

Mock Modularity In CHL Models

Ranveer Kumar Singh

NHETC

Department of Physics and Astronomy
Rutgers University

10 June, 2024

Based on arXiv: 2311.16252 with Ajit Bhand (IISER Bhopal, India)
and Ashoke Sen (ICTS Bangalore, India)

Outline

- 1 Black Hole Microstate Counting
- 2 Some Number Theory Background
- 3 Meromorphicity And Mock Modularity
- 4 Mock Modularity In CHL Models
- 5 Refinement Of DMZ result
- 6 Future Directions

Black Hole Microstate Counting

Background

- One of the great successes of string theory: microscopic description of black holes. [Strominger-Vafa (1996), Dijkgraaf-Verlinde-Verlinde (1996), Shih-Strominger-Yin (2005), Gaiotto (2005), Shih-Yin (2005), Jatkar-Sen (2005), David-Jatkar-Sen (2006)].
- $S_{BH} = \frac{A}{4G_N} = \ln d_{\text{micro}} = S_{\text{stat}}.$

Macroscopic description

Find the Bekenstein-Hawking entropy of $\frac{1}{4}$ -BPS extremal black holes carrying electric and magnetic charges $(\vec{Q}, \vec{P})$ in Type IIB supergravity compactified on $\mathcal{M} \times S^1 \times \tilde{S}^1$ with $\mathcal{M} = K3, T^4$.

Microscopic description

Count the number of $\frac{1}{4}$ -BPS configurations of various extended objects wrapping cycles in $\mathcal{M} \times S^1 \times \tilde{S}^1$ in Type IIB string theory at weak coupling.

- Perfect match at leading order! [Reviews: Sen (2007), Mandal-Sen (2010)]

Microstate Counting and Number Theory

- Microstate counting in Type IIB on $K3 \times T^2$ has fascinating new connections to number theory.
- Degeneracy of $\frac{1}{4}$ -BPS states with charges $(\vec{Q}, \vec{P})$ with $Q^2 = 2n, P^2 = 2m, Q \cdot P = \ell$ are coefficients of mock Jacobi forms of weight -10 and index m . [Dabholkar-Murthy-Zagier (DMZ) (2012)]
- Goal: generalize the story of DMZ to more general string compactifications.

Relevant String Compactifications

- $\mathbb{Z}_N$ orbifold of Type IIB string theory compactified on $\mathcal{M} \times \tilde{S}^1 \times S^1$, $\mathcal{M} = K3, T^4$.
- $\mathcal{N} = 4$ supersymmetry in 4d [Chaudhuri-Hockney-Lykken (1995)].
- Charge quantization of $\frac{1}{4}$ -BPS states [Jatkar-Sen (2005)]:

$$\frac{Q^2}{2} \in \frac{1}{N}\mathbb{Z}, \quad \frac{P^2}{2} \in \mathbb{Z}, \quad P \cdot Q \in \mathbb{Z} .$$

- $N = 1$ with $\mathcal{M} = K3$ is the DMZ case.

Degeneracy of $\frac{1}{4}$ -BPS Dyons

Degeneracy formula [Dijkgraaf-Verlinde-Verlinde (1994), David-Jatkar-Sen (2006)]

$$d(\vec{Q}, \vec{P}) = \frac{(-1)^{\ell+1}}{N} \int_{\mathcal{C}} d\tau d\sigma dz \frac{e^{-2\pi i(m\tau + n\sigma + \ell z)}}{\tilde{\Phi}_k(\tau, z, \sigma)}$$

where $\tilde{\Phi}_k$ is a Siegel modular form of weight k for some subgroup of $\text{Sp}(2, \mathbb{Z})$, the contour $\mathcal{C}$ is given by

$$\begin{aligned} \mathcal{C} : \text{Im}(\tau) &= \frac{P^2}{N\varepsilon} \equiv \frac{2n}{N\varepsilon}, & \text{Im}(\sigma) &= \frac{NQ^2}{\varepsilon} \equiv \frac{2m}{\varepsilon}, \\ & & \text{Im}(z) &= -\frac{P \cdot Q}{\varepsilon} \equiv -\frac{\ell}{\varepsilon}, \\ & & 0 \leq \text{Re}(\tau), \text{Re}(z) < 1, & 0 \leq \text{Re}(\sigma) < N, \end{aligned}$$

and $\varepsilon > 0$ is a small real number.

