)$ and

$$\delta_{\alpha,\sigma_1}X^\mu = -\delta^{(2)}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}_1\right)\partial_\alpha X^\mu(\boldsymbol{\sigma}). \tag{7.4.75}$$

We want to compute the structure constants[6]

$$\begin{aligned}\left[\delta_{\alpha,\sigma_1},\delta_{\beta,\sigma_2}\right]X^\mu(\boldsymbol{\sigma}) &= -\delta_{\alpha,\sigma_1}\left(\delta^{(2)}\left(\sigma-\sigma_2\right)\partial_\beta X^\mu(\sigma)\right) + \delta_{\beta,\sigma_2}\left(\delta^{(2)}\left(\sigma-\sigma_1\right)\partial_\alpha X^\mu(\sigma)\right)\\ &= \delta^{(2)}\left(\sigma-\sigma_1\right)\partial_\alpha\delta^{(2)}\left(\sigma-\sigma_2\right)\partial_\beta X^\mu(\sigma)-\end{aligned} \tag{7.4.76}$$

□

fill in details

Remark 7.4.5. The BRST current that we have obtained from the Noether procedure is not a conformal primary. To make it a conformal primary, we add a total derivative term to the current which does not affect the BRST charge but makes the current a primary field. From now on we will use

$$\begin{aligned} j_B &= cT^X + \frac{1}{2} : bc\partial c : + \frac{3}{2}\partial^2 c\\ \bar{j}_B &= \bar{c}\bar{T}^X + \frac{1}{2} : \bar{b}\bar{c}\bar{\partial}\bar{c} : + \frac{3}{2}\bar{\partial}^2\bar{c}. \end{aligned} \tag{7.4.77}$$

We then have the following OPEs:

Proposition 7.4.6. *We have the following OPEs*

$$\begin{aligned} T(z)j_B(w) &\sim \frac{c^{\mathrm{m}}-26}{2(z-w)^4}c(w) + \frac{j_B(w)}{(z-w)^2} + \frac{\partial j_B(w)}{(z-w)}\\ j_B(z)b(w) &\sim \frac{3}{(z-w)^3} + \frac{j^g(w)}{(z-w)^2} + \frac{T^{\mathrm{m+g}}(w)}{(z-w)}\\ j_B(z)c(w) &\sim \frac{c\partial c(w)}{(z-w)}\\ j_B(z)\mathcal{O}^{\mathrm{m}}(w,\bar{w}) &\sim \frac{h}{(z-w)^2}c(w)\mathcal{O}^{\mathrm{m}}(w,\bar{w}) + \frac{1}{(z-w)}\left[h\partial c(w)\mathcal{O}^{\mathrm{m}}(w,\bar{w}) + c(w)\partial\mathcal{O}^{\mathrm{m}}(w,\bar{w})\right] \end{aligned} \tag{7.4.78}$$

and similarly for $\bar{j}_B(\bar{z})$. Here jg is the ghost current (see eqref) and $\mathcal{O}^{\mathrm{m}}(z,\bar{z})$ is a conformal operator of dimension $(h,\bar{b})$. The OPE of $\bar{j}_B(\bar{z})$ with b, c and of $j_B(z)$ with $\bar{b},\bar{c}$ is regular respectively.

Remark 7.4.7. As we will show soon, the BRST charge Q_B is nilpotent if and only if $c^{\mathrm{m}} = 26$. This will let us conclude that j_B is a conformal primary of dimension $(1,0)$.

[6] In this example, these are not constants

Proof. □

fill in details

Now as pointed out in the general discussion around (7.4.33), gauge fixing of string leaves local conformal symmetry as residual gauge symmetry with generators L_m and $\bar{L}_m$. We have to impose

$$\langle \psi \left| L_m \right| \psi' \rangle = \langle \psi \left| \bar{L}_m \right| \psi \rangle = 0 \tag{7.4.79}$$

on the BRST Hilbert space of the string as a constraint.

BRST Hilbert space of the string

Let us start by specifying the space of all states. It is generated by Fock space constructed out of matter and ghost oscillators. The matter ground state is $|0; p\rangle$ while the ghost ground states are $|\uparrow\rangle, |\downarrow\rangle$. Again we will argue that physical states must satisfy

$$b_0 |\psi\rangle = 0. \tag{7.4.80}$$

Thus we only include $|\downarrow\rangle$ as ghost ground state in view of (7.4.58). The BRST ground state is thus $|0; p, \downarrow\rangle$ and states in the total Hilbert space is constructed using oscillators $\alpha^\mu_n, \widetilde{\alpha}^\mu_n$; $b_m, \bar{b}_m; c_n, \bar{c}_n$ for $n \le 0$. Note that these oscillators for $n > 0$ annihilate the ground state. The inner product is defined by defining Hermitian conjugate of the oscillators

$$\begin{aligned} &(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}, \quad (\widetilde{\alpha}^\mu_n)^\dagger = \widetilde{\alpha}^\mu_{-n}, \\ &b^\dagger_n = b_{-n}, \quad \bar{b}^\dagger_n = \bar{b}_{-n}, \\ &c^\dagger_n = c_{-n}, \quad \bar{c}^\dagger_n = \bar{c}_{-n}. \end{aligned} \tag{7.4.81}$$

The inner product on the full Hilbert space is then defined using above Hermitian conjugates once we define the inner product of ground states. Note that

$$\langle \uparrow | \uparrow \rangle = \langle \downarrow | \downarrow \rangle = 0 \tag{7.4.82}$$

since we can always insert $1 = b_0 c_0 + c_0 b_0$ and c_0 annihilates $|\uparrow\rangle$ while $b_0 |\downarrow\rangle = 0$. But there is no restriction on $\langle \uparrow | \downarrow \rangle$. So we define

$$\langle \downarrow | \uparrow \rangle = \langle \uparrow | \downarrow \rangle = 1 \tag{7.4.83}$$

and put the inner product of ground states to be

$$\begin{aligned} &\text{open string:} \quad \langle\!\langle 0; p, \downarrow | \, 0; p' \downarrow \rangle\!\rangle \equiv \langle 0; p, \downarrow | c_0 | \, 0; p' \downarrow \rangle = (2\pi)^D \delta^D \left(p - p'\right) \\ &\text{closed string:} \quad \langle\!\langle 0; p, \downarrow | \, 0; p' \downarrow \rangle\!\rangle \equiv \langle 0; p, \downarrow | \bar{c}_0 c_0 | \, 0; p' \downarrow \rangle = i(2\pi)^D \delta^D \left(p - p'\right). \end{aligned} \tag{7.4.84}$$

On this space of states we impose

$$b_0 |\psi\rangle = 0 \tag{7.4.85}$$

as we will argue later, is required for physical states. We now construct the BRST cohomology on these states.

Open String:

Since physical states must be Q_B-closed, we see that

$$L_0|\psi\rangle = \{Q_B, b_0\}|\psi\rangle = 0. \tag{7.4.86}$$

Using the expression for L_0, we see that

$$\begin{aligned}
&(L_0^{\mathrm{m}} + L_0^{\mathrm{g}})|\psi\rangle = 0 \\
\implies &\left(\frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_n\cdot\alpha_{-n}: - \sum_{n\in\mathbb{Z}} n :b_{-n}c_n: -1\right)|\psi\rangle = 0 \\
\implies &\left(\alpha' p^2 + \sum_{n=1}^{\infty}\sum_{\mu=0}^{25}\alpha^{\mu}_{-n}\alpha_{\mu n} - \sum_{n=1}^{\infty}(nb_{-n}c_n - nc_{-n}b_n) + c_0b_0 - 1\right)|\psi\rangle = 0.
\end{aligned} \tag{7.4.87}$$

Now using eqref, we obtain

$$m^2 = -p^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} n\left(N_{bn} + N_{cn} + \sum_{\mu=0}^{25} N_{\mu n}\right) - 1 \tag{7.4.88}$$

where we have defined the number operators

$$N_{\mu n} = n\alpha^{\mu}_{-n}\alpha_{\mu n}, \quad N_{bn} = -b_{-n}c_n, \quad N_{cn} = c_{-n}b_n \tag{7.4.89}$$

which counts the number of α^{μ}_n, b_n, c_n oscillators in $|\psi\rangle$ respectively. This can easily be checked using the commutators

$$[\alpha^{\mu}_n, \alpha^{\nu}_m] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad \{b_n, c_m\} = \delta_{m+n,0}. \tag{7.4.90}$$

Let $\widehat{\mathscr{H}}$ be the space of states satisfying egret and eqref. We claim that Q_B is a map from $\hat{\mathcal{H}}$ to $\widehat{\mathscr{H}}$. Indeed for $Q_B|\psi\rangle$, we see that

$$\begin{aligned}
b_0 Q_B|\psi\rangle &= (L_0 - Q_B b_0)|\psi\rangle \\
&= L_0|\psi\rangle - Q_B b_0|\psi\rangle \\
&= 0.
\end{aligned} \tag{7.4.91}$$

Also

$$L_0 Q_B|\psi\rangle = Q_B L_0|\psi\rangle = 0 \tag{7.4.92}$$

since $[Q_B, L_0] = 0$. To construct the BRST Hilbert space, we need to look at $\widehat{\mathscr{H}}$. The inner product egref on $\widehat{\mathscr{H}}$ is inconsistent. To see this, take the state $|0;p,\uparrow\rangle = c_0|0;p,\downarrow\rangle$. Then using the inner product, we have

$$\begin{aligned}\langle\langle 0;p,\uparrow|\, 0;p',\uparrow\rangle\rangle &= \left\langle 0;p,\downarrow\left|c_0^\dagger c_0\right|0;p',\downarrow\right\rangle\\ &= \langle 0;p,\downarrow|c_0c_0|\,0;p',\downarrow\rangle\\ &=0\end{aligned}\tag{7.4.93}$$

while the $\delta^{26}(p-p')$ has a factor $\delta(0)$ since p,p' are on the mass-shell. So we define a new inner product $\langle\cdot\|\cdot\rangle$ on $\widehat{\mathscr{H}}$ in which we ignore the ghost X^0 and the ghost zero modes. This inner product will be relevant for probabilistic interpretation. Let us now work out the first few levels of the BRST Hilbert space. Since states in $\widehat{\mathscr{H}}$ are on mass-shell we will simply denote the ground state by $|0;\boldsymbol{p},\downarrow\rangle$. At the lowest level, we have the ground state $|0;p,\downarrow\rangle$ with $m^2=-p^2=-1/\alpha'$. This is tachyon of the lightcone quantisation. This state is definitely Q_B-closed as can be checked using eqref. There are no Q_B exact states at this level.

