

Superconformal Index Of 4d $\mathcal{N} = 2$ SCFTs From Chiral Algebras And Monodromy Operator

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Superconformal Index Of 4d $\mathcal{N} = 2$ SCFTs

Definition

4d $\mathcal{N} = 2$ superconformal algebra:

- Lorentz (Euclidean): $SU(2) \times SU(2)$, Cartan generators $(j, \bar{j})$.
- Scaling Cartan generator E .
- R-symmetry $SU(2) \times U(1)$, Cartan generators (R, r) .
- Supercharges: Poincaré $Q_\alpha^I, \tilde{Q}_J^{\dot{\alpha}}$, Conformal $(Q_\alpha^I)^\dagger, (\tilde{Q}_J^{\dot{\alpha}})^\dagger$.
- Special conformal transformations: generators annihilate conformal primaries.

Superconformal index: consider the theory on $S^3 \times S^1$ and define

$$\mathcal{I}_{\text{SCI}}(p, q, t) = \text{Tr}_{\mathcal{H}(S^3)} \left[(-1)^F p^{j-\bar{j}-r} q^{j+j-r} t^{R+r} \right].$$

Kinney-Maldacena-Minwalla-Raju, Römelsberger

Can also allow flavor fugacities.

Only $\frac{1}{8}$ -BPS states contribute.

Macdonald and Schur Index

Macdonald limit: $p \rightarrow 0$:

$$\mathcal{I}_M(q, t) = \text{Tr}_{\mathcal{H}(S^3)} \left[(-1)^F q^{E-2R-r} t^{R+r} \right] .$$

Counts $\frac{1}{4}$ -BPS states.

Gadde-Rastelli-Razamat-Yan

Schur limit: $t \rightarrow q$

$$\mathcal{I}_S(q) = \text{Tr}_{\mathcal{H}(S^3)} \left[(-1)^F q^{E-R} \right] .$$

Also counts $\frac{1}{4}$ -BPS states but with less information.

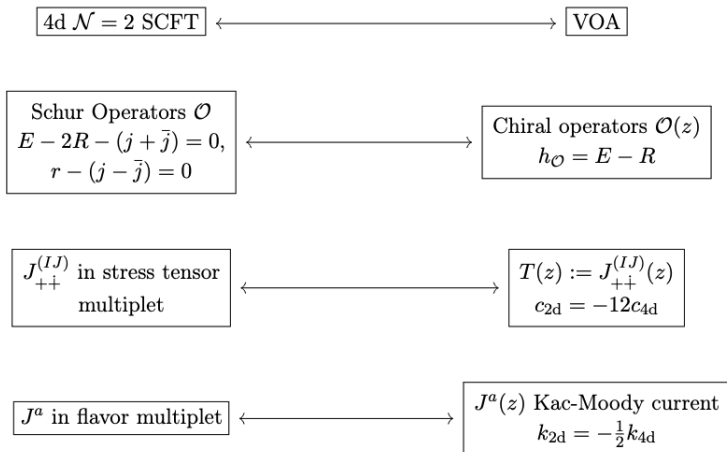
Gadde-Rastelli-Razamat-Yan

$$\mathcal{I}_S(q) = \lim_{t \rightarrow q} \mathcal{I}_M(q, t) .$$

Various indirect ways of computing the index: IR fixed point of Lagrangian theories, SCFT/VOA correspondence, Monodromy operator, class S techniques, q -deformed Yang-Mills...

Schur Index Using SCFT/VOA

SCFT/VOA Correspondence



SCFT/VOA Correspondence implies $\mathcal{I}_S(q) = \chi_V(q)$.

Example: Argyres-Douglas theories

- A class of non-Lagrangian theories labeled by two ADE Lie algebras $(\mathfrak{g}_1, \mathfrak{g}_2)$.

Cecotti-Neitzke-Vafa

- Construction using Type IIB compactified on singular Calabi-Yau.

Klemm-Lerche-Mayr-Vafa-Warner, Ooguri-Vafa

- Singularities on Coulomb branch of Lagrangian theories.

Argyres-Douglas, Argyres-Plesser-Seiberg-Witten

- IR fixed points of $\mathcal{N} = 1$ Lagrangian theories.

