Projective Representations of the Poincaré Group

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Abstract

In these set of notes, we present the mathematical theory of classification of all projective representations of the Poincaré group. We begin with the standard discussion of Lie groups and Lie algebras from manifold point of view and slowly move to matrix Lie groups which are of relevance in the discussion on Poincaré group. We present the basic ingredients of Mackey Theory required for the classification. We have omitted the proofs of mathematical results to facilitate understanding without going into the technicalities of proofs, although we have included references for all the results. Finally we end with Wigner's idea of elmentary particles in quantum field theory and its relation to projective representations of the Poincaré group.

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1 Introduction

Eugene Wigner in his seminal paper [11] introduced the idea of elementary particles in a quantum theory as the irreducible projective representations of the symmetry group of the underlying spacetime. In particular, for quantum field theory in the Minkowski space, elementary particles are the irreducible projective representations of the Poincaré group which is the symmetry group of the Minkowski space. In the same paper, Wigner introduced the little group method to classify the representations of the Poincaré group. The idea of little group which was very physical in nature sparked a flurry of work in pure mathematics, representation theory in particular. Mackey and Bargmann [3, 4, 8] completed the mathematical theory of this physical idea. which classified all unitary representations of the Poincaré group. We present this theory without going into the technical details.

We begin discussing the basic background on Lie theory and representations. We assume familiarity with topology and manifold theory.

2 Lie Groups

We begin by defining Lie groups.

Definition 2.1. A set G is called a Lie group if G is a smooth manifold endowed with a group structure such that the multiplication map

$$\begin{aligned} \mu : & G \times G \longrightarrow G \\ & (g_1, g_2) \longmapsto g_1 \cdot g_2 \end{aligned}$$

and the inverse map

$$\begin{aligned} i : & G \longrightarrow G \\ & g \longmapsto g^{-1} \end{aligned}$$

are smooth maps between manifolds.

Example 2.2. (i) The set $\mathbb{R}^n$ of n-tuple of real numbers is a group with respect to component wise addition as group operation. It is also a smooth manifold.

(ii) The set $\mathrm{GL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C})$ is the set of $n \times n$ invertible matrices with real, complex entries respectively. These are groups under matrix multiplication. It can also be shown that these groups are smooth manifolds and the matrix multiplication operation and inversion operation are smooth maps (see Example in Appendix A). Hence these are Lie groups.

(iii) Let
$$G = \mathbb{R} \times \mathbb{R} \times S^1 = \left\{(x, y, u) \mid x \in \mathbb{R}, y \in \mathbb{R}, u \in S^1 \subset \mathbb{C}\right\},$$
where S^1 is the set of complex numbers with modulus 1. Equip G with the group product given by
$$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = \left(x_1 + x_2, y_1 + y_2, e^{ix_1 y_2} u_1 u_2\right)$$
Then G is a Lie group.

Definition 2.3. A subset H of a Lie group G is called a Lie subgroup if:

(i) H is a (algebraic) subgroup of G.

(ii) H is an immersed submanifold[1] of G via the inclusion map.

(iii) the operations on H are smooth.

In particular, a Lie subgroup is itself a Lie group.

Before we present examples of Lie subgroups, we record a theorem which makes it easier to decide whether a given subgroup of a Lie group is a Lie subgroup or not.

Theorem 2.4. (Closed subgroup theorem) [7, Theorem 20.12] *Let $H \subset G$ be a subgroup of G. If H is closed in the subspace topology on H, then H is a Lie subgroup of G.*

Example 2.5. The following sets are easily checked to be subgroups of $\mathrm{GL}(n, \mathbb{C})$ or $\mathrm{GL}(n, \mathbb{R})$. Using elementary results from topology [2], it can be shown that these subgroups are indeed closed in and hence are examples of Lie subgroups and in particular Lie groups.

(i) The special linear group: the set of $n \times n$ real (complex) matrices with determinant 1, denoted by $\mathrm{SL}(n, \mathbb{R})$ $(\mathrm{SL}(n, \mathbb{C}))$is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ $(\mathrm{GL}(n, \mathbb{C}))$.

(ii) The orthogonal group: $\mathrm{O}(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T A = \mathbb{1}\}$.

(iii) The special orthogonal group: $\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n, \mathbb{R})$.

(iv) The unitary group: $\mathrm{U}(n) := \{A \in \mathrm{GL}(n, \mathbb{C}) \mid A^\dagger A = \mathbb{1}\}$.

(v) The special unitary group: $\mathrm{SU}(n) := \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$.

[1] see [10, Section 9] for a discussion on submanifolds.

[2] For example, it can be shown that the inverse image of a singleton of a continuous function between topological spaces is closed in the domain. This argument easily guarantees closedness in all of the examples. In particular for (i), it is the determinant map, for (ii), it is the map $A \longmapsto A^T A$ and so on.

2.1 Matrix Lie Groups

Of particular importance in physics are matrix Lie groups which appear as symmetry groups of physical theories. Using the above theorem, we can define matrix Lie groups without going to manifold theory. But we will keep on exploring the connections whenever necessary.

Definition 2.6. A subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ is called a matrix Lie group if given any sequence $\{A_m\}_{m=1}^{\infty} \subset G$ we have that either

$$\lim_{m \to \infty} A_m = A \in G, \quad \text{or} \quad G \notin \mathrm{GL}(n, \mathbb{C}).$$

Remark 2.7. The convergence of a sequence of matrices in above definition is component wise. We look at n^2 many sequences of real numbers and then form the corresponding limiting matrix. Moreover, the condition on G exactly means that G is closed in $\mathrm{GL}(n, \mathbb{C})$ in the subspace topology. Note that $\mathrm{GL}(n, \mathbb{C})$ is trivially a matrix Lie group. Since $\mathrm{M}(n, \mathbb{C})$ – the set of all $n \times n$ complex matrices is not even a group (inverses of many elements does not exist), so we consider matrix Lie groups to be subgroups of $\mathrm{GL}(n, \mathbb{C})$. It turns out that most of the interesting examples of Lie groups are matrix Lie groups but it is not true in general. Indeed one can prove that every compact Lie group is a matrix Lie group[3] but the converse is not in general true. We will see examples of noncompact matrix Lie groups soon.

Example 2.8. (i) All the examples in Example 2.5 are matrix Lie groups.

(ii) Generalised orthogonal group: Let n, k be two positive integers. Define a bilinear form $B : \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \longrightarrow \mathbb{R}$ by

$$B(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^{n} x_i y_i - \sum_{j=1}^{k} x_{n+j} y_{n+j},$$

where $\mathbf{x} = (x_1, \ldots, x_n, \ldots x_{n+k}), \mathbf{y} = (y_1, \ldots, y_n, \ldots y_{n+k}) \in \mathbb{R}^{n+k}$. Define the set $\mathrm{O}(n, k)$ as the set of matrices which preserve the bilinear form B:

$$\mathrm{O}(n, k) := \{A \in \mathrm{GL}(n + k, \mathbb{R}) \mid B(A\mathbf{x}, A\mathbf{y}) = B(\mathbf{x}, \mathbf{y}), \ \forall \ \mathbf{x}, \mathbf{y} \in \mathbf{R}^{n+k}\}.$$

If we write

$$\mathbb{1}_{n,k} := \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & -\mathbb{1}_k \end{pmatrix}$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix, then it is easy to see that

$$\mathrm{O}(n, k) = \left\{A \in GL(n + k, \mathbb{R}) \mid A^T \mathbb{1}_{n,k} A = \mathbb{1}_{n,k}\right\}.$$

[3] This follows from the famous Peter-Weyl theorem.

$\mathrm{O}(n,k)$ is called the generalised orthogonal group. We also define $\mathrm{SO}(n,k) := \mathrm{O}(n,k) \cap \mathrm{SL}(n+k,\mathbb{R})$. It is easily seen that the determinant of generalised orthogonal matrices is ± 1. We usually call $\mathrm{SO}(1,n)$ the *proper Lorentz group*.

(iii) Symplectic Group: Let

$$J := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}.$$

Then we define

$$\begin{aligned} &\mathrm{Sp}(n,\mathbb{R}) := \left\{A \in \mathrm{GL}(2n,\mathbb{R}) \mid A^T J A = J\right\}, \\ &\mathrm{Sp}(n,\mathbb{C}) := \left\{A \in \mathrm{GL}(2n,\mathbb{C}) \mid A^T J A = J\right\}, \\ &\mathrm{Sp}(n) := \mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(2n). \end{aligned}$$

The groups $\mathrm{Sp}(n,\mathbb{C})$ and $\mathrm{Sp}(n,\mathbb{R})$ are called the symplectic groups and $\mathrm{Sp}(n)$ is called the compact symplectic group. It turns out that[4] $\det A = 1$ for every $A \in \mathrm{Sp}(n,\mathbb{C})$.

There are two more examples which we will be interested in. Before introducing those examples, we need a definition.

Definition 2.9. (Semidirect product of groups) Let G, H be two groups and $\varphi : G \to \mathrm{Aut}(H)$ a homomorphism of groups[5], semidirect product of G and H, denoted by $G \ltimes_\varphi H$, whose underlying set is $G \times H$ and the group operation is defined by

$$\begin{aligned} \bullet :& (G \times H) \times (G \times H) \longrightarrow G \times H \\ &((g_1,h_1),(g_2,h_2)) \longmapsto (g_1,h_1) \bullet (g_2,h_2) := (g_1 g_2, h_1 \varphi(g_1)(h_2)). \end{aligned}$$

We usually omit φ from notation $\ltimes_\varphi$ whenever it is clear.

Example 2.10. (i) The Euclidean Group: It is the isometry group of the flat Euclidean space[6] $\mathbb{R}^n$. It is given by the semidirect product of rotations and translations:

$$\mathrm{ISO}(n) := \mathrm{O}(n) \ltimes \mathbb{R}^n,$$

where the homomorphism $\varphi : \mathrm{O}(n) \longrightarrow \mathrm{Aut}(\mathbb{R}^n)$ is the standard matrix multiplication action of $A \in \mathrm{O}(n)$ on $\mathbb{R}^n$ which is a automorphism since A is invertible.