At level 1, $N=1$, a generic state is of the form

$$|\psi_1\rangle = (\boldsymbol{e}\cdot\boldsymbol{\alpha}_{-1}+\beta b_{-1}+\gamma c_{-1})\,|0;\boldsymbol{p},\downarrow\rangle\tag{7.4.94}$$

where $\boldsymbol{e}=e_\mu=(e_1,\dots,e_{26})$ is a vector and $\beta,\gamma\in\mathbb{R}$. From egret, we get $m^2=-p^2=0$. The norm of this state is

$$\begin{aligned}\langle\psi_1\|\psi_1\rangle &= \langle 0;\boldsymbol{p},\downarrow\|\,(\boldsymbol{e}^*\cdot\boldsymbol{\alpha}_1+\beta^*b_1+\gamma^*c_1)\,(\boldsymbol{e}\cdot\boldsymbol{\alpha}_{-1}+\beta b_1+\gamma c_{-1})\mid 0;\boldsymbol{p}',\downarrow\rangle\\ &= (\boldsymbol{e}^*\cdot\boldsymbol{e}+\beta^*\gamma+\gamma^*\beta)\,\langle 0;\boldsymbol{p},\downarrow\|0;\boldsymbol{p}',\downarrow\rangle\,.\end{aligned}\tag{7.4.95}$$

We see that the states $\alpha^0_{-1}\,|0;\boldsymbol{p},\downarrow\rangle\,,(b_{-1}-c_{-1})\,|0;\boldsymbol{p};\downarrow\rangle$ are negative norm while $\alpha^i_{-1}|0;\boldsymbol{p},\downarrow\rangle$ and $(b_{-1}+c_{-1})\,|0;\boldsymbol{p},\downarrow\rangle$ are positive norm states. We now have to choose Q_B-closed states out of these. We have

$$\begin{aligned}Q_B\,|\psi_1\rangle &= \sqrt{2\alpha'}\,(c_{-1}\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}+c_1\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})\,|\psi_1\rangle\\ &= \sqrt{2\alpha'}\,(\boldsymbol{p}\cdot\boldsymbol{e}c_{-1}+\beta\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})\,|0;\boldsymbol{p};\downarrow\rangle\end{aligned}\tag{7.4.96}$$

see [12, Page 233] for detailed calculation. Then $Q_B\,|\psi_1\rangle=0$ implies

$$\boldsymbol{p}\cdot\boldsymbol{e}=\beta=0.\tag{7.4.97}$$

This leaves us with 26 linearly independent states, 24 out of which have positive norm and two have norm zero. The zero norm states are $c_{-1}|0;\boldsymbol{p},\downarrow\rangle$ and $\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1}|0;\boldsymbol{p},\downarrow\rangle$. To determine the BRST cohomology, we need to find the Q_B-exact states. For a state $|\chi\rangle$ of the form (7.4.94) with $\boldsymbol{e}',\beta',\gamma'$, we have

$$Q_B|\chi\rangle = \sqrt{2\alpha'}\,(\boldsymbol{p}\cdot\boldsymbol{e}'c_{-1}+\beta'\boldsymbol{p}\cdot\boldsymbol{\alpha}_{-1})\,|0;\boldsymbol{p},\downarrow\rangle.\tag{7.4.98}$$

Now since $[Q_B,L_0]=0, Q_B$ does not change the level and hence a general Q_B-exact state at this level is of the form (7.4.94). So $c_{-1}|0;\boldsymbol{p},\downarrow\rangle$ is Q_B-exact and hence a generic class in the

BRST cohomology has a representative of the form $e \cdot \alpha_{-1}|0;\boldsymbol{p},\downarrow\rangle$ for $\boldsymbol{e}' = e'_\mu = (e'_0, \ldots, e'_{25})$ satisfying $\boldsymbol{p} \cdot \boldsymbol{e} = 0$. The cohomology class is

$$[e \cdot \alpha_{-1}|0;\boldsymbol{p},\downarrow\rangle] = \{((\boldsymbol{e} + \beta'\boldsymbol{p}) \cdot \boldsymbol{\alpha}_{-1} + \gamma c_{-1})\,|0;\boldsymbol{p},\downarrow\rangle : \beta', \gamma \in \mathbb{R}\}\,. \tag{7.4.99}$$

Thus the cohomology at this level is 24 dimensional and the norm is positive-definite. One can proceed to higher levels and show that the BRST cohomology is positive-definite.

Closed String

As for open strings, we will argue later using string amplitudes that physical states must satisfy

$$b_0|\psi\rangle = \bar{b}_0|\psi\rangle = 0, \tag{7.4.100}$$

which along with the physicality condition $Q_B|\psi\rangle = 0$ implies

$$L_0|\psi\rangle = 0, \quad L_0|\psi\rangle = 0 \tag{7.4.101}$$

As before

$$L_0 = \frac{\alpha'}{4}\left(p^2 + m^2\right), \quad \bar{L}_0 = \frac{\alpha'}{4}\left(p^2 + \bar{m}^2\right) \tag{7.4.102}$$

where

$$\begin{aligned} \frac{\alpha'}{4}m^2 &= \sum_{n=1}^{\infty} n\left(N_{bn} + N_{cn} + \sum_{\mu=0}^{25} N_{\mu n}\right) - 1, \\ \frac{\alpha'}{4}\bar{m}^2 &= \sum_{n=1}^{\infty} n\left(\bar{N}_{bn} + \bar{N}_{cn} + \sum_{\mu=0}^{25} \widetilde{N}_{\mu n}\right) - 1. \end{aligned} \tag{7.4.103}$$

Repeating the same calculations as before, we obtain $m^2 = \bar{m}^2 = -4/\alpha'$ at level 0 which is the tachyon. At level 1 we obtain 24×24 massless states which are the graviton, the dilation and the B-field as we obtained in lightcone quantisation.

7.4.4 Proof of the no-ghost theorem

We saw that in Lightcone quantisation, we had no ghosts but then we lost manifest Lorentz invariance. While in covariant quantisation, we had manifest Lorentz invariance but we had ghosts and we pointed out the a necessary condition for ghosts to decouple from the physical spectrum was the critical dimension $D = 26$ and the normal ordering constant $a = 1$. In this section we prove that this is also a sufficient condition by showing an isomorphism between the Hilbert space in lightcone quantisation and covariant quantisation.

The Setup

We will prove the no-ghost theorem in the general setting of Subsection. That is, we will take the worldsheet CFT to be d scalar fields X^μ including $\mu = 0$ and a decoupled unitary

CFT K with central charge $26-d$ plus ghosts. The CFT K will be taken to be compact meaning that the conformal primaries are discrete. The Virasoro generators of the theory is

$$L_m = L_m^X + L_m^K + L_m^{\mathrm{g}}. \tag{7.4.104}$$

The Hilbert space of the theory is the linear span of states of the form $|N, I; p\rangle; |N, \bar{N}, I; p\rangle$ for the open and closed string respectively. Here N (and $\bar{N}$) denotes the combined level of the d-dimensional and ghost CFT and I is a discrete label for the spectrum of the CFT K and p is the d-momentum. Let us denote the total Hilbert space by $\mathscr{H}$. The additional physical amplitude condition

$$b_0|\psi\rangle = 0 \tag{7.4.105}$$

imposes the mass-shell condition on the states:

$$p^2 = \sum_{\mu=0}^{d-1} p_\mu p^\mu = -m^2 \tag{7.4.106}$$

where

$$\alpha' m^2 = \sum_{n=1}^{\infty} n \left(N_{bn} + N_{cn} + \sum_{\mu=0}^{d-1} N_{\mu n} \right) + L_0^K - 1 \tag{7.4.107}$$

for the open string. For the closed string we have two expression for m^2 coming from the left and right moving sectors:

$$\begin{aligned} \frac{\alpha'}{4} m^2 &= \sum_{n=1}^{\infty} n \left(N_{bn} + N_{cn} + \sum_{\mu=0}^{d-1} N_{\mu n} \right) + L_0^K - 1 \\ \frac{\alpha'}{4} \bar{m}^2 &= \sum_{n=1}^{\infty} n \left(\bar{N}_{bn} + \bar{N}_{cn} + \sum_{\mu=0}^{d-1} \widetilde{N}_{\mu n} \right) + \bar{L}_0^K - 1. \end{aligned} \tag{7.4.108}$$

This is the level-matching condition in BRST quantisation. Here L_0^K and $\bar{L}_0^K$ should be understood as the eigenvalue of the operators L_0^K and L_0^K respectively on the states. We can again construct the subspace $\hat{\mathscr{H}}$ as before, is the subspace of those states which satisfy

$$b_0|\psi\rangle = 0, \quad L_0|\psi\rangle = 0 \tag{7.4.109}$$

for open string and

$$b_0|\psi\rangle = 0, \;\; \bar{b}_0|\psi\rangle = 0 \quad L_0|\psi\rangle = 0, \;\; \bar{L}_0|\psi\rangle = 0 \tag{7.4.110}$$

for closed string. We suppose the label I is an orthonormal index to define the reduced inner product on $\widehat{\mathscr{H}}$

$$\langle 0, I; \boldsymbol{p} \| 0, J; \boldsymbol{p}' \rangle = \langle 0, 0, I; \boldsymbol{p} \| 0, 0, J; \boldsymbol{p}' \rangle = 2p^0 (2\pi)^{d-1} \delta^{d-1} \left(\boldsymbol{p} - \boldsymbol{p}' \right) \delta_{IJ}. \tag{7.4.111}$$