Maruyoshi-Song

- These constructions give us ways of computing the SCI.

$$V_{(A_1, A_{2n})} = \text{Vir}_{c_n}, \quad c_n = -\frac{2n(6n+5)}{2n+3}.$$

$$V_{(A_1, D_{2n+1})} = \widehat{\mathfrak{su}(2)}_{-\frac{4n}{2n+1}}, \quad c_n = -6n.$$

Beem-Rastelli

Schur Indices Of (A_1, A_{2n}) And (A_1, D_{2n+1}) AD Theories

$$\mathcal{I}_S^{(A_1, A_{2n})}(q) = \sum_{N_1 \geq N_2 \geq \dots \geq N_n \geq 0} \frac{q^{N_1^2 + \dots + N_n^2 + N_1 + N_2 + \dots + N_n}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{n-1} - N_n} (q; q)_{N_n}},$$

where $(a, q)_n = \prod_{k=0}^{n-1} (1 - aq^k)$.

$$\mathcal{I}_S^{(A_1, D_{2n+1})}(q) = \left(\frac{(q^{2n+1}; q^{2n+1})_\infty}{(q; q)_\infty} \right)^3, \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

Buican-Nishinaka

Matches with the vacuum character of the corresponding VOAs.

Question: Can we get the Macdonald index from VOA?

Partial answer: Refined character of the VOA

Song, Watanabe-Zhu

Macdonald Index From VOA

Strongly finitely generated vertex operator algebras

Vertex operator (super)algebra is a

- $(\frac{1}{2})\mathbb{Z}$ -graded vector space $V = \bigoplus_{k \in (\frac{1}{2})\mathbb{Z}} V_{(k)}$.
- a vertex operator map $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$, $v_n \in \text{End}(V)$.
- A conformal vector $\omega \in V_{(2)}$, such that L_n defined by $T(z) := Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ satisfies the Virasoro algebra with central charge c .
- Vacuum vector $|0\rangle \in V_{(0)}$ such that $Y(|0\rangle, z) = 1, \lim_{z \rightarrow 0} Y(v, z)|0\rangle = v$.
- Axioms: OPE, locality, commutativity...

A VOA is called *strongly finitely generated* if there exists $u^1, \dots, u^N$ such that

$$V = \text{Span}\{u_{-n_1}^{i_1} u_{-n_2}^{i_2} \cdots u_{-n_k}^{i_k} |0\rangle : n_1, n_2, \dots, n_k > 0, k \geq 0\} / \{\text{null vectors}\}$$

$u^1, \dots, u^N$ are called *strong generators*. Affine and Virasoro VOAs are strongly finitely generated.

Zhu algebra

- Define

$$\mathcal{A}(V) := V/C_2(V), \quad C_2(V) := \text{Span}\{a_{-2} \cdot b : a, b \in V\}.$$

- $\mathcal{A}(V)$ is a commutative associative algebra called *Zhu algebra*. Zhu
- It can be proved that for a strongly finitely generated VOA with N strong generators Xie-Yan, Andrews-Banerjee-RKS-Tao-Bhargava

$$\mathcal{A}(V) \cong \mathbb{C}[x^{(1)}, \dots, x^{(N)}]/I, \quad (u^i \mapsto x^{(i)})$$

where I is the ideal generated by null vector relations.

- Example:

$$\mathcal{A}\left(\text{Vir}_{-\frac{2n(6n+5)}{2n+3}}\right) \cong \mathbb{C}[x]/\langle x^{n+1} \rangle.$$

$$\mathcal{A}\left(\widehat{\mathfrak{su}(2)}_{-\frac{4n}{2n+1}}\right) \cong \mathbb{C}[x, y, z, j]/\langle f_1, f_2, f_3, f_4, f_5 \rangle,$$

where

$$f_1 = xz - y^2 - j, f_2 = j^{n+1}x, f_3 = j^{n+1}y, f_4 = j^{n+1}z, f_5 = j^{n+1}.$$

