Ramanujan's Tau Function, Lehmer's Conjecture and Mock Modular Forms

Ranveer Kumar Singh[a]

[a] *NHETC and Department of Physics and Astronomy, Rutgers University, 126 Frelinghuysen Rd., Piscataway NJ 08855, USA*

E-mail: ranveer.singh@rutgers.edu

Abstract: We review the Lehmer's conjecture and its relation to mock modular forms

Contents

1 Introduction

Let us begin by defining the Ramanujan's tau function which was introducded by Ramanujan in his seminal 1916 paper [1] called "On Certain Arithmetical Functions". Consider q as a formal variable and consider the infinite product

$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24}. \tag{1.1}$$

The function $\Delta(q)$ is called the *discriminant function*. One should first argue that this definition makes sense. Indeed, the convergence criteria in analysis says that a product $\prod a_n$ converges (absolutely) if and only if the sum $\sum \log a_n$ converges (absolutely) and $\sum \log(1+a_n)$ converges (absolutely) if and only if $\sum a_n$ converges (absolutely). Now let q be a complex number so that $\log q$ makes sense with log being the principal branch of complex logarithm. One can now immediately see that the infinite product $\Delta(q)$ in (1.1) converges absolutely if $|q| < 1$. Since the infinite product converges absolutely, we can also rearrange the terms in the product so that we can collect all powers of q^n together and write:

$$\Delta(q) := \sum_{n=1}^{\infty}\tau(n)q^n. \tag{1.2}$$

The formal equality in (1.2) defines the Ramanujan's tau function $\tau(n)$. It is clear that $\tau(n)$ is always an integer. Some values of $\tau(n)$ are given in the table below: Ramanujan observed several properties of $\tau(n)$. For example, observe that

$$\tau(2) = -24,\ \tau(3) = 252 \quad \text{and} \quad \tau(6) = -6048 = \tau(2)\tau(3).$$

Infact Ramanujan made the following three conjectures:

Conjecture 1. (Ramanujan's Conjecture)*: The tau function satisfies the following:*
(a) For $m, n \in \mathbb{Z}$ with gcd$(m,n) = 1$, we have

$$\tau(mn) = \tau(m)\tau(n).$$

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	11	534612	21	−4219488
2	−24	12	−370944	22	−12830688
3	252	13	−577738	23	18643272
4	−1472	14	401856	24	21288960
5	4830	15	1217160	25	−25499225
6	−6048	16	987136	26	13865712
7	−16744	17	−6905934	27	−73279080
8	84480	18	2727432	28	24647168
9	−113643	19	10661420	29	128406630
10	−115920	20	−7109760	30	−29211840

(b) For a prime number p and $m \geq 1$, we have

$$\tau(p^{m+1}) = \tau(p)\tau(p^m) - p^{11}\tau(p^{m-1}).$$

(c) For a prime number p, we have $\mid \tau(p) \mid \leq 2p^{\frac{11}{2}}$.

These properties are quite surprising and mysterious. Why would a function defined in such an unusual way satisfy such relations? The first two of the Ramanujan's conjectures were proved by Mordell [2] but the mathematical understanding remained a mystery until Erich Hecke in 1937 [6, 7] came up with a systemetic theory, now called *Hecke theory*, to study more general functions of this form. Infact the first two conjectures are generalised to more general functions using Hecke theory. All of the three Ramanujan's conjectures are now theorems. The third conjecture remained unresolved until 1974 when Deligne [5] proved it as a consequence of his proof of the Weil's conjectures. To date, there is no other way to prove the third Ramanujan's conjecture. The tau function satisfies many other interesting properties. For example note that the values of $\tau(p)$ in the table above is even when p (here and elsewhere) is prime. Infact one can prove that $\tau(p)$ is even for every prime p. In terms of modular arithmetic, we write $a \equiv b(\text{mod} c)$ if c divides $b - a$ and we say that a is congruent to b mod c. With this notation, the following conguences hold for the tau function [4]

1. $\tau(p) \equiv 1 + p^3 (\text{mod } 2^5)$
2. $\tau(p) \equiv 1 + p (\text{mod } 3)$
3. $\tau(p) \equiv p + p^{10} (\text{mod } 5^2)$
4. $\tau(p) \equiv p + p^4 (\text{mod } 7)$
5. $\tau(p) \equiv 1 + p^{11} (\text{mod } 691)$

Many more congurences are true for the tau function but this is not the main aim of this review. There is one other conjecture which goes by the name *Lehmer's conjecture* and first appeared in a paper by D.H. Lehmer [3] and it is still a conjecture.