[4] although somewhat difficult to prove

[5] $\mathrm{Aut}(H)$ denotes the set of all group isomorphism from H to itself which is group under composition of maps.

[6] flat Euclidean space means $\mathbb{R}^n$ endowed with the flat Euclidean metric so that the isometry group can be defined as those transformations which preserve the metric

(ii) The Poincaré group: It is the isometry group of the flat Minkowski space[7] $\mathbb{R}^{1,n}$. It is given by

$$\mathrm{ISO}(1,n) := \mathrm{O}(1,n) \ltimes \mathbb{R}^{1+n},$$

where the homomorphism $\varphi : \mathrm{O}(1,n) \longrightarrow \mathrm{Aut}(\mathbb{R}^{1+n})$ is as in above example.

2.2 Topological Properties

Since Lie groups are smooth manifolds, they are in particular topological spaces. Thus we can talk about the topological properties of Lie groups. Moreover, since matrix Lie groups are defined to be closed subsets of $\mathrm{GL}(n,\mathbb{R})$ which can be considered as subset of $\mathbb{R}^{n^2}$. Thus we can talk about the topological properties of matrix Lie groups. In particular we can talk about the compactness and connectedness of matrix Lie groups. We have the following theorem.

Theorem 2.11. [5, Section 1.3]

(i) The matrix Lie groups $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$ *and* $\mathrm{Sp}(n)$ *are compact.*

(ii) The matrix Lie groups $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})$*, the generalised orthogonal group, the symplectic group and the Poincaré group are non compact.*

Theorem 2.12. [5, Section 1.3]

(i) The matrix Lie groups $\mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C}), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$ *and* $\mathrm{Sp}(n)$ *are connected.*

(ii) The matrix Lie groups $\mathrm{O}(n), \mathrm{O}(n,k)$ *and the Poincaré group are not connected.*

Remark 2.13. Although connectedness is an important property for Lie groups, but we can get away with not having this property by restricting to the connected component of the Lie group containing the identity. This is what we will do when dealing with the representations of the Poincaré group.

Another topological property which will be important is simple connectedness[8].

Theorem 2.14. [5, Section 1.3] $\mathrm{SU}(2)$ *is simply connected while* $\mathrm{SO}(n,k)$ *is not simply connected.*

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[7] the flat Minkowski space is $\mathbb{R}^{1+n}$ endowed with the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1 \ldots, -1)$. It is denoted by $\mathbb{R}^{1,n}$.

[8] See [9, Chapter 9] for a review of simple connectedness.

2.3 Lie Group Homomorphism

Definition 2.15. Let G and H be Lie groups. A map $\Phi : G \longrightarrow H$ is called a Lie group homomorphism if

(i) Φ is a group homomorphism.

(ii) Φ is smooth as a map between manifolds.

If Φ is a diffeomorphism of manifolds, then Φ is called a Lie group isomorphism.

Example 2.16. (i) The determinant map $\det : \mathrm{GL}(n, \mathbb{C}) \longrightarrow \mathbb{C}^{\times} = \mathbb{C}\backslash\{0\}$ is a Lie group homomorphism.

(ii) The map $\Phi : \mathbb{R} \longrightarrow \mathrm{SO}(2)$ given by

$$\theta \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is easily checked to be a Lie group homomorphism.

2.3.1 The Adjoint Map

A particularly important Lie group homomorphism is the adjoint map defined as follows: for each $g \in G$, the adjoint map Ad_g is defined by

$$\begin{aligned} \mathrm{Ad}_g : G &\longrightarrow G \\ h &\longmapsto \mathrm{Ad}_g(h) := ghg^{-1}. \end{aligned} \tag{2.1}$$

This map is easily checked to be a Lie group homomorphism. Moreover note that $\mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$. We will have more to say about this map later.

3 Lie Algebra

3.1 Abstract Lie Algebra

Definition 3.1. A Lie algebra $\mathfrak{g}$ is a finite dimensional vector space $\mathfrak{g}$ over a field[9] k with an additional binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, which satisfies the following axioms:

(i) Bilinearity: $[X, aY + bZ] = a[X, Y] + b[X, Z]$ for all $X, Y, Z \in \mathfrak{g}$ and $a, b \in k$.

(ii) Antisymmetry: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.

[9] we will usually take $k = \mathbb{R}$ or $\mathbb{C}$.

(iii) Jacobi Identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for all $X, Y, Z \in \mathfrak{g}$.

Two elements X and Y of a Lie algebra $\mathfrak{g}$ commute if $[X, Y] = 0$. A Lie algebra is commutative if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.

Example 3.2. (i) Let $\mathfrak{g} = \mathbb{R}^3$ and let $[\cdot, \cdot] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$[x, y] = x \times y$$

where $x \times y$ is the cross product (or vector product). Then $\mathfrak{g}$ is a Lie algebra.

(ii) The set $\mathrm{M}(n, \mathbb{C})$ with the Lie bracket given by

$$[A, B] := AB - BA$$

is a Lie algebra.

(iii) Let $\mathfrak{sl}(n, \mathbb{C})$ denote the space of all $X \in \mathrm{M}(n, \mathbb{C})$ for which $\mathrm{tr}(X) = 0$. Then $\mathfrak{sl}(n, \mathbb{C})$ is a Lie algebra with bracket $[X, Y] = XY - YX$.

(iv) Let V be a vector space. Then it is easy to check that the space $\mathrm{End}(V)$ of all linear operators on V is again a vector space. Infact $\mathrm{End}(V)$ along with the Lie bracket $[\phi, \psi] := \phi \circ \psi - \psi \circ \phi$ is a Lie algebra.

Definition 3.3. (i) A subalgebra of a real or complex Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[H_1, H_2] \in \mathfrak{h}$ for all $H_1, H_2 \in \mathfrak{h}$. If $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{h}$ is a real vector subspace of $\mathfrak{g}$ which is closed under brackets, then $\mathfrak{h}$ is said to be a real subalgebra of $\mathfrak{g}$.

(ii) A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is said to be an ideal in $\mathfrak{g}$ if $[X, H] \in \mathfrak{h}$ for all X in $\mathfrak{g}$ and H in $\mathfrak{h}$.

Definition 3.4. (i) If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras, the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ denoted by $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, is the vector space direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, with bracket given by

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2]).$$

(ii) Let $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are subalgebras of $\mathfrak{g}$. We say that $\mathfrak{g}$ is the Lie algebra direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, and write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, if $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as vector spaces and $[X_1, X_2] = 0$ for all $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$.

Definition 3.5. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. If, in addition, ϕ is one-to-one and onto, then ϕ is called a Lie algebra isomorphism.

Example 3.6. (The Adjoint map) For each $X \in \mathfrak{g}$, we have a Lie algebra homomorphism, denoted by ad_X, defined as follows:

$$\begin{aligned} \mathrm{ad}_X : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Y &\longmapsto [X, Y]. \end{aligned} \tag{3.1}$$

ad_X is easily checked to be a linear map on $\mathfrak{g}$. The map $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is called the adjoint map.

Proposition 3.7. [5, Proposition 3.8] *The map ad* : $g \longrightarrow \mathrm{End}(\mathfrak{g})$ *is a Lie algebra homomorphism, that is*

$$\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \circ \mathrm{ad}_Y - \mathrm{ad}_Y \circ \mathrm{ad}_X = [\mathrm{ad}_X, \mathrm{ad}_Y].$$

Definition 3.8. (i) A Lie algebra $\mathfrak{g}$ is called simple if the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and $\{0\}$ and it is non commutative.

(ii) A Lie algebra $\mathfrak{g}$ is called semisimple if it is isomorphic to direct sum of simple Lie algebras.

Definition 3.9. Let $\mathfrak{g}$ be a finite dimensional real or complex Lie algebra, and let $X_1, \dots, X_d$ be a basis for the vector space $\mathfrak{g}$. Then the unique constants c_{ijk} such that

$$[X_i, X_j] = \sum_{i=1}^{d} c_{ijk} X_k$$

are called the structure constants of $\mathfrak{g}$. It depends on the chosen basis.

Remark 3.10. In terms of the structure constants, the antisymmetry and Jacobi identity of the Lie bracket takes the form:

$$\begin{aligned} c_{ijk} + c_{jik} &= 0, \\ \sum_n (c_{jkn} c_{nlm} + c_{kln} c_{njm} + c_{ljn} c_{nkm}) &= 0. \end{aligned}$$

for all j, k, l, m.

3.2 Lie Algebra of a Lie Group

Let G be a Lie group and $\Gamma(G)$ be the set of all smooth vector fields on G. Then $\Gamma(G)$ is an $\mathbb{R}$-vector space and also a $C^\infty(G)$-module where $C^\infty(G)$ is the ring of all smooth functions $f : G \longrightarrow \mathbb{R}$. There is a natural bracket operation on $\Gamma(G)$ defined as follows:

$$\begin{aligned} [\cdot, \cdot] : &\Gamma(G) \times \Gamma(G) \longrightarrow \Gamma(G) \\ &(X, Y) \longmapsto [X, Y] f := X(Yf) - Y(Xf), \quad f \in C^\infty(G). \end{aligned}$$

It is easily checked that $[\cdot,\cdot]$ is a Lie bracket but there is one obstacle. As an $\mathbb{R}$-vector space, $\Gamma(G)$ is infinite dimensional possibly of uncountable dimension. There is an easy way of extracting a finite dimensional vector subspace of $\Gamma(G)$ called the subspace $\mathcal{L}(G)$ of *left invariant* vector fields. We will not describe this subspace here and content ourselves with the following theorem.