Some Number Theory Background

Modular Forms Of Integral Weight

A holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called a modular form of weight k for $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma .$$

We also need to specify some growth condition at the cusps but it will not be necessary for this talk.

Fourier expansion at $i\infty$

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_f(n) q^{n/h} ,$$

where $h > 0$ is the smallest integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$.

Modular Form of Half-Integral Weight

A holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ is called a modular form of weight $\kappa \in \mathbb{Z} + \frac{1}{2}$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \left(\frac{c}{d}\right) \varepsilon_d^{2\kappa} (c\tau + d)^\kappa f(\tau) , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset \Gamma_0(4) ,$$

where $\left(\frac{c}{d}\right)$ is the Jacobi symbol, for odd d

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod{4}; \\ i & \text{if } d \equiv -1 \pmod{4} , \end{cases}$$

and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : c \equiv 0 \pmod{N} \right\} .$$

The vector space of modular forms of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ is denoted by $M_\kappa(\Gamma)$.

Jacobi Forms

Jacobi forms are meromorphic functions satisfying modularity transformation with respect to some subgroup $\Gamma \ltimes E$ of $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$. $\varphi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ is called a weight $k \in \mathbb{Z}$, index $m \in \mathbb{Q}$ Jacobi form if

$$\begin{aligned} & \varphi \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \\ &= (c\tau + d)^k \mathbf{e} \left(m \left(\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} - \lambda^2\tau - 2\lambda z - \lambda\mu \right) \right) \varphi(\tau, z), \\ & \mathbf{e}(x) = e^{2\pi i x}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad (\lambda, \mu) \in E \subset \mathbb{Z}^2 \end{aligned}$$

Let $h, \mu > 0$ be the smallest integers such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$, and $(0, \mu) \in E \subset \mathbb{Z}^2$, then we have Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n, r) q^{n/h} \zeta^{r/\mu}.$$

The space of such forms is denoted by $J_{k,m}^{\mathrm{mer}}(\Gamma \ltimes E)$.

Siegel Modular Forms

Siegel upper half space:

$$\mathbb{H}_2 = \left\{ \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} : \tau, \sigma \in \mathbb{H}; z \in \mathbb{C}, \det(\operatorname{Im}(\Omega)) > 0 \right\}.$$

Symplectic group

$$\operatorname{Sp}(2, \mathbb{R}) = \{ M \in \operatorname{GL}(4, \mathbb{R}) : M^t J M = J, J := \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix} \},$$

$\operatorname{Sp}(2, \mathbb{R})$ acts on $\mathbb{H}_2$ as

$$M \circ \Omega := (A\Omega + B)(C\Omega + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \operatorname{Sp}(2, \mathbb{R}).$$

Siegel Modular Form

A Siegel modular form of weight $k \in \mathbb{Z}$ for $G \subset \operatorname{Sp}(2, \mathbb{Z})$ is a meromorphic function $\Phi : \mathbb{H}_2 \rightarrow \mathbb{C}$ such that for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ we have $\Phi((A\Omega + B)(C\Omega + D)^{-1}) = \operatorname{Det}(C\Omega + D)^k \Phi(\Omega)$.

Fourier Expansion Of Siegel Modular Forms

We write $\Phi(\Omega) = \Phi(\tau, z, \sigma)$. Let N_1, N_2, N_3 be least positive integers such that

$$\begin{pmatrix} 1 & 0 & N_1 & N_2 \\ 0 & 1 & N_2 & N_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G.$$

$N_1 = N_2 = N_3 = 1$ for $G = \mathrm{Sp}(2, \mathbb{Z})$. Then

$$\Phi(\tau + N_1, z + N_2, \sigma + N_3) = \Phi(\tau, z, \sigma). \quad (1)$$

Fourier expansion

$$\Phi(\tau, z, \sigma) = \sum_{n,m,r \in \mathbb{Z}} a(n, m, r) q^{n/N_1} \zeta^{r/N_2} p^{m/N_3},$$

where $q = \mathbf{e}(\tau)$, $\zeta = \mathbf{e}(z)$, $p = \mathbf{e}(\sigma)$ and $\mathbf{e}(x) = e^{2\pi i x}$. The sum over indices n, m, r may be restricted depending on the analytic properties of Φ .