Conjecture 2. (Lehmer's Conjecture)*: For every integer $n > 0$, we have that $\tau(n) \neq 0$.*

Such an innocuous statement but notoriously difficult to prove. From the first two Ramanujan's conjectures, it can be shown that Lehmer's conjecture is equivalent to the nonvanishing of $\tau(p)$ for every prime. This conjecture has been verified for $n < 816212624008487344127999$ [8]. There are some other observations. For example Lehmer himself proved that if n_0 is the least integer such that $\tau(n_0) = 0$ then n_0 must be a prime. Secondly, one easily observe using the second congruence above that $\tau(p) \neq 0$ for every prime $p \equiv 1(\text{mod}3)$. So to prove Lehmer's conjecture, we just need to show that $\tau(p)$ does not vanish for primes $p \equiv 2(\bmod 3)$. Many other partial results are known but the conjecture remains unresolved. Infact there is a much more stronger conjecture due to Atkin and Serre.

Conjecture 3. (Atkin-Serre Conjecture)*: For any $\varepsilon > 0$ and prime p, there is a constant $C(\varepsilon) > 0$ such that $\mid \tau(p) \mid > C(\varepsilon) p^{\frac{9}{2}-\varepsilon}$.*

There is no clue about this conjecture and we leave it undisturbed here. We will now look at the systematic framework to study the discriminat function.

2 Modular forms

The theory of modular forms occupies the central position in number theory in the sense that it finds applications ranging from geometry, topology, discrete mathematics, representation theory to theoretical physics. Roughly speaking, modular forms are holomorphic functions on the upper half plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$ which satisfy certain transformation property with respect to the modular group $\mathrm{SL}_2(\mathbb{Z})$ defined below and satisfy certain growth condition. Let us make this precise now. Define the following set of matrices

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

One can easily check that $\mathrm{SL}_2(\mathbb{Z})$ is a group with respect to matrix multiplication. This group acts on $\mathbb{H}$ via linear fractional transformations as follows: for $z \in \mathbb{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have

$$\gamma \cdot z = \frac{az + b}{cz + d}.$$

We can easily check that this is a group action. We now define modular forms precisely.

Definition 2.1. (Modular Form): A function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$ on $\mathrm{SL}_2(\mathbb{Z})$ if

1. f is holomorphic.
2. $f\left(\frac{az+b}{cz+d}\right) = (cz + d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$.
3. $f(z)$ is bounded as $z \to i\infty$.

The set of all modular forms of weight k is denoted by M_k and forms a vector space. Indeed, it is easy to check that the sum of two modular forms of weight k again has correct transformation property. Something more is true. The product of two modular forms of weight k_1 and k_2 is again modular form of weight $k_1 + k_2$ and the direct sum of the vector space of modular forms of all weights forms a ring.

Using (i) of above definition for the matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ we get $f(z+1) = f(z)$. Using this periodicity and (iii) of the above definition, one can show that any modular form can be expanded in a *Fourier series* as

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \quad \text{where} \quad q = e^{2\pi i z}. \tag{2.1}$$

The complex numbers $a_f(n)$ are called the *Fourier coefficients* of the modular form f. If $a_f(0) = 0$ then f is called a *cusp form*. One can also have a more general definition, if we only require that the Fourier expansion have only finitely many negative powers of q. Such forms are called *weakly holomorphic modular forms* and the set of all such forms is denoted by $M_k^!$. We will have more to say about these forms in the next section. The first examples of modular forms are given by *Eisenstein series*. For $k \geq 4$ and even, put

$$E_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz+n)^k}, \quad z \in \mathbb{H}. \tag{2.2}$$

We can prove that $E_k(z)$ is a modular form of weight k [10]. Their Fourier expansion is given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{2.3}$$

where B_k are Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and

$$\sigma_k(n) = \sum_{d|n} d^k$$

is the kth divisor sum (here $d|n$ means that d divides n). Infact it turns out that all modular forms can be expressed in terms of Eisenstein series. One can prove that [10] the space of modular forms of weight k is a finite-dimensional vector space with a basis

$$\{E_4^\alpha E_6^\beta : 4\alpha + 6\beta = k, \alpha, \beta \in \mathbb{Z}, \alpha, \beta \geq 0\}.$$

Note that this basis allows only even k. Infact standard results in modular forms show that there are no modular forms of negative weight, odd weight and weight 2 [10]. Notice also that E_2 as defined in (2.3) makes perfect sense. Infact one can prove that the expressions for E_2 in (2.2) and (2.3) agree and for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, E_2 satisfies the following transformation property:

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) + \frac{6c(cz+d)}{\pi i}. \tag{2.4}$$

This in particular implies that E_2 is not a modular form as expected. Later we will see that E_2 is an example of what are called *mock modular forms*.