Theorem 3.11. [7, Theorem 8.37] *Let G be a Lie group and $\mathcal{L}(G)$ be the Lie algebra of left invariant vector fields. Then there is a cannonical vector space isomorphism between $\mathcal{L}(G)$ and T_eG where T_eG is the tangent space of G at identity. In particular, T_eG can be made into a Lie algebra with the bracket operation induced by the isomorphism.*

Remark 3.12. For matrix Lie groups, as we will show the tangent space at identity is isomorphic to a vector subspace of $M(n,\mathbb{R})$ or $M(n,\mathbb{C})$ depending on the Lie group (see Theorem 3.19). Moreover it can be shown that (we shall not prove this) the Lie bracket inherited from the Lie algebra of left invariant vector fields by Theorem 3.11 is simply the commutator of matrices as in Example 3.2 (ii).

We make the following definition based on Theorem 3.11.

Definition 3.13. Let G be a Lie group. The Lie algebra of G denoted by $\mathfrak{g}$ is the tangent space of G at identity with the Lie bracket induced from $\mathcal{L}(G)$ using Theorem 3.11.

We will now try to find the Lie algebra of the matrix Lie groups. Recall that the directional derivative $X_{\gamma,g}$ of a smooth curve $\gamma : \mathbb{R} \longrightarrow G$ with *initial point* $g \in G$, that is $\gamma(0) = g$ is a linear map $X_{\gamma,g} : C^\infty(G) \longrightarrow \mathbb{R}$ given by

$$X_{\gamma,g}(f) = (f \circ \gamma)'(0).$$

The directional derivative operator $X_{\gamma,g}$ is called a tangent vector at g. We next define the tangent space T_gG at g to be the set of all tangent vectors at g obtained from all smooth curves. That is

$$T_gG := \{X_{\gamma,g} \mid \gamma : \mathbb{R} \longrightarrow G \text{ is a smooth curve throught } g\}.$$

This definition simplifies a bit if we restrict ourselves to matrix Lie groups. Indeed if G is a matrix Lie groups then we can consider as being embedded in $\mathbb{R}^{n^2}$ or $\mathbb{R}^{2n^2}$ depending on whether $G \subset \mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$. Then given a curve $\gamma : \mathbb{R} \longrightarrow G$ and a function $f \in C^\infty(G)$, we can talk of their derivatives separately. In particular, by chain rule, we have

$$X_{\gamma,g}(f) = (f \circ \gamma)'(0) = \dot\gamma(0)f'(\gamma(0)) = \dot\gamma(0)f'(g).$$

This suggests the identification

$$T_gG \cong \{\dot{\gamma}(0) \mid \gamma : \mathbb{R} \longrightarrow G \text{ is a smooth curve throught } g\}.$$

Infact something more is true. Before we state the result, we need to introduce the matrix exponential.

3.2.1 The Matrix Exponential

Let X be an $n \times n$ matrix. We define the exponential of X, denoted e^X or $\exp X$, by the usual power series

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

where $X^0 = \mathbb{1}_n$ is the identity matrix and X^m is the repeated matrix product of X with itself. First of all to make sense of the infinite series definition of the matrix exponential, we must make sure that the series converges in some sense. Indeed, one can show it indeed converges in the *Hilbert-Schmidt norm* defined as

$$\|X\| := \left(\sum_{i,j} |X_{ij}|^2\right)^{1/2}.$$

We list the properties of the matrix exponential below.

Theorem 3.14. [5, Proposition 2.3, Theorem 5.3] *Let* $, Y \in \mathrm{M}(n, \mathbb{C})$. *Then the following holds:*

(i) $e^0 = \mathbb{1}$.

(ii) $\left(e^X\right)^\dagger = e^{X^\dagger}$.

(iii) $\left(e^X\right)^{-1} = e^{-X}$.

(iv) $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta Y} \quad \forall\ \alpha, \beta \in \mathbb{C}$.

(v) Baker-Campbell-Hausdorff (BCH) formula[10]

$$e^X e^Y = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots\right)$$

where $[X, Y] = XY - YX$. *In particular, if* $[X, Y] = 0$ *then* $e^X e^Y = e^{X+Y}$.

[10] The series on the right hand side may not be convergent. Given some conditions on $\|X\|, \|Y\|$, it can be shown that the series converges. We need matrix logarithm to prove this result. See — for details.

(vi) *For any* $C \in \mathrm{GL}(n, \mathbb{C})$,

$$Ce^X C^{-1} = e^{CXC^{-1}}.$$

(vii) $\det\left(e^X\right) = e^{\mathrm{tr} X}$. *In particular* $e^X \in \mathrm{GL}(n, \mathbb{C}) \quad \forall\ X \in \mathrm{M}(n, \mathbb{C})$.

(viii) $\exp : \mathbb{R} \longrightarrow \mathrm{GL}(n, \mathbb{C}), \quad t \mapsto e^{tX}$ *is a smooth map of manifolds and*

$$\frac{d}{dt} e^{tX} = X e^{tX}.$$

Last part of above theorem gives a way to construct smooth curve in $\mathrm{GL}(n, \mathbb{C})$. Infact, this gives us an example of what are called *one parameter subgroups*. We define them below:

Definition 3.15. Let G be a Lie group. A function $\gamma : \mathbb{R} \longrightarrow G$ is called a one parameter subgroup of G if

(i) γ is smooth,

(ii) $\gamma(0) = e \in G$,

(iii) $\gamma(t + s) = \gamma(t)\gamma(s)$ for all $t, s \in \mathbb{R}$.

Remark 3.16. One parameter subgroups are rather special curves to the manifold G. So we can talk about the directional derivatives of the one parameter subgroup.

For matrix Lie groups, the one parameter subgroups can be characterised using the matrix exponential.

Theorem 3.17. [5, Theorem 2.14] *If* γ *is a one parameter subgroup of* $\mathrm{GL}(n, \mathbb{C})$, *then there exists a unique* $n \times n$ *complex matrix* X *such that*

$$\gamma(t) = e^{tX}.$$

The aim of introducing these notions is to characterise the Lie algebra of matrix Lie groups without going to the vector field definition of Lie algebras. The following proposition will be useful on the way.

Proposition 3.18. [7, Proposition 20.3] *Suppose* G *is a Lie group and* $H \subseteq G$ *is a Lie subgroup. The one parameter subgroups of* H *are precisely those one parameter subgroups of* G *whose directional derivatives at* e *lie in* $T_e H$.

We are now ready to characterise the Lie algebra of a matrix Lie group.

Theorem 3.19. *Let* G *be a matrix Lie group. The Lie algebra* $\mathfrak{g}$ *of* G *is the set of all matrices* X *such that* e^{tX} *is in* G *for all real numbers* t.

Proof. We will use our observation that

$$T_{\mathbb{1}}G \cong \{\dot{\gamma}(0) \mid \gamma : \mathbb{R} \longrightarrow G \text{ is a smooth curve throught } \mathbb{1}\}.$$

First suppose $e^{tX} \in G$ for all $t \in \mathbb{R}$. Then the curve $\gamma : \mathbb{R} \longrightarrow G$ given by $\gamma(t) = e^{tX}$ is a smooth curve in G with $\gamma(0) = \mathbb{1}$. Then by Theorem 3.14, we have that

$$\dot{\gamma}(0) = Xe^0 = X.$$

Thus X already belongs to the Lie algebra. Conversely suppose $X \in \mathfrak{g}$. We want to show that $e^{tX} \in G$ for every real t. Indeed consider the curve $\gamma : \mathbb{R} \longrightarrow \mathrm{GL}(n, \mathbb{C})$ given by $t \longmapsto e^{tX}$. Then by Theorem 3.14, it is clear that γ is a one parameter subgroup in $\mathrm{GL}(n, \mathbb{C})$. Now clearly the directional derivative of γ at $\mathbb{1}$

$$\dot{\gamma}(0) = \frac{d}{dt}e^{tX}\bigg|_{t=0} = X \in T_{\mathbb{1}}G = \mathfrak{g}.$$

Thus by Proposition 3.18, γ is a one parameter subgroup of G which implies that $e^{tX} = \gamma(t) \in G$. $\square$

Remark 3.20. In physics, one usually considers the map $t \longmapsto e^{itX}$. So the computations usually differ by a factor of i. We will use the mathematicians approach here.

We now list the Lie algebras of classical matrix Lie groups. One might be thinking that if exponentiating the Lie algebra elements give elements of the corresponding Lie group, what part of the Lie group can be recovered using the Lie algebra. The answer varies depending on the topological properties of the Lie group.

Theorem 3.21. [6, Theorem 12.2, Proposition 12.5] *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$.*

(i) The exponential map $\exp : \mathfrak{g} \longrightarrow G$ *is a diffeomorphism of a neighbourhood*[11] *of* $0 \in \mathfrak{g}$ *to a neighbourhood of* $\mathbb{1} \in G$.

(ii) If G is connected then the exponential map is surjective from any neighbourhood of $0 \in \mathfrak{g}$.

3.3 Lie Group Verses Lie Algebra Homomorphism

We begin by recording an important result.

[11] The topology on $\mathfrak{g}$ is inherited from $\mathbb{R}^{n^2}$ or $\mathbb{R}^{2n^2}$ depending on whether $\mathfrak{g} \subset \mathrm{M}(n, \mathbb{R})$ or $\mathfrak{g} \subset \mathrm{M}(n, \mathbb{C})$.