Relation Between Jacobi Forms And Siegel Modular Forms

The Fourier expansion of a degree 2 Siegel modular form can be rewritten as

$$\Phi(\tau, z, \sigma) = \sum_m \varphi_m(\tau, z) p^{m/N_3}.$$

This is called the Fourier-Jacobi expansion for the following reason: using the embedding of $\mathrm{SL}(2, \mathbb{Z})$ and $\mathbb{Z}^2$ into $\mathrm{Sp}(2, \mathbb{Z})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\lambda, \mu) \mapsto \begin{pmatrix} 1 & 0 & 0 & \lambda \\ -\mu & 1 & \lambda & 0 \\ 0 & 0 & 1 & \mu \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

one can show that each φ_m transforms like a Jacobi form of weight k and index $\frac{m}{N_3}$ with respect to some subgroup of $\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$.

Relation Between Jacobi Forms And Modular Forms

A Jacobi form φ of index $\frac{m}{N} \in \mathbb{Z}/N$ for $\Gamma \ltimes (N\mathbb{Z} \times \mathbb{Z}) \subset \mathbb{Z}^2$ which is holomorphic in z for fixed τ has a *theta decomposition*

$$\varphi(\tau, z) = \sum_{\ell \in \mathbb{Z}/2m\mathbb{Z}} h_{\ell}(\tau) \vartheta_{m,\ell}(N\tau, z) ,$$

where $\vartheta_{m,\ell}$ is the index m Jacobi theta function defined by

$$\vartheta_{m,\ell}(\tau, z) := \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \ell \pmod{2m}}} q^{r^2/4m} \zeta^r = \sum_{n \in \mathbb{Z}} q^{(\ell+2mn)^2/4m} \zeta^{\ell+2mn} .$$

Then each $h_{\ell} \in M_{k-\frac{1}{2}}(\Gamma(4mN))$ where

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : \begin{array}{l} a \equiv d \equiv 1 \pmod{N} \\ b \equiv c \equiv 0 \pmod{N} \end{array} \right\} .$$

Mock Modular Forms

Idea:

- Holomorphic but not modular.
- Can be completed to a modular form by adding a nonholomorphic function.
- The completion is necessarily nonholomorphic.

Mock Modular Form

A weight $\kappa \in \frac{1}{2}\mathbb{Z}$ mock modular form is a holomorphic function $f : \mathbb{H} \rightarrow \mathbb{C}$ such that there exists $g \in M_{2-\kappa}(\Gamma)$ such that $f + g^*$ transforms like a weight- κ modular form with respect to Γ , where

$$g^*(\tau) = 2^{\kappa-1}i \int_{-\bar{\tau}}^{i\infty} d\omega \frac{\overline{g(-\bar{\omega})}}{(-i(\omega + \tau))^{\kappa}} ,$$

is the *Eichler integral* of g . The modular form g is called the *shadow* of f and $f + g^*$ is called the *completion* of f .

Mock Jacobi Forms

Mock Jacobi Form

A holomorphic function

$$\varphi(\tau, z) = \sum_{\ell \bmod 2m} h_{\ell}(\tau) \vartheta_{m,\ell}(N\tau, z) ,$$

is called a mock Jacobi form of weight k and index $\frac{m}{N} \in \mathbb{Z}/N$ if each h_{ℓ} is a weight- $(k - 1/2)$ mock modular form for some subgroup of $\mathrm{SL}(2, \mathbb{Z})$ and the completion

$$\widehat{\varphi}(\tau, z) = \sum_{\ell \bmod 2m} (h_{\ell}(\tau) + g_{\ell}^*(\tau)) \vartheta_{m,\ell}(N\tau, z)$$

transforms like a Jacobi form of weight k and index $\frac{m}{N} \in \mathbb{Z}/N$.