After all that jargon we come to the point. Consider $q \in \mathbb{C}$ as a function of $z \in \mathbb{H}$ given by $q = e^{2\pi i z}$. Then $|q| < 1$ and hence the discriminant function can be considered as a holomrophic function on $\mathbb{H}$. It turns out that [10]

$$\Delta(z) = \frac{E_4^3(z) - E_6^2(z)}{1728}. \tag{2.5}$$

(2.5) implies that $\Delta(z)$ transforms like a modular form of weight 12. Moreover since the constant term in the Fourier expansion of E_k is 1, thus $\Delta(z)$ is a cusp form of weight 12. But this review is about Lehmer's conjecture and it seems that we have lost it somewhere in discussing the theory of modular forms. But one thing that the theory gives us is an expression for the tau function. In the theory of modular forms, the second most important construction (only after Eisenstein series) are the *Poincaré series*. The Poincaré series are cusp forms and span the space of cusp forms. The construction also gives us explicit Fourier expansion of the Poincaré series. Moreover it is known that the space of cusp forms of weight 12 is one dimensional. Thus the Poincaré series of weight 12 and the discriminant function are multiples and the constant of proportionality is determined by comparing the first Fourier coefficient of the Poincaré series. The upshot of all this mumbo-jumbo is that we have the following expression for the tau function:

$$\tau(n) = \frac{2\pi n^{\frac{11}{2}}}{\beta_\Delta} \sum_{c>0} \frac{K(1,n,c)}{c} J_{11}\left(\frac{4\pi\sqrt{n}}{c}\right). \tag{2.6}$$

where $\beta_\Delta = 2.840...$(the constant of multiplicity) is a constant, $K(m,n,c)$ for $m,n \in \mathbb{Z}$ is the *Kloosterman sum* defined by

$$K(m,n;c) := \sum_{d=1}^{c-1} e^{2\pi i\left(\frac{m\bar{d}+nd}{c}\right)}$$

with $0 \leq \bar{d} \leq c-1$ defined by $d\bar{d} \equiv 1(\bmod c)$ and J_ℓ is the order ℓ Bessel function of the first kind. Does this expression say anything about the Lehmer's conjecture. Unfortunately no. Why did we discuss modular forms then? Hopefully, it turns out that the a generalisation of modular forms have a part to play in Lehmer's conjecture as we will see now. We will now discuss mock modular forms will put E_2 in perspective.

3 Harmonic Maass Forms and Mock Modular Forms

Harmonic Maass forms and mock modular forms have their origin in Ramanujan's last deathbed letter to G.H. Hardy in 1920. In this enigmatic letter, Ramanujan listed 22 functions which he called *mock theta functions*. The transformation properties and the precise definitions did not appear in literature until recently in 2002, Zwegers came up with a systematic framework to study them [15]. Ramanujan's mock theta functions are now known to be mock modular forms which we will describe in this section. Let us first discuss the general philosophy of mock modular forms. Suppose that we have a function which is holomorphic but does not have the transformation property of modular forms. If we can find a function (nonholomorphic in general) such that the sum of these two functions has correct transformation property then we say that our original function is a mock modular form. The sum which is called the completion of the mock modular form is essentially nonholomorphic but has correct transformation properties. Such forms are called nonholomorphic modular forms. We impose a further *harmonic condition* to get what are called *harmonic Maass forms*. The reader is referred to [13] for the detailed theory. Let $z = x + iy$. Define the holomorphic and antiholomorphic derivative as follows:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We now define the hyperperbolic Laplacian of weight $k \in \mathbb{R}$ as follows:

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = -4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} + 2iky\frac{\partial}{\partial \bar{z}}.$$

Let us now define harmonic Maass forms.

Definition 3.1. (Harmonic Maass Form): A real-analytic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a harmonic Maass form of weight $k \in \mathbb{Z}$ if the following conditions are satisfied:

1. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$.
2. $\Delta_k(f) = 0$.
3. There exists a polynomial P_f in variable q^{-1} (remember $q = e^{2\pi i z}$) such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ for some $\varepsilon > 0$ as $y \to \infty$.