Lie Groups	Lie Algebra
$\mathrm{GL}(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C}) = \mathrm{M}(n,\mathbb{C})$
$\mathrm{GL}(n,\mathbb{R})$	$\mathfrak{gl}(n,\mathbb{R}) = \mathrm{M}(n,\mathbb{R})$
$\mathrm{SL}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C}) = \{A \in \mathrm{M}(n,\mathbb{C}) \mid \mathrm{tr}(A) = 0\}$
$\mathrm{SL}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathrm{M}(n,\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$
$\mathrm{U}(n)$	$\mathfrak{u}(n) = \left\{A \in \mathrm{M}(n,\mathbb{C}) \mid A^\dagger = -A\right\}$
$\mathrm{SU}(n)$	$\mathfrak{su}(n) = \left\{A \in \mathrm{M}(n,\mathbb{C}) \mid A^\dagger = -A, \mathrm{tr}(A) = 0\right\}$
$\mathrm{O}(n)$	$\mathfrak{o}(n) = \{A \in \mathrm{M}(n,\mathbb{R}) \mid A^t = A\}$
$\mathrm{SO}(n)$	$\mathfrak{so}(n) = \mathfrak{o}(n)$
$\mathrm{O}(n,k)$	$\mathfrak{o}(n,k) = \{A \in \mathrm{M}(n+k,\mathbb{R}) \mid \mathbb{1}_{n,k}A^t\mathbb{1}_{n,k} = -A\}$
$\mathrm{SO}(n,k)$	$\mathfrak{so}(n,k) = \mathfrak{o}(n,k)$
$\mathrm{Sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C}) = \{A \in \mathrm{M}(2n,\mathbb{C}) \mid \Omega A^t \Omega = A\}, \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$
$\mathrm{Sp}(n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{R}) = \{A \in \mathrm{M}(2n,\mathbb{R}) \mid \Omega A^t \Omega = A\}$
$\mathrm{Sp}(n)$	$\mathfrak{sp}(n) = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{u}(n).$

Table 1: Classical Lie groups and their Lie algebras. See [5, Section 3.4] for proofs.

Theorem 3.22. [10, Theorem 16.14] *Let $\Phi : G \longrightarrow H$ be a Lie group homomorphism of Lie groups. Then the pushforward at e of this map $\Phi_{*,e} : T_eG \longrightarrow T_{\Phi(e)}H$ is a Lie algebra homomorphism. In particular, if Φ is a diffeomorphism, then $\Phi_{*,e}$ is a Lie algebra isomorphism.*

Example 3.23. Recall the adjoint map Ad_g is defined on the Lie group for each $g \in G$ (see Eq. 2.1). Since $\mathrm{Ad}_g(e) = geg^{-1} = e$, so that the pushforward $(\mathrm{Ad}_g)_{*,e} : T_eG \longrightarrow T_eG$ is a Lie algebra homomorphism. Moreover since $\mathrm{Ad}_g^{-1} = \mathrm{Ad}_{g^{-1}}$, thus $(\mathrm{Ad}_g)_{*,e}^{-1} = (\mathrm{Ad}_{g^{-1}})_{*,e}$. Thus $(\mathrm{Ad}_g)_{*,e} \in \mathrm{GL}(\mathfrak{g})$. Define the map Ad as follows:

$$\begin{aligned} \mathrm{Ad} : G &\longrightarrow \mathrm{GL}(\mathfrak{g}) \\ g &\longmapsto \mathrm{Ad}_g. \end{aligned}$$

Then clearly Ad is a Lie group homomorphism.

Proposition 3.24. [5, Proposition 3.34] *Let $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$ be the Lie group homomorphism as above. Then the induced map $\mathrm{Ad}_{*,e} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$ is given by*

$$\begin{aligned} \mathrm{Ad}_{*,e}(X) : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ Y &\longmapsto [X,Y], \quad X, Y \in \mathfrak{g}. \end{aligned}$$

In particular $\mathrm{Ad}_{*,e}(X) = \mathrm{ad}_X$ (see Eq. 3.1). We denote $\mathrm{Ad}_{*,e} = \mathrm{ad}$.

Remark 3.25. Above theorem guarantees that diffeomorphic Lie groups have isomorphic Lie algebras. The converse to this statement is not true in general. As we will see below, the converse is true under stronger assumption of simple connectedness.

In case of matrix Lie groups, we can say more about the maps in Theorem 3.22.

Theorem 3.26. [5, Theorem 3.28] *Let G and H be matrix Lie groups, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Suppose that $\Phi : G \to H$ is a Lie group homomorphism. Then there exists a unique $\mathbb{R}$-linear map $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$ such that*

$$\Phi\left(e^X\right) = e^{\phi(X)}$$

for all $X \in \mathfrak{g}$. The map ϕ has following additional properties:

(i) $\phi\left(AXA^{-1}\right) = \Phi(A)\phi(X)\Phi(A)^{-1}$*, for all* $X \in \mathfrak{g}, A \in G$.

(ii) $\phi([X,Y]) = [\phi(X), \phi(Y)]$*, for all* $X, Y \in \mathfrak{g}$.

(iii) $\phi(X) = \frac{d}{dt}\Phi\left(e^{tX}\right)\big|_{t=0}$*, for all* $X \in \mathfrak{g}$.

Example 3.27. Applying this theorem to the adjoint map, we have

$$\mathrm{Ad}_{e^X} = e^{\mathrm{ad}_X}, \quad X \in \mathrm{M}(n, \mathbb{C}).$$

As mentioned in Remark 3.25, we have the following converse to the above theorem.

Theorem 3.28. [5, Theorem 5.6] *Let G and H be matrix Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. If G is simply connected, there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\Phi\left(e^X\right) = e^{\phi(X)}$ for all $X \in \mathfrak{g}$.*

Remark 3.29. Suppose G and H are simply connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, along with a Lie algebra isomorphism $\phi : \mathfrak{g} \longrightarrow \mathfrak{h}$. Then applying above theorem for ϕ and ϕ^{-1}, we get a Lie group homomorphisms $\Phi : G \longrightarrow H$ and $\Phi^{-1} : H \longrightarrow G$. The uniqueness part of the above theorem then says that these maps are indeed inverses of each other and hence the Lie groups are diffeomorphic.

3.3.1 Universal Covers

In previous section, we saw that the Lie group homomorphisms between Lie groups G and H and Lie algebra homomorphisms between their corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively are in one to one correspondence if G is simply connected. This property will be of great importance when we go to representations of Lie groups. For simply connected Lie groups, the representations of the group and its Lie algebra are in one to one correspondence.

But it turns out that many of the important Lie groups whose representations are most important for physics applications are not simply connected, for example the Lorentz group. That is why we consider the universal cover of Lie groups. Roughly speaking, *the* universal cover of a Lie group is a simply connected Lie group[12] whose Lie algebra is *same* as the Lie algebra of the original Lie group. We make this precise now.

Definition 3.30. Let G be a connected Lie group. Then a universal cover of G is a simply connected Lie group H together with a Lie group homomorphism $\Phi : H \longrightarrow G$ such that the associated Lie algebra homomorphism (of Theorem 3.22) $\phi : \mathfrak{h} \longrightarrow \mathfrak{g}$ is a Lie algebra isomorphism. The homomorphism Φ is called the *covering map*.

Universal covers are unique in the following sense:

Theorem 3.31. [7, Theorem 7.9] *If G is a connected Lie group and (H_1, Φ_1) and (H_2, Φ_2) are universal covers of G, then there exists a Lie group isomorphism $\Psi : H_1 \longrightarrow H_2$ such that $\Phi_2 \circ \Psi = \Phi_1$.*

We have the following existence theorem for universal covers.

Theorem 3.32. [7, Theorem 7.7] *Let G be a connected Lie group then there always exists a universal cover of G.*

Universal covers are objects which are discriminative of matrix Lie group. Above theorem says that universal cover always exists but if G is a matrix Lie group then one would expect that the universal cover also to be a matrix Lie group. But it is simply not true. Consider for example the matrix Lie group $\mathrm{SL}(2, \mathbb{R})$. Its universal cover, the *metaplectic group* is not a matrix Lie group.

Note that the requirement that G be a connected Lie group is a technical condition. Given that many of the Lie groups are not connected is not a big obstacle. We can restrict the connected component containing identity due to the following theorem.

Proposition 3.33. [10, Problem 15.3] *Let G be a Lie group and G_0 be the connected component containing the identity. Then G_0 is a Lie group with the same Lie algebra as that of G.*

Example 3.34. The universal cover of $\mathrm{SO}(3)$ is $\mathrm{SU}(2)$. The universal cover of the Lorentz group $\mathrm{SO}(1, n)$ in various dimensions is called the *Spin* groups denoted by $\mathrm{Spin}(1, n)$. In particular $\mathrm{Spin}(1, 3) \cong \mathrm{SL}(2, \mathbb{C})$.

[12]It is in particular the universal cover of the topological space G.

4 Basic Representation Theory

We will begin be giving the definition and examples of representation of Lie groups and Lie algebras.

4.1 Definitions and Examples

Definition 4.1. Let V be a finite dimensional[13] vector space over $\mathbb{C}$ or $\mathbb{R}$. Let $\mathrm{GL}(V)$ denote the set of all invertible linear maps from V to V. Let $\mathrm{End}(V) =: \mathfrak{gl}(V)$ denote the space of all linear operators on V. One can see that $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{gl}(V) \cong \mathrm{M}(n, \mathbb{C})$, so that $\mathrm{GL}(V)$ can be thought of as a Lie group and $\mathfrak{gl}(V)$ as a Lie algebra with bracket $[X, Y] = XY - YX$.

Definition 4.2. Let V be a finite dimensional vector space over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

(i) A representation (Π, V) of a Lie group G is a Lie group homomorphism $\Pi : G \longrightarrow \mathrm{GL}(V)$.

(ii) A representation (π, V) of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$.

If the homomorphisms are injective then the corresponding representation is called *faithful.*

Remark 4.3. Since $\Pi(g), \pi(x)$ are operators on V, we often say (Π, V) and (π, V) to be a representation acting on V.