Examples: Modular forms of integral weight for $\mathrm{SL}(2, \mathbb{Z})$

Eisenstein series: modular forms of weight k with $4 \leq k \in 2\mathbb{Z}$

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad \sigma_k(n) = \sum_{d|n} d^k$$

and B_k are Bernoulli numbers defined by $\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$. One has

$$M_k(\mathrm{SL}(2, \mathbb{Z})) = \bigoplus_{\substack{4\ell_1 + 6\ell_2 = k \\ \ell_1, \ell_2 \geq 0}} \mathbb{C} \cdot E_4^{\ell_1} E_6^{\ell_2}.$$

Another example: Ramanujan discriminant function (weight 12)

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{1728} = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Examples: Modular forms of half-integral weight

Classical theta function: modular form of weight $1/2$ for $\Gamma_0(4)$

$$\Theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} .$$

Dedekind eta function: (not quite) modular form of weight $1/2$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) .$$

transforms as

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau) .$$

We also have $\eta^{24} = \Delta$.

Examples: Jacobi forms

Jacobi theta functions

$$\vartheta_1(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{2}(n - \frac{1}{2})^2} \zeta^{n - \frac{1}{2}}, \quad \vartheta_2(\tau, z) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n - \frac{1}{2})^2} \zeta^{n - \frac{1}{2}},$$

$$\vartheta_3(\tau, z) = \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} \zeta^n, \quad \vartheta_4(\tau, z) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} \zeta^n.$$

Define $\varphi_{k,1}$ by

$$\begin{aligned} \varphi_{-2,1}(\tau, z) &:= \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)}, \quad \varphi_{10,1}(\tau, z) := \eta^{24}(\tau) \varphi_{-2,1}(\tau, z), \\ \varphi_{0,1}(\tau, z) &:= 4 \left(\frac{\vartheta_2^2(\tau, z)}{\vartheta_2^2(\tau)} + \frac{\vartheta_3^2(\tau, z)}{\vartheta_3^2(\tau)} + \frac{\vartheta_4^2(\tau, z)}{\vartheta_4^2(\tau)} \right), \quad \varphi_{12,1} := \varphi_{0,1} \Delta, \end{aligned}$$

where $\vartheta_i(\tau) = \vartheta_i(\tau, 0)$. Then $\varphi_{k,1} \in J_{k,1}(\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2)$. In fact

$$J_{k,m}(\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2) = \bigoplus_{j=0}^m M_{k+2j}(\mathrm{SL}(2, \mathbb{Z})) \cdot A^j B^{m-j},$$

where $A = \varphi_{-2,1}$, $B = \varphi_{0,1}$.

Mock Modular Forms And Mock Jacobi Forms

Generating function of Hurwitz class numbers is a mock modular form of weight $3/2$ for $\Gamma_0(4)$ with shadow $-\frac{1}{16}\Theta$ [Zagier (1975)]:

$$\mathbf{H}(\tau) := -\frac{1}{12} + \sum_{n=1}^{\infty} H(n)q^n ,$$

Define

$$\mathbf{H}_0(\tau) = \sum_{n=0}^{\infty} H(4n)q^n , \quad \mathbf{H}_1(\tau) = \sum_{n=1}^{\infty} H(4n-1)q^{n-\frac{1}{4}} ,$$

where $H(0) = -1/12$. Then

$$\mathbf{H}(\tau, z) = \mathbf{H}_0(\tau)\vartheta_{1,0}(\tau, z) + \mathbf{H}_1(\tau)\vartheta_{1,1}(\tau, z) = \sum_{\substack{n,r \in \mathbb{Z} \\ 4n-r^2 \geq 0}} H(4n-r^2) q^n \zeta^r$$

is a mock Jacobi form of weight 2 and index 1 [DMZ (2012)].

Meromorphicity and Mock Modularity

Dabholkar-Murthy-Zagier Result

DMZ Result

Let $\varphi(\tau, z) \in J_{k,m}^{\text{mer}}(\text{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2)$ with single or double poles at $z_s = \alpha_s \tau + \beta_s \in \mathbb{Q}\tau + \mathbb{Q}$. Then there is a decomposition

$$\varphi = \varphi^F + \varphi^P ,$$

where φ^F (called the finite part) is a holomorphic (mixed) mock Jacobi form and φ^P (called polar part) has the same pole structure as φ but is not modular.