This inner product is covariant because of the $2p^0$ factor as in field theory. Let $\mathscr{H}^\perp$ be the subspace of $\widehat{\mathscr{H}}$ which do not have X^0, X^1, b or c excitations. Then $\mathscr{H}^\perp$ has positive definite inner product since these oscillators are the source of negative norm. When we take $d = D = 26$, then it is clear that

$$\mathscr{H}^\perp \cong \mathscr{H}_{\text{Lightcone}}$$

since we projected out exactly the longitudinal excitations X^0, X^{D-1} to define $\mathscr{H}_{\text{Lightcone}}$. We will prove that

$$\mathscr{H}_{\text{BRST}} \cong \mathscr{H}^\perp.$$

Finally we will complete the proof of the no-ghost theorem by proving that

$$\mathscr{H}_{\text{BRST}} \cong \mathscr{H}_{\text{Lightcone}} \cong \mathscr{H}_{\text{CQ}}. \tag{7.4.112}$$

Proof of $\mathscr{H}_{\mathbf{BRST}} \cong \mathscr{H}^\perp$

The proof proceeds as follows: we first prove that $\mathrm{H}^\perp$ is isomorphic to the cohomology of another nilpotent operator Q_1 and then prove that the cohomology of Q_1 is same as the cobomology $\mathscr{H}_{\text{BRST}}$ of Q_B. We first prove this for the open string. We begin by defining Q_1. Define the light cone oscillators

$$\alpha_n^\pm = \frac{1}{\sqrt{2}} \left(\alpha_m^0 \pm \alpha_m^1 \right). \tag{7.4.113}$$

It is easy to check that

$$\left[\alpha_m^+, \alpha_n^-\right] = -m\delta_{m,-n}, \left[\alpha_m^+, \quad \alpha_n^+\right] = \left[\alpha_m^-, \alpha_n^-\right] = 0. \tag{7.4.114}$$

The number operator

$$N^{\text{lc}} \equiv \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \frac{1}{m} : \alpha_{-m}^+ \alpha_m^- : \tag{7.4.115}$$

counts the number of α^--excitations minus the α^+-excitation. This is because for $n \neq 0$

$$\begin{aligned} \left[N^{\text{lc}}, \alpha_n^+\right] &= \sum_{\substack{m\in\mathbb{Z} \\ m\neq 0}} \frac{1}{m} : \alpha_{-m}^+ \left[\alpha_m^-, \alpha_n^+\right] + \left[\alpha_{-m}^+, \alpha_n^+\right] \alpha_m^- : \\ &= \sum_{\substack{m\neq\mathbb{Z} \\ m\neq 0}} \frac{n}{m} \delta_{n,-m} \alpha_{-m}^+ \\ &= -\alpha_n^+, \end{aligned} \tag{7.4.116}$$

and similarly

$$\left[N^{\text{lc}}, \alpha_n^-\right] = \alpha_n^-. \tag{7.4.117}$$

We want to decompose Q_B as
$$Q_B = Q_1 + Q_0 + Q_{-1} \tag{7.4.118}$$
such that
$$\left[N^{\text{lc}}, Q_j\right] = jQ_j, \quad j = 0, \pm 1. \tag{7.4.119}$$
To do this, let us write Q_B explicitly. From (7.2.89) we can write the open string Q_B as
$$Q_B = \sum_{n\in\mathbb{Z}} c_n \left(L^X_{-n} + L^K_{-n}\right) + \frac{1}{2}\sum_{m,n\in\mathbb{Z}} (m-n) : c_m c_n b_{-m-n} : -c_0. \tag{7.4.120}$$
Let us choose a Lorentz frame in which $p^+ \neq 0$. Using (7.2.89) we can write
$$\begin{aligned} L^X_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} \boldsymbol{\alpha}_{n-m}\cdot\boldsymbol{\alpha}_m \\ &= -\frac{1}{2}\sum_{m\in\mathbb{Z}}(:\alpha^+_{n-m}\alpha^-_m : + :\alpha^-_{n-m}\alpha^+_m :) + \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{i=2}^{d-1} :\alpha^i_{n-m}\alpha^i_m : . \end{aligned} \tag{7.4.121}$$
Thus
$$\begin{aligned} Q_B = \sum_{n\in\mathbb{Z}}\Bigg[-\frac{1}{2}\sum_{m\in\mathbb{Z}} c_n(:\alpha^+_{-n-m}\alpha^-_m : + :\alpha^-_{-n-m}\alpha^+_m :) + \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{i=2}^{d-1} c_n :\alpha^i_{-n-m}\alpha^i_m : + c_n L^K_{-n}\Bigg] \\ + \frac{1}{2}\sum_{m,n\in\mathbb{Z}}(m-n) : c_m c_n b_{-m-n} : -c_0. \end{aligned} \tag{7.4.122}$$
From this we see that terms with no $\alpha^\pm$ will commute with N^{lc}. Terms of the form $\alpha^\pm_m\alpha^\mp_n$ for $m, n \neq 0$ also commute with N^{lc} since this has one oscillator of α^- and α^+ each and N^{lc} gives the difference of $\alpha^-$and α^+ oscillators. Terms of the form $\alpha^\pm_m\alpha^\mp_0$ for $m \neq 0$ can be simplified to
$$\alpha^\pm_m\alpha^\mp_0 = \sqrt{\frac{\alpha'}{2}}p^\mp\alpha^\pm_m \tag{7.4.123}$$
and hence does not commute with N^{lc}. Finally
$$\alpha^\pm_0\alpha^\mp_0 = \frac{\alpha'}{2}p^\pm p^\mp$$
and hence commutes with N^{lc}. From (7.2.89), we see that to find Q_1, we need to extract terms containing $\alpha^-_m\alpha^+_0$. From first term in Q_B, the term $n = -m$ in the sum over n and from second term in Q_B, the term $m = 0$ in the sum over m contribute to Q_1. Thus we see that
$$\begin{aligned} Q_1 &= -\frac{1}{2}\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} c_{-m}\alpha^-_m\alpha^+_0 - \frac{1}{2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} c_n\alpha^-_{-n}\alpha^+_0 \\ &= -\sqrt{\frac{\alpha'}{2}}p^+\sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}}\alpha^-_{-m}c_m. \end{aligned} \tag{7.4.124}$$

Similarly

$$Q_{-1} = -\sqrt{\frac{\alpha'}{2}}p^- \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \alpha^+_{-m} c_m \tag{7.4.125}$$

and rest of the terms in Q_B contribute to Q_0. Now since $Q_B^2 = 0$, from (7.4.118) we get

$$\begin{aligned} &(Q_1 + Q_0 + Q_{-1})(Q_1 + Q_0 + Q_{-1}) = 0 \\ \Longrightarrow\ &\left(Q_1^2\right) + (\{Q_1, Q_0\}) + \left(\{Q_1, Q_{-1}\} + Q_0^2\right) + (\{Q_0, Q_{-1}\}) + \left(Q_{-1}^2\right) = 0 \end{aligned} \tag{7.4.126}$$

Now since b_m, c_m has ghost number -1 and $+1$ respectively, from (7.4.122) we see that Q_B has ghost number 1 and so does each Q_j :

$$[N^{\mathrm{g}}, Q_j] = Q_j. \tag{7.4.127}$$

Using (7.4.119)

$$\left[N^{\mathrm{lc}}, Q_i Q_j\right] = (i+j) Q_i Q_j \tag{7.4.128}$$

Thus the N^{lc}-eigenvalue of the terms in (7.4.126) is respectively $2, 1, 0, -1$ and -2 respectively. Thus each term must be zero separately. In particular Q_1 and Q_{-1} are nilpotent.

Proposition 7.4.8. *The cohomology of Q_1 is isomorphic to $\mathscr{H}^\perp$.*

Proof. Define the operators

$$R \equiv \sqrt{\frac{2}{\alpha'}} \frac{1}{p^+} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \alpha^+_{-m} b_m \tag{7.4.129}$$

and

$$S \equiv \{Q_1, R\} = -\sum_{\substack{n,m\in\mathbb{Z}\\ n,m\neq 0}} \left\{\alpha^-_{-m} c_m, \alpha^+_{-n}, b_n\right\}. \tag{7.4.130}$$

Using the identity

$$\{AB, CD\} = A[B, C]D + AC[B, D] + \{A, C\}DB + C[D, A]B, \tag{7.4.131}$$

we can simplify S as

$$\begin{aligned}
S = \{Q_1, R\} &= -\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{=-\infty\\ m\neq 0}}^{\infty}\left(c_m b_n\left[\alpha^-_{-m}, \alpha^+_{-n}\right] + \{c_m, b_n\}\,\alpha^+_{-n}\alpha^-_{-m}\right)\\
&= -\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\sum_{\substack{=-\infty\\ m\neq 0}}^{\infty}\left(-nc_m b_n\delta_{m+n,0} + \delta_{m+n,0}\alpha^+_{-n}\alpha^-_{-m}\right)\\
&= \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\left(-mc_m b_{-m} - \alpha^+_m\alpha^-_{-m}\right)\\
&= \sum_{m=-\infty}^{-1}\left(-mc_m b_{-m} - \alpha^+_m\alpha^-_{-m}\right) + \sum_{m=1}^{\infty}\left(-mc_m b_{-m} - \alpha^+_m\alpha^-_{-m}\right)\\
&= \sum_{m=1}^{\infty}\left(mc_{-m}b_m - \alpha^+_{-m}\alpha^-_m - mc_m b_{-m} - \alpha^+_m\alpha^-_{-m}\right)\\
&= \sum_{m=1}^{\infty}\left(mc_{-m}b_m - \alpha^+_{-m}\alpha^-_m - m\{c_m, b_{-m}\} + mb_{-m}c_m - \left[\alpha^+_m, \alpha^-_{-m}\right] - \alpha^-_{-m}\alpha^+_m\right)\\
&= \sum_{m=1}^{\infty}\left(mc_{-m}b_m - \alpha^+_{-m}\alpha^-_m - m + mb_{-m}c_m + m - \alpha^-_{-m}\alpha^+_m\right)\\
&= \sum_{m=1}^{\infty}\left(mc_{-m}b_m + mb_{-m}c_m - \alpha^+_{-m}\alpha^-_m - \alpha^-_{-m}\alpha^+_m\right)\\
&= \sum_{m=1}^{\infty} m\left(c_{-m}b_m + b_{-m}c_m - \frac{1}{m}\alpha^+_{-m}\alpha^-_m - \frac{1}{m}\alpha^-_{-m}\alpha^+_m\right)\\
&= \sum_{m=1}^{\infty} m\left(N_{bm} + N_{cm} + N^+_m + N^-_m\right)
\end{aligned} \tag{7.4.132}$$

where the number operator N_{bm} counts the number of b_m oscillators and so on. For example

$$\begin{aligned}
\left[N^+_m, \alpha^+_n\right] &= -\frac{1}{m}\left[\alpha^+_{-m}\alpha^-_m, \alpha^+_n\right]\\
&= -\frac{1}{m}\alpha^+_{-m}\left[\alpha^-_m, \alpha^+_n\right]\\
&= -\frac{1}{m}\left(-m\delta_{m,-n}\right)\alpha^+_{-m}\\
&= \alpha^+_n.
\end{aligned} \tag{7.4.133}$$