Example 4.4. (i) For any Lie group G and any Lie algebra $\mathfrak{g}$ and any vector space V, the map $\Pi : G \longrightarrow \mathrm{GL}(V)$ and $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ given by $\Pi(g) = \mathbb{1}_V$ and $\pi(X) = 0$ for every $g \in G$ and $X \in \mathfrak{g}$ is a representation. It is called the *trivial representation.*

(ii) Let G be a matrix Lie group. Then $(\Pi, \mathbb{C}^n)$ where Π is defined as

$$\begin{aligned} \Pi : & G \longrightarrow \mathrm{GL}(\mathbb{C}^n) \\ & A \to \Pi(A) \end{aligned}$$

where $\Pi(A)(v)$ is defined by matrix multiplication of the column vector of v by A with respect to a given basis of $\mathbb{C}^n$. This is called the *fundamental representation* of G. One can similarly define the fundamental representation of a Lie algebra of a matrix Lie group.

[13] Extending these definitions and definitions hereafter to infinite dimensional vector space requires some work. We will not pursue this in these notes.

(iii) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then we have a representation of G acting on the vector space $\mathfrak{g}$ given by $\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$, where Ad is as in Example 3.23. There is an associated representation of the Lie algebra $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g})$, where again the ad map was defined in Example 3.23. These representations are called the *adjoint representation*.

Definition 4.5. Let (Π, V) be a representation of a Lie group G acting on V.

(i) A subspace $W \subset V$ is said to be invariant subspace if $\Pi(g)(w) \in W$, $\forall\ g \in G$ and $w \in W$.

(ii) The representation (Π, V) is said to be an irreducible representation if only invariant subspace of V are $\{0\}$ and V.

These definitions are analogous for representations of Lie algebra.

Theorem 4.6. *Let $\mathfrak{g}$ be a Lie algebra. The adjoint representation of $\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}$ is simple.*

Proof. ($\Longrightarrow$) Suppose W is an invariant subspace of ad. Then $[X, W] \subset W$ for every $X \in \mathfrak{g}$ which implies that W is an ideal. Since $\mathfrak{g}$ is simple, $W = \{0\}$ or $W = \mathfrak{g}$.
($\Longleftarrow$) Suppose W is an ideal of $\mathfrak{g}$. Then $[X, W] \subset W$ for every $X \in \mathfrak{g}$ which implies that W is an invariant subspace of ad. By irreducibility of ad, $W = \{0\}$ or $W = \mathfrak{g}$ and the claim follows. □

Definition 4.7. Let (Π_1, V_1) and (Π_2, V_2) be representations of a Lie group G. A linear map $\phi : V_1 \longrightarrow V_2$ is called an intertwiner if

$$\phi\left(\Pi_1(g)v\right) = \Pi_2(g)(\phi(v)), \quad v \in V_1.$$

Intertwiner for Lie algebra is defined analogously. If ϕ is an isomorphism of vector spaces then (Π_1, V_1) and (Π_2, V_2) are said to be isomorphic representations and we write $\Pi_1 \cong \Pi_2$.

Using Theorem 3.22, we see that given a representation (Π, V) of a Lie group G, the pushforward at identity will give a representation of the associated Lie algebra since Lie algebra of $\mathrm{GL}(V)$ is precisely $\mathfrak{gl}(V)$. Infact, as a consequence of Theorem 3.26, we have a stronger statement in case of matrix Lie groups.

Theorem 4.8. [5, Proposition 4.4, Proposition 4.5] *Let (Π, V) be a representation of a matrix Lie group G. Then there exists a unique representation (Π, V) of the associated Lie algebra $\mathfrak{g}$ such that $\Pi\left(e^X\right) = e^{\pi(X)}$, $\forall\ X \in \mathfrak{g}$. Moreover (Π, V) is irreducible if and* *only if (π, V) is irreducible. If (Π_1, V_1) and (Π_2, V_2) are any two Lie group representations with corresponding Lie algebra representations π_1, π_2, then $\Pi_1 \cong \Pi_2$ if and only if $\pi_1 \cong \pi_2$.*

Example 4.9. By Theorem 4.6, the adjoint map of Lie group G is irreducible if and only if the corresponding Lie algbera is simple.

Of particular interest to us will be unitary representations.

Definition 4.10. Let V be a finite dimensional inner product space. A representation (Π, V) is called a unitary representation if $\Pi(g)$ is a unitary operator for every $g \in G$. Unitary representation of a Lie algebra is defined similarly.

4.2 Constructing New Representations from Old

We can construct new representations from old using direct sum and tensor product of representations.

Definition 4.11. Let $(\Pi_1, V_1), (\Pi_2, V_2)$ be representations of a Lie group G and $(\pi_1, V_1), (\pi_2, V_2)$ be representations of a Lie algebra $\mathfrak{g}$.

(i) The direct sum of representations (Π_1, V_1) and (Π_2, V_2), denoted by $(\Pi_1 \oplus \Pi_2, V_1 \oplus V_2)$, acts on the direct sum $V_1 \oplus V_2$ of vector spaces V_1, V_2 and is defined by

$$(\Pi_1 \oplus \Pi_2)(g)(v_1, v_2) = (\Pi_1(g)v_1, \Pi_2(g)v_2).$$

Similarly we can define direct sum of representations of Lie algebra (π_1, V_1) and (π_2, V_2).

(ii) The tensor product of representations (Π_1, V_1) and (Π_2, V_2) denoted by $(\Pi_1 \otimes \Pi_2, V_1 \otimes V_2)$, acts on the tensor product $V_1 \otimes V_2$ of vector spaces V_1, V_2 and is defined by

$$(\Pi_1 \otimes \Pi_2)(g)(v_1 \otimes v_2) = \Pi_1(g)(v_1) \otimes \Pi_2(g)(v_2).$$

For Lie algebra, it is defined by

$$(\pi_1 \otimes \pi_2)(X)(v_1 \otimes v_2) = \pi_1(X)(v_1) \otimes v_2 + v_1 \otimes \pi_2(X)(v_2).$$

Definition 4.12. A representation acting on V is said to be unitrizable if there is an inner product on V such that it becomes unitary representation.

Definition 4.13. A representation is said to be completely reducible if it is isomorphic to the direct sum of irreducible representations.

Theorem 4.14. [5, Theorem 4.27, Theorem 4.28] *(i) Every unitary representation is completely reducible.*

(ii) Let (Π, V) be a finite dimensional representation of a compact Lie group G. Then (Π, V) is unitrizable and hence completely reducible.

From above theorem, we see that we only need to study the irreducible representations of a compact Lie group because every finite dimensional representation is completely reducible.

Theorem 4.15. *Let G be a noncompact connected Lie group with the Lie algebra being simple. Then any unitary representation of G is not finite dimensional.*

Another result which is wildly useful is Schur's Lemma.

Theorem 4.16. (Schur's Lemma) [5, Corollary 4.30] *Let Π be an irreducible complex representation of a matrix Lie group G. If A is in the center of G, then $\Pi(A) = \lambda I$, for some $\lambda \in \mathbb{C}$. Similarly, if π is an irreducible complex representation of a Lie algebra $\mathfrak{g}$ and if $[X, Y] = 0$ for every $Y \in \mathfrak{g}$, then $\pi(X) = \lambda I$.*

5 Method of Induced Representations: Mackey Theory

We want to classify all irreducible projective representations (see Section 6 for precise definitions) of the Poincaré group which is a semidirect product. Now since the Poincaré group is noncompact and not simply connected, its representations are infinite dimensional and are not in one-to-one correspondence with the representations of its Lie algebra. So our strategy will be to pass on to the universal cover and classify all unitary irreducible representation and come back to Poincaré group using some standard results describes in next section (see Theorem 6.4 for the precise statement). So our first task is to classify all unitary irreducible representations of the Poincaré group. Mackey's theory comes in handy at this point. It reduces the problem of describing irreducible representation of a semidirect product to the calculation of certain orbits and stabilisers and the irreducible representations of stabilisers. This is called the method of induced representations. Let us begin by making some definitions.

5.1 Induced Representations

Definition 5.1. Let G be a Lie group and H a Lie subgroup. Suppose (Π, W) is a representation of H. Then the induced representation denoted by $(\mathrm{Ind}_G^H(\Pi), \mathrm{Ind}_G^H(W))$

acts on the vector space $\mathrm{Ind}_G^H(W)$ of functions:

$$\mathrm{Ind}_G^H(W) := \{\phi : G \longrightarrow W \mid \phi(hg) = \Pi(h)\phi(g),\ \forall\ h \in H, g \in G, \phi \text{ smooth}\},$$

and G acts on $\mathrm{Ind}_G^H(W)$ as follows: for $g \in G$, $\mathrm{Ind}_G^H(\Pi)(g) : \mathrm{Ind}_G^H(W) \longrightarrow \mathrm{Ind}_G^H(W)$ is given by

$$\left(\mathrm{Ind}_G^H(\Pi)(g)(\phi)\right)(g') = \phi\left(g'g\right), \quad \phi \in \mathrm{Ind}_G^H(W).$$

We need to show that $\mathrm{Ind}_G^H(\Pi)(g)(\phi) \in \mathrm{Ind}_G^H(W)$.Indeed

$$\begin{aligned}\left(\mathrm{Ind}_G^H(\Pi)(g)(\phi)\right)(hg') =& \phi\left(hg'g\right) = \Pi(h)\phi\left(g'g\right) \\ &= \Pi(n)\left(\mathrm{Ind}_G^H(\Pi)(g)(\phi)\right)(g').\end{aligned}$$

It is also clear that $\mathrm{Ind}_G^H(\pi)(g)(\phi)$ is smooth. So we indeed have a representation.

5.2 Representation of Semi Direct Product

Let us now go to semidirect products. Let $G = H \ltimes_\phi N$ where N is an abelian Lie group. Since we will consider these kinds of semidirect product, the next theorem is relavant.

Theorem 5.2. [5, Corollary 4.31] *Let A be an abelian group. Then all irreducible representations of A are 1 dimensional. Moreover, the irreducible characters of A form a group $\widehat{A}$:*

$$\widehat{A} := \mathrm{Hom}\left(A, \mathbb{C}^*\right)$$

which is the group of all group homomorphisms from A to $\mathbb{C}^*$.