If we just have $f(z) = O(e^{\varepsilon y})$ as $y \to \infty$ then f is called a harmonic Maass form of *manageable growth*.

The third condition is technical. Let us decode it a bit with an example. Consider a function f given by the following Fourier series:

$$f(z) = \sum_{n=1}^{N} \frac{a_f(n)}{q^n} + \sum_{n=0}^{\infty} a_f(n) q^n,$$

where N is a positive integer. Note that as $|q^{-1}| = e^{2\pi y} \to \infty$ as $y \to \infty$. This in particular implies that $|f(z)|$ grows exponentially as $y \to \infty$. Now suppose we consider the polynomial

$$P_f(z) = \sum_{n=1}^{N} \frac{a_f(n)}{q^n}.$$

Then we have

$$f(z) - P_f(z) = \sum_{n=0}^{\infty} a_f(n) q^n.$$

Thus now we only have positive powers of q in the expansion of $f(z) - P_f(z)$ which also implies that $f(z) - P_f(z)$ is at bounded as $y \to \infty$. But if f is of manageable growth then it may not be possible to cancel the growth by subtracting any polynomial in q^{-1}. This is the precise meaning of the third condition. The space of harmonic Maass forms of manageable growth is denoted by $H_k^!$ and the space of harmonic Maass forms is denoted by H_k. Ofcourse $H_k \subset H_k^!$. By the modular transformation law, we again have a Fourier expansion but slightly complicated (see [11] for details of the proof): for $k \neq 1$, harmonic Maass forms of manageable growth have Fourier expansion of the shape

$$f(z) = f(x+iy) = \sum_{n=n_0}^{\infty} c_f^+(n) q^n + c_f^-(0) y^{1-k} + \sum_{n=-\infty}^{n_0'} c_f^-(n) \Gamma(1-k, -4\pi n y) q^n, \qquad (3.1)$$

where $\Gamma(s, z)$ is the incomplete gamma function defined as

$$\Gamma(s, z) = \int_z^{\infty} e^{-t} t^s \frac{dt}{t}$$

and n_0, n_0' are integers (possibly negative, positive respectively). Note that the Fourier expansion can be cannonically broken into two parts: we call

$$f^+(z) = \sum_{n=n_0}^{\infty} c_f^+(n) q^n$$

the *holomorphic part* of f and

$$f^-(z) = c_f^-(0) y^{1-k} + \sum_{n=-\infty}^{n_0'} c_f^-(n) \Gamma(1-k, -4\pi n y) q^n$$

the *nonholomorphic part* of f. If f satisfies the first growth condition of (3) in above definition then $c_f^-(0) = 0$ and $n_0' < 0$. Observe that any weakly holomorphic modular form is a trivial example of harmonic Maass form with the nonholomorphic part being zero and the holomorphic part being the weakly holomorphic modular form itself. Thus we have the following sequence of containments:

$$M_k \subset M_k^! \subset H_k \subset H_k^!.$$

There are two operators on harmonic Maass forms which are of importance and bridge hamronic Maass forms with modular forms. These two operators are defined in a more uniform way but here we will define them in a way which will make our discussions easier to follow. Define the shadow map by

$$\xi_k : H_k^! \longrightarrow M_{2-k}^!$$

$$\xi_k(f(z)) = \xi_k(f^-(z)) = (1-k)\overline{c_f^-(0)} - (4\pi)^{1-k} \sum_{n>0} \overline{c_f^-(-n)} n^{1-k} q^n, \tag{3.2}$$

where f is given as in (3.1) and $\overline{c_f^-(-n)}$ denotes complex conjugation. The image $\xi_k(f)$ is called the *shadow* of f. One easily sees that if $f \in H_k$ then the shadow of f is a cusp form. It turns out that this map is surjective (but not injective since all weakly holomorphic modular forms map to zero). This means that we can associate a harmonic Maass form of appropriate weight to every weakly holomorphic modular form. When k is a negative integer, we can define another operator called *Bol operator*. It is defined as follows: $D^{1-k} : H_k^! \longrightarrow M_{2-k}^!$, where $D = \frac{1}{2\pi i}\frac{\partial}{\partial z}$. One can show that if $f \in H_k^!$ is given by its Fourier expansion as in (3.1) then we have

$$D^{1-k}(f)(z) = -(4\pi)^{k-1}(1-k)! c_f^-(0) + \sum_{n=n_0}^{\infty} c_f^+(n) n^{1-k} q^n. \tag{3.3}$$

But unlike ξ_k, the Bol operator is not surjective. The image $D(f)$ of f under the Bol operator is called the *ghost* of f. We now define mockm modular forms.