Beem-Rastelli

Arc space of a polynomial ring

Let $R = \mathbb{C}[x^{(1)}, \dots, x^{(N)}] / \langle f_1, \dots, f_K \rangle$ be a polynomial ring. Assign weights $d_i \geq 1$ to $x^{(i)}$ and write formal power series

$$x^{(i)}(s) = \sum_{n=d_i-1}^{\infty} x_n^{(i)} s^n.$$

The n -th jet space of R is defined by

$$\begin{aligned} R_n &:= \mathbb{C}[x_{d_1-1}^{(1)}, \dots, x_n^{(1)}, x_{d_2-1}^{(2)}, \dots, x_n^{(2)}, \dots, x_{d_N-1}^{(N)}, \dots, x_n^{(N)}] / I_n, \\ I_n &:= \langle f_1(\vec{x}(s)), \dots, f_K(\vec{x}(s)) \equiv 0 \bmod s^{n+1} \rangle. \end{aligned}$$

The n -th Jet scheme is defined as

$$X_n(R) := \operatorname{Spec} R_n.$$

The *arc space* $X_\infty(R)$ of R is defined to be

$$X_\infty(R) = \varprojlim X_n(R) \cong \operatorname{Spec} R_\infty.$$

Hilbert series of arc space

Assign bigrading $\text{wt}(x_j^{(i)}) = (j, w(x^{(i)}))$ and¹ define

$$H_{R_\infty}(q, t) := \sum_{\ell, m \geq 0} \dim(R_\infty)_{\ell, m} q^\ell t^m .$$

Example: $R = \mathbb{C}[x]/\langle x^2 \rangle$. Choosing $w(x) = 1$, the Hilbert series is given by

$$H_{R_\infty}(q, t) = \sum_{m \geq 0} t^m \frac{q^{m^2}}{(q; q)_m} .$$

Bruschek-Mourtada-Schepers, Bai-Gorsky-Kivinen

This agrees with the Macdonald index of (A_1, A_2) theory!!

Buican-Nishinaka-Bhargava

¹Warning: R-filtration!!

Illustration: $R = \mathbb{C}[x, y, z, j] / \langle xz - y^2 - j, jx, jy, jz, j^2 \rangle$

Choose $\text{wt}(x, y, z) = 1, \text{wt}(j) = 2, w(x, y, z, j) = 1$. We have the formal series

$$x(s) = \sum_{n=0}^{\infty} x_n s^n, \quad y(s) = \sum_{n=0}^{\infty} y_n s^n, \quad z(s) = \sum_{n=0}^{\infty} z_n s^n, \quad j(s) = \sum_{n=1}^{\infty} j_n s^n.$$

n -th jet space is

$$R_n := \mathbb{C}[x_0, \dots, x_n, y_0, \dots, y_n, z_0, \dots, z_n, j_1, \dots, j_n] / I_n,$$

$$I_n := \langle f_i(x(s), y(s), z(s), j(s)) \equiv 0 \bmod s^{n+1} \rangle.$$

For example:

$$I_0 = \langle x_0 z_0 - y_0^2 \rangle,$$

$$I_1 = \langle x_0 z_0 - y_0^2, x_0 z_1 - 2y_0 y_1 + z_0 x_1 - j_1, x_0 j_1, y_0 j_1, z_0 j_1 \rangle$$

Illustration continued...

$$\left. \begin{array}{l} (R_\infty)_{0,0} : 1 \\ (R_\infty)_{0,1} : x_0, y_0, z_0 \\ (R_\infty)_{n,1} : x_n, y_n, z_n, j_n \end{array} \right\} \implies H_{R_\infty}(q, t) = 1 + \left(3 + 4 \sum_{n=1}^{\infty} q^n \right) t + O(t^2).$$

Next

$$(R_\infty)_{0,2} : \mathbb{C}[x_0^2, y_0^2, z_0^2, x_0 y_0, x_0 z_0, y_0 z_0] / \langle x_0 z_0 - y_0^2 \rangle \implies \dim(R_\infty)_{0,2} = 5.$$

Counting similarly, $\dim(R_\infty)_{1,2} = 8$, $\dim(R_\infty)_{2,2} = 17$.

Systematic method: using Gröbner basis algorithm, we found a general formula for the Hilbert series. Can be evaluated using Mathematica.

$$H_{R_\infty}(q, t) = \sum_{m=0}^{\infty} \frac{t^m}{(q; q)_m} \sum_{k=0}^{2m} \frac{(q; q)_{2m}}{(q; q)_k (q; q)_{2m-k}}. \quad \text{Andrews-Banerjee-RKS-Tao-Bhargava}$$

Matches with the Macdonald index of (A_1, D_3) theory!!