Using similar calculations as above, it can be checked that

$$[Q_1, S] = 0. \tag{7.4.134}$$

Thus we can diagonalise Q_1 and S together. We look at $Q_1 = 0$ eigenspace and decompose it into eigenspaces of S :

$$\mathscr{H}^{Q_1=0} = \mathscr{H}_0^S \oplus \bigoplus_{s\neq 0} \mathscr{H}_s^S \tag{7.4.135}$$

where $\mathscr{H}_s$ is the eigenspace of S with eigenvalue s. Suppose $|\psi\rangle \in \mathscr{H}_S$, then for $s \neq 0$,

$$|\psi\rangle = \frac{1}{s} s|\psi\rangle = \frac{1}{s}\{Q, R\}|\psi\rangle = \frac{1}{s} Q_1 R|\psi\rangle \tag{7.4.136}$$

so that $\mathscr{H}_s^S$ for $s \neq 0$ is Q_1-exact space. Thus the cohomology $\mathrm{coh}(Q_1)$ of Q_1 is isomorphic to $\mathscr{H}_0^S$. Next,from the relation (7.4.132), we conclude that $\mathscr{H}_0^S$ consists of states with no b, c and α^+, α^- oscillators. Thus we see that $\mathscr{H}_0^S \subseteq \mathscr{H}^\perp$. Now since states in $\mathscr{H}^\perp$ do not contain $\alpha^\pm$ or b, c oscillators, $\mathscr{H}^\perp$ is Q_1-closed. Thus $\mathscr{H}^\perp \subseteq \mathscr{H}_0^S$ or equivalently $\mathscr{H}_0^S = \mathscr{H}^\perp$. Next note that $\mathscr{H}^\perp$ does not contain any Q_1-exact state. Indeed if $|\psi\rangle \in \mathscr{H}^\perp$ is Q_1-exact then there exists $|\chi\rangle \in \mathscr{H}^\perp$ such that

$$|\psi\rangle = Q_1|\chi\rangle = \sqrt{\frac{\alpha'}{2}} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \alpha^-_{-m} c_m |\chi\rangle. \tag{7.4.137}$$

But then for $|\psi\rangle \neq 0, |\chi\rangle$ must have b_m and α_m^+ oscillators which is a contradiction since $|\chi\rangle \in \mathscr{H}^\perp$. Thus we have

$$\mathrm{coh}(Q_1) \cong \mathscr{H}_0^S \cong \mathscr{H}^\perp.$$

□

We now show that $\mathscr{H}_{\mathrm{BRST}}$ and $\mathscr{H}_0^S$ are isomorphic which would imply that $\mathscr{H}_{\mathrm{BRST}} \cong \mathscr{H}^\perp$. To proceed let us define

$$\begin{aligned} U &= \{Q_B, R\} - S = \{Q_1 + Q_0 + Q_{-1}, R\} - S \\ &= \{Q_0 + Q_{-1}, R\}\,. \end{aligned} \tag{7.4.138}$$

It can be checked using the (anti-)commutators of $\alpha^\pm, b, c$ that S commutes with N^{lc}. Thus they can be simultaneously diagonalised. Choose the simultaneous eigenbasis so that S is a diagonal operator. Now from the definition of U, since R lowers N^{lc} eigenvalue by 1 due to α^+_{-m} factor in each term of the sum defining R and Q_0 leaves N^{lc} eigenvalue fixed and Q_{-1} also lowers N^{lc} eigenvalue by $1, U$ lowers the N^{lc} eigenvalue by one or two. Thus U is represented by a lower triangular matrix. We now claim that

$$\mathscr{H}_0^S = \mathrm{Ker}(S) \cong \mathrm{Ker}(S+U) =: \mathscr{H}_0^{S+U}. \tag{7.4.139}$$

Indeed if $|\psi_0\rangle \in \mathrm{Ker}(S)$ then

$$|\psi\rangle = \left(1 - S^{-1}U + S^{-1}US^{-1}U - \ldots\right)|\psi_0\rangle \in \mathrm{Ker}(S+U), \tag{7.4.140}$$

since

$$
\begin{aligned}
(S+U)\left(1-S^{-1}U+S^{-1}US^{-1}U-\ldots\right)|\psi_0\rangle &= \left(S-U+US^{-1}U-\ldots+U-US^{-1}U+\ldots\right)|\psi_0\rangle \\
&= S\,|\psi_0\rangle \\
&= 0.
\end{aligned} \tag{7.4.141}
$$

The factor of S^{-1} makes sense because it always acts on $U|\psi_0\rangle$ which has $N^{\mathrm{lc}}<0$ and S^{-1} is well defined on such states. This implies $\mathrm{Ker}(S)\subseteq\mathrm{Ker}(S+U)$. Since U is lower triangular, $\mathrm{Ker}(S+U)\subseteq\mathrm{Ker}(S)$. Thus we have proved the claim. We now repeat the argument for Q_1 with Q_B. We begin by observing that

$$
[S+U,Q_B]=0. \tag{7.4.142}
$$

Indeed

$$
\begin{aligned}
[S+U,Q_B] &= \left[\{Q_B,R\},Q_B\right] \\
&= Q_BRQ_B+RQ_B^2-Q_B^2R-Q_BRQ_B \\
&= 0
\end{aligned} \tag{7.4.143}
$$

where we used nilpotency of Q_B. We again diagonalise Q_B and $S+U$ together. We look at the $Q_B=0$ eigenspace and decompose it into eigenspaces of $S+U$:

$$
\mathscr{H}^{Q_B=0}=\mathscr{H}_0^{S+U}\oplus\bigoplus_{t\neq 0}\mathscr{H}_t^{S+U}. \tag{7.4.144}
$$

Note that we have proved that $\mathscr{H}_0^{S}=\mathscr{H}_0^{S+U}$. For a state $|\psi\rangle\in\mathscr{H}_t^{S+U},\quad t\neq 0$

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{t}(S+U)|\psi\rangle=\frac{1}{t}\left\{Q_BR\right\}|\psi\rangle \\
&= \frac{1}{t}Q_BR|\psi\rangle
\end{aligned} \tag{7.4.145}
$$

so that $\mathscr{H}_t^{S+U}, t\neq 0$ is Q_B-exact. Thus cohomology of Q_B is nontrivial only in $\mathscr{H}_0^{Q_B=0}$. Thus we have proved that

$$
\mathscr{H}_{\mathrm{BRST}}=\mathrm{coh}\left(Q_B\right)=\mathrm{Ker}(S+U)=\ker(S)=H_0^{Q_1=0}=\mathrm{coh}\left(Q_1\right). \tag{7.4.146}
$$

Finally we want to check that inner product is positive definite on $\mathrm{Ker}(S+U)$ using the positive definiteness of inner product on $\mathrm{Ker}(S)$. Any state in $\mathrm{Ker}(S+U)$ is of the form

$$
|\psi\rangle=\left(1-S^{-1}U+S^{-1}US^{-1}U-\ldots\right)|\psi_0\rangle,\quad \left|\psi_0\right\rangle\in\mathrm{Ker}(S). \tag{7.4.147}
$$

We claim that

$$
\langle\psi\|\psi'\rangle=\langle\psi_0\|\psi_0'\rangle\,. \tag{7.4.148}
$$

To prove this claim we observe that inner product of two states is nonzero only when their lightcone numbers N^{lc} add to zero. Now $\langle\psi\|\psi'\rangle$ is a linear combination of terms of the form

$$
\left\langle\psi_0\middle|\left(\left(S^{-1}U\right)^m\right)^\dagger\left(S^{-1}U\right)^n\middle|\psi_0'\right\rangle. \tag{7.4.149}
$$

Since S has $N^{\rm lc} = 0$ (since it commutes with $N^{\rm lc}$) and U is a sum of $N^{\rm lc} = -1$ and $N^{\rm lc} = -2$ terms, hence $(S^{-1}U)^m$ has $N^{\rm lc} < 0$ unless $m = 0$. To compute the inner product we will need to use the commutation relations between $\left((S^{-1}U)^m\right)^\dagger$and $(S^{-1}U)^n$ and the inner product vanishes unless $m = n = 0$. This proves the claim and hence the positive definiteness of the inner product.

We now complete the proof of no-ghost theorem.