Definition 3.2. (Mock Modular Form): A mock modular form of weight $(2-k)$ is the holomorphic part f^+ of a harmonic Maass form of weight $(2-k)$ for which f^- is non trivial. The weakly holomorphic modular form $\xi_{2-k}(f)$ is called the shadow of the mock modular form f^+ and the harmonic Maass form f is called the completion of f^+.

We will end this section by showing that E_2 is a mock modular form. Indeed if we put

$$\mathcal{E}_2(z) = E_2(z) - \frac{3}{\pi y},$$

then using some simple manipulations and (2.4) we can show that $\mathcal{E}_2$ satisfies (1) in the definition of harmonic Maass forms. We need to make sure that $\Delta_2(\mathcal{E}_2) = 0$ which again is a simple computation. Finally we can check that $\xi_2(\mathcal{E}_2) = \frac{3}{\pi}$. This shows that E_2 is a mock modular form of weight 2 with shadow $3/\pi$.

4 Mock Modular Form Whose Shadow Is The Discriminant Function

Since ξ_k is a surjective map, there exists (not unique) a mock modular form whose image is the discriminant function. One such mock modular form is

$$M_\Delta(z) = \sum_{n=-1}^{\infty} a_\Delta(n) q^n = \frac{39916800}{q} - \frac{2615348736000}{691} + \sum_{n=1}^{\infty} a_\Delta(n) q^n, \tag{4.1}$$

where

$$a_\Delta(n) = -(2\pi) 11! n^{-\frac{11}{2}} \cdot \sum_{c=1}^{\infty} \frac{K(-1,n,c)}{c} \cdot I_{11}\left(\frac{4\pi\sqrt{n}}{c}\right), \quad n > 0, \tag{4.2}$$

with I_ℓ being the ℓth Bessel function of the second kind. The shadow of the mock modular form M_Δ is $11\pi^{11}\beta_\Delta\Delta(z)$ (remeber β_Δ was defined in (2.6)). The proof of this fact depends on some special harmonic Maass forms which goes by the name *Maass-Poincaré* series. Let us not bother about the proof for a moment. Some of the coefficients $a_\Delta(n)$ computed numerically are

$$\begin{aligned} a_\Delta(1) &= -73562460235.68364... \\ a_\Delta(2) &= -929026615019.11308... \\ a_\Delta(3) &= -8982427958440.32917... \\ a_\Delta(4) &= -71877619168847.70781... \end{aligned} \tag{4.3}$$

The coefficients seem to be irrational. Indeed, this is a conjecture [13].

Conjecture 4. (Ken Ono)*: The coefficients $a_\Delta(n)$ are irrational for every positive integer n.*

This conjecture implies the Lehmer's conjecture. Infact Lehmer's conjecture is implied by any one of the coefficients $a_\Delta(n)$ being irrational. We will show that if $\tau(p) = 0$ for some prime p then all of the coefficients $a_\Delta(n)$ are rational. Now you can see why any of these coefficients being irrational implies the Lehmer's conjecture. Using (3.3) and the fact that $c_f^-(0) = 0$ for the completion f of the mock modular form $f^+ = M_\Delta$, we have

$$D^{11}(M_\Delta)(z) = \sum_{n=-1}^{\infty} n^{11} a_\Delta(n) q^n \tag{4.4}$$

is a weakly holomorphic modular form of weight 12. Next, Duke and Jenkins have constructed a basis for the space of weakly holomorphic modular forms [14] using the Eisenstein series, discriminant function and the j-function defined by

$$j(z) = \frac{E_4^3(z)}{\Delta(z)} = \frac{1}{q} + 744 + 196884q + \ldots.$$

The j-function transforms as weight 0 modular form but is weakly holomorphic. Using the results of Duke and Jenkins, we can write

$$D^{11}(M_\Delta)(z) = a_\Delta(-1)\left[\Delta(z)\left(j^2(z) - 1488j(z) + 713304\right)\right] + a_\Delta(1)\Delta(z). \tag{4.5}$$

One can easily prove that the j-function has integer Fourier coefficients so that $A(n)$ defined by

$$\Delta(z)\left(j^2(z) - 1488j(z) + 713304\right) = \sum_{n=-1}^{\infty} A(n) q^n$$

are all integers. Comparing the Fourier coefficients from (4.4) and (4.5), we get

$$a_\Delta(n) = \frac{11! A(n) + a_\Delta(1)\tau(n)}{n^{11}}. \tag{4.6}$$

Thus if $\tau(n) = 0$ then (4.6) implies that $a_\Delta(n)$ is rational. Now suppose $\tau(p) = 0$ for some prime p. Then using the Ramanujan's conjectures (a) and (b), we see that $\tau(p^k n) = 0$ for every integer $k \geq 1$ and n coprime to p. So we have proved that