Conjecture

Conjecture: Macdonald index from VOA Andrews-Banerjee-RKS-Tao-Bhargava

Let V_T be the 2d VOA corresponding to a 4d $\mathcal{N} = 2$ SCFT T and let $\mathcal{A}(V_T) \cong \mathbb{C}[x^{(1)}, \dots, x^{(N)}]/I$ be its Zhu algebra. Assign the grading $\text{wt}(x^{(i)}) = d_i$, where d_i is the conformal dimension of strong generator u^i . Then there exists $w(x^{(i)})$ such that

$$\mathcal{I}_M^T(q, t) = H_{\mathcal{A}(V_T)_\infty}(q, t).$$

Checked it for (A_1, A_{2n}) and (A_1, D_{2n+1}) theories.

Equating the Schur limit with the known Schur index gives new q -series identities.

- (A_1, A_{2n}) theory gives Rogers-Ramanujan identities.

Buican-Nishinaka-Bhargava

- We proved new identities coming from (A_1, D_{2n+1}) theory.

$$(A_1, D_3) : \left(\frac{(q^3; q^3)_\infty}{(q; q)_\infty} \right)^3 = \sum_{m=0}^{\infty} \frac{q^m}{(q; q)_m} \sum_{k=0}^{2m} \frac{(q; q)_{2m}}{(q; q)_k (q; q)_{2m-k}}.$$

Schur Index From Monodromy Operator

Coulomb branch geometry basics

4d $\mathcal{N} = 2$ theories generically have moduli space of vacua: Coulomb branch, Higgs branch and mixed branch.

- Coulomb branch $\mathcal{B}$ parametrized by u . Rank = $\dim_{\mathbb{C}}(\mathcal{B}) = r$.
- Charge lattice: Γ_u with antisymmetric pairing $\langle \cdot, \cdot \rangle$.
- Central charge function $\mathcal{Z} : \Gamma_u \rightarrow \mathbb{C}$.
- BPS indices $\Omega : \Gamma \rightarrow \mathbb{Z}$ which obeys *wall crossing formula*.

BPS Index

Restrict to the sublattice Γ^{BPS} of BPS charges².

- BPS particles are in irreducible representations of $\text{SU}(2)_L \times \text{SU}(2)_R$, where $\text{SU}(2)_L$ is the little group of the Lorentz group.
- Let $\mathcal{H}^\gamma$ be the Hilbert space of BPS states with charge γ . We can factor out the center-of-mass dof to write

$$\mathcal{H}^\gamma = [(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})] \otimes \mathcal{H}_{\text{int}}^\gamma.$$

- Define the Protected Spin Character

$$\Omega_\gamma(y) = \text{Tr}_{\mathcal{H}_{\text{int}}^\gamma} (-1)^R y^{J+R} = \sum_{n \in \mathbb{Z}} \Omega_n(\gamma) y^n, \quad \Omega_n(\gamma) \in \mathbb{Z}.$$

where J is the Cartan generator of $\text{SU}(2)_L$.

Gaiotto-Moore-Neitzke

- $\Omega_\gamma(y)$ jumps across walls of marginal stability and obeys a wall crossing formula.

Denef, Denef-Moore, Gaiotto-Moore-Neitzke

²Dropped the u -dependence.

Monodromy operator

Quantum torus algebra

$$\mathcal{A} = \text{Span}\{X_\gamma : \gamma \in \Gamma^{\text{BPS}}\}$$
$$X_\gamma X_{\gamma'} = q^{\frac{\langle \gamma, \gamma' \rangle}{2}} X_{\gamma+\gamma'} = q^{\langle \gamma, \gamma' \rangle} X_{\gamma'} X_\gamma.$$

Define the quantum dilogarithm

$$E_q(z) := \prod_{i=0}^{\infty} \left(1 + q^{i+\frac{1}{2}} z\right)^{-1} = \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}} z)^n}{(q; q)_n}.$$

The monodromy operator is defined as

$$\mathcal{O}(q) := \prod_{\gamma \in \Gamma^{\text{BPS}}}^{\curvearrowright} U_\gamma, \quad U_\gamma = \prod_{n \in \mathbb{Z}} E_q \left((-1)^n q^{n/2} X_\gamma \right)^{(-1)^n \Omega_n(\gamma)},$$

where the product is taken in the increasing order of $\text{Arg}(\mathcal{Z}(\gamma))$ with respect to a reference angle θ .