Recall that the composition law on $H \ltimes_\phi N$ is

$$\left(h_1, n_1\right) \cdot \left(h_2, n_2\right) = \left(h_1 h_2, n_1 \phi\left(h_1\right)\left(n_2\right)\right)$$

We usually omit ϕ so that we can write $\phi(h)(n) =: h(n)$. We often say that H acts on N via automorphisms. We will use this notation from now on. The inverse of (h, n) is

$$\left(h^{-1}, h^{-1}\left(n^{-1}\right)\right),$$

and the identity is (e_H, e_N).

Lemma 5.3. *$G = H \ltimes N$ acts naturally on N as $(h, n) \cdot n' = n \cdot h\left(n'\right)$.*

Proof. Straightforward verification. ☐

We now turn to unitary irreducible representations of the semidirect product. Observe that if H acts on N via automorphisms then it induces a natural action of H on $\widehat{N}$ (see Lemma 5.3 for this notation) as follows:

$$(h \cdot \chi)(n) = \chi\left(h^{-1}(n)\right), \quad h \in H, \chi \in \widehat{N}. \tag{5.1}$$

Let the orbits of this action be denoted by O_i with representatives χ_i. Also let $H_i = H_{\chi_i}$ be the stabiliser of χ_i.

Definition 5.4. By a section for the H-action on $\widehat{N}$ we mean a subset $S \subset \widehat{N}$ which intersects every H-orbit in precisely one point. We shall call a section σ-compact if it is a countable union of compact subsets of $\widehat{N}$.

A character $\chi \in \hat{N}$ can be thought of as a one dimensional representation of $H_\chi \ltimes N$ as follows:

$$\chi : H_\chi \ltimes N \longrightarrow \mathbb{C}^*, \quad \chi(h, n) = \chi(n).$$

We now have the following important result due to Mackey.

Theorem 5.5. (Mackey) [2, Theorem 11.6] *Let $G = H \ltimes N$ where N is an abelian Lie group. Suppose that the action of H on N allows a σ-compact section. Let $\chi \in \widehat{N}$ and ξ be a unitary irreducible representation of H_χ, the stabiliser subgroup of χ. Then the representation $\mathrm{Ind}_{H_\chi \ltimes N}^{H \ltimes N}(\xi \otimes \chi)$ is a unitary irreducible representation of G. Moreover every unitary irreducible representation of G is of this form. Furthermore*

$$\mathrm{Ind}_{H_\chi \ltimes N}^{H \ltimes N}(\xi \otimes \chi) \cong \mathrm{Ind}_{H_{\chi'} \ltimes N}^{H \ltimes N}(\xi' \otimes \chi')$$

if and only if there exists $g \in G$ such that $g \cdot \chi = \chi'$ and $\xi \cong \xi' \circ C_g$ where C_g is the conjugate representation of G on itself.

Mackey's theorem says that to find all unitary irreducible representations of $G = H \ltimes N$, find all orbits of H-action on $\widehat{N}$ and stabiliser of a representative from each orbit, and find all unitary irreducible representations of the subgroup $H_\chi \ltimes N$ and induce. We will see that this will simplify matters a lot in case of Poincaré group.

6 Irreducible Projective Representations of the Poincaré Group

As we noted earlier, the Poincaré group is the semidirect product of generalised orthogonal group and the translation group. In physics literature, it is usually denoted by

$$\mathrm{IO}(D-1,1) = \mathrm{O}(D-1,1) \ltimes \mathbb{R}^{D-1,1},$$

where D is the spacetime dimension. But we will be interested in the connected component of this group which is connected to identity – the space of inhomogenous orthochronous Lorentz transformations:

$$\mathrm{ISO}(D-1,1) = \mathrm{SO}(D-1,1)_I \ltimes \mathbb{R}^{D-1,1},$$

where

$$\mathrm{SO}(D-1,1)_I = \left\{\Lambda \in \mathrm{SO}(D-1,1) \mid \Lambda^0_0 \geq 0\right\}$$

is the component connected to identity. Clearly $\mathbb{R}^{D-1,1}$ is an abelian Lie group, so we are in the setting of Mackey's theorem.

Definition 6.1. The set of generators $\{X_i\}$ of a Lie algebra $\mathfrak{g}$ is the set of elements of $\mathfrak{g}$ such that the smallest subalgebra containing $\{X_i\}$ is $\mathfrak{g}$.

The Lie algebra of $\mathrm{ISO}(D-1,1)$ is denoted by $\mathfrak{iso}(D-1,1)$. It can be shown that $\mathfrak{iso}(D-1,1)$ is generated by $\{M_{\mu\nu}, P_\rho\}$, $\mu,\nu,\rho = 0,1,\ldots,D-1$, and the generators satisfy the *Poincaré algebra*:

$$\begin{aligned}
&iM_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\sigma\mu}M_{\rho\nu} + \eta_{\sigma\nu}M_{\rho\mu} \\
&i\,[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho, \\
&i\,[P_\mu, P_\rho] = 0.
\end{aligned}$$

Two subalgebras of $\mathfrak{iso}(D-1,1)$ are clearly visible: The Lorentz algebra generated by $\{M_\mu\}$ and the subalgebra $\mathbb{R}^D$ generated by $\{P_\mu\}$. The second commutator says that $\mathbb{R}^D$ is an ideal of $\mathfrak{iso}(D-1,1)$. Infact it can be shown that

$$\mathfrak{iso}(D-1,1) = \mathfrak{so}(D-1,1) \ni \mathbb{R}^D$$

where $\ni$ is the semidirect sum which we will not define here and hence will not be used. Let us now consider the Minkowski space.

Definition 6.2. By a pseudo-Riemannian manifold (M,g), we mean a smooth manifold M endowed with a pseudo-Riemannian metric g. If $\mathrm{Diff}(M)$ is the diffeomorphism group of M then we define the automorphism group of M by

$$\mathrm{Aut}(M) = \{\phi \in \mathrm{Diff}(M) \mid \phi^* g = g\}$$

where ϕ^* is the pullback of ϕ.

6.1 Projective Representations

Let $|\Psi\rangle$ be a state in Hilbert space $\mathcal{H}$. Note that any two states $|\Psi\rangle$ and $|\Phi\rangle$ which are nonzero and related by

$$|\Psi\rangle = \lambda|\Phi\rangle \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\} \tag{6.1}$$

are the same quantum mechanical states. So it is pertinent to consider the quotient space of $\mathcal{H}^* = \mathcal{H}\backslash\{0\}$ as $\mathbb{P}(\mathcal{H}) := \mathcal{H}^*/\sim$ where $|\Psi\rangle \sim |\Phi\rangle$ if and only if (6.1) is true. The quotient space $\mathbb{P}(\mathcal{H})$ is called the *projectivised Hilbert space.* Recall that the probability amplitude of transition from $|\Psi\rangle$ to Φ is given by

$$p(|\Psi\rangle, |\Phi\rangle) = \frac{\langle\Psi \mid \Phi\rangle}{\langle\Psi \mid \Psi\rangle\langle\Phi \mid \Phi\rangle}.$$

In the quotient topology on $\mathbb{P}(\mathcal{H})$, p induces a continuous map[14] on $\mathbb{P}(\mathcal{H})$ which we denote by $\widetilde{p}$. A homeomorphism $T : \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$ satisfying

$$\widetilde{p}(T[\Psi], T[\Phi]) = \widetilde{p}(|\Psi\rangle, |\Phi\rangle)$$

where $[\Phi], [\Psi]$ are equivalence classes in $\mathbb{P}(\mathcal{H})$, is called a *projective automorphism.* The set of all such maps, denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, is a group called *projective automorphism group.* The action of this group on $\mathbb{P}(\mathcal{H})$ leaves transition probabilities invariant. We have the following theorem regarding the automorphism group of the Minkowski space.

Theorem 6.3. [2, Lemma 13.1] *The automorphism group of the Minkowski space* $\mathbb{R}^{D-1,1}$ *is the Poincaré group* $\mathrm{ISO}(D-1,1)$.

Now consider a particle in the Minkowski space $\mathbb{R}^{D-1,1}$. By above, the symmetry group of this space is precisely the Poincaré group. Let two observers $\mathcal{O}$ and $\mathcal{O}'$, related by $\Lambda \in \mathrm{ISO}(D-1,1)$, measure the quantum mechanical particle. In general, their measurement result will reveal different states, say $[\Psi]$ and $[\Psi']$ respectively. Thus physically one expects that transition probabilities in $\mathcal{O}$ and $\mathcal{O}'$ be same. This means that the two states must be related by some projective automorphism:

$$[\Psi] = T_\Lambda[\Psi'], \quad \text{for some} \quad T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H})).$$

If $\mathcal{O} = \mathcal{O}'$ then $T_\Lambda = Id$ and we should have $T_\Lambda = T_{Id} = Id \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. Lastly if a third observer $\mathcal{O}''$, related to $\mathcal{O}'$ by Γ, measures the state then we must impose $T_\Lambda \circ T_\Gamma = T_{\Lambda\circ\Gamma}$. Thus the change of frame induces a representation $\Pi : \mathrm{ISO}(D-1,1) \longrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This is called the *projective representation.*

6.2 Wigner's Idea of Elementary Particles

The representation $(\Pi, \mathcal{H})$ of the Poincaré group is called irreducible if the only non-trivial closed invariant subspace of $\mathcal{H}$ is $\mathcal{H}$. That is $\Pi(\mathrm{ISO}(D-1,1))(V) \subseteq V$ if and only if $V = \mathcal{H}$. The closed condition is technical: we want the invariant subspace to be

[14] it is a standard result in quotient topology. See for example Topology by Munkres.

a Hilbert space in its own right which is not automatically true in infinite dimensional Hilbert space unless the subspace is closed.