Proof of $\mathscr{H}_{\mathbf{CQ}} \cong \mathscr{H}_{\mathbf{BRST}} \cong \mathscr{H}_{\mathbf{Lightcone}}$

In the previous section, we proved that $\mathscr{H}_{\rm BRST} \cong \mathscr{H}_{\rm Lightcone}$. We now prove that $\mathscr{H}_{\rm CQ} \cong \mathscr{H}_{\rm BRST}$. Consider the map

$$\begin{aligned} \mathscr{H}_{\rm CQ} &\longrightarrow \mathscr{H}_{\rm BRST} \\ |\psi\rangle &\longmapsto |\psi, \downarrow\rangle \end{aligned} \tag{7.4.150}$$

where $|\psi\rangle$ is a matter CFT state. From the expression for the BRST charge, we see that

$$Q_B|\psi, \downarrow\rangle = \sum_{n=0}^{\infty} c_{-n}\left(L_n^{\rm m} - \delta_{n,0}\right)|\psi, \downarrow\rangle = 0 \tag{7.4.151}$$

where we used the fact that $b_n|\downarrow\rangle = 0$ for $n \geq 0$ and $c_n|\downarrow\rangle = 0$ for $n > 0$. We also used the fact that $L_n^{\rm m}|\psi\rangle = 0$ for $n > 0$ and $L_0^{\rm m}|\psi\rangle = |\psi\rangle$, see (3.3.8) and recall that the normal ordering constant $a = 1$ in covariant quantisation. Thus states in $\mathscr{H}_{\rm CQ}$ gets mapped to Q_B-closed states. Note that $\mathscr{H}_{\rm CQ}$ is obtained by modding out null states and so, we need to show that the map is well-defined. Indeed if

$$|\psi\rangle - |\psi'\rangle \in \mathscr{H}_{\rm null} \tag{7.4.152}$$

then $|\psi, \downarrow\rangle - |\psi', \downarrow\rangle$ must be Q_B-exact since $|\psi, \downarrow\rangle - |\psi', \downarrow\rangle$ has norm zero and the norm is positive definite on $\mathscr{H}_{\rm BRST}$ by the isomorphism of $\mathscr{H}_{\rm BRST}$ with $\mathscr{H}_{\rm Lightcone}$. We now show that the map is injective. Indeed if $|\psi, \downarrow\rangle - |\psi', \downarrow\rangle$ is Q_B-exact, there exists $|\chi\rangle$ such that

$$|\psi, \downarrow\rangle - |\psi', \downarrow\rangle = Q_B|\chi\rangle \tag{7.4.153}$$

Now since $|\downarrow\rangle$ has ghost number $-\frac{1}{2}$ and Q_B has ghost number 1, $|\chi\rangle$ must have ghost number $-\frac{3}{2}$. Thus we can expand $|\chi\rangle$ as

$$|\chi\rangle = \sum_{n=1}^{\infty} b_{-n}\,|\chi_n, \downarrow\rangle + \cdots \tag{7.4.154}$$

where the ellipsis contains states with at least one c and two b excitations. Then we get

$$\begin{aligned}
Q_B|\chi\rangle &= \sum_{n=1}^{\infty}\left(\sum_{m=0}^{\infty} c_m \left(L^{\mathrm{m}}_{-m} - \delta_{m,0}\right) b_{-n}\left|\chi_n, \downarrow\right\rangle\right) \\
&= \sum_{m,n=1}^{\infty} c_m L^{\mathrm{m}}_{-m} b_{-n}\left|\chi_n \downarrow\right\rangle + \sum_{n=1}^{\infty}\left(c_0 L^{\mathrm{m}}_0 b_{-n} - c_0 b_{-n}\right)\left|\chi_n \downarrow\right\rangle \\
&= \sum_{m,n=1}^{\infty} c_m L^{\mathrm{m}}_{-m} b_{-n}\left|x_n, \downarrow\right\rangle + \sum_{n=1}^{\infty} c_0 b_{-n}\left(L^{\mathrm{m}}_0 - 1\right)\left|\chi_n \downarrow\right\rangle \\
&= \sum_{m,n=1}^{\infty} c_m L^{\mathrm{m}}_{-m} b_{-n}\left|\chi_n, \downarrow\right\rangle \\
&= \sum_{n=1}^{\infty} L^{\mathrm{m}}_{-n}\left|\chi_n, \downarrow\right\rangle
\end{aligned} \tag{7.4.155}$$

where we have retained terms on the RHS which correspond to ghost ground states since the LHS has only ghost ground states. Terms on the RHS which have ghost excitations must vanish independently because of different ghost numbers of each individual terms. We used the anti-commutator

$$\{b_m, c_n\} = \delta_{m,-n}. \tag{7.4.156}$$

Thus we see that

$$|\psi\rangle - |\psi'\rangle = \sum_{n=1}^{\infty} L^{\mathrm{m}}_{-n}\left|\chi_n\right\rangle. \tag{7.4.157}$$

Now since the LHS satisfies the physicality condition

$$\left(L^{\mathrm{m}}_m - \delta_{m,0}\right)\left(|\psi\rangle - |\psi'\rangle\right) = 0, \quad m \geq 0, \tag{7.4.158}$$

the right hand side is also physical. On the other hand, a state of the form

$$\sum_{n=1}^{\infty} L^{\mathrm{m}}_{-n}\left|\chi_n\right\rangle \tag{7.4.159}$$

is obviously orthogonal to all physical states and hence it is spurious as well as physical implying that it is null. This shows that $|\psi\rangle = |\psi'\rangle$ in the quotient space $\mathscr{H}_{\mathrm{CQ}}$.

We now show that the map is surjective. Take a Q_B-cohomology class and consider the operator $N' = 2N^- + N_b + N_c$ which counts the total number of (twice the) α^-_n, b and c excitations. One can check that R has $N' = -1$ and $Q_0 + Q_{-1}$ has terms with with varying N' but no term with $N' = 1$. Thus $U = \{R, Q_0 + Q_{-1}\}$ cannot increase N'. Noting the $|\psi_0\rangle \in \mathrm{Ker}(S)$ has $N' = 0$ and S has $N' = 0$ we see that the state

$$|\psi\rangle = \left(1 - S^{-1}U + S^{-1}US^{-1}U - \ldots\right)|\psi_0\rangle \tag{7.4.160}$$

has $N' \leq 0$, but by definition $N' \geq 0$ and hence $N'|\psi\rangle = 0$ and hence it has no,$\alpha_n^- b$ or c excitations implying that

$$|\psi\rangle \in \mathrm{Ker}(S+U) \tag{7.4.161}$$

is of the form $|\psi, \downarrow\rangle$ for some $|\psi\rangle \in \mathscr{H}_{\mathrm{CQ}}$ and hence has a preimage. Thus we have proved that $\mathscr{H}_{\mathrm{CQ}} \cong \mathscr{H}_{\mathrm{BRST}}$.

Chapter 8

String Compactification

8.1 Toroidal Compactification

8.1.1 T-duality of Closed Strings

For the closed string, Let us take x^{25} to be periodic. The massshell condition becomes

$$\begin{aligned} M^2 = -p^u p_\mu &= \left(p_L^{25}\right)^2 + \frac{4}{\alpha'}(N-1) \\ &= \left(p_R^{25}\right)^2 + \frac{4}{\alpha'}(\tilde{N}-1). \end{aligned}$$

Adding the two equation gives

$$M^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

and substracting them gives

$$n\omega + N - \tilde{N} = 0$$

The first term in the mass equation is the contribution from momentum n/R in x^{25} direction. The second term comes from the tension of the string stretching ω times around the s^1 picking up a contribution of $2\pi\omega RT$ to the mass, where $T = 1/2\pi\alpha'$ is the string tension. Let us look at the massless spectrum. A state becomes massless if $n = w = 0$ and $N = \tilde{N} = 1$. These are the $(24)^2$ level 1 states from string theory. Let us list them: (i) $\alpha^\mu_{-1}\tilde{\alpha}^v_{-1}|0, p\rangle, \mu, v = 0, 1, \ldots, 24$. This as before breaks as a Graviton, dilator and the anti-symmetric B-field. (ii) $\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1}|0; p\rangle$ and $\alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1}|0; p\rangle$ which are both vector fields. The fields $\alpha^\mu_{-1}\tilde{\alpha}^{25}_{-1}|0; p\rangle \pm \alpha^{25}_{-1}\tilde{\alpha}^\mu_{-1}|0; p\rangle$ can be identified with the gauge fields coming from the metric and B-field in 26 dimensions respectively. (iii) $\alpha^{25}_{-1}\tilde{\alpha}^{25}_{-1}|0; p\rangle$ is another scalar which can be identifield with the scalar σ coming from the metric.

Appendix A

Wigner's Classification of Representations of Poincaré Group

In this appendix, we will briefly review Wigner's little group method of classifying the irreducible representations of the Poincaré group. The idea is mathematically enlightening and and motivated a lot of research in representation theory. But here we will not delve into the mathematically rigorous treatment, the interested reader can look up [10] for a summary of the mathematical theory. Rather we will take a more physical approach on the lines of Weinberg's quantum theory of fields book [16]. We begin by discussing Wigner's proposal of interpreting elementary particles as irreducible representations of the Poincaré group. We assume familiarity with basic terminology of topology.

A.1 Projective Representations

Let $|\Psi\rangle$ be a state in Hilbert space $\mathcal{H}$. Note that any two states $|\Psi\rangle$ and $|\Phi\rangle$ which are nonzero and related by

$$|\Psi\rangle = \lambda|\Phi\rangle \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\} \tag{A.1.1}$$

are the same quantum mechanical states. So it is pertinent to consider the quotient space of $\mathcal{H}^* = \mathcal{H}\backslash\{0\}$ as $\mathbb{P}(\mathcal{H}) := \mathcal{H}^*/\sim$ where $|\Psi\rangle \sim |\Phi\rangle$ if and only if (A.1.1) is true. The quotient space $\mathbb{P}(\mathcal{H})$ is called the *projectivised Hilbert space*. Recall that the probability amplitude of transition from $|\Psi\rangle$ to Φ is given by

$$p(|\Psi\rangle, |\Phi\rangle) = \frac{\langle\Psi \mid \Phi\rangle}{\langle\Psi \mid \Psi\rangle\langle\Phi \mid \Phi\rangle}.$$

In the quotient topology on $\mathbb{P}(\mathcal{H})$, p induces a continuous map[1] on $\mathbb{P}(\mathcal{H})$ which we denote by $\widetilde{p}$. A homeomorphism $T : \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$ satisfying

$$\widetilde{p}(T[\Psi], T[\Phi]) = \widetilde{p}(|\Psi\rangle, |\Phi\rangle)$$

[1] it is a standard result in quotient topology. See for example Topology by Munkres.