- Easily generalizes to $\varphi(\tau, z) \in J_{k,m}^{\text{mer}}(\Gamma \ltimes \mathbb{Z}^2)$ for subgroups $\Gamma \subset \text{SL}(2, \mathbb{Z})$.
- In CHL models, we encounter meromorphic Jacobi forms in $J_{k, \frac{m}{N}}^{\text{mer}}(\Gamma \ltimes (N\mathbb{Z} \times \mathbb{Z}))$ with $\Gamma = \Gamma_0(N), \Gamma_1(N)$ with a double pole at $z = 0 \in \mathbb{Q}\tau + \mathbb{Q}$.
- The goal is to generalize DMZ result to such Jacobi forms.
- Taking $N = 1$ reduces to DMZ case.

Generalizing DMZ

- **Step 1:** Start with $\psi(\tau, z) \in J_{k, \frac{m}{N}}^{\text{mer}}(\Gamma \ltimes \mathbb{Z}^2)$ with $\Gamma = \Gamma_0(N), \Gamma_1(N)$ and construct ψ^P which is not modular and $\psi^F := \psi - \psi^P$ is holomorphic.
- **Step 2:** Prove that ψ^P admits nonholomorphic completion [Zwegers (2006)].
- **Step 3:** Use the nonholomorphic completion of ψ^P to construct nonholomorphic completion of ψ^F .

Generalizing DMZ: Step 1

- We want to construct a function ψ^P which has the same poles and residues as ψ but is not modular.
- Let $s = (\alpha, \beta) \in \mathbb{Q}^2$ and suppose $\psi(\tau, z)$ has a single or double pole at $z_s := \alpha\tau + \beta \in \mathbb{Q}\tau + \mathbb{Q}$.
- From

$$\psi(\tau, z + \lambda\tau + \mu) = e(-m(\lambda^2\tau + 2\lambda z + \lambda\mu)) \psi(\tau, z), \quad (\lambda, \mu) \in N\mathbb{Z} \times \mathbb{Z},$$

if $\psi(\tau, z)$ has a pole at $z = z_s$ then $\psi(\tau, z)$ has poles at $z = z_{s+(\lambda, \mu)} = (\alpha + \lambda)\tau + (\beta + \mu)$.

Strategy: Find functions $\tilde{\mathcal{A}}_{\frac{m}{N}}^s, \tilde{\mathcal{B}}_{\frac{m}{N}}^s$ which have simple and double poles with residue 1 at $z = z_{s+(\lambda, \mu)}$ respectively. Multiply these functions with the residue of ψ to get ψ^P .

Define the functions at $z = z_s$ by

$$e\left(\frac{m}{N}\alpha(z_s + 2\varepsilon)\right) \psi(\tau, z_s + \varepsilon) = \frac{E_s(\tau)}{(2\pi i \varepsilon)^2} + \frac{D_s(\tau)}{2\pi i \varepsilon} + O(1), \quad \varepsilon \rightarrow 0.$$

Generalizing DMZ: Step 1 Continued...

One can prove for $(\lambda, \mu) \in N\mathbb{Z} \times \mathbb{Z}$

$$\begin{aligned} D_{(\alpha+\lambda, \beta+\mu)}(\tau) &= \mathbf{e}\left(\frac{m}{N}(\mu\alpha - \lambda\beta)\right) D_{(\alpha, \beta)}(\tau), \\ E_{(\alpha+\lambda, \beta+\mu)}(\tau) &= \mathbf{e}\left(\frac{m}{N}(\mu\alpha - \lambda\beta)\right) E_{(\alpha, \beta)}(\tau) \end{aligned}$$

Start with [DMZ (2012)]

$$\mathcal{R}_c^{(1)}(\mathbf{e}(z)) = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \frac{\mathbf{e}(c(z-n))}{z-n}, \quad \mathcal{R}_c^{(2)}(\mathbf{e}(z)) = \frac{1}{(2\pi i)^2} \sum_{n \in \mathbb{Z}} \frac{\mathbf{e}(c(z-n))}{(z-n)^2},$$

These functions have single and double poles respectively with residue 1 and 0 respectively at $z = z_s$.