Cordova-Shao Conjecture

A charge $\gamma \in \Gamma^{\text{BPS}}$ is called a flavor charge if $\langle \gamma, \Gamma^{\text{BPS}} \rangle = 0$. Any flavor charge can be expanded in a basis $\gamma_f = \sum_{i=1}^{N_f} f_i(\gamma) \gamma_{f_i}$. Define

$$\text{Tr}[X_\gamma] = \begin{cases} \prod_i \text{Tr} [X_{\gamma_{f_i}}]^{f_i(\gamma)} & \langle \gamma, \Gamma^{\text{BPS}} \rangle = 0, \\ 0 & \text{else,} \end{cases}.$$

Cordova-Shao Conjecture

The Schur index of a 4d $\mathcal{N} = 2$ SCFT of rank r is given by

$$\mathcal{I}_S(q, z_1, \dots, z_{N_f}) := (q; q)_\infty^{2r} \text{Tr} \mathcal{O}(q),$$

where z_i are the flavor fugacities and related to $\text{Tr} [X_{\gamma_{f_i}}]$.

Can we compute Macdonald index from trace of monodromy operator?

Macdonald Index From Monodromy Operator

BPS quivers of 4d $\mathcal{N} = 2$ theories

The BPS spectrum of some theories in a given chamber is captured by its BPS quiver. Prominent example: $(\mathfrak{g}_1, \mathfrak{g}_2)$ theories. $(A_1, \mathfrak{g})$ theories admit source/sink chamber.

Example: (A_1, A_{2n}) theory.



Cecotti-Neitzke-Vafa

- $\text{Arg}(\mathcal{Z}(\gamma)) > \text{Arg}(\mathcal{Z}(\gamma'))$ for $\gamma \in \text{sink}$ and $\gamma' \in \text{source}$.
- $[X_\gamma, X_{\gamma'}] = 0$, for $\gamma, \gamma' \in \text{sources}$, or $\gamma, \gamma' \in \text{sinks}$.
- $\langle \gamma_i, \gamma_j \rangle$ determined by the number of arrows between i, j .

$$\mathcal{O}(q) = \prod_{i=1}^n E_q(X_{-\gamma_{2i-1}}) \prod_{i=1}^n E_q(X_{-\gamma_{2i}}) \prod_{i=1}^n E_q(X_{\gamma_{2i-1}}) \prod_{i=1}^n E_q(X_{\gamma_{2i}}) .$$

Refined monodromy operator

Define

$$E_{q,T}(X_\gamma) = \sum_{n=0}^{\infty} \frac{(-(qT)^{\frac{1}{2}} X_\gamma)^n}{(q; q)_n}, \quad \tilde{E}_{q,T}(X_\gamma) = \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}} X_\gamma)^n}{(qT; q)_n}.$$

Prescription:

$$\begin{aligned} E_q(X_{-\gamma_{\text{source}}}) &\rightarrow E_{q,T}(X_{-\gamma_{\text{source}}}), & E_q(X_{-\gamma_{\text{sink}}}) &\rightarrow \tilde{E}_{q,T}(X_{-\gamma_{\text{sink}}}) \\ E_q(X_{\gamma_{\text{source}}}) &\rightarrow E_{q,T}(X_{\gamma_{\text{source}}}), & E_q(X_{\gamma_{\text{sink}}}) &\rightarrow E_q(X_{\gamma_{\text{sink}}}). \end{aligned}$$

If there is a source with valency > 2 then exchange the roles of sources and sinks and apply the above prescription.

Conjecture Andrews-Banerjee-RKS-Tao

For a 4d $\mathcal{N} = 2$ SCFT with source/sink chamber, the Macdonald index is given by

$$\mathcal{I}_M(q, T) = (q; q)_\infty^r (qT; q)_\infty^r \text{Tr}[\mathcal{O}(q, T)].$$

$qT = t$ and Schur limit is $T \rightarrow 1$. Since $\mathcal{O}(q, 1) = \mathcal{O}(q)$, the Schur limit gives the Schur index.