where $[\Phi], [\Psi]$ are equivalence classes in $\mathbb{P}(\mathcal{H})$, is called a *projective automorphism.* The set of all such maps, denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, is a group called *projective automorphism group.* The action of this group on $\mathbb{P}(\mathcal{H})$ leaves transition probabilities invariant. Now consider a particle in the Minkowski space $\mathbb{R}^{1,D-1}$. The symmetry group of this space is precisely the Poincaré group[2] which we denote by $\mathcal{P}$. Let two observers $\mathcal{O}$ and $\mathcal{O}'$, related by $\Lambda \in \mathcal{P}$, measure the quantum mechanical particle. In general, there measurement result will reveal different states, say $[\Psi]$ and $[\Psi']$ respectively. Thus physically one expects that transition probabilities in $\mathcal{O}$ and $\mathcal{O}'$ be same. This means that the two states must be related by some projective automorphism:

$$[\Psi] = T_\Lambda[\Psi'], \quad \text{for some} \quad T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H})).$$

If $\mathcal{O} = \mathcal{O}'$ then $T_\Lambda = Id$ and we should have $T_\Lambda = T_{Id} = Id \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. Lastly if a third observer $\mathcal{O}''$, related to $\mathcal{O}'$ by Γ, measures the state then we must impose $T_\Lambda \circ T_\Gamma = T_{\Lambda\circ\Gamma}$. Thus the change of frame induces a representation $\Pi : \mathcal{P} \longrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This is called the *projective representation.*

A.2 Elementary Particles

The representation $(\Pi, \mathcal{H})$ of the Poincaré group is called irreducible if the only nontrivial closed invariant subspace of $\mathcal{H}$ is $\mathcal{H}$. That is $\Pi(\mathcal{P})(V) \subseteq V$ if and only if $V = \mathcal{H}$. The closed condition is technical: we want the invariant subspace to be a Hilbert space in its own right which is not automatically true in infinite dimensional Hilbert space unless the subspace is closed.

Wigner suggested that the irreducible projective representations of the Poincaré group correspond to elementary particles within the quantum system under consideration. Wigner's argument was as follows: an elementary particle in a quantum mechanical system is a vector in $\mathbb{P}(\mathcal{H})$. As discussed, different observers will see different vectors in $\mathbb{P}(\mathcal{H})$ corresponding to the elementary particle. All these vectors must be related by some projective automorphism. The set of all these vectors constitutes $\mathcal{P}-$invariant subspace of $\mathbb{P}(\mathcal{H})$ and hence we obtain a subrepresention of $(\Pi, \mathcal{H})$. This subrepresentation can be thought of as a subsystem which is elementary if it is irreducible (otherwise it will have more smaller subsystems). This reduces the problem of determining all relativistic free particles in Minkowski spacetime to the mathematical task of finding all irreducible projective representations of the Poincaré group.

[2] mathematically speaking, the symmetry group of a Riemannian manifold $(\mathcal{M}, g)$ is the group of all diffeomorphisms from $\mathcal{M}$ to itself whose pullback preserves the metric.

A.3 Projective Representations of the Poincaré Group

Let us now take a look at the Poincaré group more closely. We begin by defining semidirect product.

Definition A.3.1. Let H and N be groups and suppose there is a a group homomorphism $\phi : H \to \mathrm{Aut}(N)$. Then the semidirect product of H by N, denoted $H \ltimes N$ which has $H \times N$ as underlying set, and multiplication defined by $(h,n) \cdot (h', n') = (hh', n\phi(h)\,(n'))$.

Each element of the Lorentz group $SO(1, D-1)$ defines an automorphism of $\mathbb{R}^{1,D-1}$ defined by matrix multiplication. Thus we can form the semidirect product $SO(1, D-1) \ltimes \mathbb{R}^{1,D-1}$. The physically relevant Poincaré group is the semidirect product of the proper orthochronous Lorentz group and the abelian translation group. That is

$$\mathcal{P} = ISO(1, D-1) = SO(1, D-1)_I \ltimes \mathbb{R}^{1,D-1}$$

where $SO(1, D-1)_I$ is the connected component of identity in the Lorentz group. The Poincaré algebra is generated by the generators of translations and Lorentz transformations denoted by P^μ and $M^{\mu\nu}$ respectively. They satisfy the Poincaré algebra:

$$\begin{aligned}
&i\,[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\sigma\mu} M_{\rho\nu} + \eta_{\sigma\nu} M_{\rho\mu} \\
&i\,[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \\
&i\,[P_\mu, P_\rho] = 0.
\end{aligned}$$

The third commutator says that P_μ commutes among themselves. So we start with states in $\mathbb{P}(\mathcal{H})$ which are simultaneous eigenvectors of P^μ. We label all other degrees of freedom by σ. We have

$$P^\mu \psi_{q,\sigma} = q^\mu \psi_{q,\sigma}.$$

Note that infinitesimal translations are represented by $U = \mathbb{1} - iP^\mu \varepsilon_\mu$ and repeating this, we obtain finite translations

$$U(\mathbb{1}, a) = e^{-iP^\mu a_\mu}.$$

so that

$$U(\mathbb{1}, a)\psi_{q,\sigma} = e^{-iq\cdot a}\psi_{q,\sigma}.$$

These $U(\mathbb{I}, a)$ are the projective representations of the translation part of the Poincaré group. Usually the physical requirement restricts U to be unitary which restricts P^μ to be Hermitian. Recall that

$$\begin{aligned}
&(\Lambda, a) \cdot (\Lambda', a') = (\Lambda\Lambda', a' + \Lambda a) \quad \text{ in } \quad \mathcal{P} \\
&(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda a).
\end{aligned}$$

An infinitesimal Poincaré transformation with parameters ω, ε is unitarily represented as

$$U(\mathbb{1} + \omega, \varepsilon) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} - \varepsilon_\mu P^\mu + \dots.$$

So For a general $\Lambda \in SO(1, D-1)$ we have

$$U(\Lambda, a)U(\mathbb{1}+\omega, \in)U(\Lambda, a)^{-1} = U\left(\Lambda(\mathbb{1}+\omega)\Lambda^{-1}, \Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right).$$

Using infinitesimal version upto linear order in ω, we get

$$U(\Lambda, a)\left[\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - \varepsilon_\mu P^\mu\right]U(\Lambda, a)^{-1} = \mathbb{1} + \frac{i}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}M^{\mu\nu} - \left(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu.$$

Comparing coefficients of $\omega_{\mu\nu}$ and ε_μ, we get

$$\begin{aligned} U(\Lambda, a)M^{\mu\nu}U(\Lambda, a)^{-1} &= \left(\Lambda^{-1}\right)^\mu_\lambda \left(\Lambda^{-1}\right)^\nu_\rho \left(M^{\lambda\rho} - a^\lambda P^\rho + a^\rho P^\lambda\right) \\ U(\Lambda, a)P^\rho U(\Lambda, a)^{-1} &= \left(\Lambda^{-1}\right)^\rho_\mu P^\mu. \end{aligned} \tag{A.3.1}$$

Our aim now is to find the projective representation of the Lorentz part of the Poincaré group. Indeed if $U(\Lambda, 0) \equiv U(\Lambda)$ is such a representation then

$$\begin{aligned} P^\mu U(\Lambda)\psi_{p,\sigma} &= U(\Lambda)U(\Lambda)^{-1}P^\mu U(\Lambda)\psi_{p,\sigma} \\ &= U(\Lambda)\Lambda^\mu_\nu P^\nu \psi_{p,\sigma} \\ &= (\Lambda p)U(\Lambda)\psi_{p,\sigma}. \end{aligned}$$

So we must have

$$U(\Lambda)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\psi_{\Lambda p,\sigma'}. \tag{A.3.2}$$

In general, this representation is reducible. since this is a unitary representation, a theorem in representation theory says that it is completely reducible, that is it can be written as a direct sum of irreducible representations of invariant subspaces of eigenvectors of P^μ with eigenvalue Λp. Our goal is to classify all such irreducible representations. To do so, we first calculate the orbit of action of Lorentz group on $\mathbb{R}^{1,D-I}$. It is clear that $SO(1, D-1)_I$ fixes p^2 for all $p \in \mathbb{R}^{1,D-1}$ but when $p^2 \leq 0$ then it also fixes the sign of p^0. Accordingly we get the following orbits:

1. $p^2 = m^2 > 0$: one sheeted hyperboloid.
2. $p^2 = -m^2 < 0$: two sheeted hyperboloid corresponding to $p^0 > 0$ or $p^0 < 0$.
3. $p^2 = 0$: cone with vertex at the origin.

Now given any p^μ, one can choose (depending on the orbit of p^μ) a standard q^μ such that

$$p^\mu = L^\mu_\nu(p)q^\nu,$$

where $L_\nu \in SO(1, D-1)_I$. By above discussion

$$\psi_{p,\sigma} = N(p)U\left(L^\mu_\nu(p)\right)\psi_{q,\sigma},$$

where $N(p)$ is some normalesation factor. Now for any $\Lambda \in SO(1, D-1)_I$ we have

$$\begin{aligned} U(\Lambda)\psi_{p,\sigma} &= N(p)U(\Lambda)U(L(p))\psi_{q,\sigma} \\ &= N(p)U(L(\Lambda p))U\left(L^{-1}(\Lambda p)\Lambda L(p)\right)\psi_{q,\sigma}, \end{aligned}$$

where we used property of group representations. Note that

$$L^{-1}(\Lambda p)\Lambda L(p)q = L^{-1}(\Lambda p)\Lambda p = q.$$

The set of all such elements of Λ is called the *stability group* of q also called the *little group*. For any two elements $W, \overline{W}$ in the little group of q, we have

$$U(W)\psi_{q,\sigma} = \sum_{\sigma\prime} D^q_{\sigma,\sigma'}(W)\psi_{q,\sigma'}$$

and

$$\begin{aligned} U(\overline{W}W)\psi_{q,\sigma} &= \sum_{\sigma'} D^q_{\sigma,\sigma'}(W) \sum_{\sigma''} D^q_{\sigma',\sigma''}(\overline{W})\psi_{q,\sigma''} \\ &= \sum_{\sigma',\sigma''} D^q_{\sigma,\sigma'}(W) D^q_{\sigma',\sigma''}(\overline{W})\psi_{q,\sigma''} \\ &= \sum_{\sigma''} D^q_{\sigma,\sigma''}(\overline{W}W)\psi_{q,\sigma''}, \end{aligned}$$

where

$$D^q_{\sigma,\sigma''}(\overline{W}W) = \sum_{\sigma'} D^q_{\sigma,\sigma'}(W) D^q_{\sigma',\sigma''}(\overline{W}).$$

Thus we see that $D^q(W)$ is a representation of the little group. So putting $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ we have

$$U(W(\Lambda, p))\psi_{q,\sigma} = \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda, p))\psi_{q,\sigma'}.$$

So that

$$\begin{aligned} U(\Lambda)\psi_{p,\sigma} &= N(p) \sum_{\sigma'} D_{\sigma,\sigma'}(w(\Lambda, p))U(L(\Lambda p))\psi_{q,\sigma'} \\ &= \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda, p))\psi_{\Lambda p,\sigma'}. \end{aligned}$$

Hence apart from the normalisation factor, the problem of finding unitary irreducible representations of Poincaré group has been reduced to finding unitary irreducible representations of the little group corresponding to each orbit. So we first find the little group corresponding to each orbit.