Generalizing DMZ: Step 1 Continued...

Apply the averaging operator and define *universal Appell-Lerche sums*

[Bhand-Sen-RKS (2023)]

$$\tilde{\mathcal{A}}_{\frac{m}{N}}^s(\tau, z) := \mathbf{e}\left(-\frac{m}{N}\alpha z_s\right) \sum_{r \in \mathbb{Z}} q^{mNr^2} \zeta^{2mr} \mathcal{R}_{-2m\alpha/N}^{(1)}\left(\frac{q^{Nr}\zeta}{\zeta_s}\right)$$

$$\tilde{\mathcal{B}}_{\frac{m}{N}}^s(\tau, z) := \mathbf{e}\left(-\frac{m}{N}\alpha z_s\right) \sum_{r \in \mathbb{Z}} q^{Nmr^2} \zeta^{2mr} \mathcal{R}_{-2m\alpha/N}^{(2)}\left(\frac{q^{Nr}\zeta}{\zeta_s}\right), \quad \zeta_s := \mathbf{e}(z_s) .$$

Define the polar part by

$$\psi^P(\tau, z) = \sum_{s \in S(\psi)/(N\mathbb{Z} \times \mathbb{Z})} \left(D_s(\tau) \tilde{\mathcal{A}}_m^s(\tau, z) + E_s(\tau) \tilde{\mathcal{B}}_m^s(\tau, z) \right) ,$$

where $S(\psi) \subset \mathbb{Q}^2$ is the set of poles of ψ .

Generalizing DMZ: Step 3

Mock Modularity of ψ^F [Bhand-Sen-RKS (2023)]

The finite part $\psi^F := \psi - \psi^P$ is given by

$$\psi^F(\tau, z) = \sum_{\ell \bmod 2m} h_\ell(\tau) \vartheta_{m,\ell}(N\tau, z),$$

where

$$h_\ell(\tau) = q^{-\ell^2 N/4m} \int_{-\ell N\tau/2m}^{-(\ell N\tau/2m)+1} \psi(\tau, z') \mathbf{e}(-\ell z') dz' .$$

ψ^F is a (mixed) mock Jacobi form of weight k , index m/N for Γ . The completion of $h_\ell(\tau)$ is given by

$$\widehat{h}_\ell(\tau) = h_\ell(\tau) - \sum_{s \in S(\psi)/(\mathbb{N}\mathbb{Z} \times \mathbb{Z})} \left(D_s(\tau) \widetilde{\Theta}_{\frac{m}{N}, \ell}^{s*}(\tau) + E_s(\tau) \widetilde{\Xi}_{\frac{m}{N}, \ell}^{s*}(\tau) \right) ,$$

Generalizing DMZ: Step 3 Continued...

Mock Modularity of ψ^F [Bhand-Sen-RKS (2023)]

where $\tilde{\Theta}_{\frac{m}{N},\ell}^{s*}, \tilde{\Xi}_{\frac{m}{N},\ell}^{s*}(\tau)$ are Eichler integrals of

$$\tilde{\Theta}_{\frac{m}{N},\ell}^s(\tau) = \mathbf{e}\left(-\frac{m}{N}\alpha\beta\right) \sum_{r \in \mathbb{Z} + \frac{\alpha}{N} + \frac{\ell}{2m}} r \mathbf{e}(2m\beta r) q^{Nmr^2},$$

$$\tilde{\Xi}_{\frac{m}{N},\ell}^s(\tau) = \mathbf{e}\left(-\frac{m}{N}\alpha\beta\right) \sum_{r \in \mathbb{Z} + \frac{\alpha}{N} + \frac{\ell}{2m}} \mathbf{e}(2m\beta r) q^{Nmr^2},$$

respectively.