Computing the trace for (A_1, A_{2n}) theory

For (A_1, A_{2n}) , we get

$$\mathcal{O}(q, T) = \prod_{i=1}^n E_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^n \tilde{E}_{q,T}(X_{-\gamma_{2i}}) \prod_{i=1}^n E_{q,T}(X_{\gamma_{2i-1}}) \prod_{i=1}^n E_q(X_{\gamma_{2i}}) .$$

Replace the definition of $E_{q,T}, \tilde{E}_{q,T}$.

$$\begin{aligned} \mathcal{O}(q, T) = & \sum_{\substack{\ell_1, \dots, \ell_{2n} \geq 0 \\ k_1, \dots, k_{2n} \geq 0}} \frac{(-1)^{\sum_{i=1}^{2n} (k_i + \ell_i)} q^{\frac{1}{2} \sum_{i=1}^{2n} (k_i + \ell_i)} T^{\frac{1}{2} \sum_{i=1}^n (\ell_{2i-1} + k_{2i-1})}}{\prod_{i=1}^n (q)_{k_{2i-1}} (q)_{\ell_{2i-1}} \prod_{i=1}^n (q)_{k_{2i}} (qT)_{\ell_{2i}}} \\ & \times \prod_{i=1}^n X_{-\gamma_{2i-1}}^{\ell_{2i-1}} \prod_{i=1}^n X_{-\gamma_{2i}}^{\ell_{2i}} \prod_{i=1}^n X_{\gamma_{2i-1}}^{k_{2i-1}} \prod_{i=1}^n X_{\gamma_{2i}}^{k_{2i}} . \end{aligned}$$

Shortened notation $(a)_n = (a; q)_n$.

Computation of trace for (A_1, A_{2n}) theory continued...

Use quantum torus algebra to write

$$\begin{aligned} & \prod_{i=1}^n X_{-\gamma_{2i-1}}^{\ell_{2i-1}} \prod_{i=1}^n X_{-\gamma_{2i}}^{\ell_{2i}} \prod_{i=1}^n X_{\gamma_{2i-1}}^{k_{2i-1}} \prod_{i=1}^n X_{\gamma_{2i}}^{k_{2i}} \\ &= q^{-\sum_{i,j=1}^n k_{2j-1} \ell_{2i} C_{2i,2j-1}^{A_{2n}}} \prod_{i=1}^n X_{\gamma_{2i-1}}^{k_{2i-1} - \ell_{2i-1}} \prod_{i=1}^n X_{\gamma_{2i}}^{k_{2i} - \ell_{2i}} \end{aligned}$$

where $C_{ij}^{A_{2n}}$ is the Cartan matrix of A_{2n} . Use quantum torus algebra again

$$\mathrm{Tr} \left[\prod_{i=1}^n X_{\gamma_{2i-1}}^{k_{2i-1} - \ell_{2i-1}} \prod_{i=1}^n X_{\gamma_{2i}}^{k_{2i} - \ell_{2i}} \right] = q^{\#} \mathrm{Tr} \left[X_{\sum_{i=1}^{2n} (k_i - \ell_i) \gamma_i} \right] = \prod_{i=1}^{2n} \delta_{k_i, \ell_i}.$$

Last step follows from absence of flavor charges. After some simplification, our conjecture gives

$$\mathcal{I}_{\mathrm{M}}^{(A_1, A_{2n})}(q, T) = (q)_{\infty}^n (qT)_{\infty}^n \sum_{k_1, \dots, k_{2n} \geq 0} \frac{q^{\sum_{i=1}^{2n} k_i + \frac{1}{2} \sum_{i=1}^{2n-1} k_i k_{i+1}} T^{\sum_{i=1}^n k_{2i-1}}}{\prod_{i=1}^n (q)_{k_{2i-1}}^2 \prod_{i=1}^n (q)_{k_{2i}} (qT)_{k_{2i}}}.$$