1. $q^2 = m^2 > 0$: by going to rest frame, we can set q^μ to $q^\mu = (0, 0, \cdots, 0, m)$. Looking at the form of this vector, we can see that the little group is $SO(1, D-2)_I \hookrightarrow SO(1, D-1)_I$.

2. $q^2 = -m^2 < 0$: , by going to rest frame, we can take q^μ to be $q^\mu = (m, \vec{0})$. Clearly the little group is $SO(D-1)$.

3. $q^2 = 0$: the little group computation is not so obvious. Although it turns out to be the Euclidean group $E(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}$. This is the isometry group of $\mathbb{R}^{D-2}$ with the Euclidean metric.

In $q^2 = 0$, one case is $q^\mu = 0$ whose stabiliser is the whole Poincaré group $\mathcal{P}$.

Gender	Orbit	Little Group	Unitary Representation
$q^2 = -m^2$	Mass shell	$SO(D-1)$	Massive
$q^2 = -m^2$	Hyperboloid	$SO(1, D-1)_I$	Tachyonic
$q^2 = 0$	Lightcone	$E(D-2)$	Massless
$q^\mu = 0$	Origin	$\mathcal{P}$	Zero Momentum

Physically, Tachyonic representations are not accepted. So we will only deal with the other two. One can use the little group method to find all irreducible representations of the Euclidean group. The idea is to go to the Lie algebra of $E(D-2)$ and identify the "translations" generators and repeat the procedure above. The upshot of this computation is that we get two orbits and the corresponding little groups are called *short little groups*. The corresponding unitary irreducible representations are labelled as *helicity* and *infinite spin*. The analogue of the Lorentz group here is obviously $SO(D-2)$. The short Little group corresponding to infinite spin is $S0(D-3)$ and that for infinite spin is $SO(D-2)$.

Next one can use Young Tableau to embed the irreducible representations of the Little groups in all cases into tensorial representations. For the particular case that we will be dealing with, we would like to find the massless irreducible representations of dimension $(D-2)^2$ of the Poincaré group. It turns out that it is the direct sum of three irreducible parts:

$$\begin{array}{lccccc} & \textbf{Traceless symmetric} & \oplus & \textbf{Antisymmetric} & \oplus & \textbf{Trace (Scalar)} \\ \text{Dim:} & \frac{(D-2)(D-1)}{2} - 1 & & \frac{(D-2)(D-3)}{2} & & 1 \end{array}$$

Appendix B

Symmetry Generators: Generators of the Conformal Algebra

In this appendix, we describe the general principle of symmetry transformations at classical and quantum level and the general method of finding the generators of symmetry transformation. We also describe the consequences of symmetries in classical and quantum theories namely, Noether's theorem and Ward identities respectively. As an application, we compute the generators of conformal symmetry and the corresponding charges and describe the Ward identities.

B.1 Continuous Symmetry Transformations

Let $\Phi : \mathbb{R}^{1,D-1} \longrightarrow \mathcal{M}$ be a field from the spacetime to some target manifold. Its dynamics is governed by an action by virtue of the Euler-Lagrange equations. The action generally is a functional of Φ and its first derivatives:

$$S[\Phi] = \int d^D x \mathcal{L}(\Phi, \partial_\mu \Phi),$$

where $\mathcal{L}(\Phi, \partial_\mu \Phi)$ is the Lagrange density. Suppose we transform the field as $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$. Then the action also transforms as $S[\Phi] \longrightarrow S[\Phi'] =: S'[\Phi]$.

Definition B.1.1. (i) A transformation of field $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ is called a *symmetry* of the action $S[\Phi]$ is under the transformation of field the action remains invariant in the sense that $S'[\Phi] = S[\Phi]$.

(ii) A symmetry $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ of the action is called a *continuous symmetry* is the transformation of the field is parametrised by a continuous parameter α. A symmetry which is not continuous is called *discrete*.

(iii) A symmetry $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ is called a *spacetime symmetry* if the field transformation results from a spacetime transformation. Otherwise it is called *internal symmetry*.

(iv) A symmetry $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ is called *local symmetry* if the field transforms differently at different spacetime points. If the field transforms in exactly the same way at every spacetime point, then it is called *global symmetry.*

Since we will mostly be considering symmetries with respect to conformal transformations which is a spacetime transformation, we will restrict our attention to spacetime symmetries.

B.1.1 Spacetime Symmetries

Consider a spacetime transformation

$$\begin{aligned} &x \longrightarrow x'(x) \\ &\Phi(x) \longrightarrow \Phi'\left(x'\right) \end{aligned} \tag{B.1.1}$$

Under such a transformation, the field Φ changes in two ways: first by the functional change $\Phi' = \mathcal{F}(\Phi)$ where we have expressed the new field Φ' as a function of the old field Φ, and second by the change of argument $x \longrightarrow x'$. Expressing the new field at x', we see that

$$\Phi'\left(x'\right) = \mathcal{F}(\Phi(x)). \tag{B.1.2}$$

This way of looking at symmetry transformations is called *active transformation.* Under such a transformation, the action transforms as

$$\begin{aligned} S' &= \int d^D x \mathcal{L}\left(\Phi'(x), \partial_\mu \Phi'(x)\right) \\ &= \int d^D x' \mathcal{L}\left(\Phi'\left(x'\right), \partial'_\mu \Phi'\left(x'\right)\right) \\ &= \int d^D x' \mathcal{L}\left(\mathcal{F}(\Phi(x)), \partial'_\mu \mathcal{F}(\Phi(x))\right) \\ &= \int d^D x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\Phi(x)), \left(\partial x^\nu / 2x'^\mu\right) \partial_\nu \mathcal{F}(\Phi(x))\right), \end{aligned}$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of variable change. We have changed variables $x \longrightarrow x'$ according to the transformation (B.1.1) and used (B.1.2) and in first two steps finally again made a change of variables in last step.

Example B.1.2. (i) Translation: it is defined as

$$\begin{aligned} &x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu \\ &\Phi'(x+a) = \Phi(x) \end{aligned}$$

It is clear that $S' = S$. The action is invariant under translations, unless it depends explicitly on position.

(ii) Lorentz Transformation: under Lorentz transformation $\Lambda \in \mathrm{SO}(1, D-1)$,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$
$$\Phi'(\Lambda x) = L_{\Lambda} \Phi(x),$$

where we have assumed that the fields transform *linearly* with respect to Lorentz transformation, so that the operators L_{Λ} furnish a representation of the Lorentz group. Depending on the action and the representation Φ of the Lorentz group, the action may or may not be invariant under Lorentz transformation.

(iii) Scale Transformations: it is defined as

$$x' = \lambda x$$
$$\Phi'(\lambda x) = \lambda^{-\Delta} \Phi(x)$$

where λ is the dilation factor and Δ is called the *scaling dimension* of the field Φ. Note that the Jacobian of this transformation is $|\partial x'/\partial x| = \lambda^{D}$. Thus we have

$$S' = \lambda^{D} \int d^{D}x \mathcal{L}\left(\lambda^{-\Delta}\Phi, \lambda^{-1-\Delta}\partial_{\mu}\Phi\right)$$

As an example, consider the action of a massless scalar field φ in spacetime dimension D:

$$S[\varphi] = \int d^{D}x \partial_{\mu}\varphi\partial^{\mu}\varphi.$$

It is easily checked that this action is scale invariant if we make the choice

$$\Delta = \frac{1}{2}D - 1.$$

B.2 Infinitesimal Transformation and Noether's Theorem

We now consider continuous transformations and study their effect when the parameter is very small. We will keep the parameter only upto linear order. Such a general transformation may be written as

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\ \Phi'\left(x'\right) &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x), \end{aligned} \tag{B.2.1}$$

where $\{\omega_a\}$ is a set of infinitesimal parameters.