Wigner suggested that the irreducible projective representations of the Poincaré group correspond to elementary particles within the quantum system under consideration. Wigner's argument was as follows: an elementary particle in a quantum mechanical system is a vector in $\mathbb{P}(\mathcal{H})$. As discussed, different observers will see different vectors in $\mathbb{P}(\mathcal{H})$ corresponding to the elementary particle. All these vectors must be related by some projective automorphism. The set of all these vectors constitutes $\mathrm{ISO}(D-1,1)$-invariant subspace of $\mathbb{P}(\mathcal{H})$ and hence we obtain a subrepresention of $(\Pi, \mathcal{H})$. This subrepresentation can be thought of as a subsystem which is elementary if it is irreducible (otherwise it will have more smaller subsystems). This reduces the problem of determining all relativistic free particles in Minkowski spacetime to the mathematical task of finding all irreducible projective representations of the Poincaré group.

6.3 Using Mackey's Theorem

We have to find all projective irreducible representations of the Poincaré group. We need to pass on to the universal cover of the Poincaré group. Recall that the universal cover of Lorentz group is the spin group $\mathrm{Spin}(D-1,1)$. Let $\Phi : \mathrm{Spin}(D-1,1) \longrightarrow \mathrm{SO}(D-1,1)$ be the covering map. One can then show that the universal cover of the Poincaré group is $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$ where now in the semidirect product, $\mathrm{Spin}(D-1,1)$ acts on $\mathbb{R}^{D-1,1}$ via a the covering map:

$$A \cdot v := \Phi(A)v, \quad A \in \mathrm{Spin}(D-1,1), v \in \mathbb{R}^{D-1,1},$$

where the later action $\Phi(A)v$ is the usual action by matrix multiplication. The following theorem will be of utmost importance.

Theorem 6.4. [2, Theorem 14.3, Corollary 14.4] *Every irreducible unitary representation of* $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$ *in a complex Hilbert space* $\mathcal{H}$ *naturally induces an irreducible projective representation of the Poincaré group in* $\mathcal{H}$*. Moreover every projective representation of the Poincaré group lifts to a unique unitary representation of* $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$ *and the latter is irreducible if and only if the formar is irreducible. This sets up a bijective correspondence between the projective irreducible representations of the Poincaré group and unitary irreducible representations of* $\mathrm{Spin}(1, D-1) \ltimes \mathbb{R}^{1,D-1}$

We stress that this is not true in general. It has to do something with the second cohomology group being trivial. We shall not comment on this any further. Thus our strategy will be to classify all irreducible unitary representations of $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$

using Mackey's theorem and then use the above theorem to get all irreducible projective representations of the Poincaré group. Before we go on the task of using Mackey's theorem, let us see how to get a projective representation of the Poincaré group from unitary representation of the universal cover.

Indeed suppose $\Pi : \mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1} \longrightarrow \mathrm{GL}(\mathcal{H})$ be a representation. Now it is known that $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$ is the double cover of the Poincaré group with the kernel $\{e, e'\}$ of the covering map Φ isomorphic to $\mathbb{Z}_2$. It also happens to be the center of $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$. Thus by Schur's lemma, kernel acts as scalars. Suppose $\Pi(e) = \mathbb{1}_{\mathcal{H}}$ and $\Pi(e') = \lambda \mathbb{1}_{\mathcal{H}}$ for some $\lambda \in \mathbb{C}^*$. It is clear that $\Pi(e) = \Pi(e') = \mathbb{1}_{\mathbb{P}(\mathcal{H})}$ on the projectivised Hilbert space. Thus the representation Π induces a projective representation Π' of $\mathrm{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$. Now using the inverse of the covering map and the fact that Π' is trivial on kernel of Φ, it is easy to see that the the map $\widetilde{\Pi} = \Pi' \circ \Phi^{-1}$ furnishes a projective representation of the Poincaré group on the Hilbert space $\mathcal{H}$. In this way we get a projective representation of the Poincaré corresponding to each unitary representation of the universal cover and the above theorem makes sure that these are all the projective representations of the Poincaré group.

Now the next task is to find orbits and stabilisers. We first reduce this to a simpler problem.

Theorem 6.5. [1, Proposition 4.8] *The map $T : \mathbb{R}^{D-1,1} \longrightarrow \widehat{\mathbb{R}}^{D-1,1}$ given by $T(v)(x) = e^{2g(v,x)}$, where $g(v,x) = v \cdot x$ is the Minkowski metric of the Minkowski space, is an isomorphism. Moreover $v \in \mathbb{R}^{D-1,1}$ is $\mathrm{Spin}(D-1,1)$-stable ($\mathrm{Spin}(D-1,1)$ acts on $\mathbb{R}^{D-1,1}$ via automorphism coming from the semidirect product structure) if and only if $\chi = T(v)$ is $\mathrm{Spin}(D-1,1)$-stable in the action defined in Eq. 5.1.*

This theorem says that if $\chi \in \widehat{\mathbb{R}}^{D-1,1}$ then there exists $v \in \mathbb{R}^{D-1,1}$ such that $T(v) = \chi$ and that for $A \in \mathrm{Spin}(D-1,1)$

$$A \cdot v = \Phi(A)v \longleftrightarrow A \cdot \chi = \chi$$

where

$$(A \cdot \chi)(x) = \chi(A^{-1} \cdot x) = \chi(\Phi(A)^{-1}x).$$

Now since Φ is surjective by definition, the problem reduces to finding the orbits a stabilisers of the action on $\mathbb{R}^{D-1,1}$. The computation of orbit is a bit technical, so we just mention the result here for the ease of understanding. The orbits are Labelled by $c \in \mathbb{R}$.

(i) $g(v,v) = v^2 = m^2 > 0$. The orbit is a one-sheeted hyperboloid and the stabiliser is $\mathrm{Spin}(D-2,1) \hookrightarrow \mathrm{Spin}(D-1,1)$.

(ii) $g(v,v) = v^2 = 0$. The orbit is a cone with vertex at the origin – the lightcone of special relativity. The stabiliser is isomorphic to the Euclidean group $\mathrm{ISO}(D-2)$.

(iii) $g(v,v) = v^2 = -m^2 < 0$. The orbit is a two sheeted hyperboloid corresponding to $v^0 > 0$ or $v^0 < 0$. The stability group is $\mathrm{Spin}(D-1)$.

Note that $\mathrm{SO}(D-1,1)_I$ also preserves the sign of v^0 if $v^2 < 0$. We now need to classify

Gender	Orbit	Little Group	Unitary Representation
$v^2 = -m^2$	Mass shell	$\mathrm{Spin}(D-1)$	Massive
$v^2 = m^2$	Hyperboloid	$\mathrm{Spin}(D-2,1)$	Tachyonic
$v^2 = 0$	Lightcone	$\mathrm{ISO}(D-2)$	Massless
$v = 0$	Origin	$\mathrm{Spin}(D-1,1)$	Zero Momentum

Table 2: Orbits and Stabilisers

all unitary irreducible representations of the stability groups. For physical applications, Tachyonic representations are irrelevant. So we only have to find unitary irreducible representations of the remaining three. We analyse them case by case.

(i) For the zero momentum representation corresponding to orbit $v = 0$, the stability group is $\mathrm{Spin}(D-1,1)$ which is simply connected but not compact with simple Lie algebra and hence its unitary irreducible representations are all infinite dimensional. Hence analysing its irreducible representations is hard and we shall not pursue it any further.

(ii) For the massive representation, the stability group is $\mathrm{Spin}(D-1)$ which is compact and simply connected, so all its unitary irreducible representations are finite dimensional. Thus we can classify its unitary irreducible representations by going to its Lie algebra and using weights and roots. We omit all further details of this classification as it deserves another set of discussions altogether.

(iii) The massless representation corresponds to orbit $v^2 = 0$ whose stability group is $\mathrm{ISO}(D-2) = \mathrm{SO}(D-2) \ltimes \mathbb{R}^{D-2}$. We can again use Mackey's theorem here to get all unitary irreducible representations. The $\mathrm{SO}(D-2)$-action breaks $\mathbb{R}^{D-2}$ into two orbits:

- A sphere of radius $\mu^2 > 0$ with stability group $\mathrm{SO}(D-3)$ is compact. The unitary irreducible representation corresponding to this orbit is called *Infinite spin*.

- The origin with stability group $\mathrm{SO}(D-3)$ which is again compact. The unitary irreducible representation corresponding to this orbit is called *Helicity*.

We will again not delve into the details.

6.4 Four Dimensional Spacetime

We now specialise to the $D = 4$ case. This case is standard for quantum field theory and Particle physics applications. In this case the massive and massless representations are well known:

(i) $v^2 = -m^2 < 0$, the stability group is $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$ which is a compact, simply connected Lie group. All its unitary irreducible representations are labelled by $s \in \frac{1}{2}\mathbb{N} \cup \{0\}$ called *spin* of particle. The representation is realised on $\mathbb{C}^{2s+1}$ and hence it is $2s + 1$ dimensional.

(ii) $v^2 = 0$ with stability group $\mathrm{SO}(2) \ltimes \mathbb{R}^2$. The orbits of $\mathrm{SO}(2)$-action on R^2 has two orbits.

- a circle with radius $r > 0$ with stability group $\mathrm{SO}(1)$. It gives rise to the trivial representation.

- the origin with stability group $\mathrm{SO}(2)$. The irreducible representation of the abelian group $\mathrm{SO}(2)$ are all one dimensional. They are Labelled by $n \in \mathbb{Z}$. It is customary to write $n = 2s$ with $s \in \frac{1}{2}\mathbb{Z}$. The modulus $|s|$ is called the spin of the representation and the sign of s is called the polarisation.

6.4.1 The Mass-Squared Parameter

The orbits are labelled by m^2 which has a natural physical interpretation of mass squared of the particle. We will investigate this in this section. To do so, we first introduce the universal enveloping algebra and Casimir element.