New identity

Equating with the known Macdonald index gives

$$\begin{aligned} (q)_\infty^n (Tq)_\infty^n \sum_{k_1, \dots, k_{2n} \geq 0} \frac{T^{k_1+k_3+\dots+k_{2n-1}} q^{k_1k_2+k_2k_3+\dots+k_{2n-1}k_{2n}+k_1+\dots+k_{2n}}}{\prod_{i=1}^n (q)_{k_{2i-1}}^2 \prod_{i=1}^n (q)_{k_{2i}} (Tq)_{k_{2i}}} \\ = \sum_{\ell_1, \dots, \ell_n \geq 0} \frac{T^{\ell_1+\dots+\ell_n} q^{\ell_1^2+\dots+\ell_n^2+\ell_1+\dots+\ell_n}}{(q)_{\ell_1-\ell_2} (q)_{\ell_2-\ell_3} \cdots (q)_{\ell_{n-1}-\ell_n} (q)_{\ell_n}}. \end{aligned}$$

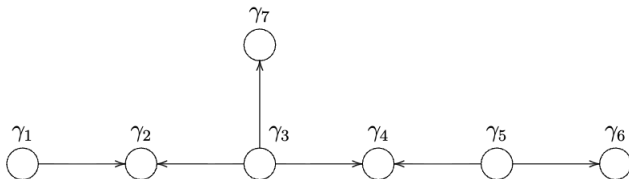
Can be proved rigorously.

Andrews-Banerjee-**RKS**-Tao

Reproduces the Macdonald index of all $(A_1, \mathfrak{g})$ theories
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Andrews-Banerjee-**RKS**-Tao

Macdonald index of (A_1, E_7) theory



Cecotti-Neitzke-Vafa

Source with valency > 2 . With our prescription

$$\mathcal{O}(q, T) = \prod_{i=1}^3 E_{q,T}(X_{-\gamma_{2i}}) \prod_{i=1}^4 \tilde{E}_{q,T}(X_{-\gamma_{2i-1}}) \prod_{i=1}^3 E_{q,T}(X_{\gamma_{2i}}) \prod_{i=1}^4 E_q(X_{\gamma_{2i-1}})$$

The theory has $U(1)$ flavor symmetry. Let z be the flavor fugacity. Choose the flavor charge to be $\gamma_f = \gamma_4 - \gamma_6 - \gamma_7$ and $\text{Tr}[X_{\gamma_f}] = -z$.

Macdonald index of (A_1, E_7) theory continued...

The Macdonald index is given by

$$\begin{aligned}
 & \mathcal{I}_M^{(A_1, E_7)}(q, T, z) \\
 &= (q)_\infty^3 (qT)_\infty^3 \sum_{\substack{\ell_1, \dots, \ell_7 \geq 0 \\ k_1, \dots, k_7 \geq 0}} \frac{(-1)^{\sum_{i=1}^7 (k_i + \ell_i)} T^{\frac{1}{2}(k_7 + \ell_7) + \frac{1}{2} \sum_{i=1}^3 (k_{2i} + \ell_{2i})}}{(q)_{k_7} (q)_{\ell_7} \prod_{i=1}^3 (qT)_{k_{2i-1}} (q)_{\ell_{2i-1}}} \\
 & \quad \times \frac{q^{\frac{1}{2} \sum_{i=1}^7 (k_i + \ell_i)} q^{k_2(\ell_1 + \ell_3) + k_4(\ell_3 + \ell_5) + k_6 \ell_5 + k_7 \ell_3}}{\prod_{i=1}^3 (q)_{k_{2i}} (q)_{\ell_{2i}}} \\
 & \quad \times (-z)^{-\ell_7 + k_7} \delta_{k_4 + k_7, \ell_4 + \ell_7} \delta_{k_6 - k_7, \ell_6 - \ell_7} \prod_{i \in \{1, 2, 3, 5\}} \delta_{k_i, \ell_i}
 \end{aligned}$$

Limitations

- The VOA approach is incomplete because of R-filtration problem, i.e., determining the weights $w(x^{(i)})$. Song, Beem-Maneghelli-Rastelli
- The prescription from monodromy operator is only for theories with a source/sink chamber.
- The Cordova-Shao prescription works even for non-conformal Lagrangian theory. Our proposal does not seem to work for non-conformal theories.

Future directions

- Generalize the prescription to general 4d $\mathcal{N} = 2$ SCFTs.
- Proof using 3d TFTs?
- Generalization of wall-crossing formula?

Gang-Kim-Stubbs, Kim-Kim-Song

Thank You