Mock Modularity In CHL Models

Dyon Partition Function In CHL Models

The Dyon partition function $\tilde{\Phi}_k^{-1}$ is a Siegel modular form of weight $-k$ with respect to $G \subset \mathrm{Sp}(2, \mathbb{Z})$. For $\mathcal{M} = K3$, the weight k is given by [David-Jatkar-Sen (2006)]

$$k = \begin{cases} \frac{24}{N+1} - 2, & N = 1, 2, 3, 5, 7; \\ 3, & N = 4; \\ 2, & N = 6; \\ 1, & N = 8. \end{cases}$$

For $\mathcal{M} = K3$, $N = 1$ (DMZ case), the partition function $\tilde{\Phi}_k^{-1} = \Phi_{10}^{-1}$ is the inverse of the well-known Igusa cusp form.

Poles Of $\tilde{\Phi}_k^{-1}$

Zeros of $\tilde{\Phi}_k$ [David-Jatkar-Sen (2006)]

The partition function $\tilde{\Phi}_k$ has double zeros on the hypersurface

$$n_2 (\sigma\tau - z^2) + jz + n_1\sigma - m_1\tau + m_2 = 0$$

where

$$m_1 \in N\mathbb{Z}, \quad n_1 \in \mathbb{Z}, \quad j \in 2\mathbb{Z}+1, \quad m_2, n_2 \in \mathbb{Z}, \quad m_1 n_1 + m_2 n_2 + \frac{j^2}{4} = \frac{1}{4}.$$

The behaviour near $z = 0$ is

$$\tilde{\Phi}_k(\tau, z, \sigma) = -4\pi^2 z^2 f^{(k)}(N\tau) g^{(k)}(\sigma/N) + O(z^4),$$

for some modular forms $f^{(k)}, g^{(k)}$.

Fourier-Jacobi Expansion

Fourier-Jacobi expansion of $\tilde{\Phi}_k^{-1}$ [Bhand-Sen-RKS (2023)]

$$\frac{1}{\tilde{\Phi}_k(\tau, z, \sigma)} = \sum_{m=-\tilde{\alpha}}^{\infty} \phi_m(\sigma, z) q^m, \quad q = \mathbf{e}(\tau), \quad \text{Im}(\tau) \rightarrow \infty$$

$$\frac{1}{\tilde{\Phi}_k(\tau, z, \sigma)} = \sum_{n=-\tilde{\gamma}N}^{\infty} \psi_n(\tau, z) p^{n/N}, \quad p = \mathbf{e}(\sigma), \quad \text{Im}(\sigma) \rightarrow \infty,$$

$\tilde{\alpha}, \tilde{\gamma}N \in \mathbb{Z}$.

- $\phi_m \in J_{-k, m}^{\text{mer}}(\Gamma^1(N) \ltimes \mathbb{Z}^2)$, poles at $z \in \mathbb{Z}\sigma + \mathbb{Z}$
- $\psi_n \in J_{-k, \frac{n}{N}}^{\text{mer}}(\Gamma_1(N) \ltimes (N\mathbb{Z} \times \mathbb{Z}))$, poles at $z \in N\mathbb{Z}\tau + \mathbb{Z}$.
- The finite parts ϕ_m^F, ψ_n^F are mock Jacobi forms of appropriate type.

Degeneracy Of Single-Centered Black Holes

For charges $Q^2 = \frac{2m}{N}$, $P^2 = 2n$, $Q \cdot P = \ell$ with $(n, \ell, m) \in \mathbb{Z}^3$,

$$d(\vec{Q}, \vec{P}) = \frac{(-1)^{\ell+1}}{N} \int_{\mathcal{C}} d\tau d\sigma dz \frac{e^{-2\pi i(m\tau + n\sigma + \ell z)}}{\tilde{\Phi}_k(\tau, z, \sigma)}$$

with $\mathcal{C} : \text{Im}(\tau) = \frac{2n}{N\varepsilon}$, $\text{Im}(\sigma) = \frac{2m}{\varepsilon}$, $\text{Im}(z) = -\frac{\ell}{\varepsilon}$ and
 $0 \leq \text{Re}(\tau), \text{Re}(z) < 1$, $0 \leq \text{Re}(\sigma) < N$.