Theorem 4.1. *If $\tau(p) = 0$ for some prime p then $a_\Delta(p^k n)$ is rational for every positive integer k and n coprime to p.*

Using Hecke theory, Ono shows that [13] for every prime p, the Fourier series

$$\sum_{n=-p}^{\infty} \left(p^{11} a_\Delta(pn) - \tau(p) a_\Delta(n) + a_\Delta(n/p)\right) q^n$$

is a weakly holomorphic modular form of weight -10 with integer Fourier coefficients. Here it is understood that $a_\Delta(n/p) = 0$ if $n/p \notin \mathbb{Z}$. With $\tau(p) = 0$, the $(np)^{\text{th}}$ Fourier coefficient of the above weakly holomorphic modular form is $p^{11} a_\Delta(p^2 n) + a_\Delta(n) \in \mathbb{Z}$. Now if n is coprime to p then Theorem 4.1 along with the fact that $p^{11} a_\Delta(p^2 n) + a_\Delta(n) \in \mathbb{Z}$ implies that $a_\Delta(n)$ is rational. Thus we have proved that

Theorem 4.2. *If $\tau(p) = 0$ for a prime p then $a_\Delta(n)$ is rational for every integer n coprime to p.*

Combining the two theorems, we get the following theorem.

Theorem 4.3. *If $\tau(p) = 0$ for some prime p then $a_\Delta(n)$ is rational for every positive integer n.*

So we just need to show that any one of the coefficients $a_\Delta(n)$ defined in (4.2) is irrational to prove the Lehmer's conjecture.

References

[1] Srinivasa Ramanujan, *On certain arithmetical functions*, Trans. Camb. Philos. Soc., **22** (9): 159–184 (1916).

[2] Louis J. Mordell, *On Mr. Ramanujan's empirical expansions of modular functions*, Proceedings of the Cambridge Philosophical Society, 19: 117–124 (1917).

[3] D.H. Lehmer, *The vanishing of Ramanujan's function $\tau(n)$*, Duke Math. J., **14** (2): 429–433 (1947), doi:10.1215/s0012-7094-47-01436-1.

[4] O. Kolberg, *Congruences for Ramanujan's function $\tau(n)$*, Arbok Univ. Bergen Mat.-Natur. Ser. (11) (1962).

[5] Pierre Deligne, *La conjecture de Weil: I* Publications Mathématiques de l'IHÉS. **43**: 273–307 (1974), doi:10.1007/bf02684373.

[6] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung I*, Mathematische Annalen (in German), **114**: 1–28 (1937), doi:10.1007/BF01594160.

[7] E. Hecke, *Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung II*, Mathematische Annalen (in German), **114**: 316–351 (1937), doi:10.1007/BF01594180.

[8] Maarten Derickx, Mark van Hoeij, Jinxiang Zeng, *Computing Galois representations and equations for modular curves $X_H(\ell)$*, ArXiv: 1312.6819 (2013).

[9] M. Ram Murty and V. Kumar Murty, The Mathematical Legacy of Srinivasa Ramanujan, Springer (2013).

[10] M. Ram Murty, Michael Dewar and Hester Graves, Problems in the Theory of Modular Forms, Lecture notes in Mathematics, Springer (2016).

[11] R. K. Singh, Applications of number theory in string theory, MS Thesis (2020) (https://ranveer14.github.io/MS_Thesis.pdf), IISER Bhopal (India).

[12] Kathrin Bringmann, Amanda Folsom, Ken Ono and Larry Rolen, Harmonic Maass Forms and Mock Modular Forms: Theory and Applications, American Mathematical Society, (2017).

[13] Ken Ono, A Mock Theta Function for Delta-Function, Proceedings of the 2007 Integers Conference.

[14] W. Duke, Paul Jenkins, On the Zeros and Coefficients of Certain Weakly Holomorphic Modular Forms, Pure and Applied Mathematics Quarterly, Volume 4, Number 4, Pages 1327 – 1340, (2008).

[15] S. Zwegers, *Mock Theta Functions*, PhD. Thesis, Universiteit Utrecht, The Netherlands, (2002).