Definition 6.6. An algebra A is a vector space over a field k along with a multiplication map $\mu : A \times A \longrightarrow A, \quad (a, b) \longmapsto \mu(a, b) =: a \cdot b$ satisfying: for all $a, b \in A$ and $r \in k$,

1. (Associativity) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
2. (Distributivity) $a \cdot (b + c) = a \cdot b + a \cdot c$.
3. (Homogeneity) $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$.

A is called an algebra with identity if there exists an identity with respect to the multiplication operation μ.

Due the associativity, we sometimes call A an associative algebra. The universal enveloping algebra is defined by the universal property as below:

Theorem 6.7. [5, Theorem 9.7] *Let $\mathfrak{g}$ be a Lie algebra, then there exists an associative algebra $\mathcal{U}(\mathfrak{g})$ with identity together with a linear map $i : \mathfrak{g} \longrightarrow \mathfrak{U}(g)$ such that the following properties hold:*

(i) For every $X, Y \in \mathfrak{g}$, we have

$$i\left([X, Y]\right) = i(X)i(Y) - i(Y)i(X).$$

(ii) $\mathcal{U}(\mathfrak{g})$ is generated by the set $i(X)$ $X \in g$ in the sense that the smallest subalgebra with identity of $\mathcal{U}(\mathfrak{g})$ containing every $i(X)$ is $\mathcal{U}(\mathfrak{g})$.

(iii) (Universal property) If A is an associative algebra with identity linear map $j : \mathfrak{g} \longrightarrow A$ such that $j\left([X, Y]\right) = j(X)j(Y) - j(Y)j(X)$, then there exists a unique algebra homomorphism $\phi : \mathcal{U}(\mathfrak{g}) \longrightarrow A$ with $\phi(1_{\mathfrak{g}}) = 1_A$ and $\phi(i(X)) = j(X)$ $\forall$ $X \in \mathfrak{g}$. A pair $(\mathcal{U}(\mathfrak{g}), i)$ with the above three properties is called the universal enveloping algebra*.*

The following proposition will be useful. Let $\mathrm{End}(V)$ denote the vector space of linear operators from V to V. It is easy to show that $\mathrm{End}(V)$ is an associative algebra under composition.

Proposition 6.8. *If $\pi : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is a representation of a Lie algebra (possibly infinite dimensional), there is a unique algebra homomorphism $\widetilde{\pi} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$ such that $\widetilde{\pi}(1) = \mathbb{1}_V$ and $\widetilde{\pi} \circ i = \pi$.*

Proof. Immediate from the Universal Property of universal enveloping algebra. □

Theorem 6.9. [5, Theorem 9.10](Poincaré-Birkhoff-Witt (PBW) Theorem) *If $\mathfrak{g}$ is a finite dimensional Lie algebra with basis $X_1, \ldots, X_k$, then elements of the form*

$$i\left(X_1\right)^{n_1} i\left(X_2\right)^{n_2} \cdots i\left(X_k\right)^{n_k}$$

where each n_k is a nonnegative integer, span $\mathcal{U}(\mathfrak{g})$ and are linearly independent. In particular, the elements $i\left(X_1\right), \ldots, i\left(X_k\right)$ are linearly independent, meaning that the map $i : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ is injective.

Definition 6.10. Let $\mathfrak{g}$ be a Lie algebra with the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The Casimir elements of $\mathfrak{g}$ are homogeneous polynomials in the generators of $\mathfrak{g}$ which commute with all elements of $\mathfrak{g}$ hence with all elements of $\mathcal{U}(\mathfrak{g})$ (by PBW theorem). Thus the Casimir elements lie in the center $Z(\mathcal{U}(\mathfrak{g}))$ of $\mathfrak{g}$.

As a consequence of Schur's Lemma, we have the following theorem.

Theorem 6.11. *Let $\mathfrak{g}$ be a Lie algebra. For any irreducible representation, the Casimirs of $\mathfrak{g}$ act as scalars.*

Proof. Immediate from Schur's Lemma. $\square$

For the Lie algebra $\mathfrak{so}(D-1,1)$ of the Lorentz group, the quadratic Casimir elements are

$$\mathcal{C}_2(\mathfrak{so}(D-1,1)) = \frac{1}{2} M^{\mu\nu} M_{\mu\nu}$$

where we have summation over μ, ν is as follows:

$$M^{\mu\nu} M_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma} M^{\mu\nu} M^{\rho\sigma} \quad \text{(sum over repeated index)},$$

where $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{dia}(-1, \underbrace{1, 1, \ldots 1}_{D-1})$ is the Minkowski metric. For the Poincaré algebra $\mathfrak{iso}(D-1,1)$, the quadratic Casimir element is

$$C_2(\mathfrak{iso}(D-1,1)) = P^\mu P_\mu = P^2$$

and the quartic Casimir elements are

$$C_4(\mathfrak{iso}(D-1,1)) = \frac{1}{2} P^2 M_{\mu\nu} M^{\mu\nu} + M_{\mu\rho} P^\rho M^{\mu\sigma} P_\sigma.$$

Suppose now that Π is a representation of the Poincaré group on a Hilbert space $\mathcal{H}$. Recall that it induces a representation π of the Lie algebra on the $\mathcal{H}$. Moreover recall that

$$\Pi\left(e^{iX}\right) = e^{i\pi(x)}$$

for every X in the Lie algebra (we used the physicists convention of putting an i). In particular for $P^\mu \in \mathfrak{iso}(D-1,1)$, we have

$$\Pi\left(e^{iP_\mu}\right)|\psi\rangle = e^{i\pi(P_\mu)}|\psi\rangle = \pi(T_\mu)|\psi\rangle,$$

where T_μ is the unit translation. The last equality comes from the fact that P_μ generates translations in spacetime. We now apply Mackey's theorem to the universal cover of the Poincaré group, i.e. to $G = \text{Spin}(D-1,1) \ltimes \mathbb{R}^{D-1,1}$. Any unitary irreducible representation of G is of induced type. Thus we can identify $\mathcal{H}$ with the space of continuous functions on G with some transformation property with respect to the stabiliser subgroup:

$$\mathcal{H} = \text{Ind}_{G_\chi}^{G}(\xi \otimes \chi) = \{\phi : G \longrightarrow V_\xi \mid \phi((h,n)g) = \chi(n)\xi(h)\phi(g); \phi \text{ is continuous}\}$$

$G_\chi := \mathrm{Spin}(D-1,1)_\chi \ltimes \mathbb{R}^{D-1,1}$ with $\mathrm{Spin}(D-1,1)_\chi$ being the stabiliser of χ and (ξ, V_ξ) is a representation of $\mathrm{Spin}(D-1,1)_\chi$.

Using Theorem 3.26 (iii), we get

$$\begin{aligned}(\pi(P_\mu)\phi)(g) &= \left[\left(\frac{d}{dt}\Pi\left(e^{tP_\mu}\right)\bigg|_{t=0}\right)\phi\right](g) \\ &= \frac{d}{dt}\left(\phi\left(g \cdot e^{tP_\mu}\right)\right)\bigg|_{t=0}\end{aligned}$$

where we used the fact that evaluation of ϕ at g is linear and hence commutes with time derivative. Now by the property of universal enveloping algebra, the representation π lifts to a homomorphism $\widetilde{\pi} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{End}(\mathcal{H})$ of associative algebras. Then for $X \in \mathfrak{g}$ and the identity $e \in G$,

$$\begin{aligned}(\pi(X)\phi)(e) &= \frac{d}{dt}\phi(e\exp(tX))\bigg|_{t=0} \\ &= \frac{d}{dt}\phi(\mathrm{ext}(tX)e)\bigg|_{t=0} \\ &= \frac{d}{dt}\chi(\exp(tX))\xi(e)\phi(e)\bigg|_{t=0} \\ &= \frac{d}{dt}\chi(\exp(tX))\phi(e) \\ &= \chi_*(X)\phi(e)\end{aligned}$$

where χ_* is the Lie algebra homomorphism corresponding to the Lie group homomorphism χ. Using these we have

$$\widetilde{\pi}\left(P^\mu P_\mu\right) = -\pi\left(P^0\right)^2 + \sum_{i=1}^{D-1}\pi(P^i)^2$$

which gives

$$\left(\widetilde{\pi}\left(P^\mu P_\mu\right)\phi\right)(e) = \left(-\chi_*(P^0)^2 + \sum_{i=1}^{D-1}\chi_*(P^i)^2\right)\phi(e). \tag{6.2}$$

But now by Theorem 6.11, since $(\Pi, \mathcal{H})$ is an irreducible representation, $P^\mu P_\mu$ acts as scalar. So we have

$$(\widetilde{\pi}(P^\mu P_\mu)\phi = \lambda\phi, \quad \text{for some } \lambda \in \mathbb{C}.$$

So we evaluate it at identity to get λ. Eq. 6.2 gives

$$\lambda = -\chi_*\left(P^0\right)^2 + \sum_{i=1}^{D-1}\chi_*\left(P^i\right)^2.$$

Now recall that for character χ, there exists $v \in \mathbb{R}^{D-1,1}$ such that

$$\chi(\omega) = e^{ig(v,\omega)}.$$

so

$$\chi_*(X) = ig(v, X),$$

so that $\chi_*\left(P^\mu\right) = ig\left(v, P^\mu\right)$. Note that $P^\mu \in \mathbb{R}^D$ is just the μ^{th} basis vector multiplied by $-i$ (due to i in e^{iP^μ}). Thus $ig\left(v, P^\mu\right) = v^\mu$, so we have

$$\lambda = -\left(v^0\right)^2 + \sum_{i=1}^{D-1}\left(v^i\right)^2 = g(v, v) = m^2.$$

Thus $\widetilde{\pi}\left(P^\mu P_\mu\right) = m^2$. In physics literature, $\pi\left(P^\mu P_\mu\right)$ is simply denoted by $P^\mu P_\mu$ and we identify $P^\mu P_\mu = m^2$. We recognise this as the total relativistic energy. Thus m is identified with the mass of the particle.

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