- Suppose we can deform $\mathcal{C}$ from $\text{Im}(\tau) = \frac{2n}{N\varepsilon}$ to $\text{Im}(\tau) \rightarrow \infty$. Then using the Fourier-Jacobi expansion:

$$d(\vec{Q}, \vec{P}) = \frac{(-1)^{\ell+1}}{N} \int d\sigma \int dz e^{-2\pi i \frac{n}{N} \sigma} e^{-2\pi i \ell z} \phi_m^F(\sigma, z) .$$

- Suppose we can deform $\mathcal{C}$ from $\text{Im}(\sigma) = \frac{2m}{\varepsilon}$ to $\text{Im}(\sigma) \rightarrow \infty$. Then using the Fourier-Jacobi expansion:

$$d(\vec{Q}, \vec{P}) = \frac{(-1)^{\ell+1}}{N} \int d\tau \int dz e^{-2\pi i m \tau} e^{-2\pi i \ell z} \psi_n^F(\tau, z) .$$

Deformations Of Contour

Contour Deformation [Bhand-Sen-RKS (2023)]

(1) Deformation of $\mathcal{C}$ from $\text{Im}(\tau) = \frac{2n}{N\varepsilon}$ to $\text{Im}(\tau) \rightarrow \infty$ does not get contribution from poles of $\tilde{\Phi}_k^{-1}$ if and only if $(n, \ell, m) \in A$ where $A \subset \mathbb{Z}^3$ is a set of triples (n, ℓ, m) with $n, m > 0, 4mn - N\ell^2 > 0$ and

$$\left[\left(\frac{m_1 \ell + 2ms + m}{m} \right)^2 + \frac{m_1^2}{m^2} \left(4 \frac{n}{N} m - \ell^2 \right) \right] \geq 1,$$

$$\forall m_1 \in N\mathbb{Z}, \quad s \in \mathbb{Z}, \quad s(s+1) = 0 \bmod m_1,$$

(2) Deformation of $\mathcal{C}$ from $\text{Im}(\sigma) = \frac{2m}{\varepsilon}$ to $\text{Im}(\sigma) \rightarrow \infty$ does not get contribution from poles of $\tilde{\Phi}_k^{-1}$ if and only if $(n, \ell, m) \in B$ where $B \subset \mathbb{Z}^3$ is a set of triples (n, ℓ, m) with $n, m > 0, 4mn - N\ell^2 > 0$ and

$$\left[\left(\frac{n_1 N \ell - 2ns - n}{n} \right)^2 + \frac{n_1^2 N^2}{n^2} \left(4 \frac{n}{N} m - \ell^2 \right) \right] \geq 1,$$

$$\forall n_1, s \in \mathbb{Z}, \quad s(s+1) = 0 \bmod n_1.$$

Refinement Of DMZ Result

Restriction On Charges

- For $\mathcal{M} = K3, N = 1$, the DMZ claim that for any charge (n, ℓ, m) , the degeneracy is given by Fourier coefficient of the mock Jacobi forms ϕ_m^F when $n > m$ and ψ_n^F when $n < m$.
- Our analysis refines the condition $n > m$ to $(n, \ell, m) \in A$ and $n < m$ to $(n, \ell, m) \in B$.
- Checked using wall crossing formula (story for another time).
- Example: $(n, \ell, m) = (10, 14, 5)$ satisfies $n > m$ but does not belong to either A or B . The degeneracy differs from the coefficient of the mock Jacobi form ϕ_5^F by a wall crossing term.
- Another such example is $(n, \ell, m) = (13, 19, 7)$.

Future Directions

Open Questions

- Positivity of degeneracy: the degeneracy is given by Fourier coefficient of interesting number theoretic object. Physics predicts the positivity of these Fourier coefficients [Sen (2010)]. Prove this mathematically. Proved for $\mathcal{M} = K3, N = 1$ [Bringmann-Murthy (2017), Rosselló (2024)].
- The degeneracy of single-centered black holes is not captured by mock Jacobi forms for all charges. Formulate the notion of “mock Siegel forms” which captures the degeneracy for all charges [Bhand-Sen-RKS work in progress].

Thank You