Definition B.2.1. The *generator* G_a of a symmetry transformation is defined by the following expression

$$\delta_{\omega}\Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i\omega_a G_a \Phi(x).$$

Observe that to first order in ω_a,

$$\begin{aligned}\Phi'(x') &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \\ &= \Phi(x') - \omega_a \frac{\delta x^\mu}{\delta \omega_a}\partial_\mu \Phi(x') + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x').\end{aligned}$$

This gives an explicit expression for the generators:

$$iG_a\Phi = \frac{\delta x^\mu}{\delta \omega_a}\partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{B.2.2}$$

Example B.2.2. (i) Infinitesimal translation: for an infinitesimal translation by a vector ε^μ (the index a becomes here a spacetime index), we have $\delta x^\mu / \delta \varepsilon^\nu = \delta^\mu_\nu$ and $\mathcal{F}$ is trivial. Thus using (B.2.2), we see that the generator is simply

$$P_\mu = -i\partial_\mu.$$

(ii) Infinitesimal Lorentz transformation: an infinitesimal Lorentz transformation has the form

$$\begin{aligned}x'^\mu &= x^\mu + \omega^\mu{}_\nu x^\nu \\ &= x^\mu + \omega_{\rho\nu}\eta^{\rho\mu}x^\nu.\end{aligned}$$

Using (1.2.1), we can easily see that $\omega_{\mu\nu} = -\omega_{\nu\mu}$. This antisymmetry gives the following variation of coordinates:

$$\frac{\delta x^\mu}{\delta \omega_{\rho\nu}} = \frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\nu\mu}x^\rho\right).$$

The field Φ transforms as

$$\mathcal{F}(\Phi) = L_\Lambda \Phi, \qquad L_\Lambda \approx 1 - \frac{1}{2} i\omega_{\rho\nu}S^{\rho\nu}$$

where $S^{\rho\nu}$ is some Hermitian matrix obeying the Lorentz algebra (generator of Lie algebra in the particular representation in which Φ belongs). Using (B.2.2), we get

$$\frac{1}{2} i\omega_{\rho\nu}L^{\rho\nu}\Phi = \frac{1}{2}\omega_{\rho\nu}\left(x^\nu\partial^\rho - x^\rho\partial^\nu\right)\Phi + \frac{1}{2} i\omega_{\rho\nu}S^{\rho\nu}\Phi$$

where $L^{\rho\nu}$ is the generator. The factor of $\frac{1}{2}$ preceding $\omega_{\rho\nu}$ in the definitions of $L^{\rho\nu}$ and $S^{\rho\nu}$ cancels the double counting of transformation parameters. The generators of Lorentz transformations are thus

$$L^{\rho\nu} = i\left(x^\rho\partial^\nu - x^\nu\partial^\rho\right) + S^{\rho\nu}.$$

In particular, if we take $\Phi(x) = x$, then $\mathcal{F}$ is trivial and the generator is simply $L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right)$ which is just the angular momentum operator. Thus on spacetime, Lorentz transformation is generated by the angular momentum operator.

(iii) Infinitesimal dilatation: let $x \longrightarrow x' = (1+\alpha)x$ be an infinitesimal scaling then it is clear that $\delta x/\delta\alpha = x$. Next, the field transforms as $\Phi'((1+\alpha)x) = (1+\alpha)^{-\Delta}\Phi(x)$ where Δ is the scaling dimension of the field Φ. Thus we have

$$\frac{\delta \mathcal{F}}{\delta \alpha} = \frac{\delta}{\delta \alpha}((1-\Delta\alpha)\Phi(x)) = -\Delta\Phi(x).$$

Thus using (B.2.2), we find that the generator of infinitesimal dilation in the representation Φ is

$$D := -ix^\mu\partial_\mu - i\Delta. \tag{B.2.3}$$

B.2.1 Generators of Conformal Transformations

As described in Subsection 5.2.1, conformal transformation in dimensions $D \geq 3$ include four different kinds of transformations. We will now find the *spacetime* generators of those transformations which we directly indicated in Subsection 5.2.1.

(i) Translation: we already computed the generator for any field Φ. In particular, for the field $\Phi(x) = x$, the generator is

$$P_\mu = -i\partial_\mu.$$

(ii) Lorentz transformation: the spacetime generator for Lorentz transformation was computed to be

$$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right).$$

(iii) Dilatation: let α be an infinitesimal dilatation parameter, then $x \longrightarrow x' = (1+\alpha)x$. Thus we have

$$\frac{\delta x^\mu}{\delta \alpha} = x^\mu.$$

Hence the generator D of dilatation can be directly read off from (B.2.2),

$$D = -ix^\mu\partial_\mu.$$

(iv) Special conformal transformation: let b^μ be an infinitesimal SCT parameter. Then spacetime transforms as

$$x'^\mu = x^\mu + 2(x\cdot b)x^\mu - (x\cdot x)b^\mu.$$

Thus we quickly find that

$$\frac{\delta x^\mu}{\delta b^\nu} = 2x_\rho\delta^\rho{}_\nu x^\mu - (x\cdot x)\delta^\mu{}_\nu.$$

Hence the generator is

$$K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu).$$

(v) Two dimensional infinitesimal conformal transformations: in Subsection 5.3.1, we concluded that infinitesimal conformal transformations in complex coordinates are given by

$$
\begin{aligned}
z' &= z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n \left(-z^{n+1}\right) \\
\bar{z}' &= \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right)
\end{aligned}
$$

So the generator corresponding to $z' = z - \varepsilon_n z^{n+1}$ and $\bar{z}' = \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1}$ can be computed similar to above cases. Indeed for the transformation of z, we have

$$\frac{\delta z}{\delta \varepsilon_n} = -z^{n+1}.$$

Using (B.2.2)[1] we get,

$$l_n = -z^{n+1}\partial_z.$$

Similarly we get the conjugated generator.

B.2.2 Noether's Theorem

In classical field theory, the dynamics is governed by the action of the classical field. A *classical symmetry* is a symmetry of the action under some transformation of the field. Noether's theorem is a statement about a particularly fruitful consequence of continuous symmetries.

Theorem B.2.3. *Let Φ be a classical field and $S[\Phi]$ be its action. Given a continuous symmetry parametrised by ω_a of the action, there exists a conserved current j_a^μ in the sense that when the classical equations of motion are satisfied then*

$$\partial_\mu j_a^\mu = 0.$$

We will not prove this theorem here as it is a standard result covered in any quantum field theory course. If we explicitly write the transformation of field as

$$
\begin{aligned}
x'^\mu &= x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \\
\Phi'\left(x'\right) &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x),
\end{aligned}
$$

then the conserved current j_a^μ is given by

$$j_a^\mu = \left\{ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta^\mu{}_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a}, \tag{B.2.4}$$

[1] we have to remove i since we are already in complex coordinates.

where $\mathcal{L}$ is the Lagrangian density. Under the transformation, the action changes as follows:

$$\delta S = -\int dx j_a^\mu \partial_\mu \omega_a, \tag{B.2.5}$$

which after integration by parts yields the conservation of current under the assumption that the classical equations of motions are satisfied. Observe that we can add any antisymmetric tensor to the current without affecting its conservation. Indeed for

$$j_a^\mu \longrightarrow j_a^\mu + \partial_\nu B_a^{\nu\mu} \quad , \quad B_a^{\nu\mu} = -B_a^{\mu\nu}$$

$\partial_\mu \partial_\nu B_a^{\nu\mu} = 0$ by antisymmetry. Thus although the expression (B.2.4) for j_a^μ is cannonical, it is somewhat ambiguous.

B.3 Quantum Symmetries: Ward Identity

In previous sections, we discussed classical symmetries which has nothing to do with quantum field theory. We now discuss what it means for a classical symmetry to also be a symmetry of the corresponding quantum theory.

In quantum theory, the most important objects are correlation functions. Consider a theory with field Φ with action $S[\Phi]$. The n-point correlation function is given by

$$\langle \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle = \frac{\int [\mathcal{D}\Phi]\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\exp(-S[\Phi])}{\int [\mathcal{D}\Phi]\exp(-S[\Phi])}.$$

We can say something about the classical symmetry in quantum theory by looking at these correlation function. Indeed, suppose $\Phi \longrightarrow \Phi'$ be a symmetry of the action $S[\Phi]$, that is a classical symmetry. Thus we see that in quantum theory, we need the exponential $\exp(-S[\Phi])$ to be invariant under the transformation. But this is not it. Under this transformation, the integral measure $[\mathcal{D}\Phi]$ may change non trivially and may not remain invariant. Then even if the field transformation is a classical symmetry, it may not be a *quantum symmetry* in the sense that the correlation functions change under the transformation and hence the quantum theory may change entirely.

We have the following theorem if we assume that the integral measure is also invariant under a continuous classical symmetry transformation.

Theorem B.3.1. *Suppose* (B.1.1) *and* (B.1.2) *be a classical symmetry of the action* $S[\Phi]$*. Suppose also that the functional integration measure* $[\mathcal{D}\Phi]$ *is also invariant under* (B.1.2)*. Then we have*

$$\langle \Phi\left(x_1'\right)\cdots\Phi\left(x_n'\right)\rangle = \langle \mathcal{F}\left(\Phi\left(x_1\right)\right)\cdots\mathcal{F}\left(\Phi\left(x_1\right)\right)\rangle .$$

Proof. The proof is straightforward using change of variables. □

Example B.3.2. (i) Under translation $x \longrightarrow x + a$, we have

$$\langle \Phi(x_1 + a) \cdots \Phi(x_n + a) \rangle = \langle \Phi(x_1) \cdots \Phi(x_n) \rangle .$$

(ii) Lorentz invariance of scalar fields results in the following identity

$$\langle \Phi(\Lambda^{\mu}{}_{\nu} x_1^{\nu}) \cdots \Phi(\Lambda^{\mu}{}_{\nu} x_n^{\nu}) \rangle = \langle \Phi(x_1^{\mu}) \cdots \Phi(x_n^{\mu}) \rangle .$$

(iii) Scale invariance of scalar fields ϕ_i with scaling dimensions Δ_i gives

$$\langle \phi_1(\lambda x_1) \cdots \phi_n(\lambda x_n) \rangle = \lambda^{-\Delta_1} \dots \lambda^{-\Delta_n} \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle .$$

Another consequence of a classical symmetry and the invariance of functional integral measure is the following theorem:

Theorem B.3.3. (Ward Identities) *Let* (B.2.1) *be a classical infinitesimal symmetry of the action* $S[\Phi]$ *with generators* G_a *and corresponding classical conserved current* j_a^{μ}. *Suppose also that functional integral measure is invariant under the symmetry transformation of fields. Then we have*

$$\frac{\partial}{\partial x^{\mu}} \langle j_a^{\mu}(x) \Phi(x_1) \cdots \Phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \Phi(x_1) \cdots G_a \Phi(x_i) \cdots \Phi(x_n) \rangle .$$

Ward identity is the quantum version of Noether's theorem. Given a classical symmetry, it survives quantisation if the corresponding Ward identity holds and we say that the classical symmetry is also a quantum symmetry.

Definition B.3.4. A classical symmetry $\Phi \longrightarrow \Phi'$ is called an anomaly if the corresponding Ward identity does not hold. In this case we say that we have a quantum symmetry breaking.

There are other aspects of symmetry braking a quantum level, *spontaneous symmetry breaking* for example in which case the projective unitary representation of the classical symmetry group does not keep the ground state of the quantum theory invariant. We will not delve further into these topics.

We will use Ward identities in conformal field theory very often. Ward identity of fields characterises them as primary or non primary fields.

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